Canad. J. Math. Vol. 52 (1), 2000 pp. 92–118

A Stochastic Calculus Approach for the Brownian Snake

Jean-Stéphane Dhersin and Laurent Serlet

Abstract. We study the "Brownian snake" introduced by Le Gall, and also studied by Dynkin, Kuznetsov, Watanabe. We prove that Itô's formula holds for a wide class of functionals. As a consequence, we give a new proof of the connections between the Brownian snake and super-Brownian motion. We also give a new definition of the Brownian snake as the solution of a well-posed martingale problem. Finally, we construct a modified Brownian snake whose lifetime is driven by a path-dependent stochastic equation. This process gives a representation of some super-processes.

1 Introduction

The aim of this paper is to develop a notion of stochastic calculus for the Brownian snake. The Brownian snake is a Markov process which takes its values in the set of stopped paths in $\mathbb{R}^d$, that will be rigorously defined later. This process, introduced by Le Gall [7], [6], is closely connected with super-Brownian motion, introduced by Watanabe, and has been intensively studied this last decade by Dynkin, Dawson, Le Gall, Perkins, and many other authors. The Brownian snake has already been used successfully to investigate various properties of super-Brownian motion: It is a way to give an explicit construction of super-Brownian motion, and to obtain many of its path properties. It also shares with super-Brownian motion connections with a class of semi-linear partial differential equations.

Heuristically, super-Brownian motion models the behavior in space and time of a cloud of particles with spatial evolution a Brownian motion in $\mathbb{R}^d$, and that may split or die according to a certain rate. For a fixed time t, super-Brownian motion gives the "density" of the cloud. One problem of this model is that though super-Brownian motion gives a good description of this density for each time, it does not describe the underlying genealogical structure of the particles. For this reason, Dynkin, Dawson and Perkins have introduced the notion of historical process [2], [3]. The values of the historical process at time t are measures which describe the paths used by the particles from time 0 to time t. One other way to describe the whole evolution of the particles is given by the Brownian snake, which gives a parameterization of the tree of the paths of the particles. This is a nice tool to study properties of super-Brownian motion. Moreover, it gives a slightly easier expression for the probabilistic representation of solutions of the semi-linear partial differential equation $\Delta u = u^2$. However, if we exclude Le Gall and Le Jan's recent Levy snake construction, the Brownian snake only enables us to study superprocesses with a quadratic branching mechanism.

Connections between the Brownian snake and super-Brownian motion have been established by Le Gall [6]. The idea of the proofs is to compare the discrete approximations

Received by the editors November 9, 1998.
AMS subject classification: 60J25, 60G44, 60J80, 60J60.

of the Brownian snake—using a parameterization of Brownian excursions by trees—and the discrete approximations of super-Brownian motion. One interest of our work is that it gives a direct proof of these connections, without speaking of discrete approximations: It only uses stochastic calculus for the Brownian snake.

In the first part, we give an explicit expression of the generator of the Brownian snake, in terms of the generator of the Markov process which describes the spatial motion. This naturally leads to an Itô's formula, and a definition of the Brownian snake as a solution of a well-posed martingale problem. We then use this new characterization to give a direct rebuilding of super-Brownian motion from the Brownian snake. Finally, we define a modified Brownian snake which allows us to obtain a larger class of super-processes.

Before giving a rigorous expression of our results, let us briefly recall the definition of the Brownian snake. For a more detailed presentation, one can refer to Le Gall [6], [7].

Let $x \in \mathbb{R}^d$ be a fixed point. We denote by $\mathcal{W}_x$ the set of all stopped paths in $\mathbb{R}^d$ started at x. A stopped path is a couple (w, ζ), where $\zeta \geq 0$ is called the lifetime of the path, and $w\colon \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous mapping, which is constant on $[\zeta, +\infty)$, such that $w(0) = x$. We denote by $\mathcal{W} = \bigcup_{x\in\mathbb{R}^d} \mathcal{W}_x$ the set of all stopped paths. When there is no risk of confusion, we often write w for (w, ζ), and $\zeta(w)$ or ζ_w for the lifetime. The distance on $\mathcal{W}$ is $d(w, w') = \sup_{t\geq 0} |w(t) - w'(t)| + |\zeta_w - \zeta_{w'}|$, so $\mathcal{W}$ is a Polish space. We denote by $\hat{w} = w(\zeta_w)$ the endpoint of w, and $\tilde{x}$ the path of lifetime 0 started at x. Finally, we denote by $w_{\leq r}$ or $w^{\leq r}$ the path of lifetime $\zeta_w \wedge r$ such that for $u \geq 0$, $w_{\leq r}(u) = w(u \wedge r)$.

Let us fix $(\xi_t, t \geq 0)$ a Feller diffusion with values in $\mathbb{R}^d$, and let us denote by A its generator. The Brownian snake started at x with spatial motion $(\xi_t, t \geq 0)$ is the strong Markov continuous process $W = (W_s, s \geq 0)$ with values in $\mathcal{W}_x$ characterized by the following properties:

1. The lifetime process $\zeta_s = \zeta(W_s)$ is a reflecting Brownian motion in $\mathbb{R}_+$;
2. Conditionally on $(\zeta_s, s \geq 0)$, the distribution of $(W_s, s \geq 0)$ is that of an inhomogeneous Markov process whose transition kernels are described as follows: For every $s < s'$,
   - $W_{s'}(t) = W_s(t)$ for every $t \leq m(s, s') = \inf_{[s,s']} \zeta_r$.
   - $\Big(W_{s'}\big(m(s, s') + t\big), 0 \leq t \leq \zeta_{s'} - m(s, s')\Big)$ is independent of W_s conditionally on $W_s\big(m(s, s')\big)$ and has the law of a diffusion in $\mathbb{R}^d$ with generator A, starting from $W_s\big(m(s, s')\big)$ and stopped at time $\zeta_{s'} - m(s, s')$.

Heuristically, the path W_s can be seen as a path in $\mathbb{R}^d$ with random lifetime ζ_s evolving like reflecting Brownian motion. When ζ_s "decreases", the path is erased. When it "increases", we put small independent pieces of process $\xi_\cdot$ at its tip.

In what follows, we may and will assume that the process W is the canonical process on the space $\mathcal{C}(\mathbb{R}_+, \mathcal{W})$ of all continuous functions on $\mathbb{R}_+$ with values in $\mathcal{W}$. We denote by $(\mathcal{F}_s, s \geq 0)$ the associated σ-algebra completed the usual way. For every $w \in \mathcal{W}$, $\mathbb{P}_w$ is the law of the Brownian snake started at w, and when $w \neq \tilde{x}$, $\mathbb{P}^*_w$ the law of the Brownian snake killed when its lifetime process hits 0. As defined above, $W_s^{\leq t}$ is the path W_s truncated at time t.

Throughout the paper, we will use the notion of infinitesimal generator of path valued Markov processes. The easiest example of this kind of process—and that will be important—is the so-called A-path associated with the $\mathbb{R}^d$-valued Markov process ξ. This process has been widely studied by Wentzell [11]. See also Dawson [1, p. 202]. The law P_w of the A-path associated with ξ, started at $w \in \mathcal{W}$, is characterized by: Under P_w,

- $W_0 = w$;
- if $s \geq 0$, then $\zeta_s = \zeta_w + s$, $W_s^{\leq \zeta_w} = w$ and the distribution of process $\big(W_s(\zeta_w + u), u \in [0, s)\big)$ is that of the process ξ started at $\hat{w}$.

We will denote by L the generator of the A-path. For certain regular functions F, LF can be easily described using A. For example, if we suppose that for $(w, \zeta) \in \mathcal{W}$, $F(w) = h(\zeta, \hat{w})$ where $h\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a bounded continuous function such that $\big(\frac{\partial}{\partial t} + A\big)\, h$ exists and is also bounded, then it is easy to show that

$$LF(w) = \left(\frac{\partial}{\partial t} + A\right) h(\zeta, \hat{w}).$$

If F is defined by $F(w) = \int_0^\zeta g(w_{\leq r})\, dr$ where $g\colon \mathcal{W} \to \mathbb{R}$ is a continuous function then $LF(w) = g(w)$.

Then the infinitesimal generator of the Brownian snake can be given in terms of the generator L. This is the next result, which is also given in Theorem 1 in a more precise setting.

Notation We denote by $\mathcal{D}$ the set of functions $F\colon \mathcal{W} \to \mathbb{R}$ defined by

$$F(w) = \int_0^{\zeta(w)} g(w_{\leq r})\, dr,$$

such that the function $g\colon \mathcal{W} \to \mathbb{R}$ is in the domain of L and g and Lg are bounded continuous functions. We denote $\mathcal{D}_x$ the set of $F \in \mathcal{D}$ such that $g(\bar{x}) = 0$. Then, for every $F \in \mathcal{D}$, the generator of the Brownian snake at F is $\frac{1}{2} Lg$. In particular, the process

$$M_s(F) = F(W_s) - F(W_0) - \frac{1}{2}\int_0^s Lg(W_r)\, dr$$

is a $(\mathcal{F}_s)$-martingale. More precisely, Theorem 2 gives the following Itô's formula: For all $F \in \mathcal{D}$, under $\mathbb{P}_w$, if $s \geq 0$,

$$F(W_s) = F(W_0) + \int_0^s g(W_r)\, d\zeta_r + \frac{1}{2}\int_0^s Lg(W_r)\, dr.$$

An other problem is the definition of the Brownian snake as a solution of a martingale problem. More precisely, if a path-valued process satisfies: For all $F \in \mathcal{D}_{w(0)}$, the process

$$M_s(F) = F(W_s) - F(w) - \frac{1}{2}\int_0^s Lg(W_r)\, dr$$

is a martingale with quadratic variation

$$\langle M(F)\rangle_s = \int_0^s g(W_r)^2\, dr,$$

is it the Brownian snake starting from w? The answer to this question is yes, as explained in Theorem 3.

One interesting application of the previous results is the rebuilding of super-Brownian motion from the Brownian snake. To this aim, we recall the definition of super-Brownian motion, and the associated historical process, as solutions of well-posed martingale problems. For a more detailed presentation of this definition of super-Brownian motion as solution of a martingale problem, one can refer to Dawson [1]. An (A, z^2)-historical process is a process $(H_t, t \geq 0)$ with values in the set $M_F(\mathcal{W})$ of all finite measures on $\mathcal{W}$ which is the solution of the well-posed following martingale problem: For every function $g\colon \mathcal{W} \to \mathbb{R}_+$ in the domain of L such that g and Lg are bounded, then the process

$$M_t(g) = \langle H_t, g\rangle - \langle H_0, g\rangle - \int_0^t ds\langle H_s, Lg\rangle, \quad t \geq 0$$

is a martingale with quadratic variation

$$\langle M(g)\rangle_t = 4\int_0^t ds\langle H_s, g^2\rangle.$$

(In the usual definition of the historical process, there is no 4. However, our definition will give results which are easier to write.)

