A GEOMETRIC CONDITION FOR THE UNIFORM STABILITY OF LINEAR MAGNETOELASTICITY

THOMAS DUYCKAERTS[1]

ABSTRACT. We give a necessary and sufficient condition, of geometrical type, for the uniform decay of energy of solutions of the linear system of magnetoelasticity in a bounded domain with smooth boundary. A Dirichlet-type boundary condition is assumed. Our strategy is to use microlocal defect measures to show suitable observability inequalities on high-frequency solutions of the Lamé system.

1. INTRODUCTION

Let Ω be a bounded, simply connected domain of $\mathbb{R}^3$, with a smooth boundary. Let us consider the following system, modelling the displacement of an elastic solid in a magnetic field:

$$
(1) \quad
\left.
\begin{aligned}
\partial_t^2 v - \varsigma \Delta v - (\lambda + \varsigma)\nabla \operatorname{div} v - \kappa \operatorname{curl} h \wedge \boldsymbol{B} &= 0 \\
\beta \partial_t h + \operatorname{curl} \operatorname{curl} h - \beta \operatorname{curl}(\partial_t v \wedge \boldsymbol{B}) &= 0 \\
\operatorname{div} h &= 0
\end{aligned}
\right\}
(t, y) \in (0, \infty) \times \Omega
$$

$$
v = 0, \quad h.\mathbf{n} = 0, \quad \operatorname{curl} h \wedge \mathbf{n} = 0 \quad (t, y) \in (0, \infty) \times \partial\Omega,
$$

where $v = (v_1, v_2, v_3)$ is the displacement vector of the solid, and $h = (h_1, h_2, h_3)$ the magnetic field. The system is located in a constant exterior magnetic field $\boldsymbol{B} = (B, 0, 0)$. We have denoted by Δ, ∇, div, curl respectively the Laplace operator, gradient, divergence and curl operators according to the space variable y, in the Euclidean metric of $\mathbb{R}^3$. The positive constants κ and β are coupling constants, and $\mathbf{n}$ is the external normal vector to the boundary of Ω. The real Lamé constants λ and ς are such that $\lambda + 2\varsigma > 0$, $\varsigma > 0$ and $\lambda + \varsigma \neq 0$.

The system (1) has a natural time-decreasing energy:

$$
E(t) = \frac{1}{2} \int_\Omega |\partial_t v|^2 + \varsigma |\nabla v|^2 + (\lambda + \varsigma)|\operatorname{div} v|^2 + \kappa |h|^2 dy.
$$

When Ω is simply connected, G. Perla Menzala and E. Zuazua have shown that this energy tends to zero as time tends to infinity, which is a simple consequence, using La Salle invariance principle, of the non-existence of stationary solution for (1).

The system (1) may be seen as a coupling between the Lamé system

$$
(2) \qquad \partial_t^2 u - \varsigma \Delta u - (\lambda + \varsigma)\nabla \operatorname{div} u = 0,
$$

with Dirichlet boundary conditions, which is a conservative system, and the following heat equation:

$$
\beta \partial_t g - \Delta g = 0.
$$

The energy decay is produced by this strongly dissipative equation. From the point of view of v, the dissipation is caused by the coupling term: $R(v) := \operatorname{curl}(\partial_t v \wedge \boldsymbol{B})$.

In this paper, we are interested by the possible uniform decay of the energy:

$$
(3) \qquad E(t) \leq f(t)E(0), \quad f(t) \underset{t \to +\infty}{\longrightarrow} 0,
$$

where f if independent of the initial condition. In this case it is easy to show, using the semi-group property of the equation (1), that f may be taken as a negative exponential function.

We will give, with a technical hypothesis on Ω, a necessary and sufficient condition on the geometry of the problem for (3) to hold. When this condition is not fulfilled, there exist rays on Ω, named $\boldsymbol{B}$-resistant rays, along which the energy of some solutions of (1) concentrates, and the dissipative term $R(v)$ is very small. Indeed, when such a ray exists, there is a sequence of solutions of (1) concentrating on the ray and which is in first approximation parallel to $\boldsymbol{B}$.

Before giving more explicit results, let us mention some earlier works on related subjects. As it was already stated, the convergence to 0 for the energy of magnetoelasticity in a bounded, simply connected domain was shown by G. Perla Menzala and E. Zuazua in [20], but their method does not give any information on the rate of convergence. By energy methods, Muñoz Rivera and Racke [18], Muñoz Rivera and de Lima Santos [9] have shown the rate of convergence to be at least polynomial, in dimension 2 or 3, but only for some precise types of domains. Andreou and Dassios [1] have examined the same system on the entire space $\mathbb{R}^3$, showing again polynomial decay for some initial conditions. In [5], Charão, Oliveira and Perla Menzala have studied the system of magnetoelasticity in dimension 3 with an additional (linear or semilinear) damping term, and proved exponential or polynomial decay depending on the nature of the decay. See also [6] for a study in unbounded domains. Ma, Rivera, Oquendo and Suárez have studied the system of magnetoelastic plate (analogous to (1), where the wave equation is replaced by a plate equation) in [16].

The linear system of thermoelasticity is better understood. In this system, the Lamé system is coupled with a scalar heat equation. The dissipation is caused by the longitudinal part of the Lamé equation (the curl-free part of v). G. Lebeau and E. Zuazua [15] and N. Burq and G. Lebeau [2] have given (under a spectral assumption) a necessary and sufficient condition on Ω, of geometrical nature, for the uniform decay in dimension 2 or 3. Namely, this decay is equivalent to the non-existence of rays, called "transversal polarization rays", carrying the transversal component of v (the divergence-free component), which resists to the dissipation. In [15], Lebeau and Zuazua have also proved the polynomial decay in dimension 2, under the same spectral assumption, which is namely that the operator associated to the equation does not admit any real eigenvalue. As shown in [20], this spectral condition is always fulfilled for the system of magnetoelasticity in a bounded, simply-connected domain.

The comparison of the two systems of thermo and magnetoelasticity show that thermoelasticity is slightly less dissipative (the coupling of the Lamé system with the heat equation is weaker), and more difficult to describe, because of the non-trivial polarization of transversal waves.

Let us also mention the work [14] about the controllability of the system of thermoelasticity, and the articles [19], and [7] on the stability of the system of magneto-thermo-elasticity that take into accounts both thermic and magnetic effects.

The system of thermoelasticity of type III, where the classical heat equation is replaced by an hyperbolic (Cattaneo-type) heat equation was studied by Quintanilla and Racke [21] and Zhang and Zuazua in [24]. In this last article, the authors proved polynomial decay for most domain in dimension 2, and investigated uniform decay in dimension 3, relying on the results of [15].

