

KOLMOGOROV'S TEST FOR THE BROWNIAN SNAKE

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We present a Kolmogorov's test for the Brownian snake. This result was conjectured by Le Gall in 1998. It has to be compared with Kolmogorov's test for super Brownian motion by Dhersin and Le Gall.

1. Introduction. Super-Brownian motion $X = (X_t, t \geq 0)$ is a measure valued process which appears as a limit of branching particles systems (see [2]). Dhersin and Le Gall [4] studied the growth of the radius ρ_t^X of the support of X_t : $\rho_t^X = \inf\{R > 0; \text{supp } X_t \subset B(0, R)\}$, where $B(0, R)$ is the ball centered at 0 with radius R . In particular for the super-Brownian motion started at the Dirac mass at $0 \in \mathbb{R}^d$, they consider the exit time T^X of the domain $Q = \{(t, x); t > 0, x \in \mathbb{R}^d, |x| < \sqrt{t}h(t)\}$, where h is a nonincreasing measurable function: $T^X = \inf\{t > 0; \rho_t^X > \sqrt{t}h(t)\}$. They proved that $T^X = 0$ a.s. if and only if the integral

$$(1) \quad \int_{0+} \frac{h(t)^{d+2}}{t^2} e^{-h(t)^2/2} dt \quad \text{is divergent.}$$

This result has been extended in [3] for a general domain $Q \subset (0, +\infty) \times \mathbb{R}^d$ using a parabolic capacity. Super-Brownian motion is an infinitely divisible process. A description of the canonical measure \mathbb{N}_0 was given by Le Gall [13] using a path valued process called the Brownian snake. Roughly speaking, the clumps described by Brownian snakes are the contributions of infinitesimal ancestor particles located at 0. Since super-Brownian motion has countably many such independent clumps, one can expect a different behavior if one looks only at one clump. For example, the speed of growth of its radius ρ_t should be slower. Indeed we prove that $T = \inf\{t > 0; \rho_t > \sqrt{t}h(t)\}$ is zero a.e. if and only if the integral

$$(2) \quad \int_{0+} \frac{h(t)^{d+2}}{t} e^{-h(t)^2/2} dt \quad \text{is divergent.}$$

Notice the power of $1/t$ is 1 instead of 2 for super-Brownian motion.

Super-Brownian motion and the Brownian snake are used to represent solutions to nonlinear PDE (see [5, 6, 7, 12] and references therein). In particular, if X is under $\mathbb{P}_{t,x}$ a super Brownian motion started at time t at the Dirac mass at point $x \in \mathbb{R}^d$, then $u(t, x) = -\log \mathbb{P}_{t,x}[T^X = \infty]$ is the maximal nonnegative solution to $\partial_t u + (\Delta/2)u = 2u^2$ in Q . Therefore if h satisfies the integral test (1) then the function u blows up at $(0, 0)$. We prove here that if h satisfies the stronger integral test (2) then a.s. $\int_{0+} u(t, \gamma_t) dt = +\infty$, where γ is a

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Brownian motion started at 0 (see Corollary 2). There are other zero–one laws for finiteness of Brownian integrals: for example, see [14] for $\int_{\mathbb{R}^+} v(|\gamma_t|) dt$ or more generally the zero–one law of Engelbert–Schmidt on the finiteness of $\int_{0^+} v(\gamma_t) dt$. See [8] and in a parabolic setting see the results from Jeulin on the integrals $\int_{\mathbb{R}^+} v(|\gamma_t|/\sqrt{t})\varphi(t) dt$ ([10] or [9], page 44) for a more general result.

2. Kolmogorov’s test for the Brownian snake. In this section, we briefly recall the basic facts concerning the Brownian snake and give a rigorous version of our main result.

The Brownian snake is a Markov process with values in the set of stopped paths. A stopped path is a pair (w, ζ) , where $\zeta \geq 0$ and $w: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is a continuous mapping such that $w(t) = w(\zeta)$ for every $t \geq \zeta$. The real ζ is called the lifetime of the path. We always abuse notation and simply write w for (w, ζ) . We also use the notation $\hat{w} = w(\zeta)$ for the tip of the path. We endow the set \mathscr{W} of all stopped paths with the distance

$$d(w, w') = \sup_{t \geq 0} |w(t) - w'(t)| + |\zeta - \zeta'|.$$

We shall write ζ_w for the lifetime of w . Let $x \in \mathbb{R}^d$ be a fixed point. We denote by \mathscr{W}_x the set of all stopped paths with initial point $w(0) = x$.

The Brownian snake with initial point x is the continuous strong Markov process $W = (W_s, s \geq 0)$ in \mathscr{W}_x whose law is characterized as follows:

1. If ζ_s denotes the lifetime of W_s , the process $(\zeta_s, s \geq 0)$ is a reflecting Brownian motion in \mathbb{R}_+ .
2. Conditionally on $(\zeta_s, s \geq 0)$, the process W is a time-inhomogeneous Markov process whose transition kernels are characterized by the following properties: if $0 \leq s < s'$,
 - (a) $W_{s'}(t) = W_s(t)$ for every $t \leq m(s, s') := \inf_{[s, s']} \zeta_r$;
 - (b) $(W_{s'}(m(s, s') + t) - W_{s'}(m(s, s')), 0 \leq t \leq \zeta_{s'} - m(s, s'))$ is a Brownian motion in \mathbb{R}^d , independent of W_s .

