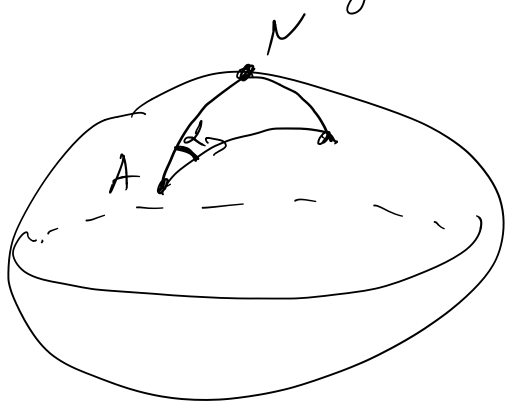
Géométrie sphérique

géo inversive ⟷ géo affine

géo sphérique   géo euclidienne

But: savoir naviguer sur mer

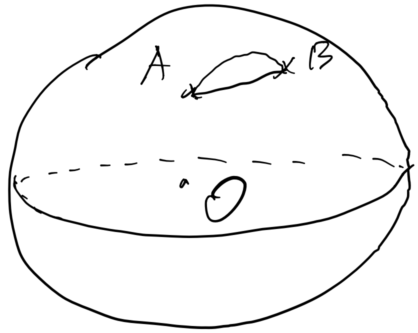


Cap pour aller de A vers B

⇝ trigonométrie des triangles sur S^2

Loi des cosinus
Loix des sinus
Cas d'égalités des triangles

Métrique sur S^2



$d(A,B)$

$\cap$

$\mathbb{R}^3$ $\quad$ AB distance dans $\mathbb{R}^3$

Souci : courbe sur S^2 allant de A vers B

le chemin le plus court $\rightsquigarrow$ sa longueur soit $d(A,B)$

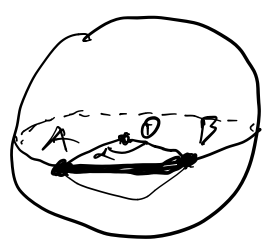
Chemin le + court $A \rightsquigarrow B$

plan contenant O, A, B

$\underbrace{\cap S_2}$

grand cercle

arc de cercle $\widehat{AB}$



$R=1$

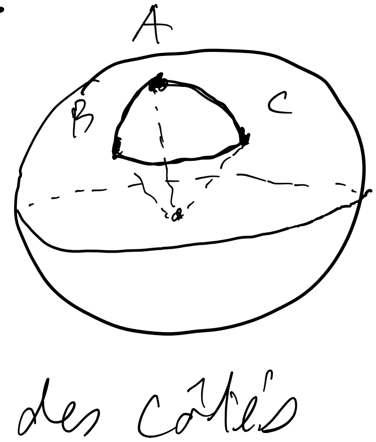
Def : $d(A,B) = \alpha = \widehat{AOB}$

$$\vec{OA} \cdot \vec{OB} = \|\vec{OA}\| \cdot \|\vec{OB}\| \cdot \cos(\alpha) = \cos\alpha$$

$$d(A,B) = \arccos(\vec{OA} \cdot \vec{OB}) \in [0, \pi]$$

Le chemin le + court $A \rightsquigarrow B$ est de longueur $d(A,B)$

Triangles



A, B, C

côtés = arcs de grands cercles

- longueurs des côtés OK
- angles aux sommets

courbe $\rightsquigarrow$ plan tangent à S^2 en M

$\mathscr{C}_M \qquad\qquad T_M$

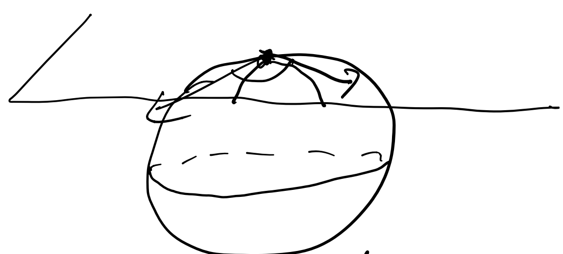
⇝ vecteur tangent à la courbe C_M

2 courbes passant par M ⇝ 2 vecteurs tangents

$\Rightarrow$ plan T_M

T_M : muni du produit scalaire euclidien

⇝ angles entre les vecteurs



⇝ l'angle au sommet ⇝ 3 angles

a, b, c = longueurs

α, β, γ = angles

Relations

géo euclidienne $\alpha+\beta+\gamma=\pi$

géo sphérique

Thm : (Formule de Girard)

$\alpha+\beta+\gamma-\pi=$ aire du triangle

Rem : cas particulier de la formule de Gauss-Bonnet

variété Riemannienne dim 2

$$\alpha+\beta+\gamma-\pi=\int_T K\,d\sigma$$

$K=$ courbure

$d\sigma=$ mesure canonique

ex : plan euclidien $K=0 \leadsto \alpha+\beta+\gamma=\pi$

S^2 $K=1 \leadsto$ aire

Idem

Fuseau:

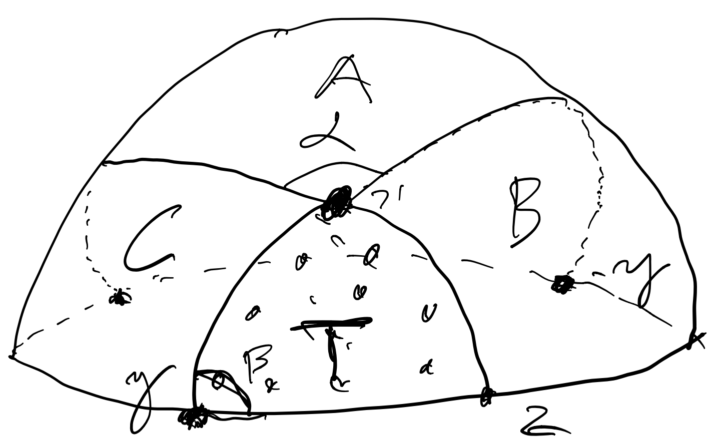
(tranche de mandarine)