To state our main result, we need some geometrical preliminaries. Assume that $\partial\Omega$ has no contact of infinite order with its tangents. Thus, the Hamiltonian flow of the symbol of a wave operator $\partial_t^2 - c^2\Delta$, which is defined locally in $S^*(\mathbb{R} \times \Omega)$ (the spherical cotangent bundle of Ω), may be extended to the boundary of this bundle to a global flow, the generalized bicharacteristic flow, which may be seen as a continuous flow on the spherical compressed cotangent bundle $S_b^*(\mathbb{R} \times \overline{\Omega})$ (cf [11, chap. 24.3]). We shall call bicharacteristic rays or just rays the characteristic curves of this flow. Such a curve γ will be said parallel to $\boldsymbol{B}$ if its direction of propagation is always parallel to $\boldsymbol{B}$ and orthogonal to $\boldsymbol{B}$ if its direction of propagation is always orthogonal to $\boldsymbol{B}$. We refer to Section 3 for the exact definitions of $S_b^*(\mathbb{R} \times \overline{\Omega})$ and of the generalized bicharacteristic flow.

The Lamé system (2) may be written as the sum of two wave equations known as the longitudinal and transversal wave equations, of respective speed $c_L := \sqrt{\lambda + 2\varsigma}$ and $c_T := \sqrt{\varsigma}$ (cf §3.5). The assumption $\lambda + \varsigma \neq 0$ is equivalent to $c_L \neq c_T$.

Definition 1.1. One calls **longitudinal ray** (respectively **transversal ray**) any bicharacteristic ray for the operator $\partial_t^2 - c_L^2 \Delta$ (respectively $\partial_t^2 - c_T^2 \Delta$). One calls **B-resistant ray** any **continuous** application γ from an open interval $I = (s_0, s_n)$ to $S_b^*(\mathbb{R} \times \overline{\Omega})$ such that there exists a finite number of real numbers $s_0 < s_1 < ... < s_n$ such that

- on (s_{j-1}, s_j), $j \in \{1, ..., n\}$, γ is a longitudinal ray parallel to $\boldsymbol{B}$, or a transversal ray orthogonal to $\boldsymbol{B}$;
- if $j \in \{1, ..., n-1\}$, $\gamma(s_j)$ is an hyperbolic point for the longitudinal and transversal waves (cf §3.1.7) and one of the following assertions is true:
  - $(L \to T)$ case: γ is a longitudinal ray on $]s_{j-1}, s_j[$, and a transversal ray on $]s_j, s_{j+1}[$;
  - $(T \to L)$ case: γ is a transversal ray on $]s_{j-1}, s_j[$, and a longitudinal ray on $]s_j, s_{j+1}[$.

Some $\boldsymbol{B}$-resistant rays are drawn in Figure 1. The rays of this figure are all planar, but this is not a general property of $\boldsymbol{B}$-resistant rays.

Near s_j, $1 \leq j \leq n-1$, the continuity imposed by the definition of γ gives a condition on the angles of incidence and refraction. In the case $(L \to T)$, if we denote by α_L the angle between the longitudinal incoming ray and the tangent to $\partial\Omega$ in the plane of incidence, and by β_T the angle between the transversal outcoming ray and this tangent, we have $\tan \alpha_L = \frac{c_T}{c_L}$ and $\tan \beta_T = \frac{c_L}{c_T}$ (which implies $\alpha_L + \beta_T = \pi/2$). In the case $(T \to L)$, and with similar notations, we have $\tan \alpha_T = \frac{c_L}{c_T}$ and $\tan \beta_L = \frac{c_T}{c_L}$.

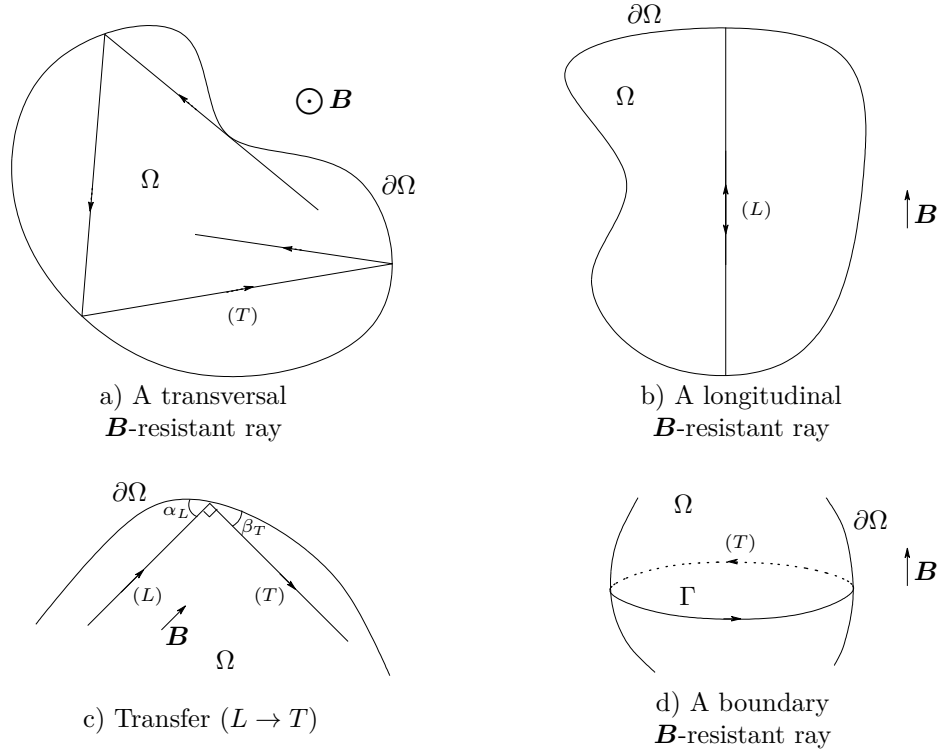a) A transversal
$\boldsymbol{B}$-resistant ray

b) A longitudinal
$\boldsymbol{B}$-resistant ray

c) Transfer $(L \to T)$

d) A boundary
$\boldsymbol{B}$-resistant ray

FIGURE 1. Examples of $\boldsymbol{B}$-resistant rays

Theorem 1. *Let Ω be a bounded, simply connected domain of $\mathbb{R}^3$, with a smooth boundary, having no contact of infinite order with its tangents. The energy of the system of magnetoelasticity in Ω decays uniformly if and only if there exists an $\ell > 0$ such that the length of every $\boldsymbol{B}$-resistant ray on Ω is lesser than ℓ.*

Remark 1.2. As it will be shown in the proof, at high frequency, the transversal rays carry the component of v which is orthogonal to the direction of propagation, and the longitudinal rays the component of v which is parallel to this direction. A $\boldsymbol{B}$-resistant ray, whose direction of propagation is orthogonal to $\boldsymbol{B}$ in the transversal case and parallel to $\boldsymbol{B}$ in the longitudinal case carries essentially the component of v which is parallel to $\boldsymbol{B}$, thus cancelling the dissipative term $R(v) := \operatorname{curl}(\partial_t v \wedge \boldsymbol{B})$. From this point of view, Theorem 1 is very natural.