Heuristically, we can see W_s as a Brownian path in \mathbb{R}^d whose random lifetime ζ_s evolves like reflecting Brownian motion. Furthermore, when ζ_s decreases, the path W_s is “erased”; when ζ_s increases, the path W_s is extended by “adding” independent pieces of d -dimensional Brownian motion at its tip.

From now on we shall consider the canonical realization of the process W defined on the space $C(\mathbb{R}^+, \mathscr{W}_x)$ of all continuous functions from \mathbb{R}_+ into \mathscr{W}_x . The law of W started at $w \in \mathscr{W}_x$ is denoted by \mathcal{E}_w . We shall write \mathcal{E}_w^* for the law of the process W killed when its lifetime reaches zero. The distribution of W under \mathcal{E}_w^* can be characterized as above, except that in 1 the lifetime process is distributed as a linear Brownian motion killed at its first hitting time of $\{0\}$. The state space for (W, \mathcal{E}_w^*) is the space $\mathscr{W}_x^* = \mathscr{W}_x \cup \partial$, where ∂ is a cemetery point.

The trivial path \mathbf{x} such that $\zeta_{\mathbf{x}} = 0, \mathbf{x}(0) = x$ is clearly a regular point for the process (W, \mathcal{E}_w) . Following [1], Chapter 3, we can consider the excursion

measure, \mathbb{N}_x , outside $\{\mathbf{x}\}$, normalized by

$$\mathbb{N}_x \left[\sup_{s \geq 0} \zeta_s > 1 \right] = \frac{1}{2}.$$

The distribution of W under \mathbb{N}_x can be characterized as above, except that in 1 the lifetime process ζ is distributed according to Itô measure of positive excursions of linear Brownian motion. Using this remark, it is easy to see that \mathbb{N}_x satisfies the following useful scaling property: if $\lambda > 0$, we define $W_s^{(\lambda)} \in \mathscr{W}_x$ by

$$\zeta_s^{(\lambda)} = \lambda^{-2} \zeta_{\lambda^4 s}, \quad W_s^{(\lambda)}(t) - x = \lambda^{-1}(W_{\lambda^4 s}(\lambda^2 t) - x), \quad s \geq 0, t \geq 0;$$

then the law under \mathbb{N}_x of the process $W^{(\lambda)}$ is $\lambda^{-2}\mathbb{N}_x$.

Let $\sigma = \inf\{s > 0; \zeta_s = 0\}$ denote the length of the excursion for the lifetime process under \mathbb{N}_x .

Let h be a measurable nonincreasing nonnegative function defined on $(0, \infty)$. Let $Q = \{(t, x); |x| < \sqrt{t}h(t)\}$ be a domain of $\mathbb{R}^+ \times \mathbb{R}^d$. We define T as the lowest level at which the Brownian snake hit Q^c :

$$T = \inf\{t > 0; \rho_t > \sqrt{t}h(t)\},$$

where $\rho_t = \sup\{|\widehat{W}_s|; \zeta_s = t\}$ is the radius of the smallest ball centered at the origin that contains the snake at level t . We use the convention $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

THEOREM 1. *We have the following 0–1 law type for the Brownian snake started at $0 \in \mathbb{R}^d$:*

- (i) $\mathbb{N}_0[T = 0] = 0$ if there exists $r > 0$, s.t. $\int_0^r (h(t)^{d+2}/t)e^{-h(t)^2/2} dt < +\infty$.
- (ii) $\mathbb{N}_0[T > 0] = 0$ if for all $r > 0$, $\int_0^r (h(t)^{d+2}/t)e^{-h(t)^2/2} dt = +\infty$.

As a direct consequence of Section 4 we can deduce the following corollary. Let u be the maximal solution to $\partial_t u + (\Delta/2)u = 2u^2$ in Q . Let γ be a Brownian motion in \mathbb{R}^d started at 0 under \mathbb{P} .

COROLLARY 2. *We have a 0–1 law for the finiteness of $\int_{0+} u(t, \gamma_t) dt$: \mathbb{P} -a.s. there exists $r > 0$ s.t. $\int_0^r u(t, \gamma_t) dt$ is finite if and only if the integral $\int_0^s (h(t)^{d+2}/t)e^{-h(t)^2/2} dt$ is finite for some $s > 0$.*

3. More details on the snake. We introduce the law of the process W started from a Brownian path with lifetime $\zeta_0 = \delta > 0$ and killed when its lifetime reaches zero,

$$\mathcal{E}_{(\delta)}^*(dW) = \int \mathbb{P}_x^\delta(d\gamma) \mathcal{E}_\gamma^*(dW),$$

where $\mathbb{P}_x^\delta(d\gamma)$ is the law of Brownian motion started at $x \in \mathbb{R}^d$ and stopped at time δ , considered as a probability measure on \mathscr{W}_x^* .