An (A, z^2)-super-process is the process (X_t) with values in the space $M_F(\mathbb{R}^d)$ of all finite measure on $\mathbb{R}^d$, associated with the process (H_t) by:

$$\langle X_t, \varphi\rangle = \int H_t(dw)\varphi(\hat{w}), \quad t \geq 0$$

for every bounded measurable function $\varphi\colon \mathbb{R}^d \to \mathbb{R}_+$. In fact, one can prove that the distribution of the (A, z^2)-super-Brownian motion is the solution of the well-posed following martingale problem: If $\varphi\colon \mathbb{R}^d \to \mathbb{R}_+$ is in the domain of A, and the functions φ and $A\varphi$ are bounded, then

$$M_t(\varphi) = \langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \int_0^t ds\langle X_s, A\varphi\rangle, \quad t \geq 0$$

is a martingale with quadratic variation

$$\langle M(\varphi)\rangle_t = 4\int_0^t ds\langle X_s, \varphi^2\rangle.$$

Let us now recall how these processes can be rebuilt from the Brownian snake. Under the probability measure $\mathbb{P}_{\bar{x}}$, for $s \geq 0$, $t \geq 0$, we denote by $L_s^t(\zeta)$ the local time of the lifetime process $(\zeta_\cdot)$ at level t, and time s, and $\tau_1 = \inf\{s \geq 0; L_s^0(\zeta) > 1\}$ the first time the local time at level 0 hits 1. For $t \geq 0$, we introduce the measures defined on $\mathbb{R}^d$ and $\mathcal{W}$ by:

$$X_t = \int_0^{\tau_1} d_s L_s^t(\zeta)\delta_{\hat{W}_s} \quad \text{and} \quad H_t = \int_0^{\tau_1} d_s L_s^t(\zeta)\delta_{W_s}$$

where $d_s L_s^t(\zeta)$ is the measure associated with the increasing process $s \mapsto L_s^t(\zeta)$. We give in Theorem 7 a new proof that, under $\mathbb{P}_{\bar{x}}$, the process (H_t) is a (A, z^2)-historical process, and the process (X_t) is the associated super-Brownian motion started at δ_x.

In Proposition 12 we give the definition of the modified Brownian snake. The lifetime of a such snake (ζ_s, W_s) is no longer a reflecting Brownian motion, but a diffusion such that on $\{\zeta_s > 0\}$,

$$d\zeta_s = \frac{1}{c(\hat{W}_s)} d\gamma_s - \theta(W_s)\, ds$$

where (γ_s) is a linear Brownian motion, and c, θ two nonnegative functions on $\mathbb{R}^d$ and $\mathcal{W}$ respectively. Heuristically, thinking at the process (W_s) as a model for a tree of paths of particles, the coefficient $1/c$ may be seen as a modification of the branching rate of the particles. Moreover, this modification depends on where the particles are. The term $-\theta(W_s)\, ds$ may be seen as a killing coefficient, with intensity depending on the path the particle has used. Notice that existence of such processes does not follow from usual general theorems on stochastic differential equations: The lifetime process does depend on the spatial evolution of the whole process, so that it does not seem possible to construct the lifetime process first, and then to construct the spatial evolution. For these reasons, we have to define this snake as the solution of a well-posed martingale problem. Its spatial evolution is then not as clear as for the standard Brownian snake, but a powerful byproduct is that we have an Itô's formula, which is well adapted for studying the associated super-processes. The effective construction of this snake uses an idea by Watanabe [10]—which is based on a change of time for each path—and a Girsanov Theorem.

We finally prove that this modified Brownian snake can be used to build a more general set of super-processes—that will be called $\big(A, c(x)z^2, b\big)$-super-processes. Heuristically, these super-processes are models for the evolution of a cloud of particles which have a spatial motion described by the generator A, each particle branching according to a rate which depends of its position, and each particle being killed with a rate b which depends on the path the particle has used to reach its position. More precisely, the associated historical process is the solution of the following well-posed martingale problem: For every bounded function ψ in the domain of L, the process

$$M_t(\psi) = \langle Z_t, \psi\rangle - \int_0^t ds\langle Z_s, L\psi\rangle - \int_0^t ds\langle Z_s, bc\psi\rangle, \quad t \geq 0 \tag{1}$$

is a martingale with quadratic variation

$$\langle M(\psi)\rangle_t = 4\int_0^t ds\langle Z_s, c\psi^2\rangle. \tag{2}$$

In (1) and (2), the function c is extended on $\mathcal{W}$ by $c(w) = c(\hat{w})$.

2 Generator of the Brownian Snake

Let us recall that L denotes the generator of the A-path process, and $\mathcal{D}$ the set of test functions defined in the introduction.

Theorem 1 *Let $F \in \mathcal{D}$. Then, for all $w \in \mathcal{W}$ such that $\zeta(w) > 0$, we have*

$$\lim_{s \downarrow 0} \frac{1}{s}\left(\mathbb{E}_w[F(W_s)] - F(w)\right) = \frac{1}{2} Lg(w).$$

This result still holds if $\zeta(w) = 0$ and $g(w_{\leq 0}) = 0$.

Proof Let us fix $w \in \mathcal{W}$ such that $\zeta_w > 0$. Under $\mathbb{P}_w$, we have $W_s(u) = w(u)$ for $0 \leq u \leq m_s = \inf_{[0,s]} \zeta$. Hence

$$F(W_s) - F(w) = \int_{m_s}^{\zeta_s} g(W_s^{\leq r})\, dr - \int_{m_s}^{\zeta} g(w_{\leq r})\, dr,$$

and then

$$\mathbb{E}_w[F(W_s)] - F(w) = \int \mathbb{P}_w(\zeta_s \in db, m_s \in da) \times \left(\int_a^b dr \mathbb{E}_w[g(W_s^{\leq r}) \mid \zeta_s = b, m_s = a] - \int_a^{\zeta} g(w_{\leq r})\, dr \right).$$

We know the explicit law of the process (W_s) knowing (ζ_s), as recalled in introduction: Under $\mathbb{P}_w(\cdot \mid \zeta_s = b, m_s = a)$, for $r \in [a, b]$, the path $W_s^{\leq r}$ is obtained by extending the path $w_{\leq a}$ with a path which generator is A, until time r. It is then the law of a A-path (W'_u) at time $r - a$, starting from the path $w_{\leq a}$. Hence we get

$$\begin{aligned} \mathbb{E}_w[g(W_s^{\leq r}) \mid \zeta_s = b, m_s = a] &= E_{w_{\leq a}}[g(W'_{r-a})] \\ &= g(w_{\leq a}) + \int_0^{r-a} E_{w_{\leq a}}[Lg(W'_u)]\, du. \end{aligned}$$

Then,

$$\begin{aligned} &\mathbb{E}_w[F(W_s) - F(w)] \\ &\quad = \int \mathbb{P}_w[\zeta_s \in db, m_s \in da] \int_a^b dr \int_0^{r-a} du\, E_{w_{\leq a}}[Lg(W'_u)] + R_1 - R_2, \end{aligned} \tag{3}$$

with

$$R_1 = \int \mathbb{P}_w[\zeta_s \in db, m_s \in da] \int_a^b dr\, g(w_{\leq a}),$$

and

$$R_2 = \int \mathbb{P}_w[m_s \in da] \int_a^{\zeta} dr\, g(w_{\leq r}).$$

The law of (ζ_s, m_s) is easy to get (see [6]):

$$\begin{aligned}
&\mathbb{P}_w[\zeta_s \in db, m_s \in da] \\
&= \frac{2}{\sqrt{2\pi s^3}}(\zeta + b - 2a)\exp -\left(\frac{(\zeta+b-2a)^2}{2s}\right)\mathbf{1}_{\{0<a<\zeta\wedge b\}}\,da\,db \qquad (4)\\
&\quad + \frac{2}{\sqrt{2\pi s}}\exp -\left(\frac{(\zeta+b)^2}{2s}\right)\delta_0(da)\,db.
\end{aligned}$$

Then, we obtain for R_1:

$$\begin{aligned}
R_1 = &\int_0^\zeta da\, g(w_{\le a})\int_a^{+\infty} dr\frac{2}{\sqrt{2\pi s}}\exp -\left(\frac{(\zeta+r-2a)^2}{2s}\right)\\
&+ g(w_{\le 0})\int_0^{+\infty} db\frac{2}{\sqrt{2\pi s}} b\exp -\left(\frac{(\zeta+b)^2}{2s}\right)
\end{aligned}$$

From the law of (ζ_s, m_s), we immediately deduce the law of m_s, and then

$$R_2 = \int_0^\zeta dr\, g(w_{\le r})\int_{\zeta - r}^{+\infty} dv\frac{2}{\sqrt{2\pi s}}\exp -\left(\frac{v^2}{2s}\right).$$

Hence, for every $p \ge 0$,

$$R_1 - R_2 = g(w_{\le 0})\int_0^{+\infty} db\frac{2}{\sqrt{2\pi s}} b\exp -\left(\frac{(\zeta+b)^2}{2s}\right) = O(s^p).$$

(We have used that $\zeta > 0$.) Then, it follows from equation (3) that

$$\begin{aligned}
\frac{1}{s}\big(\mathbb{E}_w[F(W_s)] - F(w)\big) &= \frac{1}{s}\mathbb{E}_w\left[\int_{m_s}^{\zeta_s} dr\int_0^{r-m_s} du\, E_{w_{\le m_s}}[Lg(W'_u)]\right] + O(s)\\
&= \frac{1}{s}\mathbb{E}_w\left[\frac{(\zeta_s - m_s)^2}{2}\right] Lg(w) + O(s)\\
&\quad + \frac{1}{s}\mathbb{E}_w\left[\int_{m_s}^{\zeta_s} dr\int_0^{r-m_s} du\, E_{w_{\le m_s}}[Lg(W'_u) - Lg(w)]\right].
\end{aligned}$$

It is easy to see that the last term of this formula tends to 0 by dominated convergence. It remains to estimate $\mathbb{E}_w[(\zeta_s - m_s)^2]$. If $(\zeta_s, s \ge 0)$ was a standard non-reflecting Brownian motion $(B_s, s \ge 0)$, Lévy's Theorem would lead to the result:

$$E[(B_s - \inf_{[0,s]} B)^2] = E[B_s^2] = s.$$

In our case, (ζ_s) is a reflected Brownian motion started from $\zeta > 0$. It is quite easy to see that $E[(\zeta_s - m_s)^2] = s + o(s)$, which achieves the proof when $\zeta(w) > 0$. The case $\zeta(w) = 0$ is easier. ■

3 Itô's Formula for the Brownian Snake

Theorem 2 *Let $F \in \mathcal{D}$, and $w \in \mathcal{W}$. Then, under $\mathbb{P}_w$, we have for every $s \geq 0$,*

$$F(W_s) = F(W_0) + \int_0^s g(W_r)\, d\zeta_r + \frac{1}{2} \int_0^s Lg(W_r)\, dr.$$

Proof We can and will assume that $g(w_{\leq 0}) = 0$. The general case can be immediately deduced from this particular situation by considering $g(\cdot) - g(w_{\leq 0})$. Using Theorem 1, we get that

$$F(W_s) - F(w) - \frac{1}{2} \int_0^s Lg(W_r)\, dr, \quad s \geq 0$$

is a $(\mathcal{F}_s)$-martingale. To prove the theorem, we only have to show that this martingale is equal to

$$\int_0^s g(W_r)\, d\zeta_r.$$

Notice that as the process $(\zeta_s, s \geq 0)$ is a reflecting Brownian motion, this stochastic integral can be defined. Moreover, using that the finite variation part of the semi-martingale $(\zeta_s, s \geq 0)$ is its local time, and that $g(w_{\leq 0}) = 0$, we get that this stochastic integral is in fact a martingale. Let us introduce the martingale $(N_s, s \geq 0)$ defined by

$$N_s = F(W_s) - \int_0^s g(W_r)\, d\zeta_r.$$

To prove that $N = 0$, it is sufficient to obtain that its quadratic variation satisfies $\langle N \rangle_s = 0$, for every $s \geq 0$.