Remark 1.3. It is essential to assume $c_L \neq c_T$. Otherwise, the first equation in (1) would be a wave equation with speed $c_L = c_T$. Every solution of (1) such that

$$v_{\restriction t=0} \perp \boldsymbol{B}, \quad \partial_t v_{\restriction t=0} \perp \boldsymbol{B}, \quad h_{\restriction t=0} = 0$$

would be of constant energy.

Remark 1.4. If Ω is not simply connected, there exists a finite dimensional space E of stationary solutions of (1), whose components along v are null. The study of the decay to zero of the solutions may be replaced by the study of their convergence to the eigenfunctions corresponding to the space E (cf [20, chap. 5]). We won't develop this aspect here.

Remark 1.5. The condition of uniform decay is not fulfilled in simple cases, like the one of a ball, or an ellipsoid which has an axis parallel to the magnetic field. However it seems generic in the class of C^∞ open sets. When this condition is not fulfilled, one can prove polynomial decay of regular solution (see the arXiv preprint [8]).

The proof uses propagation arguments on microlocal defect measures. This strategy of proof was initiated by G. Lebeau (see [12]).

The remainder of the paper is organized as follows. In Section 2, we reduce Theorem 1 to an high-frequency observability inequality on the Lamé system (2). This is based on two arguments: the setting aside of low frequencies, which is a consequence of the non-existence of stationary solution for the equation (1) shown in [20], and the decoupling, by simple calculations, of the two equations (the Lamé system and the heat equation) which compose (1). In Section 3, we introduce microlocal defect measures (an object due to P. Gérard [10] and L. Tatar [22], and in this particular setting to N. Burq and G. Lebeau [2]), in order to study the lack of compactness of a sequence of high-frequency solutions of the Lamé system. The main result of this section (apart from the existence of the measures), is a propagation theorem which was stated and shown in [2]. In Section 4, we prove the observability inequality on solutions of the Lamé system (2) which implies Theorem 1. The method of proof is to introduce, in a contradiction argument, a sequence of high frequency solutions of (2) which contradicts this inequality, and to use propagation arguments on the defect measures of this sequence. Section 5 is devoted to the necessary condition of Theorem 1, and is inspired by [4]: defect measures are used to construct a sequence of solutions of (2) concentrating on a $\boldsymbol{B}$-resistant ray and contradicting an observability inequality.

This paper is a corrected, updated and abbreviated[1] version of the unpublished preprint [8] (2004), which is a translation of a chapter of the French PhD thesis of the author. The author would like to thank his former PhD supervisor, Nicolas Burq, for introducing him to this subject and more generally to microlocal analysis. This work is also indebted to Enrique Zuazua, who played an important role in the mathematical study of linear elasticity, especially with microlocal analysis techniques (see his articles with Gilles Lebeau [14] and [15]), and also has introduced system (1) in the mathematical community, in his work with G. Perla Menzala [20].

---

[1]A section on polynomial decay was taken off for the sake of brevity

2. Observability inequality for the Lamé system

2.1. **Notations and preliminary results.** In this subsection are gathered a few basic facts about equations (1) and (2), as well as some notations. The main results of Section 2 are stated in the next subsection.

If U is an open set of $\mathbb{R}^3$ or $\mathbb{R}^4$ we set

$$\boldsymbol{H}^s(U) := H^s(U, \mathbb{C}^3), \quad \boldsymbol{L}^2(U) := L^2(U, \mathbb{C}^3).$$

2.1.1. *Magnetoelasticity.* Consider the following spaces:

$$H := \left\{ g \in \boldsymbol{L}^2(\Omega), \ \mathrm{div}\, g = 0 \ \mathrm{in}\ \Omega, \ g.n = 0 \ \mathrm{in}\ \partial\Omega \right\}, \quad \boldsymbol{H}_0^1 := \{ f \in \boldsymbol{H}^1(\Omega), \ f = 0 \ \mathrm{in}\ \partial\Omega \}$$

$$H_2 := \left\{ f \in H \cap \boldsymbol{H}^2(\Omega), \ \mathrm{curl}\, f \wedge n = 0 \ \mathrm{in}\ \partial\Omega \right\},$$

and the norms $\|g\|_H^2 := \kappa \|g\|_{\boldsymbol{L}^2(\Omega)}^2$, $\|f\|_{\boldsymbol{H}_0^1(\Omega)}^2 := (\lambda + \varsigma)\|\mathrm{div}\, f\|_{L^2(\Omega)}^2 + \varsigma \|\nabla f\|_{(\boldsymbol{L}^2(\Omega))^3}^2$. Let $\mathcal{A}$ be the unbounded operator on $X := \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times H$, with domain $D(\mathcal{A})$, defined by

$$\mathcal{A}(V_0) := \begin{pmatrix} -v_1 \\ -\Delta_e v_0 - \kappa(\mathrm{curl}\, h_0) \wedge \boldsymbol{B} \\ -\mathrm{curl}\,(v_1 \wedge \boldsymbol{B}) + \frac{1}{\beta}\mathrm{curl}\,\mathrm{curl}\, h_0 \end{pmatrix}, \quad D(\mathcal{A}) := (\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1) \times \boldsymbol{H}_0^1 \times H_2.$$

where $V_0 = (v_0, v_1, h_0)$ denotes an element of X. Equation (1) may be rewritten

$$\text{(4)} \qquad\qquad \partial_t V + \mathcal{A}V = 0, \quad V = (v, \partial_t v, h).$$

The following proposition is due to G. Perla Menzala and E. Zuazua [20]:

Proposition 2.1. *a) The operator $\mathcal{A}$ is maximal accretive. For any initial data $V_0 \in X$, there exists a unique weak solution $V(t) = (v(t), \partial_t v(t), h(t)) \in C^0([0, +\infty[; X)$ of (4) such that $V(0) = V_0$.*
b) The energy $E(t)$

$$\text{(5)} \qquad\qquad E(t) := \frac{1}{2}\|V(t)\|_X^2 = \frac{1}{2}\int_\Omega |\partial_t v|^2 + \varsigma|\nabla v|^2 + (\lambda + \varsigma)|\mathrm{div}\, v|^2 + \kappa |h|^2 dy$$

is decreasing. More precisely,

$$\text{(6)} \qquad\qquad \forall t \geq 0, \quad E(t) - E(0) = -\frac{\kappa}{\beta}\int_0^t \int_\Omega |\mathrm{curl}\, h|^2 dy.$$

c) If Ω is simply connected, for all V_0 in X, $\lim_{t \to +\infty} E(t) = 0$.