We also recall a particular case of the strong Markov property for the snake under \mathbb{N}_x (this can be deduced from [13]). Let τ be a stopping time of the natural filtration generated by the lifetime process. Assume $\tau > 0$ \mathbb{N}_x -a.e., and let H be a nonnegative measurable functional on $C(\mathbb{R}^+, \mathscr{W}_x^*)$. Then if θ denotes the usual shift operator, we have

$$(3) \quad \mathbb{N}_x[\tau < \infty; H \circ \theta_\tau] = \mathbb{N}_x[\tau < \infty; \mathcal{E}_{(\zeta_\tau)}^*[H]].$$

Eventually we recall the Poissonian decomposition of the process W under \mathcal{E}_w^* (see [13]). Let $(\bar{\zeta}_s = \inf_{u \in [0, s]} \zeta_u, s \geq 0)$ be the infimum process associated to the lifetime process. Let $((\alpha_i, \beta_i), i \in \mathcal{I})$ be the excursion intervals of the process $(\zeta_s - \bar{\zeta}_s, s \geq 0)$ above zero. For $s \in (\alpha_i, \beta_i)$, the paths W_s coincide up to time $\zeta^i = \zeta_{\alpha_i} = \zeta_{\beta_i}$. Let W^i denote the corresponding excursion of the snake: for $s \geq 0$,

$$\zeta_s^i = \zeta_{(\alpha_i+s) \wedge \beta_i} \quad \text{and} \quad W_s^i(t) = W_{(\alpha_i+s) \wedge \beta_i}(t), \quad t \in [\zeta^i, \zeta_{(\alpha_i+s) \wedge \beta_i}].$$

For $(r, x) \in \mathbb{R} \times \mathbb{R}^d$, let $\mathbb{N}_{r, x}$ denote the law of the process $(W'_s, s \geq 0)$ defined by $W'_s(u) = W_s(u - r), u \geq r$ under \mathbb{N}_x . Under $\mathbb{N}_{r, x}$, the lifetime process is a Brownian excursion above level r . Then we have the following decomposition of the Brownian snake.

The random measure $\sum_{i \in \mathcal{I}} \delta_{\zeta^i, W^i}$ is under \mathcal{E}_w^* a Poisson point measure with intensity

$$2 \times \mathbf{1}_{[0, \zeta_w]}(t) dt \mathbb{N}_{t, w(t)}[dW].$$

From the definition of $\mathcal{E}_{(\delta)}^*$, we see that for any $\eta \geq \delta$, the law of $((\zeta^i, W^i), i \in \mathcal{I}$ such that $\zeta^i \leq \delta)$ under $\mathcal{E}_{(\eta)}^*$ is the law of $((\zeta^i, W^i), i \in \mathcal{I})$ under $\mathcal{E}_{(\delta)}^*$.

4. Preliminaries estimates. We fix the starting point of the snake at $x = 0$.

We will prove in this section that either $\mathbb{N}_0[T = 0] = 0$ or $\mathbb{N}_0[T > 0] = 0$.

Let $\delta > 0$ and $\tau_\delta = \inf\{s > 0; \zeta_s = \delta\}$, with the convention $\inf \emptyset = +\infty$. The time τ_δ is a stopping time with respect to the filtration generated by the lifetime process of the snake. Let I be a Borel subset of $[0, \infty)$. Thanks to (3), we have

$$\begin{aligned} & \mathbb{N}_0 \left[\tau_\delta < \infty, \sup_{t \in I} \frac{\rho_t}{\sqrt{t}h(t)} > 1 \right] \\ & \geq \mathbb{N}_0 \left[\tau_\delta < \infty, \exists s \in [\tau_\delta, \sigma] \text{ s.t. } \zeta_s \in I \text{ and } |\widehat{W}_s| > \sqrt{\zeta_s}h(\zeta_s) \right] \\ & = \mathbb{N}_0 \left[\tau_\delta < \infty, \mathcal{E}_{(\delta)}^* \left[\sup_{t \in I} \frac{\rho_t}{\sqrt{t}h(t)} > 1 \right] \right] \\ & = \frac{1}{2\delta} \mathcal{E}_{(\delta)}^* \left[\sup_{t \in I} \frac{\rho_t}{\sqrt{t}h(t)} > 1 \right]. \end{aligned}$$

The time-reversal invariance property of the Itô measure and the characterization of the excursion measure \mathbb{N}_x readily imply that the latter itself enjoys the same invariance property. Let $L_\delta = \sup\{s > 0; \zeta_s = \delta\}$. Thus we have

$$\begin{aligned} & \mathbb{N}_0 \left[\tau_\delta < \infty, \sup_{t \in I} \frac{\rho_t}{\sqrt{th}(t)} > 1 \right] \\ & \leq \mathbb{N}_0 \left[\tau_\delta < \infty, \exists s \in [\tau_\delta, \sigma] \text{ s.t. } \zeta_s \in I \text{ and } |\widehat{W}_s| > \sqrt{\zeta_s} h(\zeta_s) \right] \\ & \quad + \mathbb{N}_0 \left[\tau_\delta < \infty, \exists s \in [0, L_\delta] \text{ s.t. } \zeta_s \in I \text{ and } |\widehat{W}_s| > \sqrt{\zeta_s} h(\zeta_s) \right] \\ & = 2\mathbb{N}_0 \left[\tau_\delta < \infty, \exists s \in [\tau_\delta, \sigma] \text{ s.t. } \zeta_s \in I \text{ and } |\widehat{W}_s| > \sqrt{\zeta_s} h(\zeta_s) \right] \\ & = \frac{1}{\delta} \mathcal{E}_{(\delta)}^* \left[\sup_{t \in I} \frac{\rho_t}{\sqrt{th}(t)} > 1 \right]. \end{aligned}$$