Let us fix $s \geq 0$, and $\Delta_n = [0 = s_0^n \leq s_1^n \leq \cdots \leq s_{k(n)}^n \leq s_{k(n)+1}^n = s]$ be a sequence of subdivisions such that the modulus of Δ_n tends to 0 and

$$\langle N \rangle_s = \lim_{n \to +\infty} \sum_{i=0}^{k(n)} (N_{s_{i+1}^n} - N_{s_i^n})^2.$$

In order to make expressions more readable, we write s_i for s_i^n. We note that

$$\begin{aligned} N_{s_{i+1}} - N_{s_i} &= F(W_{s_{i+1}}) - F(W_{s_i}) - \int_{s_i}^{s_{i+1}} g(W_r)\, d\zeta_r \\ &= \int_{m_i}^{\zeta_{s_{i+1}}} g(W_{s_{i+1}}^{\leq r})\, dr - \int_{m_i}^{\zeta_{s_i}} g(W_{s_i}^{\leq r})\, dr - \int_{s_i}^{s_{i+1}} g(W_r)\, d\zeta_r, \end{aligned}$$

with $m_i = \inf_{u \in [s_i, s_{i+1}]} \zeta_u$. We obviously have

$$\int_{s_i}^{s_{i+1}} g(W_{s_i})\, d\zeta_r = \int_{m_i}^{\zeta_{s_{i+1}}} g(W_{s_i})\, dr - \int_{m_i}^{\zeta_{s_i}} g(W_{s_i})\, dr.$$

Hence, we can write $N_{s_{i+1}} - N_{s_i} = A_i - B_i - C_i$ where

$$A_i = \int_{m_i}^{\zeta_{s_{i+1}}} \big(g(W_{s_{i+1}}^{\leq r}) - g(W_{s_i})\big)\, dr,$$

$$B_i = \int_{m_i}^{\zeta_{s_i}} \big(g(W_{s_i}^{\leq r}) - g(W_{s_i})\big)\, dr,$$

$$C_i = \int_{s_i}^{s_{i+1}} \big(g(W_r) - g(W_{s_i})\big)\, d\zeta_r.$$

Notice that $(N_{s_{i+1}} - N_{s_i})^2 \leq 3(A_i^2 + B_i^2 + C_i^2)$. We thus have three terms to bound.

For the third term, we notice that

$$\mathbb{E}_w\Big[\sum_i C_i^2\Big] = \mathbb{E}_w\Big[\sum_i \int_{s_i}^{s_{i+1}} \big(g(W_r) - g(W_{s_i})\big)^2\, dr\Big]$$

which tends to 0 by dominated convergence.

For the first term, we write

$$\mathbb{E}_w\Big[\sum_i A_i^2\Big] \leq \sum_i \mathbb{E}_w[M(\zeta_{s_{i+1}} - m_i)^2]$$

where

$$M = \sup_i \sup_{r \in [m_i, \zeta_{s_{i+1}}]} |g(W_{s_{i+1}}^{\leq r}) - g(W_{s_i})|^2.$$

By the Cauchy-Schwarz inequality, we have

$$\mathbb{E}_w\Big[\sum_i A_i^2\Big] \leq (\mathbb{E}_w[M^2])^{1/2} \sum_i \big(\mathbb{E}_w[(\zeta_{s_{i+1}} - m_i)^4]\big)^{1/2}.$$

The second term of the right hand side is bounded. Notice that M is bounded, so it remains to prove that M almost surely tends to 0 when n tends to ∞. But it is obvious that $\{W_u^{\leq r}; u \in [0, s], r \geq 0\}$ is compact. Using that the function g is uniformly continuous on this compact set, we get that M tends to 0, a.s.

The proof for the upper bound for $\mathbb{E}_w[\sum_i B_i^2]$ is similar. Hence we have proved that $\mathbb{E}_w[\sum_i (N_{s_{i+1}} - N_{s_i})^2]$ tends to 0. Using Fatou's Lemma this implies that for $s \geq 0$, $\mathbb{E}_w[\langle N \rangle_s] = 0$, which achieves the proof. ■

4 Martingale Problem for the Brownian Snake

In this section, we prove that we can give a characterization of the Brownian snake using a martingale problem.

For $x \in \mathbb{R}^d$, let us recall that $\mathcal{D}_x$ is the set of functions $F\colon w \to \int_0^\zeta g(w_{\leq r})\, dr$ such that $F \in \mathcal{D}$ and $g(\tilde{x}) = 0$.

Theorem 3 *Let $w_0 \in \mathcal{W}$ and $x_0 = w_0(0)$. Let us assume that the process $(W_s, s \geq 0)$ with values in $\mathcal{W}_{x_0}$ satisfies $W_0 = w_0$ and the following assumption, denoted by (H): For every function $F \in \mathcal{D}_{x_0}$, the process*

$$M(F)_s = F(W_s) - \frac{1}{2}\int_0^s Lg(W_r)\,dr$$

is a martingale with quadratic variation

$$\langle M(F)\rangle_s = \int_0^s g^2(W_r)\,dr.$$

Then the law of $(W_s, s \geq 0)$ is $\mathbb{P}_{w_0}$, i.e., it is a Brownian snake starting from w_0.

Proof First of all, let us notice that it is easy to prove that the process $(\zeta_s, s \geq 0)$ is a reflecting Brownian motion. Indeed, if $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a function of class C^2 such that $f'(0) = 0$, then applying assumption (H) with $g(w) = f'\big(\zeta(w)\big)$, we get that the process

$$f(\zeta_s) - \frac{1}{2}\int_0^s f''(\zeta_r)\,dr, \quad s \geq 0$$

is a martingale. This proves (see *e.g.* [5]) that the process $(\zeta_s, s \geq 0)$ is a reflecting Brownian motion.

Let us denote by $(G_s, s \geq 0)$ the semi-group of the Brownian snake. We first suppose proved the following result.

Lemma 4 *For every $T > 0$, and every bounded continuous function $H\colon \mathcal{W} \to \mathbb{R}$, the process $\big(G_{T-s}H(W_s), 0 \leq s \leq T\big)$ is a martingale.*

Then, if $T > 0$, $s \in [0, T]$ and $w \in \mathcal{W}$, the martingale property at time T gives

$$G_T H(w) = \mathbb{E}_w[G_T H(w)] = \mathbb{E}_w[G_0 H(W_T)] = \mathbb{E}_w[H(W_T)],$$

which proves that the process $(W_s, s \geq 0)$ and the Brownian snake have the same finite-dimensional distributions. This implies (see [4, th. 4.2 p. 184]) that the process $(W_s; s \geq 0)$ is a Brownian snake. Hence, Theorem 3 will be proved as soon as we have proved Lemma 4.

We first give an explicit expression of the semi-group of the Brownian snake. To simplify notations, we introduce

$$p^{(1)}(s,x) = \frac{1}{\sqrt{2\pi s}}\exp -\frac{x^2}{2s}, \quad s > 0, x \in \mathbb{R},$$

the transition densities of the linear Brownian motion, and denote by $(Q_b, b \geq 0)$ the semi-group of the A-path.

Lemma 5 *The semi-group $(G_s, s \geq 0)$ of the Brownian snake is characterized by: if $s \geq 0$, $H\colon \mathcal{W} \to \mathbb{R}$ is a bounded continuous function, and $w \in \mathcal{W}$, then*

$$G_s H(w) = -2 \int_0^{\zeta} da \int_0^{\infty} db\, \partial_2 p^{(1)}(s, \zeta + b - a) Q_b H(w_{\leq a}) + 2 \int_0^{\infty} db\, p^{(1)}(s, \zeta + b) Q_b H(\tilde{x}).$$

Proof Following Section 2, if $s \geq 0$ we denote by $m_s = \inf_{r \in [0,s]} \zeta_r$ the minimum of the lifetime process before time s. Then

$$\begin{aligned} G_s H(w) &= \mathbb{E}_w[H(W_s)] \\ &= \int \mathbb{P}_w[\zeta_s \in db, m_s \in da] \mathbb{E}_w[H(W_s) \mid \zeta_s = b, m_s = a] \\ &= \int \mathbb{P}_w[\zeta_s \in db, m_s \in da] Q_{b-a} H(w_{\leq a}). \end{aligned}$$

Now, notice that formula (4) can be written

$$\begin{aligned} &\mathbb{P}_w[\zeta_s \in db, m_s \in da] \\ &\quad = -2\partial_2 p^{(1)}(s, \zeta + b - 2a) \mathbf{1}_{\{0 < a < \zeta \wedge b\}}\, da\, db + 2p^{(1)}(s, \zeta + b) \mathbf{1}_{\{b > 0\}}\, \delta_0(da)\, db. \end{aligned}$$

This achieves the proof of the Lemma. ■

Proof of Lemma 4 Using Lemma 5, we see that if $T > 0$ and $s \in [0, T]$, we have

$$G_{T-s} H(w) = \mathbb{E}_w[H(W_{T-s})] = \int_0^{\zeta} da\, h(s, \zeta, w_{\leq a}) + \theta(s, \zeta),$$

where

$$\begin{aligned} h(s, \zeta, w') &= -2 \int_0^{\infty} db\, \partial_2 p^{(1)}(T - s, \zeta + b - \zeta') Q_b H(w') \\ \theta(s, \zeta) &= 2 \int_0^{\infty} db\, p^{(1)}(T - s, \zeta + b) Q_b H(\tilde{x}). \end{aligned} \tag{5}$$

Hence,

$$G_{T-s} H(W_s) = \int_0^{\zeta_s} da\, h(s, \zeta_s, W_s^{\leq a}) + \theta(s, \zeta_s), \tag{6}$$

We first study the process $\big(\theta(s, \zeta_s), s \geq 0\big)$. To this aim, we introduce the Tanaka representation of the reflecting Brownian motion $(\zeta_s, s \geq 0)$

$$\zeta_s = \beta_s + \frac{1}{2} L_s^0(\zeta), \quad s \geq 0,$$

where the process $(\beta_s, s \geq 0)$ is a linear Brownian motion and $L_s^0(\zeta)$ is the local time at time s and level 0 of the process $(\zeta_\cdot)$. Hence, using Itô's formula, we get

$$\theta(s,\zeta_s) = \theta(0,\zeta_0) + \int_0^s \left(\partial_1\theta + \frac{1}{2}\partial_{2,2}^2\theta\right)(r,\zeta_r)\,dr + \int_0^s \partial_2\theta(r,\zeta_r)\,d\beta_r$$
$$+ \frac{1}{2}\int_0^s \partial_2\theta(r,0)\,dL_r^0(\zeta).$$

As we have

$$\left(\partial_1 + \frac{1}{2}\partial_{2,2}^2\right) p^{(1)}(T-s,\zeta+b) = 0, \tag{7}$$

it remains

$$\begin{aligned}\theta(s,\zeta_s) &= \theta(0,\zeta_0) + \int_0^s \partial_2\theta(r,\zeta_r)\,d\beta_r + \frac{1}{2}\int_0^s \partial_2\theta(r,0)\,dL_r^0(\zeta)\\ &= \theta(0,\zeta_0) + \int_0^s \partial_2\theta(r,\zeta_r)\,d\beta_r\\ &\quad + \int_0^s dL_r^0(\zeta)\int_0^\infty db\,\partial_2 p^{(1)}(T-r,b)Q_bH(\tilde{x}).\end{aligned} \tag{8}$$

Now, let us study the process $\left(\int_0^{\zeta_s} da\, h(s,\zeta_s,W_s^{\leq a}), s \geq 0\right)$. First of all, let us prove that assumption (H) is in fact satisfied for a larger class of functions.