aire du fuseau:

coordonnées sphériques

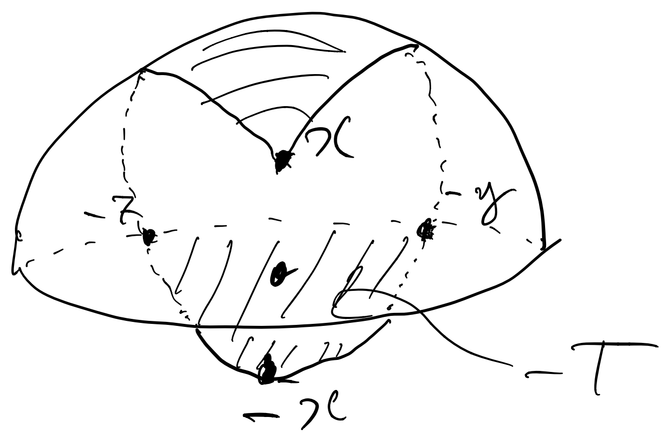
$x = \cos\theta$
$y = \sin\theta$
$z = \sin\varphi$

$\leadsto$ aire: $\displaystyle\int_0^{\alpha}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi\right) d\theta = \underline{\underline{2\alpha}}$



Fuseaux : $\begin{cases} T \cup B = \text{fuseau}(\pm y, \beta) \\ T \cup C = \text{fuseau}(\pm z, \gamma) \\ (-T) \cup A = \text{fuseau}(\pm x, \alpha) \end{cases}$

$-T : \rightsquigarrow -y, -z, \underline{-x}$



$$\underbrace{2\pi}_{\text{aire demi-sphère}} = \mathcal{A}(T) + \mathcal{A}(A) + \mathcal{A}(B) + \mathcal{A}(C)$$

$\mathcal{A}(?) =$ aire de ?

$$\mathcal{A}(A) + \underbrace{\mathcal{A}(-T)}_{=\mathcal{A}(T)} = 2\alpha$$

$$\mathcal{A}(B) + \mathcal{A}(T) = 2\beta$$

$$\mathcal{A}(C) + \mathcal{A}(T) = 2\gamma$$

$$2\pi = \mathcal{A}(T) + 2\alpha - \mathcal{A}(T) + 2\beta - \mathcal{A}(T) + 2\gamma - \mathcal{A}(T)$$

$$\cancel{2}\mathcal{A}(T) = \cancel{2}(\alpha+\beta+\gamma) - \cancel{2}\pi$$

$$\boxed{\begin{array}{l}\underline{\text{Rem}}: \text{ la somme des angles d'un polygone sphérique (sur } S^2\text{)} \\ \displaystyle\sum_{i=1}^{n} \alpha_i - (n-2)\pi = \text{aire du polygone}\end{array}}$$

Relations : $a, b, c, \alpha, \beta, \gamma$

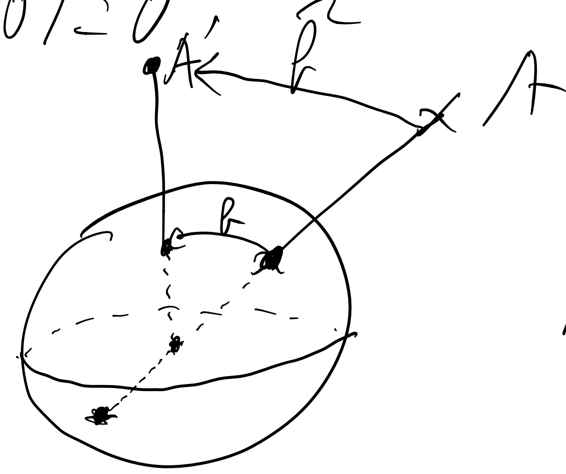
Groupe des isométries (de S^2 qui conserve les distances)

$A, B \in S^2$

f : $d(A,B) = 2 \iff B = -A$

$\rightsquigarrow \tilde{f}(\underset{\neq 0}{A}) = f\left(\frac{A}{\|\vec{OA}\|}\right) \cdot \|\vec{OA}\|$

$\tilde{f}(0) = 0$



$\forall M \in S^2$

$\underline{f(-M) = -f(M)}$

$\tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$ conserve les distances

$0 \mapsto 0$

$\rightsquigarrow \tilde{f} \in O_3(\mathbb{R})$

et réciproquement $\left| \begin{array}{l} g \in O_3(\mathbb{R}) \\ g_{|S^2} \ \underline{\text{OK}} \end{array} \right.$

Notre groupe : $\underline{\underline{O_3(\mathbb{R})}} \ni f \mapsto f_{|S^2}$

$\cdot$ $SO_3(\mathbb{R}) \rightsquigarrow$ axe + angle

homothétie

3 coordonnées / ~~bond~~ 1 facteur

2 paramètres réels

1 paramètre réel

$O_3(\mathbb{R}) \longleftrightarrow$ 3 paramètres réels

Triangle : $a, b, c, \alpha, \beta, \gamma \rightsquigarrow$ | 3 relations indépendantes

Rem : $\forall$ A, B / A', B' $\in S^2$ $d(A,B) = d(A',B')$

$\Rightarrow \exists f : A \mapsto B$ / $A' \mapsto B'$

$O_3(\mathbb{R})$ $\nearrow$

$\rightsquigarrow$ la distance est un invariant
c'est le seul invariant associé à
1 couple de points