So, we get for any Borel set $I \subset [0, \infty)$,

$$\begin{aligned} (4) \quad \frac{1}{2\delta} \mathcal{E}_{(\delta)}^* \left[\sup_{t \in I} \frac{\rho_t}{\sqrt{th}(t)} > 1 \right] & \leq \mathbb{N}_0 \left[\tau_\delta < \infty, \sup_{t \in I} \frac{\rho_t}{\sqrt{th}(t)} > 1 \right] \\ & \leq \frac{1}{\delta} \mathcal{E}_{(\delta)}^* \left[\sup_{t \in I} \frac{\rho_t}{\sqrt{th}(t)} > 1 \right]. \end{aligned}$$

In particular, taking $I = (0, \varepsilon]$ in the above inequality and letting $\varepsilon \downarrow 0$, we deduce from the dominated convergence theorem that

$$(5) \quad \frac{1}{2\delta} \mathcal{E}_{(\delta)}^*[T = 0] \leq \mathbb{N}_0[\tau_\delta < \infty, T = 0] \leq \frac{1}{\delta} \mathcal{E}_{(\delta)}^*[T = 0].$$

Notice that if $t \geq \varepsilon$, then $\mathbb{N}_{t,x}[T < \varepsilon] = 0$. From the Poissonian representation of the snake and the definition of $\mathcal{E}_{(\delta)}^*$, we have for $\varepsilon > 0$,

$$(6) \quad \mathcal{E}_{(\delta)}^*[T < \varepsilon] = \mathbb{E} \left[1 - \exp \left\{ -2 \int_0^{\delta \wedge \varepsilon} dt \mathbb{N}_{t, \gamma_t}[T < \varepsilon] \right\} \right],$$

where $(\gamma_t, t > 0)$ is under \mathbb{P} a Brownian motion in \mathbb{R}^d started at 0.

We recall the function $u(t, x) = \mathbb{N}_{t,x}[T < \infty]$ is the maximal nonnegative solution of $\partial_t u + (\Delta/2)u = 2u^2$ in $\{(t, x); |x| < \sqrt{th}(t)\}$ (see [7] for the representation of the maximal solution using super-Brownian motion and [11, 13] for the Poissonian representation of super-Brownian motion using the Brownian snake). By the 0–1 law for Brownian motion we deduce that either:

- 1. \mathbb{P} -a.s. for $r > 0$ small enough, $\int_0^r u(t, \gamma_t) dt < +\infty$, or
- 2. \mathbb{P} -a.s. for all $r > 0$, $\int_0^r u(t, \gamma_t) dt = +\infty$.

Assume we are in case 1. Then from (6), we deduce that

$$\mathcal{E}_{(\delta)}^*[T < \varepsilon] \leq \mathbb{E} \left[1 - e^{-2 \int_0^{\delta \wedge \varepsilon} dt \mathbb{N}_{t, \gamma_t}[T < \infty]} \right] = \mathbb{E} \left[1 - e^{-2 \int_0^{\delta \wedge \varepsilon} dt u(t, \gamma_t)} \right].$$

By dominated convergence, we get, by letting $\varepsilon \downarrow 0$, that

$$\forall \delta > 0, \quad \mathcal{E}_{(\delta)}^*[T = 0] = 0.$$

Assume we are in case 2. Notice that for $\varepsilon > 0$,

$$\mathbb{N}_{t, \gamma_t}[T \geq \varepsilon + t, T < \infty] \leq \mathbb{N}_0[\tau_\varepsilon < \infty] = \frac{1}{2\varepsilon}.$$

Thus we have

$$\int_0^{\delta \wedge \varepsilon} dt \mathbb{N}_{t, \gamma_t}[T < \varepsilon + t] \geq \int_0^{\delta \wedge \varepsilon} dt u(t, \gamma_t) - \frac{1}{2}.$$

If \mathbb{P} -a.s. for all $r > 0$, $\int_0^r u(t, \gamma_t) dt = +\infty$, then we get that \mathbb{P} -a.s. $\int_0^{\delta \wedge \varepsilon} dt \mathbb{N}_{t, \gamma_t}[T < \varepsilon + t] = +\infty$. This implies in turn that for all $\varepsilon > 0$, $\delta > 0$, we have $\int_0^{\delta \wedge \varepsilon} dt \mathbb{N}_{t, \gamma_t}[T < 2\varepsilon] = +\infty$ and thanks to (6) (with ε replaced by 2ε), $\mathcal{E}_{(\delta)}^*[T < 2\varepsilon] = 1$. By letting $\varepsilon \downarrow 0$, we get $\mathcal{E}_{(\delta)}^*[T = 0] = 1$.