Lemma 6 *Let*

$$\begin{aligned} h\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{W}_x &\to \mathbb{R}\\ (s,\zeta,w') &\mapsto h(s,\zeta,w')\end{aligned}$$

be a bounded function such that, if $w' \in \mathcal{W}_x$, the function $h(\cdot,\cdot,w')$ is of class $\mathcal{C}^2$, and if $(s,\zeta) \in \mathbb{R}_+\times\mathbb{R}_+$, the function $h(s,\zeta,\cdot)$ is in the domain of the generator L and Lh is bounded. If $s \geq 0$, let us introduce the function F_s defined on $\mathcal{W}$ by

$$F_s(w) = \int_0^\zeta da\, h(s,\zeta,w_{\leq a}).$$

Then, the process

$$F_s(W_s) - \int_0^s dr\int_0^{\zeta_r} da\left(\partial_1 h + \frac{1}{2}\partial_{2,2}^2 h\right)(r,\zeta_r,W_r^{\leq a}) - \frac{1}{2}\int_0^s dL_r^0(\zeta)h(r,0,\tilde{x}_0)$$
$$- \int_0^s dr\left(\partial_2 h + \frac{1}{2}Lh\right)(r,\zeta_r,W_r), \quad s \geq 0$$

is a martingale.

Proof We can and will suppose that the function h can be written

$$h(s,\zeta,w') = f(s)\varphi(\zeta)g(w')$$

where f, φ and g are very regular functions. The general case is then obtained by using the monotone class Theorem. Let us introduce the function

$$F(w) = \int_0^\zeta g(w_{\le a})\, da = \int_0^\zeta \big(g(w_{\le a}) - g(\tilde{x})\big)\, da + g(\tilde{x})\zeta.$$

Applying assumption (H) with the function $w \mapsto \int_0^\zeta \big(g(w_{\le a}) - g(\tilde{x})\big)\, da$, and using the decomposition $\zeta_s = \beta_s + (1/2)L_s^0(\zeta)$, we get:

$$F(W_s) = M_s + \frac{1}{2}\int_0^s Lg(W_r)\, dr + \frac{1}{2}g(\tilde{x})L_s^0(\zeta),$$

where $(M_s, s \ge 0)$ is a martingale.

Now, let us apply Itô's formula (15) for $F_s(W_s) = f(s)\varphi(\zeta_s)F(W_s)$.

$$\begin{aligned} d\big(F_s(W_s)\big) &= f'(s)\varphi(\zeta_s)F(W_s)\, ds + f(s)\varphi'(\zeta_s)F(W_s)\, d\zeta_s + f(s)\varphi(\zeta_s)\, d\big(F(W_s)\big) \\ &\quad + \frac{1}{2}f(s)\varphi''(\zeta_s)F(W_s)\, d\langle\zeta\rangle_s + f(s)\varphi'(\zeta_s)\, d\langle\zeta, F(W_.)\rangle_s \\ &= f(s)\varphi(\zeta_s)\, dM_s + f(s)\varphi'(\zeta_s)F(W_s)\, d\beta_s + f'(s)\varphi(\zeta_s)F(W_s)\, ds \\ &\quad + \frac{1}{2}f(s)\varphi'(0)F(W_s)\, dL_s^0(\zeta) + \frac{1}{2}f(s)\varphi(\zeta_s)Lg(W_s)\, ds \\ &\quad + \frac{1}{2}f(s)\varphi(\zeta_s)g(\tilde{x})\, dL_s^0(\zeta) + \frac{1}{2}f(s)\varphi''(\zeta_s)F(W_s)\, ds + f(s)\varphi'(\zeta_s)g(W_s)\, ds \\ &= f(s)\varphi(\zeta_s)\, dM_s + f(s)\varphi'(\zeta_s)F(W_s)\, d\beta_s \\ &\quad + \left[\int_0^{\zeta_s} da\left(f'(s)\varphi(\zeta_s)g(W_s^{\le a}) + \frac{1}{2}f(s)\varphi''(\zeta_s)g(W_s^{\le a})\right)\right] ds \\ &\quad + \frac{1}{2}f(s)\varphi(\zeta_s)g(\tilde{x}_0)\, dL_s^0(\zeta) + \left(f(s)\varphi'(\zeta_s)g(W_s) + \frac{1}{2}f(s)\varphi(\zeta_s)Lg(W_s)\right) ds, \end{aligned}$$

which are the terms announced in Lemma 6. ∎

Let us apply this result with the function h defined in equation (5). First, notice that thanks to equation (7), we have $\partial_1 h + \frac{1}{2}\partial_{2,2}^2 h = 0$. We then use the explicit expression of Lh

and $\partial_2 h$, that L is the generator of (Q_b), and finally an integration by parts, to get:

$$\begin{aligned}\frac{1}{2}Lh(s,\zeta,w')+\partial_2 h(s,\zeta,w') &= \int_0^\infty db\, \partial_{2,2}^2 p^{(1)}(T-s,\zeta+b-\zeta')Q_bH(w') \\ &\quad - \int_0^\infty db\, \partial_2 p^{(1)}(T-s,\zeta+b-\zeta')L\big(w'\to Q_bH(w')\big) \\ &\quad - 2\int_0^\infty db\, \partial_{2,2}^2 p^{(1)}(T-s,\zeta+b-\zeta')Q_bH(w') \\ &= -\int_0^\infty db\, \partial_{2,2}^2 p^{(1)}(T-s,\zeta+b-\zeta')Q_bH(w') \\ &\quad - \int_0^\infty db\, \partial_2 p^{(1)}(T-s,\zeta+b-\zeta')\frac{\partial}{\partial b}Q_bH(w') \\ &= -\partial_2 p^{(1)}(T-s,\zeta-\zeta')H(w').\end{aligned}$$

If we apply this result when $w'=w$ we get

$$\frac{1}{2}Lh(s,\zeta,w)+\partial_2 h(s,\zeta,w)=0.$$

Hence, using Lemma 6, there exists a martingale $(M_s, 0\le s\le T)$ such that, if $s\in[0,T]$,

$$\begin{aligned}\int_0^{\zeta_s} da\, h(s,\zeta_s,W_s^{\le a}) &= \frac{1}{2}\int_0^s dL_r^0(\zeta)h(r,0,\tilde{x})+M_s \\ &= -\int_0^s dL_r^0(\zeta)\int_0^\infty db\, \partial_2 p^{(1)}(T-r,b)Q_bH(\tilde{x})+M_s.\end{aligned}$$

Combining this formula with equations (6) and (8), we get that the process $\big(G_{T-s}H(W_s),$ $0\le s\le T\big)$ is a martingale, which completes the proof of Lemma 4. ■

5 Construction of the (A,z^2)-Super-Process

Let us fix $x\in\mathbb{R}^d$. We recall that $\tilde{x}$ is the path starting from x with lifetime 0. Hence, under the probability measure $\mathbb{P}_{\tilde{x}}$, the process (ζ_s,W_s) is a Brownian snake starting from $\tilde{x}$, and its lifetime process is a reflecting Brownian motion starting from 0. We also recall the notation $\tau_1=\inf\{s; L_s^0(\zeta)>1\}$ for the first hitting time of 1 by the local time at level 0, and we introduce the change of time:

$$A_r^t=\inf\left\{s>0; \int_0^s \mathbf{1}_{\{\zeta_u\le t\}}\,du>r\right\}\wedge\tau_1,$$

and the filtration $(\mathcal{G}_t)$ defined by

$$\mathcal{G}_t=\sigma(W_{A_r^t}, r\ge 0).$$

The definition of the (A,z^2)-historical super-process as solution of a martingale problem is recalled in the introduction.

Theorem 7 *Under $\mathbb{P}_{\bar{x}}$, let us introduce the process $(H_t, t \geq 0)$ which takes values in $M_F\big(C(\mathbb{R}_+, \mathbb{R}^d)\big)$ defined by: for every $t \geq 0$, and every nonnegative measurable function $\varphi\colon C(\mathbb{R}^d, \mathbb{R}_+) \to \mathbb{R}$,*

$$\langle H_t, \varphi \rangle = \int_0^{\tau_1} d_{(s)} L_s^t(\zeta) \varphi(W_s).$$

Then the process $(H_t, t \geq 0)$ is a (A, z^2)-historical super-process adapted to the filtration $(\mathcal{G}_t)$ with initial condition $H_0 = \delta_{\bar{x}}$.

Before proving this Theorem, we need three lemmas.

Lemma 8 *Let $(\zeta_s, s \geq 0)$ be a reflecting Brownian motion on $\mathbb{R}_+$, starting from $a \in \mathbb{R}_+$ under P_a.*

1. *Let us denote by $\tau_1 = \inf\{s \geq 0; L_s^o(\zeta) > 1\}$ the first hitting time of 1 by the local time at level 0 of the process (ζ_s). Then, for every $t > 0$ and every nonnegative real $\theta \leq \frac{\pi}{4t} \wedge 1$, we have*

$$E_0\left[\exp\left(\frac{\theta^2}{2}\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_s)\, ds\right)\right] \leq e \tag{9}$$

$$E_0\left[\exp\left(\frac{\theta}{2}\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_s)\, d\zeta_s\right)\right] \leq \exp\frac{1}{2}. \tag{10}$$

2. *Let r, a and t be such that $0 \leq r < a < t$, and $T_r = \inf\{s \geq 0; \zeta_s = r\}$ be the first hitting time of r by the process (ζ_s). Then, for every nonnegative real $\theta \leq \frac{\pi}{3t}$ we have*

$$E_a\left[\exp\left(\frac{\theta^2}{2}\int_0^{T_r} \mathbf{1}_{(r,t]}(\zeta_s)\, ds\right)\right] \leq 2. \tag{11}$$

Proof 1. First of all, notice that the process $\left(\int_0^s \mathbf{1}_{(0,t]}(\zeta_u)\, d\zeta_u, s \geq 0\right)$ is a $(\mathcal{F}_s)$-martingale with quadratic variation $\int_0^s \mathbf{1}_{(0,t]}(\zeta_u)\, du$. Hence, if $\theta \geq 0$, the process

$$\exp\left(\theta \int_0^{\tau_1 \wedge s} \mathbf{1}_{(0,t]}(\zeta_u)\, d\zeta_u - \frac{\theta^2}{2}\int_0^{\tau_1 \wedge s} \mathbf{1}_{(0,t]}(\zeta_u)\, du\right), \quad s \geq 0$$

is a local martingale. Moreover, using Novikov's criterion (see *e.g.* [9, p. 318]) it is in fact a martingale if (9) is satisfied. Hence, using the Cauchy-Schwarz inequality, assertion (10) is true. It remains to prove that assertion (9) is satisfied. Using the occupation times formula, and Ray-Knight Theorem, we get

$$\begin{aligned} E_0\left[\exp\left(\frac{\theta^2}{2}\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\, dr\right)\right] &= E_0\left[\exp\left(\frac{\theta^2}{2}\int_0^{t} L^x_{\tau_1}(\zeta)\, dx\right)\right] \\ &= \left(E\left[\exp\left(\frac{\theta^2}{2}\int_0^{t} R_x\, dx\right)\right]\right)^2 \end{aligned}$$

where $(R_x, x \geq 0)$ is the square of a 0-dimensional Bessel Process starting from 1. But we know the exact Laplace transforms of the process $\int_0^t R_x\,dx$ (see *e.g.* [9, cor. II.1.8 p. 425]): if $\lambda \geq 0$,

$$E\left[\exp\left(-\lambda \int_0^t R_x\,dx\right)\right] = \exp[-2\sqrt{2\lambda}\,\mathrm{th}(t\sqrt{2\lambda})].$$

The right part of this equality is an analytic function of λ, at least when $|\lambda|$ is small. As $\int_0^t R_x\,dx$ is positive, using a monotone convergence argument, we get that on the domain of analyticity of the right side, the left side is also analytic. Then, if $0 \leq \theta \leq \frac{\pi}{4t} \wedge 1$,

$$E_0\left[\exp\left(\frac{\theta^2}{2}\int_0^t R_x\,dx\right)\right] = \exp\bigl(2\theta\tan(t\theta)\bigr) \leq \exp\frac{1}{2},$$

which proves (9).