The assertions a) and b) are straightforward applications of the semi-group theory for the operator $\mathcal{A}$. The assertion c) is a consequence of the non-existence of stationary solutions for the system.

2.1.2. *Lamé system.* Let us now consider the Lamé system with Dirichlet boundary conditions:

$$\partial_t^2 u - \Delta_e u = 0 \ \mathrm{in}\ \mathbb{R} \times \Omega$$

$$\text{(7)} \qquad\qquad u_{\restriction \partial\Omega} = 0$$

$$(u_{\restriction t=0}, \partial_t u_{\restriction t=0}) = (u_0, u_1).$$

Let X_e be the space $\boldsymbol{H}_0^1 \times \boldsymbol{L}^2$ and $\mathcal{L}$ the unbounded operator on X_e defined by

$$\text{(8)} \qquad\qquad \mathcal{L} := \begin{bmatrix} 0 & -\mathrm{Id} \\ -\Delta_e & 0 \end{bmatrix} \qquad D(\mathcal{L}) := (\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1) \times \boldsymbol{L}^2.$$

Taking (u_0, u_1) in the energy space X_e, the equation (7) may be written

$$\text{(9)} \qquad\qquad \partial_t U + \mathcal{L}U = 0, \ U(t) = (u, \partial_t u).$$

Proposition 2.2. *The operator $\mathcal{L}$ is maximal and unitary. For any initial data $U_0 = (u_0, u_1) \in X_e$, the system (9) has a unique weak solution $U \in C^0(\mathbb{R}, X_e)$. Furthermore, the function u is a solution of (7) in the distributional sense. At last, the energy*

$$\frac{1}{2}\left(\|u(t)\|_{\boldsymbol{H}_0^1}^2 + \|\partial_t u\|_{\boldsymbol{L}^2}^2\right) = \frac{1}{2}\|U(t)\|_{X_e}^2$$

of this solution is constant.

2.1.3. *Two useful lemma.* The two following standard lemma will be of great help in all this paper. The first one is due to the fact that Ω is simply connected (cf [23, Appendix I, Lemma 1.6]):

Lemma 2.3. *The $\boldsymbol{H}^1$ norm on $H \cap \boldsymbol{H}^1$ is equivalent to the norm $\|u\| := \|\operatorname{curl} u\|_{\boldsymbol{L}^2}$.*

The second lemma is a elementary energy estimate on solutions of the non-homogeneous Lamé system. If $w(t)$ is a function with values in some Hilbert space, we set $(w_0, w_1) := (w, \partial_t w)_{\restriction t=0}$.

Lemma 2.4. *Let $T > 0$, $W \in C^0((0, T), X_e)$ and $F \in L^2((0, T), X_e)$ such that*

$$(10) \qquad\qquad \partial_t W + \mathcal{L}W = F, \quad t \in (0, T).$$

Then

$$\int_0^T \|W(t)\|_{X_e}^2 \, dt \le C \left\{\|W(0)\|_{X_e}^2 + \int_0^T \|F(t)\|_{X_e}^2 dt\right\},$$

where C only depends on T. In particular, if

$$w \in C^1((0, T), \boldsymbol{L}^2(\Omega)) \cap C^0((0, T), \boldsymbol{H}_0^1(\Omega)), \quad f \in \boldsymbol{L}^2((0, T) \times \Omega)$$
$$\partial_t^2 w - \Delta_e w = f,$$

then

$$\|w\|_{\boldsymbol{H}^1((0,T)\times\Omega)}^2 \le C\left(\|w_0\|_{\boldsymbol{H}^1(\Omega)}^2 + \|w_1\|_{\boldsymbol{L}^2(\Omega)}^2 + \|f\|_{\boldsymbol{L}^2((0,T)\times\Omega)}^2\right).$$

This is a standard energy inequality (using $\operatorname{Re}(\mathcal{L}W, W)_{X_e} = 0$) and we omit the proof.

2.2. **Necessary and sufficient condition for uniform decay.**

2.2.1. *Statement of the condition.*

Proposition 2.5. *Let Ω be a smooth, simply connected, bounded domain of $\mathbb{R}^3$.*

   a) *Assume that there exist $T > 0$ and $C > 0$ such that for any solution U of (9),*

$$(11) \qquad \|u_0\|_{\boldsymbol{H}_0^1}^2 + \|u_1\|_{\boldsymbol{L}^2}^2 \le C\left(\|\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 + \|u_0\|_{\boldsymbol{L}^2}^2 + \|u_1\|_{\boldsymbol{H}^{-1}}^2\right).$$

   Then the energy of solutions of the system of magnetoelasticity (1) decays uniformly.
   b) *Conversely, if the energy of solutions of (1) decays uniformly, then there exist $T > 0$ and $C > 0$ such that for any solution of (7) of finite energy,*

$$(12) \qquad\qquad \|u_0\|_{\boldsymbol{H}_0^1}^1 + \|u_1\|_{\boldsymbol{L}^2}^2 \le C\|\partial_t u(t) \wedge \boldsymbol{B}\|_{\boldsymbol{L}^2((0,T)\times\Omega)}^2.$$

Remark 2.6. Proposition 2.5 is a fairly classical result in this setting. To prove it, we shall avoid the usual abstract decoupling argument (see [14]) but rather use simple energy estimates on the systems of magnetoelasticity and Lamé.

To prove Proposition 2.5, we first write a necessary and sufficient condition of uniform decay for solutions of a general dissipative equation. The second step of the proof consists in applying this condition to the system of magnetoelasticity, furthermore decoupling it in the system of Lamé and a heat equation.