We then deduce from (5) that

$$\begin{aligned} \mathbb{P}\text{-a.s. } \exists r > 0 \text{ s.t. } \int_0^r u(t, \gamma_t) dt < +\infty &\Leftrightarrow \mathcal{E}_{(\delta)}^*[T = 0] = 0 \quad \forall \delta > 0 \\ &\Leftrightarrow \exists \delta > 0 \text{ s.t. } \mathcal{E}_{(\delta)}^*[T > 0] > 0 \\ &\Leftrightarrow \mathbb{N}_0[T = 0] = 0 \text{ and} \\ &\quad \mathbb{N}_0[T > 0] = +\infty, \end{aligned}$$

$$\begin{aligned} \mathbb{P}\text{-a.s. for all } r > 0, \int_0^r u(t, \gamma_t) dt = +\infty &\Leftrightarrow \mathcal{E}_{(\delta)}^*[T > 0] = 0 \quad \forall \delta > 0 \\ &\Leftrightarrow \exists \delta > 0 \text{ s.t. } \mathcal{E}_{(\delta)}^*[T = 0] > 0 \\ &\Leftrightarrow \mathbb{N}_0[T = 0] = +\infty \text{ and} \\ &\quad \mathbb{N}_0[T > 0] = 0. \end{aligned}$$

5. Proof of part (i) of Theorem 1. We assume $\int_{0^+} (dt/t) h(t)^{d+2} e^{-h(t)^2/2}$ is convergent. We will prove that $\mathcal{E}_{(\delta)}^*[T = 0] = 0$. Thanks to the previous section, it will imply part (i) of Theorem 1.

We set $I_n = [2^{-n}, 2^{-n+1}]$ and $h_n = h(2^{-n})$. We consider the event

$$A_n = \left\{ \sup_{u \in I_n} \frac{\rho_u}{\sqrt{u}} > h_{n-1} \right\}.$$

By the Poissonian representation of the snake and the definition of $\mathcal{E}_{(\delta)}^*$, we have

$$\mathcal{E}_{(\delta)}^*[A_n] = \mathbb{E} \left[1 - \exp \left\{ -2 \int_0^\delta dt \mathbb{N}_{t, \gamma_t}[A_n] \right\} \right].$$

Notice that for $t > 2^{-n+1}$, $\mathbb{N}_{t, \gamma_t}[A_n] = 0$. Hence for n such that $2^{-n+1} < \delta$, we have

$$\begin{aligned} \mathcal{E}_{(\delta)}^*[A_n] &= \mathbb{E}\left[1 - e^{-2\int_0^{2^{-n+1}} dt \mathbb{N}_{t, \gamma_t}[A_n]}\right] \\ &= \mathcal{E}_{(2^{-n+1})}^*[A_n] \\ &\leq 2^{-n+2} \mathbb{N}_0[\tau_{2^{-n+1}} < \infty, A_n] \\ &\leq 2^{-n+2} \mathbb{N}_0[A_n] \\ &= 4\mathbb{N}_0\left[\sup_{t \in [1, 2]} \frac{\rho_t}{\sqrt{t}} > h_{n-1}\right]. \end{aligned}$$

The first inequality is a consequence of (4) with $I = I_n$ and $h = h_{n-1}$. The last equality is a consequence of the scaling property of the Brownian snake.

Let us now recall the following result due to [4], Lemma 4: there exists a constant β such that if $a \geq 1$,

$$\mathbb{N}_0\left[\sup_{t \in [1, 4]} \frac{\rho_t}{\sqrt{t}} > a\right] \leq \frac{\beta}{4} a^{d+2} e^{-a^2/2}.$$

Hence we get, if $h_{n-1} \geq 1$,

$$(7) \quad \mathcal{E}_{(\delta)}^*[A_n] \leq 4\mathbb{N}_0\left[\sup_{t \in [1, 4]} \frac{\rho_t}{\sqrt{t}} > h_{n-1}\right] \leq \beta h_{n-1}^{d+2} e^{-h_{n-1}^2/2}.$$

Since the integral $\int_{0+} (dt/t) h(t)^{d+2} e^{-h(t)^2/2}$ is convergent, the series $\sum_{n \geq 1} h_n^{d+2} e^{-h_n^2/2}$ is convergent and $\lim_{n \rightarrow \infty} h_n = +\infty$. This implies that the series $\sum_{n \geq 1} \mathcal{E}_{(\delta)}^*[A_n]$ is finite. From the Borel–Cantelli lemma, we get that $\mathcal{E}_{(\delta)}^*[A_n \text{ i.o.}] = 0$, hence $\mathcal{E}_{(\delta)}^*[T = 0] = 0$ since the function h is nonincreasing.

6. Proof of part (ii) of Theorem 1. We assume $\int_{0+} (dt/t) h(t)^{d+2} e^{-h(t)^2/2} = +\infty$. This implies the series $\sum_{n \geq 1} h_n^{d+2} e^{-h_n^2/2}$ is divergent. We will prove that $\mathcal{E}_{(\delta)}^*[T = 0] > 0$. Thanks to Section 4, this will imply part (ii) of Theorem 1.