2. This part of the Lemma is easier to prove. We have to show that the time the process (ζ_s) stay in $(r,t]$ before time r has an exponential moment. But if we "erase" the parts of the process above t it remains a Brownian motion reflecting when it hits t. Hence,

$$\begin{aligned} E_a\left[\exp\left(\frac{\theta^2}{2}\int_0^{T_r}\mathbf{1}_{(r,t]}(\zeta_s)\,ds\right)\right] &\leq E_0\left[\exp\left(\frac{\theta^2}{2}T_{t-r}\right)\right] \\ &\leq E_0\left[\exp\left(\frac{\theta^2}{2}T_t\right)\right]. \end{aligned}$$

But (see *e.g.* [9, exercise II.3.10]), if $0 \leq \theta \leq \frac{\pi}{6t}$, we have

$$E_0\left[\exp\left(\frac{\theta^2}{2}T_t\right)\right] \leq \frac{1}{\cos(\theta t)} \leq 2,$$

which achieves to prove the result. ∎

Lemma 9 *Let $(\zeta, w) \in \mathcal{W}$, and r, t and ε such that $t > \zeta = r + \varepsilon > r \geq 0$. Under $\mathbb{P}_w$, we denote by $T_r = \inf\{s \geq 0; \zeta_s = r\}$ the first hitting time of r by the process (ζ_s). Then, if $0 \leq \theta \leq \frac{\pi}{3t}\|\varphi\|_\infty^{-1}$,*

$$\mathbb{E}_w\left[\exp\left(\theta\int_0^{T_r}\mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)\,d\zeta_s - \frac{\theta^2}{2}\int_0^{T_r}\mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)^2\,ds\right)\right] = 1.$$

Proof The process $\bigl(\int_0^{T_r\wedge s}\mathbf{1}_{(r,t]}(\zeta_u)\varphi(W_u)\,d\zeta_u, s \geq 0\bigr)$ is a $(\mathcal{F}_s)$-martingale with quadratic variation $\int_0^{T_r\wedge s}\mathbf{1}_{(r,t]}(\zeta_u)\varphi(W_u)^2\,du$, hence

$$\exp\left(\theta\int_0^{T_r\wedge s}\mathbf{1}_{(r,t]}(\zeta_u)\varphi(W_u)\,d\zeta_u - \frac{\theta^2}{2}\int_0^{T_r\wedge s}\mathbf{1}_{(r,t]}(\zeta_u)\varphi(W_u)^2\,du\right), \quad s \geq 0,$$

is a $(\mathcal{F}_s)$-local martingale. Moreover, using Novikov's criterion again, we know that it is a martingale if

$$\mathbb{E}_w\left[\exp\left(\frac{\theta^2}{2}\int_0^{T_r}\mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)^2\,ds\right)\right] < \infty,$$

which is true if $0 \leq \theta \leq \frac{\pi}{3t}\|\varphi\|_\infty^{-1}$ (see Lemma 8). ∎

Lemma 10 *Under $\mathbb{P}_{\bar{x}}$, the process*

$$M_t(\varphi) = \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\varphi(W_r)\, d\zeta_r, \quad t \geq 0$$

is a $(\mathcal{G}_t)$-martingale with quadratic variation

$$A_t(\varphi) = \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\varphi(W_r)^2\, dr.$$

Proof Using [8, proposition 1.17], it is sufficient to prove that

(i) the processes $\big(M_t(\varphi), t \geq 0\big)$ and $\big(A_t(\varphi), t \geq 0\big)$ and $(\mathcal{G}_t)$-adapted;
(ii) for every $t > 0$, there exists $\theta_0 > 0$ such that for every $\theta \in [0, \theta_0]$, the process

$$X_r^\theta = \exp\left(\theta M_r(\varphi) - \frac{\theta^2}{2} A_r(\varphi)\right), \quad r \in [0, t]$$

is a $(\mathcal{G}_r)$-martingale which satisfies $\mathbb{E}_{\bar{x}}\big[\exp\big(\theta M_t(\varphi)\big)\big] < \infty$.

(i) First of all, let us verify that the process $\big(M_t(\varphi), t \geq 0\big)$ is $(\mathcal{G}_t)$-adapted. Let us fix $\varepsilon > 0$, and introduce the sequence of $(\mathcal{F}_s)$-stopping times $(U_k^\varepsilon, k \geq 0)$ and $(V_k^\varepsilon, k \geq 0)$ defined by $U_0^\varepsilon = 0$, and for $k \geq 0$,

$$V_k^\varepsilon = \inf\{s \in (U_k^\varepsilon, \tau_1); \zeta_s = t\},$$
$$U_{k+1}^\varepsilon = \inf\{s \in (V_k^\varepsilon, \tau_1); \zeta_s = t - \varepsilon\},$$

with by convention $\inf \varnothing = \tau_1$. We set $K_\varepsilon = \max\{k; U_k^\varepsilon < \tau_1\}$. A second order moment calculus proves that

$$M_t(\varphi) = L^2 - \lim_{\varepsilon \to 0} \sum_{k=0}^{K_\varepsilon} \int_{U_k^\varepsilon}^{V_k^\varepsilon} \varphi(W_u)\, d\zeta_u.$$

Hence, we can find a sequence of positive numbers ε_n decreasing to 0, such that $\mathbb{P}_{\bar{x}}$-p.s.,

$$M_t(\varphi) = \lim_{\varepsilon_n \to 0} \sum_{k=0}^{K_{\varepsilon_n}} \int_{U_k^{\varepsilon_n}}^{V_k^{\varepsilon_n}} \varphi(W_u)\, d\zeta_u.$$

But the random variables of the right hand side are measurable for the σ-algebra generated by the processes $(W_{(U_k^\varepsilon + \cdot) \wedge V_k^\varepsilon}; k \geq 0)$, which is included in the σ-algebra $\mathcal{G}_t$. Hence $M_t(\varphi)$ is $\mathcal{G}_t$-measurable. A similar argument proves that $A_t(\varphi)$ is also $\mathcal{G}_t$-measurable.

(ii) Let us prove that for every fixed $t_0 > 0$, there exists $\theta_0 > 0$ such that if $\theta \in [0, \theta_0]$, the process

$$X_r^\theta = \exp\left(\theta M_r(\varphi) - \frac{\theta^2}{2} A_r(\varphi)\right), \quad 0 \leq r \leq t_0,$$

is a $(\mathcal{G}_r)$-martingale. To this aim, it is sufficient to prove that if $0 \le r \le t \le t_0$, then

$$\mathbb{E}_{\bar{x}}\left[\exp\left(\theta \int_0^{\tau_1} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)\, d\zeta_s - \frac{\theta^2}{2}\int_0^{\tau_1} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)^2\, ds\right) \Bigm| \mathcal{G}_r\right] = 1. \tag{12}$$

In a first step, we mimic the proof of the previous part. If $\varepsilon > 0$, we introduce the sequences of $(\mathcal{F}_s)$-stopping times $(S_k^\varepsilon, k \ge 0)$ and $(T_k^\varepsilon, k \ge 0)$ defined by $T_0^\varepsilon = 0$, and if $k \ge 1$,

$$S_k^\varepsilon = \inf\{s \in (T_{k-1}^\varepsilon, \tau_1); \zeta_s = r + \varepsilon\},$$
$$T_k^\varepsilon = \inf\{s \in (S_k^\varepsilon, \tau_1); \zeta_s = r\},$$

with the same convention $\inf \varnothing = \tau_1$. We set $N_\varepsilon = \max\{k; T_k^\varepsilon < \tau_1\}$. The integer N_k^ε is in fact the number of excursions of the process (ζ_s) above level r, which hit level $r + \varepsilon$. For $k \in \{1, \dots, N_\varepsilon\}$, we set

$$B_k^\varepsilon = \exp\left(\theta \int_{S_k^\varepsilon}^{T_k^\varepsilon} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)\, d\zeta_s - \frac{\theta^2}{2}\int_{S_k^\varepsilon}^{T_k^\varepsilon} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)^2\, ds\right)$$

the contribution of the k-th excursion. Let us prove that

$$\mathbb{E}_{\bar{x}}\Big[\prod_{k=1}^{N_\varepsilon} B_k^\varepsilon \mid \mathcal{G}_r\Big] = 1. \tag{13}$$

This can be proved using the monotone class Theorem, and applying the following result by induction: For every $\mathcal{F}_{S_k^\varepsilon}$-measurable bounded random variable U and for every G measurable nonnegative function defined on $C(\mathbb{R}_+, \mathcal{W})$,

$$\mathbb{E}_{\bar{x}}[U\mathbf{1}_{\{S_k^\varepsilon<\tau_1\}} B_k^\varepsilon G(W_{T_k^\varepsilon+\cdot})] = \mathbb{E}_{\bar{x}}[U\mathbf{1}_{\{S_k^\varepsilon<\tau_1\}} G(W_{T_k^\varepsilon+\cdot})]. \tag{14}$$

Using the strong Markov property at time T_k^ε, we can write $\mathbb{E}_{W_{T_k^\varepsilon}}(G)$ instead of $G(W_{T_k^\varepsilon+\cdot})$ in the left hand side. As $W_{T_k^\varepsilon} = W_{S_k^\varepsilon}^{\le r}$, if we apply one more time the strong Markov property, at time S_k^ε, we get

$$\begin{aligned}
&\mathbb{E}_{\bar{x}}[U\mathbf{1}_{\{S_k^\varepsilon<\tau_1\}} B_k^\varepsilon G(W_{T_k^\varepsilon+\cdot})] \\
&\quad = \mathbb{E}_{\bar{x}}\Bigg[U\mathbf{1}_{\{S_k^\varepsilon<\tau_1\}}\mathbb{E}_{W_{S_k^\varepsilon}^{\le r}}[G]\mathbb{E}_{W_{S_k^\varepsilon}}\Bigg[\exp\Bigg(\theta\int_0^{T_r} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)\, d\zeta_s \\
&\qquad\qquad - \frac{\theta^2}{2}\int_0^{T_r} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)^2\, ds\Bigg)\Bigg]\Bigg].
\end{aligned}$$