$\boxed{\text{Formules}}$

$$\begin{cases} \cos a = (\vec{y} \mid \vec{z}) \\ \cos b = (\vec{z} \mid \vec{x}) \\ \cos c = (\vec{x} \mid \vec{y}) \end{cases}$$

définition de a, b, c

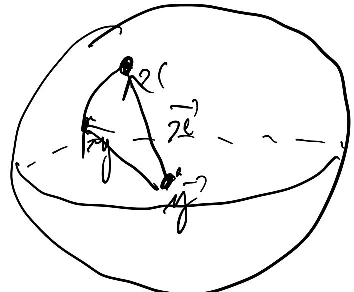
$$\begin{cases} \cos \alpha = (\vec{x} \wedge \vec{y} \mid \vec{x} \wedge \vec{z}) \\ \cos \beta = (\vec{y} \wedge \vec{z} \mid \vec{y} \wedge \vec{x}) \\ \cos \gamma = (\vec{z} \wedge \vec{x} \mid \vec{z} \wedge \vec{y}) \end{cases}$$

$\leftarrow$ OK

α

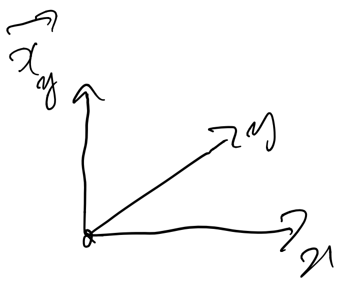
Dem :

Notation



$\vec{x}_y$: $(\vec{x}, \vec{y}) \xrightarrow[\text{Schmidt}]{\text{orthonormalisation}}$

$\vec{x}$



$\vec{x}_y = \alpha \vec{x} + \beta \vec{y}$

$\cancel{\|\vec{x}_y\| = 1 = \alpha^2 + \beta^2 + 2\alpha\beta \langle \vec{x} | \vec{y} \rangle}$

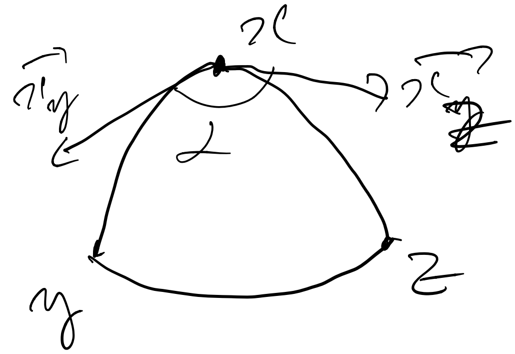
$\cancel{\langle \vec{x}_y | \vec{x} \rangle = 0 = \alpha + \beta \langle \vec{x} | \vec{y} \rangle}$

ou alors $\vec{x}'_y = \vec{y} - \lambda \vec{x}$

$\langle \vec{x}'_y | \vec{x} \rangle = \langle \vec{x} | \vec{y} \rangle - \lambda = 0$

$\vec{x}'_y = \vec{y} - \langle \vec{x} | \vec{y} \rangle \vec{x}$

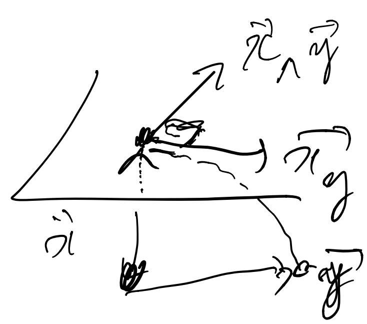
$\vec{x}_y = \dfrac{\vec{x}'_y}{\|\vec{x}'_y\|}$



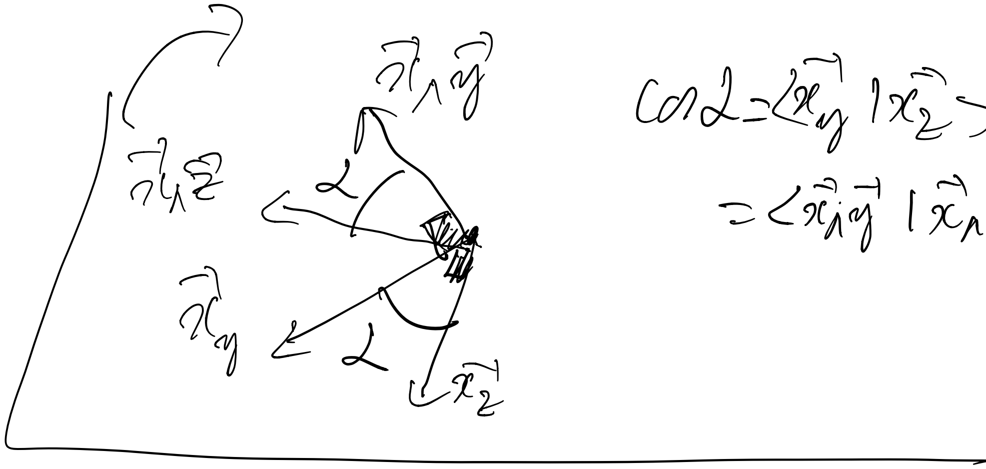
$$\cos\alpha = \langle \vec{x_y} \mid \vec{x_z} \rangle$$

$\vec{x} \wedge \vec{y} \in$ plan tangent à S^2 en $\vec{x}$
directᵗ orthogonal à $\vec{x_y}$

plan tangent



$\vec{x} \wedge \vec{z} \in$ plan tgt à S^2 en $\vec{x}$
directᵗ orthogonal à $\vec{x_z}$



$\cos\alpha = \langle \vec{x_y} \mid \vec{x_z} \rangle$

$= \langle \vec{x} \wedge \vec{y} \mid \vec{x} \wedge \vec{z} \rangle$

idem pour β et γ

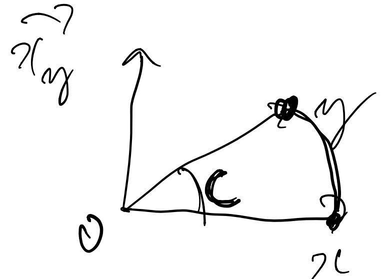
Thm : (Al Kashi sphérique)