By standard arguments we may and will assume that h satisfies: if $n \geq 3$, then

$$(8) \quad 1 \leq \sqrt{\log n} \leq h(2^{-n}) \leq 2\sqrt{\log n}.$$

We use the Poissonian representation of the snake. We define the radius at time t of the excursion W^i : $\rho_t^i = \sup\{|\widehat{W}_s|; \zeta_s = t, s \in (\alpha_i, \beta_i)\}$. Notice that $\rho_t^i = 0$ if $t < \zeta^i$. Of course we have $\rho_t^i \leq \rho_t$.

We set $J_n = [2^{-n+2}, 2^{-n+3}]$. Let $\delta > 0$ fixed. For $n \geq 6$ such that $2^{-n+1} < \delta$, we define under $\mathcal{E}_{(\delta)}^*$ the event

$$A_n = \left\{ \exists i \in \mathcal{I}; \zeta^i \in [2^{-n-1}, 2^{-n+1}] \text{ and } \sup_{u \in J_n} \frac{\rho_u^i}{\sqrt{u}} > h_{n-3} \right\}.$$

Since $\mathcal{E}_{(\delta)}^*[A_n \text{ i.o.}] \leq \mathcal{E}_{(\delta)}^*[T = 0]$, it is enough to prove that $\mathcal{E}_{(\delta)}^*[A_n \text{ i.o.}] > 0$. Thanks to the properties of Poisson point measures, the events A_n are independent under \mathcal{E}_γ^* . Unfortunately, they are not independent under $\mathcal{E}_{(\delta)}^*$. But we will use a refined version of the Borel–Cantelli lemma which one can find in [15], page 317: *If there exists n_0 such that*

$$(C1) \quad \sum_{n=n_0}^{\infty} \mathcal{E}_{(\delta)}^*[A_n] = \infty;$$

$$(C2) \quad \liminf_{N \rightarrow \infty} \frac{\sum_{n_0 \leq n, m \leq N} \mathcal{E}_{(\delta)}^*[A_n \cap A_m]}{\sum_{n_0 \leq n, m \leq N} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m]} < \infty,$$

then $\mathcal{E}_{(\delta)}^*[A_n \text{ i.o.}] > 0$.

The proof of part (ii) of Theorem 1 will be complete once (C1) and (C2) are checked.

First, we check Condition (C1). Let $n_1 \geq 6$ such that $2^{-n_1+1} < \delta$. For $n \geq n_1$, we have

$$\begin{aligned} \mathcal{E}_{(\delta)}^*[A_n] &= \mathcal{E}_{(\delta)}^* \left[\exists i \in \mathcal{I}; \zeta^i \in [0, 2^{-n+1}] \text{ and } \sup_{u \in J_n} \frac{\rho_u^i}{\sqrt{u}} > h_{n-3} \right] \\ &\quad - \mathcal{E}_{(\delta)}^* \left[\exists i \in \mathcal{I}; \zeta^i \in [0, 2^{-n-1}] \text{ and } \sup_{u \in J_n} \frac{\rho_u^i}{\sqrt{u}} > h_{n-3} \right] \\ &= \mathcal{E}_{(2^{-n+1})}^* \left[\sup_{u \in J_n} \frac{\rho_u}{\sqrt{u}} > h_{n-3} \right] - \mathcal{E}_{(2^{-n-1})}^* \left[\sup_{u \in J_n} \frac{\rho_u}{\sqrt{u}} > h_{n-3} \right] \\ &\geq 2^{-n+1} \mathbb{N}_0 \left[\tau_{2^{-n+1}} < \infty; \sup_{u \in J_n} \frac{\rho_u}{\sqrt{u}} > h_{n-3} \right] \\ &\quad - 2 \cdot 2^{-n-1} \mathbb{N}_0 \left[\tau_{2^{-n-1}} < \infty; \sup_{u \in J_n} \frac{\rho_u}{\sqrt{u}} > h_{n-3} \right] \\ &= 2^{-n} \mathbb{N}_0 \left[\sup_{u \in J_n} \frac{\rho_u}{\sqrt{u}} > h_{n-3} \right]. \end{aligned}$$

We used the remark at the end of Section 3 for the second equality; (4) for the first inequality; and the fact that $\tau_\eta = +\infty$ implies $\rho_\eta = 0$ for the third equality.

From Lemma 4 of [4], there exists a constant α such that if $a \geq 1$,

$$4\alpha a^{d+2} e^{-a^2/2} \leq \mathbb{N}_0 \left[\sup_{u \in [1,2]} \frac{\rho_u}{\sqrt{u}} > a \right].$$

Hence, using the scaling property of the snake, $h_{n-3} \geq 1$ and this result, we get

$$(9) \quad \mathcal{E}_{(\delta)}^*[A_n] \geq \alpha (h_{n-3})^{d+2} e^{-(h_{n-3})^2/2}.$$

Since the series $\sum_{n \geq 1} h_n^{d+2} e^{-h_n^2/2}$ is divergent, we have $\sum_{n \geq 1} \mathcal{E}_{(\delta)}^*[A_n] = \infty$.