Now, applying Lemma 9, and the strong Markov property (in the other side this time), we obtain equality (14), and hence equality (13). To get equality (12) from equality (13), we use the following result. If U and V are random variables, then

$$\begin{aligned}
&\mathbb{E}_{\bar{x}}\big[|\mathbb{E}_{\bar{x}}[\exp(U) \mid \mathcal{G}_r] - \mathbb{E}_{\bar{x}}[\exp(V) \mid \mathcal{G}_r]|\big] \\
&\quad \le \left(\mathbb{E}_{\bar{x}}[|U - V|^2]\right)^{1/2} \left(\mathbb{E}_{\bar{x}}[\exp(2U) + \exp(2V)]\right)^{1/2}.
\end{aligned}$$

Let us set

$$U = \theta \int_0^{\tau_1} \mathbf{1}_{(r,t]}(\zeta_r)\varphi(W_s)\, d\zeta_s - \frac{\theta^2}{2}\int_0^{\tau_1} \mathbf{1}_{(r,t]}(\zeta_s)\varphi^2(W_s)\, ds$$

and

$$V = \sum_{k=1}^{N_\varepsilon} \left\{ \theta \int_{S_k^\varepsilon}^{T_k^\varepsilon} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)\, d\zeta_s - \frac{\theta^2}{2}\int_{S_k^\varepsilon}^{T_k^\varepsilon} \mathbf{1}_{(r,t]}(\zeta_s)\varphi(W_s)^2\, ds \right\}.$$

Using Lemma 8, if $\theta \le \theta_0 = \frac{1}{2}\left(\frac{\pi}{4t} \wedge 1\right) \|\varphi\|_\infty^{-1}$, then

$$\mathbb{E}_{\bar{x}}[\exp(2U) + \exp(2V)] \le 2\mathbb{E}_{\bar{x}}\left[\exp\left(\frac{1}{2}\left(\frac{\pi}{4t}\wedge 1\right)^2 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_s)\, ds\right)\right] \le 2\mathrm{e}.$$

Moreover, using the occupation times formula, and second Ray-Knight Theorem, we have

$$\begin{aligned}
\mathbb{E}_{\bar{x}}[|U-V|^2] &\le 2\left(\theta\|\varphi\|_\infty + \frac{\theta^2}{2}\|\varphi\|_\infty^2\right)\mathbb{E}_{\bar{x}}\left[\int_0^{\tau_1} \mathbf{1}_{(r,r+\varepsilon]}(\zeta_s)\, ds\right] \\
&\le 2\left(\|\varphi\|_\infty + \frac{1}{2}\|\varphi\|_\infty^2\right)\int_r^{r+\varepsilon} \mathbb{E}_{\bar{x}}[L^u_{\tau_1}(\zeta)]\, du \\
&\le 2\left(\|\varphi\|_\infty + \frac{1}{2}\|\varphi\|_\infty^2\right)\varepsilon.
\end{aligned}$$

Letting ε tends to 0, we get inequality (10). Hence we have proved that the process $(X_r^\theta, 0 \le r \le t)$ is a $(\mathcal{G}_r)$-martingale if $\theta \le \theta_0$. To achieve the proof of the Lemma, it is enough to prove that for $\theta \le \theta_0/2$, we have $\mathbb{E}_{\bar{x}}\big[\exp\big(\theta M_t(\varphi)\big)\big] < \infty$. But we have proved that the process $(X_r^{2\theta}, 0 \le r \le t)$ is a $(\mathcal{G}_r)$-martingale. Hence, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}_{\bar{x}}\big[\exp\big(\theta M_t(\varphi)\big)\big] &\le (\mathbb{E}_{\bar{x}}[X_t^{2\theta}])^{1/2}\Big(\mathbb{E}_{\bar{x}}\big[\exp\big(2\theta^2 A_t(\varphi)\big)\big]\Big)^{1/2} \\
&\le \left(\mathbb{E}_{\bar{x}}\left[\exp\left(\frac{\theta^2}{2}\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_s)\, ds\right)\right]\right)^{1/2} \le \mathrm{e}^{1/2},
\end{aligned}$$

thanks to Lemma 8, which achieves the proof of the Lemma. ■

Proof of Theorem 7 For $t > 0$ and $\varepsilon > 0$, let us consider the continuous function equal to 1 on $[\varepsilon, t]$ and 0 on $(-\infty, 0] \cup [t+\varepsilon, +\infty)$ and which is linear on $[0, \varepsilon]$ and on $[t, t+\varepsilon]$, and denote by g_ε an approximation of this function which is of class $\mathcal{C}^1$. We denote by φ a bounded function defined on $\mathcal{W}$ which belongs to the domain of the generator L. We then apply Itô's formula to the function

$$F_\varepsilon(w) = \int_0^{\zeta(w)} g_\varepsilon(r)\varphi(w_{\le r})\, dr$$

between times 0 and τ_1. We get

$$0 = \int_0^{\tau_1} g_\varepsilon(\zeta_r)\varphi(W_r)\, d\zeta_r + \frac{1}{2}\int_0^{\tau_1} g'_\varepsilon(\zeta_r)\varphi(W_r)\, dr + \frac{1}{2}\int_0^{\tau_1} g_\varepsilon(\zeta_r) L\varphi(W_r)\, dr.$$

We will now make ε decrease to 0. Using the well-known result of approximation of the local time, we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_0^{\tau_1} g'_\varepsilon(\zeta_r)\varphi(W_r)\,dr &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\tau_1}\mathbf{1}_{[0,\varepsilon]}(\zeta_r)\varphi(W_r)\,dr - \frac{1}{\varepsilon}\int_0^{\tau_1}\mathbf{1}_{[t,t+\varepsilon]}(\zeta_r)\varphi(W_r)\,dr\\ &= \int_0^{\tau_1} dL^0_s(\zeta)\varphi(W_s) - \int_0^{\tau_1} dL^t_s(\zeta)\varphi(W_s) = \langle H_0,\varphi\rangle - \langle H_t,\varphi\rangle.\end{aligned}$$

Hence, when $\varepsilon \downarrow 0$ we obtain

$$0 = \int_0^{\tau_1}\mathbf{1}_{(0,t]}(\zeta_r)\varphi(W_r)\,d\zeta_r + \frac{1}{2}\langle H_t,\varphi\rangle - \frac{1}{2}\langle H_0,\varphi\rangle + \frac{1}{2}\int_0^{\tau_1}\mathbf{1}_{[0,t]}(\zeta_r)L\varphi(W_r)\,dr.$$

Moreover, using the occupation times formula, we get that for every bounded continuous function ψ,

$$\int_0^t \langle H_s,\psi\rangle\,ds = \int_0^{\tau_1}\mathbf{1}_{[0,t]}(\zeta_r)\psi(W_r)\,dr.$$

We also obtain that

$$\langle H_t,\varphi\rangle - \langle H_0,\varphi\rangle - \int_0^t \langle H_s, L\varphi\rangle\,ds = M_t(\varphi)$$

where

$$M_t(\varphi) = 2\int_0^{\tau_1}\mathbf{1}_{(0,t]}(\zeta_r)\varphi(W_r)\,d\zeta_r.$$

But using Lemma 10, we know that the process $\big(M_t(\varphi)\big)$ is a $(\mathcal{G}_t)$-martingale with quadratic variation $4\int_0^{\tau_1}\mathbf{1}_{(0,t]}(\zeta_r)\varphi(W_r)^2\,dr$. This achieves the proof of Theorem 7. ■

6 Modified Brownian Snake

First of all, let us introduce some notations. Let us fix $x_0 \in \mathbb{R}^d$, $c\colon \mathbb{R}^d \to \mathbb{R}$ a continuous function which is bounded above and below by two positive constants, and $b\colon \mathcal{W} \to \mathbb{R}$ a bounded measurable function. For $w \in \mathcal{W}$, we set

$$\phi(w,s) = \int_0^s \frac{dr}{c\big(w(r)\big)},\quad \varphi(w,s) = \int_0^s c\big(w(r)\big)\,dr.$$

For every $w \in \mathcal{W}$, the function $\phi(w,\cdot)$ is a strictly increasing continuous function(in fact of class $\mathcal{C}^1$), which is then one to one from $\mathbb{R}_+$ into $\mathbb{R}_+$. For $r \geq 0$, let us introduce $\phi^{-1}(w,r) = \inf\{s \geq 0, \phi(w,s) = r\}$ and $\varphi^{-1}(w,r) = \inf\{s \geq 0, \varphi(w,s) = r\}$. For $w_1 \in \mathcal{W}$, we write $w_2 = w_1 \circ \phi^{-1}(w_1,\cdot)$ for the path defined by $w_2(r) = w_1\big(\phi^{-1}(w_1,r)\big)$. Notice that it is a one to one transformation, as it is described below:

Lemma 11 *The two following assertions are equivalent:*

(i) $w_2 = w_1 \circ \phi^{-1}(w_1,\cdot)$ *and* $\zeta_2 = \phi(w_1,\zeta_1)$;

(ii) $w_1 = w_2 \circ \varphi^{-1}(w_2, \cdot)$ *and* $\zeta_1 = \varphi(w_2, \zeta_2)$.

Proposition 12 *Let* $w \in \mathcal{W}_{x_0}$. *There exists a unique probability measure* $\mathbb{P}_w^{b,c}$ *on* $C(\mathbb{R}_+, \mathcal{W})$ *such that, under* $\mathbb{P}_w^{b,c}$, *the canonical process* (ζ_s, W_s) *is the only solution of the following martingale problem, denoted by* $\mathbf{PM}(b, c)$*: for every function* $g \colon \mathcal{W} \to \mathbb{R}$, *let the function* $F \colon \mathcal{W} \to \mathbb{R}$ *be defined by*

$$F(w) = \int_0^\zeta g(w_{\le r})\, dr;$$

if

$$h(w) = \frac{g\big(w \circ \phi^{-1}(w, \cdot)\big)}{c(\hat{w})}$$

is a bounded function which belongs to the domain of L and satisfies $h(\tilde{x_0}) = 0$, *and if* Lh *is also a bounded continuous function, then*

$$M(F)_s = F(W_s) - F(W_0) - \frac{1}{2}\int_0^s Lh\big(W_r \circ \varphi^{-1}(W_r, \cdot)\big)\, dr - 2\int_0^s \frac{g(W_r) b(W_r)}{c(\hat{W}_r)}\, dr$$

is a $(\mathcal{F}_s)$*-martingale with quadratic variation*

$$\langle M(F)\rangle_s = \int_0^s \frac{g^2(W_r)}{c^2(\hat{W}_r)}\, dr.$$

Moreover, under $\mathbb{P}_w^{b,c}$, *the lifetime process* (ζ_s) *is a diffusion which solves*

$$d\zeta_s - \frac{1}{2} dL_s^0(\zeta) = \frac{1}{c(\hat{W}_s)}\, d\gamma_s + \theta(W_s)\, ds,$$

where (γ_s) *is a linear Brownian motion, and the function* θ *is defined by*

$$\theta(w) = \frac{1}{2c(\hat{w})} A\left(\frac{1}{c}\right)(\hat{w}) - 2b(w).$$

Proof There will be two steps in this proof. In the first step, we prove the result for $b = 0$. To this aim, we follow an idea due to Watanabe [10]: we make a change of time for each path W_s. This modification of speed is, in terms of a particle system, a modification of the branching rate of the system. In the equation solved by the lifetime process, the consequence of this modification is the appearance of the coefficient $1/c(\hat{W}_s)$. In a second step, we introduce a killing term. For that, we use Girsanov Theorem, which adds a drift $\theta(\hat{W}_s)$ for the lifetime process.