$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos\beta$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos\gamma$$

dem : $\cos a = \langle \vec{y} \mid \vec{z} \rangle$



$$\begin{cases} \vec{y} = \cos c\, \vec{x} + \sin c\, \vec{x_y} \\ \vec{z} = \cos b\, \vec{x} + \sin b\, \vec{x_z} \end{cases}$$

$$\langle \vec{y} \mid \vec{z} \rangle = \langle (\cos c\, \vec{x} + \sin c\, \vec{x_y}) \mid (\cos b\, \vec{x} + \sin b\, \vec{x_z}) \rangle$$

$$\cos a = \cos c \cos b + \sin c \sin b \underbrace{\langle \vec{x_y} \mid \vec{x_z} \rangle}_{= \cos \alpha}$$

$$\langle \vec{x} \mid \vec{x_y} \rangle = \langle \vec{x} \mid \vec{x_z} \rangle = 0$$

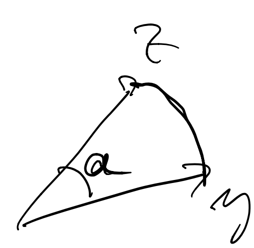
idem pour les autres $\square$

Def : Triangle polaire

$$A, B, C \rightsquigarrow A^*, B^*, C^*$$

$$\vec{x} \quad \vec{y} \quad \vec{z} \qquad \vec{x}^* \quad \vec{y}^* \quad \vec{z}^*$$

$$\sin a\, \vec{x}^* = \vec{y} \wedge \vec{z}$$



ainsi $\vec{y}^* = \vec{z} \wedge \vec{x}$

ainsi $\vec{z}^* = \vec{x} \wedge \vec{y}$

Notation : $\Delta(ABC) = 1$ si $(\vec{x}, \vec{y}, \vec{z})$ direct

$= -1$ ——— indirect

Lemme : $(\vec{x}^*)^* = \Delta\, \vec{x}$

$(\vec{y}^*)^* = \Delta\, \vec{y}$

$(\vec{z}^*)^* = \Delta\, \vec{z}$

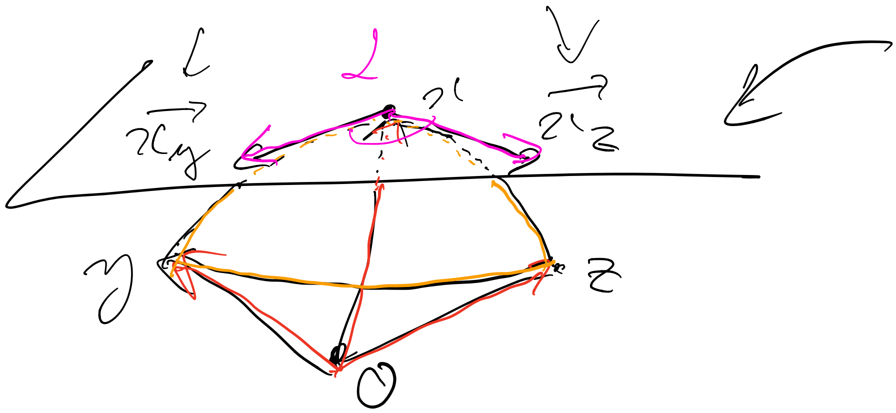
dem : ainsi ainsi $\vec{y}^* \wedge \vec{z}^* = (\vec{z} \wedge \vec{x}) \wedge (\vec{x} \wedge \vec{y})$

formule du double produit

$= ((\vec{z} \wedge \vec{x}) \cdot \vec{y})\, \vec{x} - \cancel{(\vec{z} \wedge \vec{x} \cdot \vec{x})\, \vec{y}}$

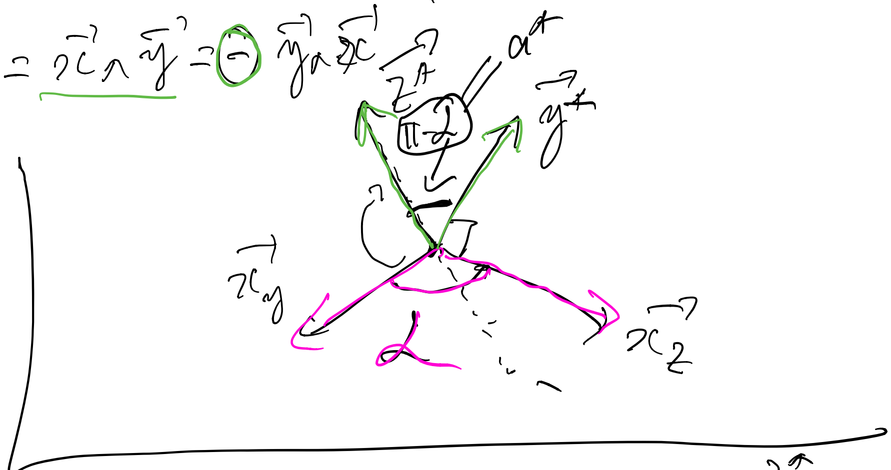
$= \det(\vec{x}, \vec{y}, \vec{z})\, \vec{x}$

idem pour les autres

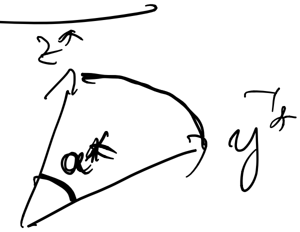


$$\text{mab } y^* = \vec{z} \wedge \vec{x} \quad \perp \vec{x}, \vec{z} \Rightarrow \perp \vec{x_z} \in \langle \vec{x}, \vec{z} \rangle$$