Let us now check Condition (C2). Let $n_1 \geq 6$ such that $2^{-n_1+1} < \delta$. For $n \geq n_1$, we define under $\mathcal{E}_{(\delta)}^*$ the event

$$B_n = \{\exists t \in [0, 2^{-n+1}]; |\gamma_t| > 8.2^{-n/2} \sqrt{\log n}\}.$$

Let us recall there exists a universal constant β' independent of n such that

$$(10) \quad \mathcal{E}_{(\delta)}^*[B_n] = \mathbb{E} \left[\sup_{u \in [0, 1]} |\gamma_u| > 4\sqrt{2} \sqrt{\log n} \right] \leq \beta' e^{-(4\sqrt{2} \sqrt{\log n})^2/2} = \beta' \frac{1}{n^{16}}.$$

Let $n \geq m \geq n_1$. We give an upper bound for $\mathcal{E}_{(\delta)}^*[A_n \cap A_m]$. We have

$$\mathcal{E}_{(\delta)}^*[A_n \cap A_m] \leq \mathcal{E}_{(\delta)}^*[B_n] + \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m].$$

First notice that (9) and the monotonicity of the function $\phi(x) = x^{d+2} \times \exp(-x^2/2)$ for x large imply

$$\begin{aligned} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m] &\geq \left(\alpha(h_{n-3})^{d+2} e^{-(h_{n-3})^2/2} \right) \left(\alpha(h_{m-3})^{d+2} e^{-(h_{m-3})^2/2} \right) \\ &\geq \left(\alpha h_n^{d+2} e^{-h_n^2/2} \right)^2 \\ &\geq \frac{\alpha^2 (4 \log n)^{d+2}}{n^4}, \end{aligned}$$

where we used the monotonicity of ϕ and $h_n \leq 2\sqrt{\log n}$ for the last inequality. Hence, it follows from (10) that there exists $n_2 \geq n_1$ such that if $n \geq m \geq n_2$,

$$\mathcal{E}_{(\delta)}^*[B_n] \leq \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m].$$

Let us now give an upper bound for $\mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m]$. We assume that $n - m \geq 3$, so that $2^{-n+1} < 2^{-m-1}$. We have

$$\begin{aligned} \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] &= \int \mathbb{P}^\delta(d\gamma) \mathcal{E}_\gamma^*[A_n \cap A_m] \mathbf{1}_{B_n^c} \\ &= \int \mathbb{P}^\delta(d\gamma) \mathcal{E}_\gamma^*[A_n] \mathcal{E}_\gamma^*[A_m] \mathbf{1}_{B_n^c} \\ &= \mathbb{E}^\delta[\mathcal{E}_\gamma^*[A_n] \mathbf{1}_{B_n^c} \mathbb{E}^\delta[\mathcal{E}_\gamma^*[A_m] | \sigma(\gamma_s, s \leq 2^{-n+1})]], \end{aligned}$$

where we used the independence of A_n and A_m under \mathcal{E}_γ^* . Now, notice that on B_n^c , we have $|\gamma_{2^{-n+1}}| \leq 8.2^{-n/2} \sqrt{\log n}$. Hence, on B_n^c , we get by space-time

translation,

$$\begin{aligned} & \mathbb{E}^\delta[\mathcal{E}_\gamma^*[A_m]|\sigma(\gamma_s, s \leq 2^{-n+1})] \\ & \leq \mathcal{E}_{(\delta)}^* \left[\exists i \in \mathcal{I}; \zeta^i \in [2^{-m-1} - 2^{-n+1}, 2^{-m+1} - 2^{-n+1}] \right. \\ & \quad \left. \text{and } \sup_{u \in [2^{-m+2} - 2^{-n+1}, 2^{-m+3} - 2^{-n+1}]} \frac{\rho_u^i + 8 \cdot 2^{-n/2} \sqrt{\log n}}{\sqrt{u + 2^{-n+1}}} > h_{m-3} \right] \\ & \leq \mathcal{E}_{(\delta)}^* \left[\sup_{s \in [2^{-m+1}, 2^{-m+3}]} \frac{\rho_s + 8 \cdot 2^{-n/2} \sqrt{\log n}}{\sqrt{s}} > h_{m-3} \right] \\ & \leq \mathcal{E}_{(\delta)}^* \left[\sup_{s \in [2^{-m+1}, 2^{-m+3}]} \frac{\rho_s}{\sqrt{s}} > h_{m-3} - 8 \cdot 2^{(m-1)/2} 2^{-n/2} \sqrt{\log n} \right]. \end{aligned}$$