First Step We suppose that $b = 0$. To prove the existence of solutions of the martingale problem $\mathbf{PM}(b, c)$, we will give an explicit construction using the standard Brownian snake.

Let us consider under $\mathbb{P}_w$ a Brownian snake (ζ_s^*, W_s^*) with spatial motion a Markov process of generator $\frac{1}{c}A$. The reason for the coefficient $1/c$ will appear later. We then define

$$\zeta_s = \phi(W_s^*, \zeta_s^*), \quad W_s = W_s^* \circ \phi^{-1}(W_s^*, \cdot).$$

Notice that thanks to Lemma 11 this is equivalent to

$$\zeta_s^* = \varphi(W_s, \zeta_s), \quad W_s^* = W_s \circ \varphi^{-1}(W_s, \cdot).$$

Now, using the change of variable $r = \phi(W_s^*, u)$ in the definition of F, we get

$$F(W_s) = \int_0^{\zeta_s} g(W_s^{\leq r})\, dr = F^*(W_s^*)$$

where

$$F^*(w) = \int_0^{\zeta} h(w_{\leq r})\, dr.$$

Applying Itô's formula given by Theorem 2 for the standard Brownian snake (W_s^*), we obtain

$$F(W_s^*) = F(W_0^*) + \frac{1}{2}\int_0^s Lh(W_r^*)\, dr + \int_0^s h(W_r^*)\, d\zeta_r^*.$$

This implies

$$F(W_s) = F(W_0) + \frac{1}{2}\int_0^s Lh\big(W_r \circ \varphi^{-1}(W_r, \cdot)\big)\, dr + \int_0^s \frac{g(W_r)}{c(\hat{W}_r)}\, d\zeta_r^* \tag{15}$$

Hence the process (ζ_s, W_s) solves the martingale problem $\mathbf{PM}(0, c)$ given by Proposition 12.

Conversely, if the process (ζ_s, W_s) solves the martingale problem $\mathbf{PM}(0, c)$ then the process (ζ_s^*, W_s^*) defined by the one to one transform

$$\zeta_s^* = \varphi(W_s, \zeta_s), \quad W_s^* = W_s \circ \varphi^{-1}(W_s, \cdot)$$

solves the martingale problem for the standard Brownian snake given in Theorem 3. As it is well posed, this implies that the martingale problem $\mathbf{PM}(0, c)$ is also well posed.

Second Step Let us denote by $\mathbb{P}_w^c$ the law of the process (ζ_s, W_s) starting from w we have built in the previous step. Let us introduce one more time the semi-martingale decomposition of the reflecting Brownian snake (ζ_s^*): $\zeta_s^* = \beta_s + \frac{1}{2}L_s^0(\zeta^*)$ where (β_r) is a $(\mathcal{F}_s)$-linear Brownian motion under $\mathbb{P}_w^c$, and $L_s^0(\zeta^*)$ is the local time at level 0 of the reflecting Brownian motion (ζ_s^*). Notice that we have supposed $g(\bar{x}_0) = 0$. Hence we get that $\{\zeta_r = 0\} = \{\zeta_r^* = 0\}$ which implies that

$$\int_0^s \frac{g(W_r)}{c(\hat{W}_r)}\, d\zeta_r^* = \int_0^s \frac{g(W_r)}{c(\hat{W}_r)}\, d\beta_r. \tag{16}$$

Let us introduce the $(\mathcal{F}_s)$-martingale $L_s = -2\int_0^s b(W_r)\,d\beta_r$, $s \geq 0$. Its quadratic variation is given by $\langle L\rangle_s = 4\int_0^s b^2(W_r)\,dr$. Hence we can (see *e.g.* [9, appendice (6.1) p. 521]) define a probability measure $\mathbb{P}_w^{b,c}$ on $C(\mathbb{R}_+, \mathcal{W})$ locally equivalent to $\mathbb{P}_w^c$, by setting, for every $s > 0$,

$$\left.\frac{d\mathbb{P}_w^{b,c}}{d\mathbb{P}_w^c}\right|_{\mathcal{F}_s} = \mathcal{E}(L)_s = \exp -2\left(\int_0^s b(W_r)\,d\beta_r + \int_0^s b^2(W_r)\,dr\right).$$

Using Girsanov Theorem, we then know that the process defined by

$$\gamma_s = \beta_s - \langle L, \beta\rangle_s = \beta_s + 2\int_0^s b(W_r)\,dr$$

is under $\mathbb{P}_w^{b,c}$ a $(\mathcal{F}_s)$-linear Brownian motion. Hence if we write $d\gamma_s - 2b(W_s)\,ds$ for $d\beta_s$ in formula (16), we get

$$F(W_s) = F(W_0) + \frac{1}{2}\int_0^s Lh\big(W_r \circ \varphi^{-1}(W_r,\cdot)\big)\,dr \\ - 2\int_0^s \frac{g(W_r)b(W_r)}{c(\hat{W}_r)}\,dr + \int_0^s h\big(W_r \circ \varphi^{-1}(W_r,\cdot)\big)\,dr.$$

This implies that $\mathbb{P}_w^{b,c}$ is solution of the martingale problem $\mathbf{PM}(b,c)$.

Conversely, using again the same Girsanov transform, but with the function $-b$ instead of b, we can build from any solution of the martingale problem $\mathbb{P}^{b,c}$ a solution of the martingale problem $\mathbf{PM}(0,c)$. The uniqueness of solutions of this problem immediately prove that the problem $\mathbb{P}^{b,c}$ is well posed. ■

Proposition 13 *The probability measure $\mathbb{P}_w^{b,c}$ is absolutely continuous with respect to the probability measure $\mathbb{P}_w^c$, on $\mathcal{F}_{\tau_1}$ (a case we will denote by* **(AC)***) if at least one of these conditions is true:*

1. $\big(\mathcal{E}(L)_{s\wedge\tau_1}; s \geq 0\big)$ *is uniformly integrable;*
2. $\mathbb{E}_w^c\big[\exp\big(2\int_0^{\tau_1} b^2(W_r)\,dr\big)\big] < +\infty$*;*
3. $\exists T > 0, \forall w \in \mathcal{W}, b(w) \leq \frac{\pi}{8T}\mathbf{1}_{\{\zeta(w)\leq T\}}$*;*
4. $\Big(\exp -2\big(\int_0^{\tau_1\wedge s} b(W_r)\,d\beta_r\big); s \geq 0\Big)$ *is uniformly integrable;*
5. $\forall w \in \mathcal{W}, b(w) = b_0$ *with* $b_0 \geq 0$.

Proof It is well known that the first property is equivalent to absolute continuity **(AC)**. The second point is a version of Novikov's criterion. The third point is an example of function b such that Novikov's criterion can be used. The fourth point is a version of Kazamaki's criterion. The last point is an example of function b such that 1 holds. Indeed, for such a function b,

$$\mathcal{E}(L)_{s\wedge\tau_1} = \exp\big(-2b_0\zeta^*_{s\wedge\tau_1} + b_0 L^0_{s\wedge\tau_1}(\zeta^*) - 2b_0(s\wedge\tau_1)\big) \leq \exp(b_0).$$

7 Construction of a $\big(A, c(x)z^2, b\big)$ Super-Process

We have already explained how to give an explicit construction of a (A, z^2)-super-process using the Brownian snake. In this section, we will see that we can also construct a more general class of super-processes—that will be denoted by $\big(A, c(x)z^2, b\big)$ super-processes—using the modified Brownian snake. The rigorous definition of these processes has been given in introduction. To construct it from the modified Brownian snake, we proceed as we did for super-Brownian motion from the standard Brownian snake. Let us reuse the notations introduced in the previous section. Under the probability measure $\mathbb{P}^c_w$, we have

$$\zeta_s = \zeta_0 + \int_0^s \frac{1}{c(\hat{W}_r)}\, d\zeta^*_r + \frac{1}{2}\int_0^s \frac{1}{c(\hat{W}_r)} A\left(\frac{1}{c}\right)(\hat{W}_r)\, dr. \tag{17}$$

This result is an immediate consequence of the definition of ζ_r, applying Itô's formula (15). In particular, under the probability measure $\mathbb{P}^c_w$, the lifetime process (ζ_s) is a semi-martingale. Hence it is also a semi-martingale under the probability measure $\mathbb{P}^{b,c}_w$, thanks to Girsanov Theorem. This proves that we can define a family of local times $L^t_s(\zeta)$. Moreover, we have

$$L^t_s(\zeta) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int_0^s \mathbf{1}_{\{\zeta_r \in (t, t+\varepsilon)\}}\, d\langle \zeta\rangle_r.$$

Notice that under $\mathbb{P}^c_w$, or under $\mathbb{P}^{b,c}_w$, the quadratic variation is given by $d\langle\zeta\rangle_r = dr/c^2(\hat{W}_r)$. In fact, following Watanabe [10], it is more relevant to use a slightly different increasing process. Let us introduce

$$l(s,t) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^s \mathbf{1}\big(\varphi(W^*_r, t) \le \varphi(W^*_r, \zeta_r) \le \varphi(W^*_r, t) + \varepsilon\big)\, dr.$$

This process is closely connected with the process $\big(L^t_s(\zeta), s \ge 0, t \ge 0\big)$: Under $\mathbb{P}^c_w$ or $\mathbb{P}^{b,c}_w$, we have

$$l(s,t) = \int_0^s d_{(r)} L^t_r(\zeta) c(\hat{W}_r)\, dr.$$

Let us introduce the first time the process $(l(s,0), s \ge 0)$ hits 1:

$$\tau_1 = \inf\{s; l(s,0) > 1\}$$

We define under $\mathbb{P}^{b,c}_w$ the process $(Z_t, t \ge 0)$ with values in $M_F\big(C(\mathbb{R}_+, \mathbb{R}^d)\big)$ by: For every bounded continuous function $\psi\colon C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}_+$,

$$\begin{aligned} \langle Z_t, \psi\rangle &= \int_0^{\tau_1} d_{(s)} l(s,t)\psi(W_s) \\ &= \int_0^{\tau_1} dL^t_s(\zeta) c(\hat{W}_s)\psi(W_s). \end{aligned} \tag{18}$$

Theorem 14 *Let $x \in \mathbb{R}^d$, c a continuous real function on $\mathbb{R}^d$ bounded above and below by positive constants, and b a bounded measurable function such that $b(\tilde{x}) = 0$ and the condition* **(AC)** *given in Proposition 13 is satisfied.*

Then, under the probability measure $\mathbb{P}^{b,c}_{\tilde{x}}$, the process $(Z_t, t \ge 0)$ with values in $M_F\big(C(\mathbb{R}_+, \mathbb{R}^d)\big)$ defined by (18) is a $\big(A, c(x)z^2, b\big)$-historical super-process starting from $X_0 = \delta_{\tilde{x}}$.