$$\text{sinc } z^* = \vec{x} \wedge \vec{y} = - \vec{y} \wedge \vec{x}$$



$$\langle \vec{z}^* \mid \vec{y}^* \rangle = \cos(\pi - \alpha)$$



$$\cos a^* = -\cos \alpha$$

$$\boxed{a^* + \alpha = \pi}$$

idem pour les autres

$(x^*, y^*, z^*) \to$

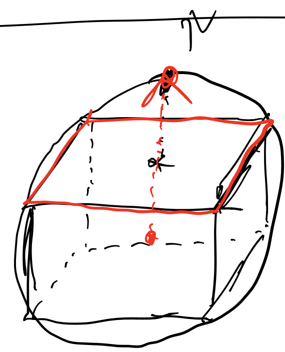
$\cos a^* = \cos b^* \cos c^* + \sin b^* \sin c^* \cos \alpha^*$

$$\boxed{\cos \alpha = -\cos\beta \cos\gamma + \sin\beta \sin\gamma \cos a}$$

les autres idem

$$\boxed{\begin{array}{ll} \underline{\text{Al Kashi}}: & \cos a = \cos b \cos c + \sin b \sin c \cos \alpha \\ \text{Loi } \underline{\text{des cosinus}} & \cos \alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a \end{array}}$$

<u>rem</u>:



inscrit dans S^2

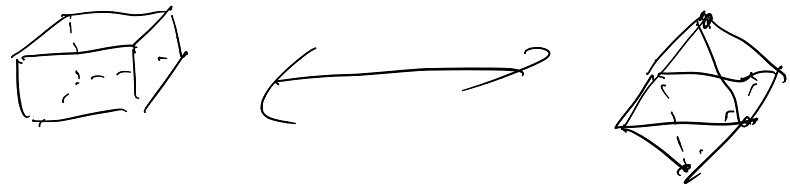
face $\rightsquigarrow$ vecteur normal $\rightsquigarrow$ 1 point de $\mathcal{E}_2$

face $\longrightarrow$ 1

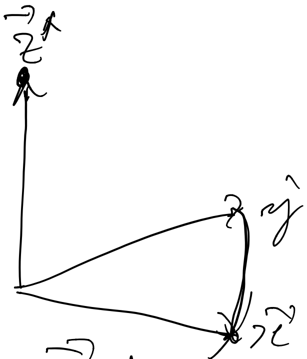
polyèdre $\longrightarrow$ polyèdre dual

<u>cube</u>:

6 faces 8 sommets $\longrightarrow$ 6 sommets, 8 faces



Loi des sinus

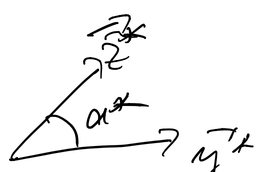


$$\det(\vec{x}^*, \vec{y}^*, \vec{z}^*) = \vec{x}^* \cdot (\vec{y}^* \wedge \vec{z}^*)$$

$$\sin a \, \vec{x}^* = \vec{y} \wedge \vec{z}$$

$$\vec{y}^* \wedge \vec{z}^* = s \underset{\uparrow}{\sin a^*} \, \vec{x}$$

sin de l'angle



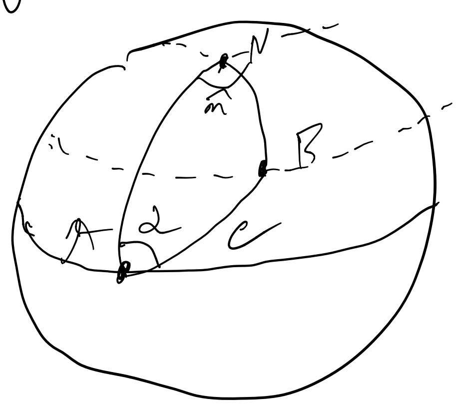
$$\det(\vec{x}^*, \vec{y}^*, \vec{z}^*) = s \, \frac{\sin a^*}{\sin a} \; \vec{x} \cdot (\vec{y} \wedge \vec{z})$$

$$= s \, \frac{\sin a^*}{\sin a} \; \det(\vec{x}, \vec{y}, \vec{z})$$

$$\frac{\sin a^*}{\sin a} = \frac{\det(\vec{x}^*, \vec{y}^*, \vec{z}^*)}{\det(\vec{x}, \vec{y}, \vec{z})} = \frac{\sin \alpha}{\sin a}$$

$$\frac{\sin a^*}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}$$

Navigation



$c = d(A, B)$

position de $\left| \begin{array}{l} A \\ B \end{array} \right.$

* différence entre les 2 longitudes de B et A $\Rightarrow \hat{n}$
* $d(A, N) = a, \quad d(B, N) = b$

Al-Kashi pour les côtés

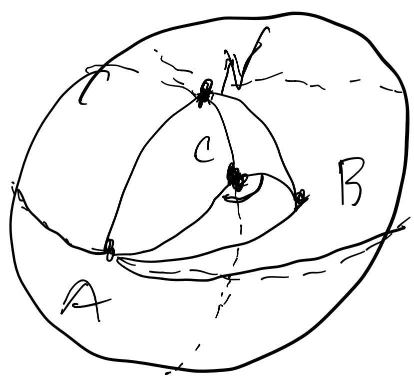
$$\boxed{\cos c} = \cos a \cos b + \sin b \sin a \cos \hat{n}$$

$\rightsquigarrow$ $c = d(AB)$

$\hat{A}$ Loi des sinus

$$\frac{\sin \hat{n}}{\sin c} = \frac{\sin \alpha}{\sin b} \rightsquigarrow \alpha = ?$$

Triangulation



dans le triangle ABC : $a, b, c, \alpha, \beta, \gamma$

Connus : c, α, β

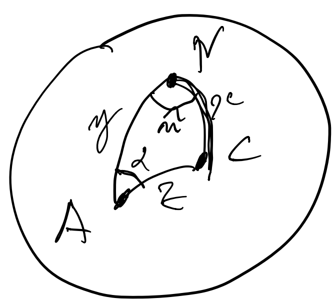
Al Kashi pour les angles :