2.2.2. *Abstract framework.* Let $\mathcal{P}$ be a maximal, accretive operator on an Hilbert space X, with dense domain $D(\mathcal{P})$. Denote by $\|..\|$ the norm of X, $\|..\|_1$ the natural norm on $D(\mathcal{P})$ and $\|..\|_{-1}$ the norm on its dual space, with respect to the pivot space X. Assume that the embedding

$$X \longrightarrow D(\mathcal{P})'$$

is compact. For $z_0 \in X$, we will denote by $z(t)$ the solution (obtained for example by standard semi-group theory) of

$$(13) \qquad \frac{d\,z}{dt} + \mathcal{P}z = 0, \quad z_{t=0} = z_0.$$

By accretivity of $\mathcal{P}$, the energy $\frac{1}{2}\|z\|^2$ is time-decreasing. The following uniqueness-compactness argument is by now classical (cf [3]):

Lemma 2.7. *The two following assertions (i) and (ii) are equivalent:*

(i) $\qquad\qquad \exists C > 0,\ \exists a > 0,\ \forall z_0 \in X,\ \forall t > 0,\ \|z(t)\|^2 \leq C\|z_0\|^2 e^{-at}$

$\qquad\qquad$ *(the energy is uniformly decreasing)*

(ii) $\qquad$ *a)* $\exists T > 0,\ \exists C > 0,\ \forall z_0 \in X,\ \|z(T)\|^2 \leq C \left(\|z(0)\|^2 - \|z(T)\|^2 + \|z(0)\|_{-1}^2 \right)$

$\qquad$ *b) There is no non-zero solution of (13) of constant energy on $[0, +\infty[$.*

Corollary 2.8. *The energy of (4) is uniformly time-decreasing if and only if*

$$\exists T > 0,\ \exists C > 0,\ \forall V_0 \in X,\ E(T) \leq C \left\{ \int_0^T \int_\Omega |\operatorname{curl} h|^2 \, dy \, dt + \|V_0\|_{D(\mathcal{A})'}^2 \right\}.$$

Indeed, the non-existence of stationary solution (the condition ii,b of Lemma 2.7 has been proved in [20, p.356]), which shows the corollary.

Proof of Lemma 2.7. It is easy to see that (i) may be replaced by

(i') $\qquad\qquad \exists T > 0,\ \exists C > 0,\ \forall z_0 \in X, \quad \|z(T)\|^2 \leq C \left(\|z(0)\|^2 - \|z(T)\|^2 \right)$

Clearly (i') implies (ii).

Assume (ii). For some $T > 0$, set

$$q_T(z) := \|z(0)\|^2 - \|z(T)\|^2, \quad G^T := \{z_0 \in X,\ q_T(z) = 0\},$$

which is the kernel of a positive, bounded, quadratic form on X, thus a closed subspace of X.

According to (ii), a) and the compactness of the embedding from X to $D(P)'$, $G(T)$ is locally compact, and thus of finite dimension, for large T. Combining with assumption b), we obtain that $G^T = \{0\}$ if T is large enough.

Let us fix such a T. We prove (i') by contradiction. Assume that there exists a sequence (z_0^k) of elements of X such that

$$(14) \qquad\qquad 1 = \|z^k(T)\|^2, \qquad \lim_{k \to +\infty} q_T(z^k) = 0.$$

Extracting subsequences, we may assume that (z_0^k) converges weakly in X. Since $G^T = \{0\}$, its weak limit must be 0. The compactness of the embedding of X in $D(P)'$ yields $\lim_{k \to +\infty} \|z_0^k\|_{-1} = 0$, Contradicting a) and (14). $\qquad\square$

2.2.3. *Proof of Proposition 2.5.* Assume the uniform time-decay of the energy of solutions of (4). Then, by (6), there exist $T > 0$ and $C > 0$ such that the following estimates hold for any solution v of (4)

$$(15) \qquad \|v_0\|_X^2 \leq C \int_0^T \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2(\Omega)}^2 \, dt.$$

Let U be a solution of the Lamé system with initial data $U_0 = (u_0, u_1) \in D(\mathcal{L})$ and V the solution of the system of magnetoelasticity with initial data

$$V_0 = (v, \partial_t v, h)_{\restriction t=0} = (u_0, u_1, 0).$$

Set: $W(t) := V(t) - \big(u(t), \partial_t u(t), 0\big)$. Then

$$\partial_t W + \mathcal{A}W = \big(0, 0, -\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\big)$$

Take the scalar product in X with W of the two sides of this equality, then integrate the real part with respect to time between 0 and T. Using

$$\operatorname{Re}(\mathcal{A}W, W)_X = \frac{\kappa}{\beta} \|\operatorname{curl} h\|_{\boldsymbol{L}^2(\Omega)}^2$$

$$\big(\operatorname{curl}(\partial_t u \wedge \boldsymbol{B}), h\big)_{\boldsymbol{L}^2(\Omega)} = (\partial_t u \wedge \boldsymbol{B}, \operatorname{curl} h)_{\boldsymbol{L}^2(\Omega)},$$

the fact that $W_{\restriction t=0} = 0$, and in the second line, the inequality (15), we get

$$\|W(T)\|_X^2 + \int_0^T \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2(\Omega)}^2 \, dt \leq C \int_0^T \|\partial_t u(t) \wedge \boldsymbol{B}\|_{\boldsymbol{L}^2(\Omega)}^2 \, dt$$

$$\|u_0\|_{\boldsymbol{H}_0^1(\Omega)}^2 + \|u_1\|_{\boldsymbol{L}^2(\Omega)}^2 \leq C \int_0^T \|\partial_t u(t) \wedge \boldsymbol{B}\|_{\boldsymbol{L}^2(\Omega)}^2 \, dt$$

This shows point b). To prove $a)$, assume that inequality (11) holds. Consider a solution $V = (v, \partial_t v, h)$ of (4) with initial data $V_0 = (v_0, v_1, h_0)$, and the solution u of the Lamé system with initial data

$$(u, \partial_t u)_{\restriction t=0} = (v_0, v_1).$$

Thus, by (11):

$$(16) \qquad \|v_0\|_{\boldsymbol{H}_0^1}^2 + \|v_1\|_{\boldsymbol{L}^2}^2 \leq C \left(\|\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 + \|v_0\|_{\boldsymbol{L}^2}^2 + \|v_1\|_{\boldsymbol{H}^{-1}}^2 \right).$$

Furthermore, the energy inequality on the non-homogeneous Lamé system (Lemma 2.4) yields

$$\|u - v\|_{\boldsymbol{H}^1((0,T)\times\Omega)}^2 \leq C \int_0^T \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2}^2 \, dt$$

$$\|\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 \leq C \left\{ \int_0^T \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2}^2 \, dt + \|\operatorname{curl}(\partial_t v \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 \right\},$$

which implies, using (16), Lemma 2.3 and the equation $\beta \partial_t h + \operatorname{curl}\operatorname{curl} h = \beta \operatorname{curl}(\partial_t v \wedge \boldsymbol{B})$,

$$(17) \qquad \|v_0\|_{\boldsymbol{H}_0^1}^2 + \|v_1\|_{\boldsymbol{L}^2}^2 \leq C \left\{ \int_0^T \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2}^2 dt + \|v_0\|_{\boldsymbol{L}^2}^2 + \|v_1\|_{\boldsymbol{H}^{-1}}^2 \right\}.$$