Proof Let us introduce a bounded continuous function $\psi\colon C(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}_+$. First, we work under $\mathbb{P}^c_{\bar{x}}$. We apply Itô's formula (15) with the function $F(w)=\int_0^\zeta g_\varepsilon(r)c\big(w(r)\big)\psi(w_{\le r})\,dr$ for $s=\tau_1$ and g the function g_ε which is the approximation of the function $\mathbf{1}_{(0,t]}$ already used for the proof of Theorem 7. We get

$$(19)\qquad 0=\int_0^{\tau_1} g_\varepsilon(\zeta_r)\psi(W_r)\,d\zeta^*_r+\frac{1}{2}\int_0^{\tau_1} Lh\big(W_r\circ\varphi^{-1}(W_r,\cdot)\big)\,dr$$

where

$$h(w)=g_\varepsilon\big(\phi(w,\zeta)\big)\psi\big(w\circ\phi^{-1}(w,\cdot)\big).$$

It is easy to see that

$$Lh(w)=g'_\varepsilon\big(\phi(w,\zeta)\big)\frac{1}{c(w)}\psi\big(w\circ\phi^{-1}(w,\cdot)\big)+g_\varepsilon\big(\phi(w,\zeta)\big)\frac{1}{c(w)}L\psi\big(w\circ\phi^{-1}(w,\cdot)\big).$$

Let us recall that the derivate of the function g_ε is an approximation of the function $\frac{1}{\varepsilon}(\mathbf{1}_{(0,\varepsilon)}-\mathbf{1}_{(t,t+\varepsilon)})$. Using that $d\langle\zeta\rangle_r=dr/c^2(\hat{W}_r)$, the occupation time formula applied to the semi-martingale (ζ_s) gives

$$\int_0^{\tau_1}\frac{1}{\varepsilon}\mathbf{1}_{(0,\varepsilon)}(\zeta_r)\frac{\psi(W_r)}{c(\hat{W}_r)}\,dr=\frac{1}{\varepsilon}\int_0^\varepsilon ds\int_0^{\tau_1} d_{(r)}L^s_r(\zeta)c(\hat{W}_r)\psi(W_r)$$
$$\xrightarrow{\varepsilon\downarrow 0}\int_0^{\tau_1} d_{(r)}L^0_r(\zeta)c(\hat{W}_r)\psi(W_r)=\langle Z_0,\varphi\rangle.$$

Similar arguments prove that

$$\int_0^{\tau_1}\frac{1}{\varepsilon}\mathbf{1}_{(t,t+\varepsilon)}(\zeta_r)\frac{\psi(W_r)}{c(\hat{W}_r)}\,dr\xrightarrow{\varepsilon\downarrow 0}\langle Z_t,\psi\rangle.$$

Using again the occupation time formula, we also get

$$(20)\qquad \int_0^t ds\langle Z_s,L\psi\rangle=\int_0^{\tau_1}\mathbf{1}_{(0,t)}(\zeta_r)\frac{L\psi(\hat{W}_r)}{c(\hat{W}_r)}\,dr.$$

Hence, letting ε decrease to 0 in (19), we obtain

$$\langle Z_t,\psi\rangle-\langle Z_0,\psi\rangle-\int_0^t ds\langle Z_s,L\psi\rangle=2\int_0^{\tau_1}\mathbf{1}_{(0,t)}(\zeta_r)\psi(W_r)\,d\zeta^*_r.$$

Let us denote by $N_t(\psi)$ the right hand side. We want to prove that under $\mathbb{P}^c_{\bar{x}}$, the process $\big(N_t(\psi),t\ge 0\big)$ is a $(\mathcal{G}_t)$-martingale, with quadratic variation

$$\langle N(\psi)\rangle_t=4\int_0^{\tau_1}\mathbf{1}_{(0,t)}(\zeta_r)\psi^2(W_r)\,dr.$$

Notice that using equation (17), we can write $d\zeta^*_r=c(\hat{W}_r)\,d\zeta_r+\theta(\hat{W}_r)\,dr$, where $\theta(w)=\frac{1}{2}A(\frac{1}{c})(\hat{w})$. The $\mathcal{G}_t$-measurability of $N_t(\psi)$ can be obtained as in the proof of Lemma 10.

In order to follow again the proof of Lemma 10, we have to prove that the strong Markov property is still true under $\mathbb{P}^c_{\bar{x}}$. This is clear, because $\mathbb{P}^c_{\bar{x}}$ is the law of a process obtained as a deterministic functional of the process (ζ^*_s, W^*_s), which is a strong Markov process. The next step is the analogue of Lemma 9: We only have to use that there exists two constants α_1 and α_2 such that, for every $x \in \mathbb{R}^d$, $0 < \alpha_1 \leq 1/c(x) \leq \alpha_2 < +\infty$. Using this remark again, and mimicking Lemma 10, we end the proof that the process $\big(N_t(\psi)\big)$ is a martingale. Hence under $\mathbb{P}^c_{\bar{x}}$ we have proved that the process (Z_t) is a solution of the martingale problem given by (1) and (2) in introduction with $b = 0$.

We now study the general case for b. Let us recall that the assumption of absolute continuity **(AC)**implies that the probability measure $\mathbb{P}^{b,c}_{\bar{x}}$ can be obtained from $\mathbb{P}^c_{\bar{x}}$ by

$$\begin{aligned} \frac{d\mathbb{P}^{b,c}_{\bar{x}}}{d\mathbb{P}^c_{\bar{x}}}\Big|_{\mathcal{F}_{\tau_1}} &= \mathcal{E}(L)_{\tau_1} = \exp\left(L_{\tau_1} - \frac{1}{2}\langle L\rangle_{\tau_1}\right) \\ &= \exp\left(-2\int_0^{\tau_1} b(W_r)\,d\beta_r - 2\int_0^{\tau_1} b^2(W_r)\,dr\right). \end{aligned}$$

But the process

$$N_t(b) = 2\int_0^{\tau_1} \mathbf{1}_{(0,t)}(\zeta_r) b(W_r)\,d\zeta^*_r, \quad t \geq 0$$

is a $\big(\mathbb{P}^c, (\mathcal{G}_t)\big)$-martingale. Hence the process

$$\mathcal{E}\big(-N(b)\big)_t = \exp\left(-N_t(b) - \frac{1}{2}\langle N(b)\rangle_t\right), \quad t \geq 0$$

is a $\big(\mathbb{P}^c, (\mathcal{G}_t)\big)$-local martingale. As it is nonnegative, it is in fact a super-martingale. Moreover, the Lebesgue Dominated Convergence Theorem for stochastic integrals (see *e.g.* [9, th. 2.12 p. 134]) implies that there exists an increasing sequence of positive reals t_n which tends to ∞ such that $N_{t_n}(b)$ tends to $2\int_0^{\tau_1} b(W_r)\,d\beta_r$ almost surely. Hence, the sequence $\mathcal{E}\big(-N(b)\big)_{t_n}$ almost surely tends to $\mathcal{E}(L)_{\tau_1}$, and we note that the expectation of this random variable is 1. We deduce that the martingale $\mathcal{E}\big(-N(b)\big)$ is uniformly integrable. In particular, if $t \geq 0$, we have

$$\mathcal{E}\big(-N(b)\big)_t = \mathbb{E}^c[\mathcal{E}(L)_{\tau_1} \mid \mathcal{G}_t].$$

This implies that

$$\frac{d\mathbb{P}^{b,c}_{\bar{x}}}{d\mathbb{P}^c_{\bar{x}}}\Big|_{\mathcal{G}_t} = \mathbb{E}^c[\mathcal{E}(L)_{\tau_1} \mid \mathcal{G}_t] = \mathcal{E}\big(-N(b)\big)_t.$$

But we have already proved that the process

$$N_t(\psi) = \langle Z_t, \psi\rangle - \langle Z_0, \psi\rangle - \int_0^t ds\langle Z_s, L\psi\rangle, \quad t \geq 0$$

is a $\big(\mathbb{P}^c, (\mathcal{G}_t)\big)$-martingale. Hence, using Girsanov Theorem, we get that the process

$$M_t(\psi) = N_t(\psi) + \langle N(\psi), N(b)\rangle_t, \quad t \geq 0$$

is a $\left(\mathbb{P}^{b,c}, (\mathcal{G}_t)\right)$-martingale. Moreover,

$$\langle N(\psi), N(b)\rangle_t = 4\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r) b(W_r)\psi(W_r)\, dr$$
$$= \int_0^t ds \langle Z_s, bc\psi\rangle.$$

Finally, we have proved that the process

$$M_t(\psi) = \langle Z_t, \psi\rangle - \langle Z_0, \psi\rangle - \int_0^t ds\langle Z_s, L\psi\rangle - \int_0^t ds\langle Z_s, bc\psi\rangle, \quad t \geq 0$$

is a $\left(\mathbb{P}^{b,c}, (\mathcal{G}_t)\right)$-martingale with quadratic variation

$$\langle M(\psi)\rangle = \int_0^t ds\langle Z_s, c\psi^2\rangle,$$

which achieves the proof. ∎

Acknowledgements We would like to thank Prof. J-F. Le Gall for helpful discussions during the preparation of this paper.

References

[1] D. A. Dawson, *Measure-valued Markov processes.* École d'été de probabilités de Saint-Flour 1991, Lecture Notes in Math. **1541**(1993), 1–260.

[2] D. A. Dawson and E. A. Perkins, *Historical processes.* Mem. Amer. Math. Soc. (454) **93**(1991).

[3] E. B. Dynkin, *Path processes and historical superprocesses.* Probab. Theory Related Fields **90**(1991), 1–36.

[4] S. N. Ethier and T. G. Kurtz, *Markov processes, characterization and convergence.* Wiley, New York, 1986.

[5] W. Feller, *The parabolic differential equations and the associated semi-groups of transformations.* Ann. Math. **55**(1952), 458–519.

[6] J-F. Le Gall, *A class of path-valued Markov processes and its applications to superprocesses.* Probab. Theory Related Fields **95**(1993), 25–46.

[7] ——, *A path-valued Markov process and its connections with partial differential equations.* Proceedings in First European Congress of Mathematics, vol. II, Progr. Math. **120**(1994), 185–212.

[8] P. Priouret, *Processus de diffusion et équations différentielles stochastiques.* École d'été de probabilités de Saint-Flour 1973, Lecture Notes in Math. **390**(1974), 37–113.

[9] D. Revuz and M. Yor, *Continuous martingales and Brownian motion.* 2nd edn, Springer Verlag, 1995.

[10] S. Watanabe, *Branching diffusions (superdiffusions) and random snakes.* Trends in Probability and Related Analysis (Taipei, 1996), World Scientific Publishing, 1997, 289–304.

[11] A. D. Wentzell, *Infinetesimal characteristics of Markov processes in a function space which describes the past.* Theory Probab. Appl. **30**(1985), 625–639.

UFR de Mathématiques et d'Informatique
Université René Descartes
45 rue des Saint Pères
75270 Paris Cedex 06
France
email: dhersin@math-info.univ-paris5.fr
serlet@math-info.univ-paris5.fr