Following the proof of [7] we get for $n_3 \geq 3$ large enough and $n - m \geq n_3$,

$$\begin{aligned} & \mathcal{E}_{(\delta)}^* \left[\sup_{s \in [2^{-m+1}, 2^{-m+3}]} \frac{\rho_s}{\sqrt{s}} > h_{m-3} - 8 \cdot 2^{(m-1)/2} 2^{-n/2} \sqrt{\log n} \right] \\ & \leq 4\mathbb{N}_0 \left[\sup_{t \in [1, 4]} \frac{\rho_t}{\sqrt{t}} > h_{m-3} - 8 \cdot 2^{(m-1)/2} 2^{-n/2} \sqrt{\log n} \right] \\ & \leq \beta \left(h_{m-3} - 4\sqrt{2} 2^{-(n-m)/2} \sqrt{\log n} \right)^{d+2} \\ & \quad \times \exp\left(-\left(h_{m-3} - 4\sqrt{2} 2^{-(n-m)/2} \sqrt{\log n}\right)^2 / 2\right) \\ & \leq \beta (h_{m-3})^{d+2} e^{-(h_{m-3})^2/2} \exp\left(4\sqrt{2} 2^{-(n-m)/2} \sqrt{\log n} 2\sqrt{\log(m-3)}\right) \\ & \leq \beta (h_{m-3})^{d+2} e^{-(h_{m-3})^2/2} \exp(8\sqrt{2} 2^{-(n-m)/2} \log n). \end{aligned}$$

Hence, we deduce that

$$\mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] \leq \mathcal{E}_{(\delta)}^*[A_n] \beta (h_{m-3})^{d+2} e^{-(h_{m-3})^2/2} \exp(8\sqrt{2} 2^{-(n-m)/2} \log n).$$

Case 1. $n - m \geq 4 \log \log n$. Then

$$\begin{aligned} \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] & \leq \frac{\beta}{\alpha} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m] e^{8\sqrt{2} 2^{-(n-m)/2} \log n} \\ & \leq e^{8\sqrt{2}} \frac{\beta}{\alpha} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m]. \end{aligned}$$

Case 2. $n_3 \leq n - m \leq 4 \log \log n$. Then using (8) and the monotonicity of the function $\phi(x) = x^{d+2} \exp(-x^2/2)$ for x large we get for m large enough,

$$\phi(h_{m-3}) \leq \phi\left(\sqrt{\log(n - 4 \log \log n - 3)}\right) \leq \phi\left(\sqrt{(\log n)/2}\right).$$

So we have

$$\begin{aligned} &\mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] \\ &\leq \beta \mathcal{E}_{(\delta)}^*[A_n] (\log n)^{(d+2)/2} \exp(-(\log n)/4 + 8\sqrt{2} 2^{-(n-m)/2} \log n). \end{aligned}$$

Hence there exists $n_0 \geq \sup(n_2, n_3)$ such that if $n_0 \leq n - m \leq 4 \log \log n$, we have at least

$$\mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] \leq \frac{1}{4 \log \log n} \mathcal{E}_{(\delta)}^*[A_n].$$

Finally, using the previous results, we get

$$\begin{aligned} &\frac{1}{2} \sum_{n_0 \leq n, m \leq N} \mathcal{E}_{(\delta)}^*[A_n \cap A_m] \\ &\leq \sum_{n_0 \leq m \leq n \leq N} \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] + \sum_{n_0 \leq m \leq n \leq N} \mathcal{E}_{(\delta)}^*[B_n] \\ &= \sum_{\substack{n_0 \leq m \leq n \leq N \\ n-m \geq 4 \log \log n}} \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] + \sum_{\substack{n_0 \leq m \leq n \leq N \\ n_0 \leq n-m < 4 \log \log n}} \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] \\ &\quad + \sum_{\substack{n_0 \leq m \leq n \leq N \\ n-m < n_0}} \mathcal{E}_{(\delta)}^*[(A_n \cap B_n^c) \cap A_m] + \sum_{n_0 \leq m \leq n \leq N} \mathcal{E}_{(\delta)}^*[B_n] \\ &\leq \sum_{\substack{n_0 \leq m \leq n \leq N \\ n-m \geq 4 \log \log n}} e^{8\sqrt{2}} \frac{\beta}{\alpha} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m] + \sum_{\substack{n_0 \leq m \leq n \leq N \\ n_0 \leq n-m < 4 \log \log n}} \frac{1}{4 \log \log n} \mathcal{E}_{(\delta)}^*[A_n] \\ &\quad + \sum_{\substack{n_0 \leq m \leq n \leq N \\ n-m < n_0}} \mathcal{E}_{(\delta)}^*[A_n] + \sum_{n_0 \leq m \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m] \\ &\leq e^{8\sqrt{2}} \frac{\beta}{\alpha} \sum_{n_0 \leq m \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m] + \sum_{n_0 \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n] \\ &\quad + (n_0 - 1) \sum_{n_0 \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n] + \sum_{n_0 \leq m \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n] \mathcal{E}_{(\delta)}^*[A_m] \\ &\leq \left(1 + e^{8\sqrt{2}} \frac{\beta}{\alpha}\right) \left(\sum_{n_0 \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n]\right)^2 + n_0 \sum_{n_0 \leq n \leq N} \mathcal{E}_{(\delta)}^*[A_n]. \end{aligned}$$

This inequality and the divergence of the series $\sum \mathcal{E}_{(\delta)}^*[A_n]$ immediately imply assumption (2) of the Borel–Cantelli lemma. Hence part (ii) of Theorem 1 is proved.

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