In order to use Corollary 2.8, we need to add to the left-hand side of inequality (17) the $\boldsymbol{L}^2$-norm of $h(T)$. We may do so by taking a larger T. Indeed, consider $s \in [0, T]$ such that

$$\|h(s)\|_{\boldsymbol{L}^2}^2 = \min_{t \in [0,T]} \|h(t)\|_{\boldsymbol{L}^2}^2.$$

By Lemma 2.3,

$$\|h(s)\|_{\boldsymbol{L}^2(\Omega)}^2 \leq \frac{1}{T} \int_0^T \|h(t)\|_{\boldsymbol{L}^2(\Omega)}^2 dt \leq \frac{C}{T} \int_0^T \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2(\Omega)}^2 dt.$$

Inequality (17) taken at the initial time $t = s$ yields

$$E(s) = \frac{1}{2}\|V(s)\|_X^2 \leq C\left(\int_0^{2T} \|\operatorname{curl} h(t)\|_{\boldsymbol{L}^2(\Omega)}^2 dt + \|V(s)\|_{D(\mathcal{A})'}^2\right).$$

Since E and $\|V\|_{D(\mathcal{A})'}$ are non-increasing functions of time, this implies the inequality of Corollary 2.8, and thus the uniform decay of solutions of (1). The proof of b) is complete.

3. Defect measures

Let $N \geq 1$ be an integer (below we will take $N = 1$ or $N = 3$). For an open subset U of an Euclidean space, we set

$$\boldsymbol{L}^2(U) := L^2(U, \mathbb{C}^N), \quad \boldsymbol{H}^s(U) := H^s(U, \mathbb{C}^N).$$

We consider an open subset Ω of $\mathbb{R}^n$, $n \geq 1$ (below we will take $n = 3$), and a sequence (u^k) of functions on $\mathbb{R}_t \times \Omega_y$ such that

$$(18) \qquad\qquad u^k \underset{k \to +\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\text{loc}}(\mathbb{R} \times \overline{\Omega}),$$

(in the sense that (φu^k) converges weakly to 0 in $\boldsymbol{H}^1(\mathbb{R} \times \overline{\Omega})$ for all $\varphi \in C_0^\infty(\mathbb{R} \times \overline{\Omega})$). We assume that every u^k is solution of a wave equation in Ω:

$$(19) \qquad\qquad (\nu^2 \partial_t^2 - \Delta)u^k = 0, \text{ in } \mathbb{R} \times \Omega.$$

We shall introduce in this section a measure describing, from a microlocal point of view, the defect of compactness in $\boldsymbol{H}^1$ of the sequence (u^k). This description is of fundamental importance to show the observability inequalities of the preceding section, since the Lamé system decomposes into two waves equation (see §3.5.1). Micro-local defect measures have been independently introduced by P. Gérard and L. Tatar [10, 22]. We shall follow the construction of N. Burq and G. Lebeau, which describes the defect of convergence up to the boundary of Ω.

We assume, for the sake of simplicity that the functions u^k are smooth, so that their traces on the boundary are always defined. In the sequel we shall always reduce to this case.

In Subsection 3.1 we will give a few definitions and notations. In Subsection 3.2 we will state an existence theorem of microlocal defect measures and set out their first properties. Subsection 3.3 is devoted to the propagation theorem of the measure (proved in [2]), and Subsection 3.4 to some important properties of the traces of u^k on the boundary. Finally, in Subsection 3.5, we shall apply the construction of the measure to the case of a sequence of solutions of the Lamé system.

3.1. **Notations.**

3.1.1. *Local coordinates.* Consider an open cover of Ω: $\Omega = \bigcup_{j=0}^J \Omega_j$, where $\overline{\Omega}_0 \subset \Omega$ and, for all $j \geq 1$, Ω_j is a small neighbourhood of a point of $\partial\Omega$, such that on Ω_j, there are geodesic normal coordinates

$$z \in \Omega_j \mapsto (y', x_n) \in Y := Y' \times]0, l[,$$

where x_n is the distance to the boundary, and Y' an open subset of $\mathbb{R}^{n-1}$. Most objects introduced here are global objects but we will mainly use local coordinates. For a large part of this section we choose one of the open set Ω_j, $j \geq 1$.

Set $X := \mathbb{R} \times Y$, $X' := \mathbb{R} \times Y'$, and denote the elements of X and Y by

$$x = (\underbrace{x_0, x_1, .., x_{n-1}}_{x'}, x_n), \quad x_0 = t, \quad y = (\underbrace{x_1, x_2, .., x_{n-1}}_{y'}, x_n) \in Y,$$

Let

$$\mathbb{R}^{n+1}_+ := \left\{(x', x_n) \in \mathbb{R}^{n+1}, \ x_n > 0\right\}, \quad \overline{\mathbb{R}}^{n+1}_+ := \overline{\mathbb{R}^{n+1}_+}, \quad \partial X = X' \times \{0\}, \quad \overline{X} := X' \times [0, l[.$$

The set $\overline{X}$ is an open subset of $\overline{\mathbb{R}}_+^n$. Let g be the natural metric on Y, induced by the change of coordinates, and $\mathbf{g} = \det g$. In a geodesic system of coordinates, g is of the form

$$g(y) = \begin{bmatrix} g'(y) & 0 \\ 0 & 1 \end{bmatrix}.$$

3.1.2. *Bundles on X.* Let us consider $T^*X = X \times \mathbb{R}^{n+1}$ the cotangent bundle of X and S^*X the spherical cotangent bundle, which is defined to be the quotient

$$S^*X := (T^*X \backslash \{|\xi| = 0\})/\mathbb{R}_+^*,$$

by the action of $\mathbb{R}_+^* : (\lambda, \xi) \mapsto \lambda \xi$. The elements of those two bundles will be denoted by

$$\rho = (x, \xi), \quad x \in X, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n \times \mathbb{R}, \quad \xi = (\tau, \eta) = (\tau, \eta', \xi_n).$$

There is a natural Euclidean norm for the η-component of T^*X: $\|\eta\|^2 := {}^t\eta g^{-1}\eta$.

We will also consider $T^*\overline{X} = \overline{X} \times \mathbb{R}^{n+1}$, $T^*\partial X := \partial X_{x'} \times \mathbb{R}_{\xi'}^n$ the boundary cotangent bundle and $S^*\partial X$ the associated spherical bundle.