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c$$

$\hookrightarrow \boxed{\gamma = }$

$$\frac{\sin\gamma}{\sin c} = \frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} \Rightarrow \boxed{\begin{array}{l} a = \\ b = \end{array}}$$

Triangle ACN



Connu : y, z, α

Al Kashi pour les côtés :

$$\cos x = \cos z\cos y + \sin z\sin y\cos\alpha$$

$\hookrightarrow \boxed{x = }$

Loi des sinus : $\frac{\sin \hat{a}}{\sin \hat{\alpha}} = \frac{\sin \hat{c}}{\sin \hat{\gamma}}$ $\boxed{\hat{n} = \ldots}$

Dans la pratique

~~radians~~ ⇝ degré et secondes

$\pi = 180°$ $60 s = 1$ degré

Longitude $15°$ ~~W~~ E ⇝

Latitude : $30° 15 s$ N

Cas d'égalité des triangles (isométriques)

- a, b, c
- a, b, γ
- a, β, γ
- α, β, γ

en utilisant triangle polaire

Ex: $a = a'$ ABC $A'B'C'$

$b = b'$

$c = c'$ $A' = A$ $\vec{x} = \vec{x}'$

$\vec{x}_y, \vec{x}_z$ $\vec{x}'_{y'}, \vec{x}'_{z'}$

.....

$\rightsquigarrow$ construire matrice $\in \underline{O_3(\mathbb{R})}$

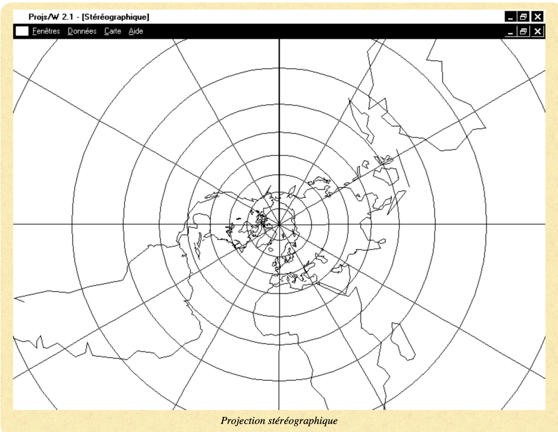
Cartographies (de S^2)

cartes planaires de S^2 ou d'une partie de S^2

Proj stéréographique

avantage: • conserve cercle $\rightsquigarrow$ cercle/droite

• conserve les angles entre les courbes



Projection stéréographique

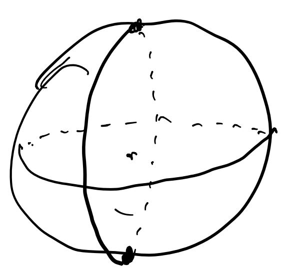
⚠ les distances !

d'autres projections

* équivalente : préserve les surfaces (cadastre)

* conformes : conserve les angles (navigation ; cap)

* équidistante : préserve les distances sur les méridiens ....

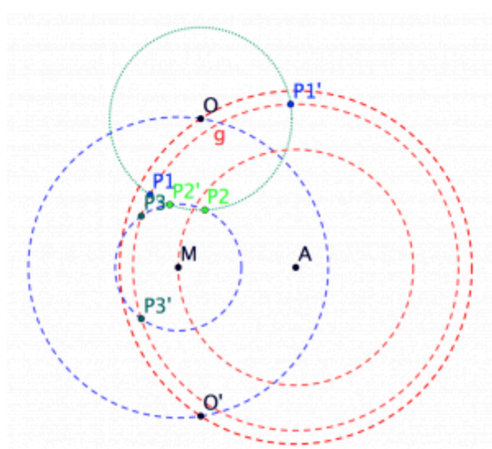


* préserve les géodésiques à partir d'1 point

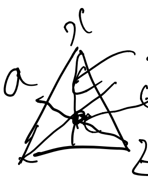
Prop: U un ouvert de S^2

$\nexists\ f : U \to \mathbb{R}^2$ qui soit isométrique

	Athènes	Madrid	Paris	Oslo
Athènes	0 km	1743 km	2414 km	2612 km
Madrid	1743 km	0 km	936,6 km	2224 km
Paris	2414 km	936,6 km	0km	1351 km
Oslo	2612 km	2224 km	1351 km	0km



dem



$\varepsilon' \subset \mathbb{R}^2$

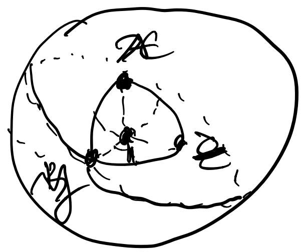
Triangle équilatéral

r

$\varepsilon' x' = r$

$y' z' = a$

$$\boxed{\frac{\varepsilon' x'}{y' z'} = \frac{1}{\sqrt{3}}}$$



$t \in$ médiatrice $[x,y]$

$\cap$ celle de $[y,z]$

$\Rightarrow$ celle de $[x,z]$

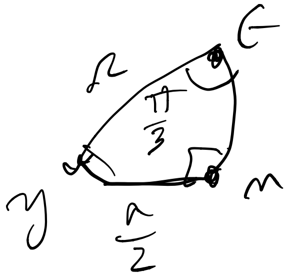
$a = d(x,y) = d(y,z) = d(x,z)$

$r = d(x,t) = d(y,t) = d(z,t)$

m = milieu de $[y,z]$

<u>Triangle</u> : . mty rectangle en m.

. angle en t : par symétrie $= \frac{2\pi}{6} = \frac{\pi}{3}$



<u>Loi des sinus</u>

$$\frac{\sin r}{\sin \frac{\pi}{2}} = \frac{\sin \frac{a}{2}}{\sin \pi/3}$$

$$\boxed{\sin \frac{a}{2} = \frac{\sqrt{3}}{2} \sin r}$$

on prend $0 < r < \frac{\pi}{2}$

$\frac{\sin x}{x} : \left]0, \frac{\pi}{2}\right[\to \mathbb{R}$ décroissante

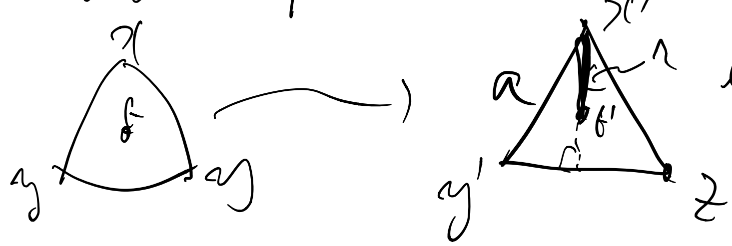
$\sin \frac{a}{2} < \sin r \Rightarrow 0 < \frac{a}{2} < r$

$$\frac{\sin r}{r} < \frac{\sin (a/2)}{a/2}$$

$$\frac{\sin r}{\sin a/2} = \frac{2}{\sqrt{3}} < \frac{2r}{a} \Rightarrow \boxed{a < \sqrt{3}\, r}$$

Contradiction

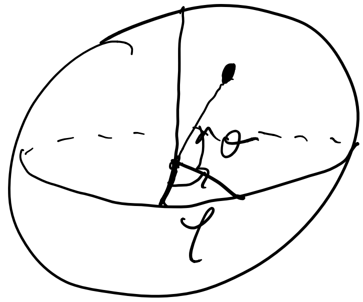
On raisonne par l'absurde



et ici $a = \sqrt{3}\, r$

1 projection équivalente

$\theta, \varphi \rightsquigarrow$
latitude, longitude



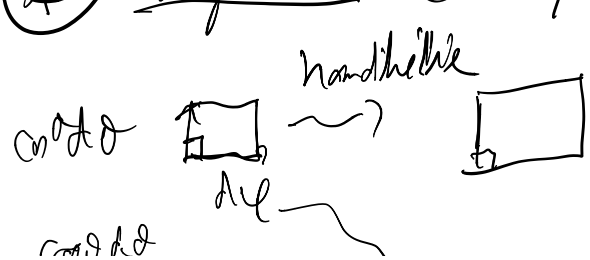
élément de surface : $\boxed{d\varphi \, d\theta \cos\theta}$

$\theta, \varphi \longmapsto (x(\theta,\varphi), y(\theta,\varphi))$

$$\left[\begin{vmatrix} \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} d\varphi \, d\theta = \pm d\varphi \, d\theta \cos\theta \right]$$

$$\boxed{\left| \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \varphi} \right| = \cos\theta \quad \text{cond } 1}$$

2 conforme (respecte les formes)



cons

(1) $$\begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \end{pmatrix} \perp \begin{pmatrix} \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \varphi} \end{pmatrix} = 0 \Longleftrightarrow \boxed{\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial y} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \varphi} = 0}$$

$\varphi = \text{cte}$ $\qquad$ $\theta = \text{cte}$

$d\varphi, \quad d\theta \cos\theta$

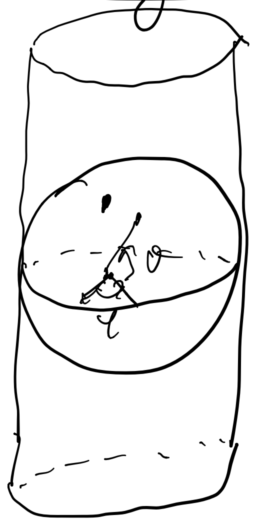
(2) rapport des longueurs

$$\boxed{(\cos\theta)^2\left[\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2\right] = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}$$

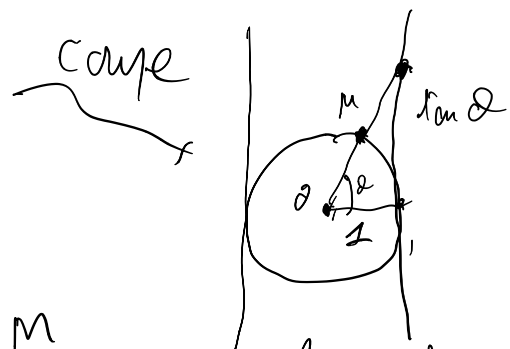
(3) équidistante (sur les méridiens)

$\varphi = \text{cte}$ $$\boxed{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = 1}$$

Projections cylindriques

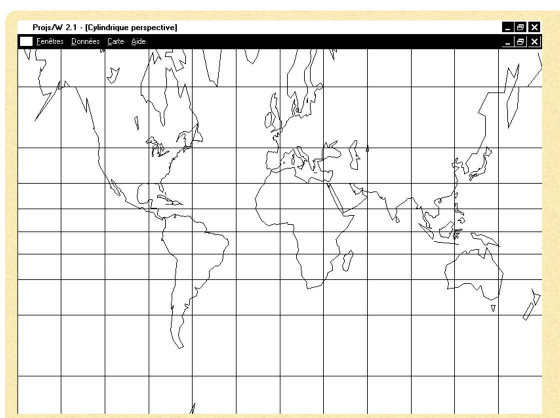


$$\theta, \varphi \longmapsto \begin{cases} x = \varphi \\ y = f(\theta) \end{cases}$$



ex : perspective : $(OM) \cap$ cylindre

$f(\theta) = \tan\theta$



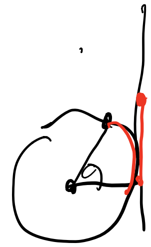
Projection cylindrique perspective

Rem : les pôles sont envoyés à l'infini !