3.1.3. *Operators in the interior of Ω.* Le S_i^m the set of matrix symbols of degree m with compact support in X, which are the functions

$$a(x, \xi) \in C^\infty(X \times \mathbb{R}^{n+1}, M_N(\mathbb{C})),$$

such that there is a compact K of X with $\operatorname{supp} a \subset K \times \mathbb{R}^{n+1}$, and satisfying the following estimates:

$$(20_m) \qquad \forall \alpha, \; \forall \beta, \; \exists C_{\alpha\beta}, \; \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \le C_{\alpha\beta} \left(1 + |\xi|\right)^{m-|\beta|},$$

and which have a principal symbol $a_m(x, \xi)$, homogeneous function of degree m in ξ, such that $a - a_m$ satisfies (20_{m-1}) for large $|\xi|$. The operator of symbol a, $A = a(x, D)$, is defined by

$$Av(x) := \frac{1}{(2\pi)^{n+1}} \int a(x, \xi)\hat{v}(\xi)e^{ix\cdot\xi}d\xi.$$

In order to act on functions which are only defined in X, it is convenient to consider only the set $\mathcal{A}_i^m$ consisting of operators A which are of compact support in X, in the sense that $A = \varphi A \varphi$ for a function $\varphi \in C_0^\infty(X)$. An operator in $\mathcal{A}_i^m$ maps a distribution in X to a compactly supported distribution in X. We shall denote by $\sigma_m(A)$ the principal symbol of an operator A of degree m.

3.1.4. *Operators near the boundary.* Let S_b^m be the set of **tangential** matrix symbols of degree m with compact support in X, defined as the functions

$$a(x, \xi') \in C^\infty(\overline{X} \times \mathbb{R}^n, M_N(\mathbb{C})),$$

such that there is a compact K of $\overline{X}$ with $\operatorname{supp} a \subset K \times \mathbb{R}^n$, and satisfying the estimates

$$(21_m) \qquad \forall \alpha, \; \forall \beta, \; \exists C_{\alpha\beta}, \; \left|\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi')\right| \le C_{\alpha\beta} \left(1 + |\xi'|\right)^{m-|\beta|},$$

and which have a principal symbol $a_m(x, \xi')$, homogeneous of degree m in ξ' and such that $a - a_m$ satisfies the inequalities (21_{m-1}) for large $|\xi'|$. We define the operator of symbol a, $A = a(x, D')$, by

$$Av(x) := \frac{1}{(2\pi)^n} \int a(x, \xi')\hat{v}(\xi', x_n)e^{ix'\cdot\xi'}d\xi'.$$

Here, the Fourier transform of v is only taken with respect to the tangential variable x'. As in the interior case, we introduce the set $\mathcal{A}_b^m$ of tangential operators A with compact support in $\overline{X}$, i.e such that $A = \varphi A \varphi$ for a compactly supported function $\varphi \in C_0^\infty(\overline{X})$.

The set of all pseudo-differential operators of interest for us will be denoted by

$$\mathcal{A}^m := \{a = A_i + A_b, \; A_i \in \mathcal{A}_i^m, \; A_b \in \mathcal{A}_b^m\}.$$

3.1.5. *Sobolev spaces.* Let $s \in \mathbb{R}$ and ω be an open set of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. As before, we denote by $\boldsymbol{H}^s(\omega)$ the Sobolev space of $\mathbb{C}^N$-valued distributions (which is the set of restrictions to ω of elements of $\boldsymbol{H}^s(\mathbb{R}^n)$), endowed with the quotient norm). We also consider the space $\boldsymbol{H}_{\mathrm{loc}}^s(\omega)$, the space of vector-valued distributions such that

$$\forall \varphi \in C_0^\infty(\omega), \quad \varphi u \in \boldsymbol{H}^s,$$

and $\boldsymbol{H}_{\mathrm{comp}}^s(\omega)$, the space of distributions in $\boldsymbol{H}^s(\omega)$ compactly supported in ω. The notation $\boldsymbol{H}_{\mathrm{loc}}^s(Z)$, will also be used when Z is not open ($Z = \mathbb{R} \times \overline{\Omega}$, or $Z = \overline{X}$) as follows:

$$u^k \underset{k \to +\infty}{\longrightarrow} u \ (\text{ or } = O(1)) \text{ in } \boldsymbol{H}_{\mathrm{loc}}^s(Z) \iff \forall \varphi \in C_0^\infty(Z), \quad \varphi u^k \underset{k \to +\infty}{\longrightarrow} \varphi u \ (= O(1)) \text{ in } \boldsymbol{H}^s,$$

where $C_0^\infty(Z)$ is the space of C^∞ functions whose support is a compact of Z. We will also consider the following spaces, suitable for boundary-value problems:

$$\boldsymbol{H}_{\mathrm{loc}}^{0,s}(\overline{X}) = L^2(0,l; \boldsymbol{H}_{\mathrm{loc}}^s(X')), \quad \boldsymbol{H}_{\mathrm{comp}}^{0,s}(\overline{X}) = \left\{ u \in \boldsymbol{H}_{\mathrm{loc}}^{0,s}(\overline{X}), \ \exists \varphi \in C_0^\infty(\overline{X}), \ u = \varphi u \right\}.$$

The elements of $\mathcal{A}_i^m$ are continuous maps from $\boldsymbol{H}_{\mathrm{loc}}^s(X)$ to $\boldsymbol{H}_{\mathrm{comp}}^{s-m}(X)$, and those of $\mathcal{A}_b^m$ are continuous maps from $\boldsymbol{H}_{\mathrm{loc}}^{0,s}\left(\overline{X}\right)$ to $\boldsymbol{H}_{\mathrm{comp}}^{0,s-m}\left(\overline{X}\right)$.

Definition 3.1. Let $\rho \in S^*X$. The sequence (v^k) is said to be **bounded (respectively converging to** 0) **in** $\boldsymbol{H}_\rho^s$ when there exists $A \in \mathcal{A}_i^s$, whose principal symbol is invertible near ρ and such that (Av^k) is bounded in $\boldsymbol{H}^s$ (respectively converges to 0 in $\boldsymbol{H}^s$).

Let $\rho' \in S^*\partial X$. The sequence (v^k) is said to be **bounded (respectively converging to** 0) **in** $\boldsymbol{H}_{\rho'}^{0,s}$ when there exists $A \in \mathcal{A}_b^s$, whose principal symbol is invertible near ρ' and such that (Av^k) is bounded in $\boldsymbol{H}^{0,s}$ (respectively converges to 0 in $\boldsymbol{H}^{0,s}$).