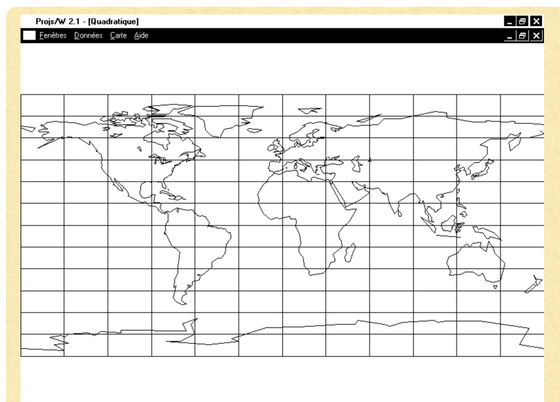
* <u>proj cylindrique équidistante</u>

$\rightsquigarrow \quad f'(\theta)^2 = 1 \qquad f(\theta) = \pm\theta$

$$\underline{f(\theta) = \theta}$$

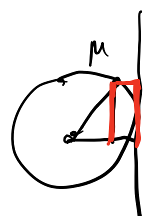


<u>Cartes plates carrées</u>



Projection quadratique

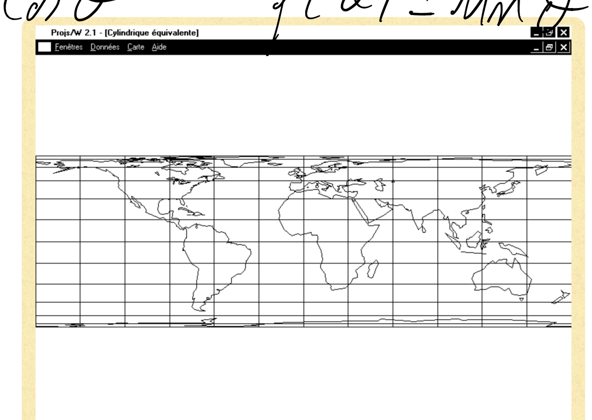
* <u>proj cylindri équivalente</u>



$\rightsquigarrow \quad f'(\theta) = \cos\theta \qquad f(\theta) = \sin\theta$

<u>de Lambert</u>

(proj ⊥ sur le cylindre)

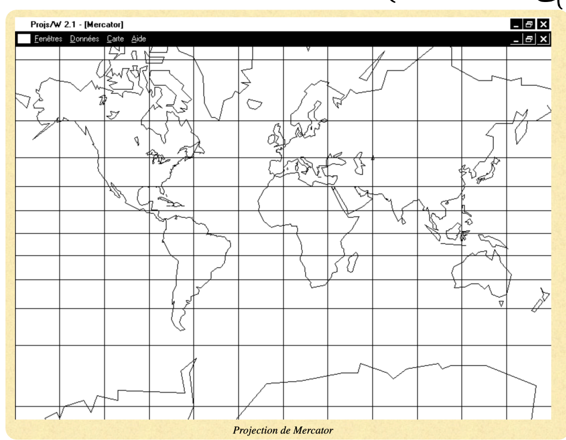


Projection cylindrique équivalente

* <u>Conforme</u> (Mercator)

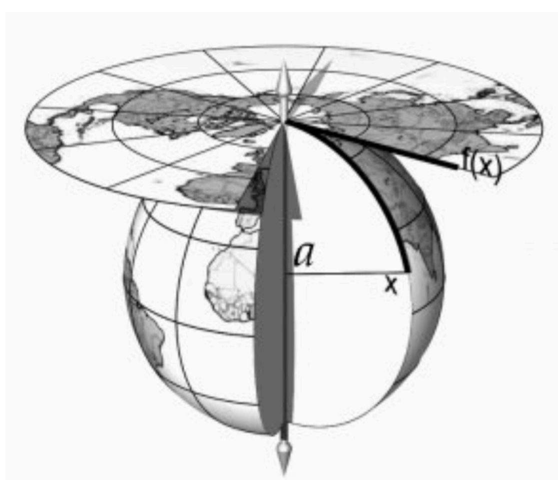
$$\rightsquigarrow f'(\theta) = 1/\cos\theta$$

$$\rightsquigarrow f(\theta) = \ln\left(\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right)$$



Projection de Mercator

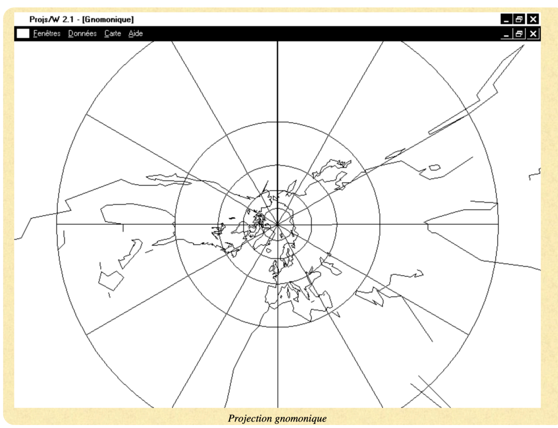
<u>Proj azimutales</u>



$$M(\varphi, \theta) \longmapsto \rho e^{i\varphi}$$

$$\rho = g(\theta)$$

ex : gnomonique : $\rho = \tan z$
$= \tan(\sin \theta)$



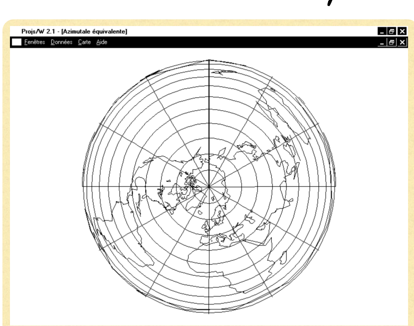
Projection gnomonique

* orthographique : $\rho = \sin z$

* équidistante : $\rho(z) = z$

* équivalente : $-\rho(z)\rho'(z) = \sin z$

$$\rho(z) = \sqrt{2 - 2\cos z}$$



Projection azimutale équivalente

* <u>conforme</u> : (proj stéréographique)