Note that, according to Proposition A.1 of the appendix, for a sequence of solutions of (19), the convergence in $\boldsymbol{H}^{0,1}$ and $\boldsymbol{H}^1$ are equivalent. The spaces $\boldsymbol{H}^{0,1}$ and the tangential operators are thus well fitted for the description of the $\boldsymbol{H}^1$ convergence of (u^k).

3.1.6. *Melrose's compressed cotangent bundle.* We shall now introduce a bundle which naturally contains as subbundles both bundles T^*X and $T^*\partial X$. For this purpose, set $T_b^*X := \overline{X} \times \mathbb{R}^{n+1}$, endowed with its canonical topology and consider

$$\begin{array}{ccc} T^*\overline{X} & \xrightarrow{j} & T_b^*X \\ (x, \xi', \xi_n) & \longmapsto & (x, \xi', \theta = x_n \xi_n). \end{array}$$

The mapping j restricts to a continuous map:

$$T^*X \longrightarrow T_b^*X \cap \{x_n > 0\},$$

which identifies T^*X to a subbundle of dimension $2(n+1)$ of the interior of T_b^*X. Furthermore, the restriction of j to $x_n = 0$ defines a map from $T^*\overline{X} \cap \{x_n = 0\}$ to $T_b^*X \cap \{x_n = 0\}$, whose kernel is the set $\{\xi' = 0\}$. This clearly identifies

$$T^*\partial X \approx (T^*\overline{X} \cap \{x_n = 0\})/\mathbb{R}_{\xi_n},$$

(quotient taken by identifying all the points $(\tilde{x}', \tilde{\xi}', \xi_n)$, $\xi_n \in \mathbb{R}$) with a $2n$-dimensional subbundle of T_b^*X. The set of all sections of T_b^*X, with the above identifications, may be seen as the dual bundle of the bundle of all vector fields on X tangent to ∂X. It is called the compressed cotangent bundle.

We will also consider S_b^*X the spherical bundle of T_b^*X, which naturally contains the spherical bundles S^*X and $S^*\partial X$.

3.1.7. *Symbol of P and related manifolds.* The equation (19) takes the following form in local coordinates:

$$\text{(22)} \qquad Pu^k = 0, \quad P := -\mathbf{g}^{-1/2}\partial_{x_n}\mathbf{g}^{1/2}\partial_{x_n} + Q,$$

where Q is a scalar tangential differential operator of degree 2. Let $q(x,\xi')$ be the principal symbol of Q, and $p(x,\xi) = \xi_n^2 + q(x,\xi')$ the principal symbol of P. They are both scalar, homogeneous polynomials of degree 2 with respect to ξ. Let

$$\text{Char}\, P := \left\{(x,\xi) \in T^*\overline{X},\ p(x,\xi) = 0\right\}, \quad Z := j\,(\text{Char}\, P), \quad \widehat{Z} := j\,(\text{Char}\, P) \cup j\,(\{x_n = 0\}),$$

and $S\text{Char}\, P$, SZ, $S\widehat{Z}$ the corresponding spherical bundles.

Denoting $q_0(x',\xi') = q(x',0,\xi')$, we decompose $T^*(\partial X)$ (and $S^*(\partial X)$) into the disjoint union of the elliptic region $\mathcal{E}$, the glancing region $\mathcal{G}$ and the hyperbolic region $\mathcal{H}$:

$$\mathcal{E} := \{q_0 > 0\}, \quad \mathcal{G} := \{q_0 = 0\}, \quad \mathcal{H} := \{q_0 < 0\}.$$

3.1.8. *Global measure.* The defect measure is at first constructed in each of the preceding local coordinate systems. The objects obtained are then pieced together to $M = \mathbb{R} \times \Omega$. It is easy to define from local objects global Sobolev spaces and bundles on M, such as Melrose's compressed cotangent bundle T_b^*M. We shall use the same notations ($\text{Char}\, P$, Z, $\widehat{Z}$, $S\text{Char}\, P$, SZ, $S\widehat{Z}$,...) for the local and global objects. The definition of global operators is less natural in our setting. The symbol $\mathcal{A}^m$ will denote the set of operators A acting on functions on M, which are of the form

$$A = \sum_{j=0}^{J} A_{(j)}.$$

where $A_{(0)}$ is a classical pseudo-differential operator of order m with compact support in M and each $A_{(j)}$ is an operator of the sets $\mathcal{A}^m$ defined in each system of local coordinates. The global space $\mathcal{A}^m$ depends of the coordinate patches chosen, which shall not cause any problem in the remaining of the article. For a totally intrinsic construction, we could have used Melrose's totally characteristic operators (see [11, chap 18.3]).

3.2. **Existence of the measure.**

3.2.1. *The existence theorem.* The next elementary proposition (cf Lemmas 2.3 and 2.4 of [2]) shows that for any $A \in \mathcal{A}^0$, the behaviour of Au^k in $\boldsymbol{H}^1$ only depends upon the restriction of its principal symbol to $S\widehat{Z}$:

Proposition 3.2. *Let $A_b \in \mathcal{A}_b^{-\varepsilon}$. Then*

$$A_b u^k \xrightarrow[k \to 0]{} 0 \ \text{in } \boldsymbol{H}^1.$$

Let $A_i \in \mathcal{A}_i^0$, whose principal symbol vanishes on $\text{Char}\, P$. *Then*

$$A_i u^k \xrightarrow[k \to 0]{} 0 \ \text{in } \boldsymbol{H}^1.$$

According to Proposition 3.2, it is sufficient to describe the $\boldsymbol{H}^1$ convergence of (u^k) near $S\widehat{Z}$, in the sense given by Definition 3.1. Let $\mathcal{M}$ be the set of matrix-valued measures on $S\widehat{Z}$, i.e. the dual space of

$$\mathcal{C} := C_0^0\left(S\widehat{Z}, M^N\,(\mathbb{C})\right),$$

and $\mathcal{M}^+$ the subset of all positive measures in $\mathcal{M}$, i.e. measures μ which satisfy

$$\forall b \in \mathcal{C}, \quad \left(\forall z \in S\widehat{Z}, \quad b(z) \geq 0\right) \implies \langle \mu, b \rangle \geq 0.$$

($M \geq 0$ means M is positive Hermitian).

Before coming to the main theorem of this paragraph, we shall introduce a technical condition on u^k:

Definition 3.3. Let the sequence (u^k) satisfies (18) and (19). We shall say that (u^k) is **regular on the boundary** when one the following equivalent assumptions is satisfied:

$$(23a) \qquad\qquad\qquad u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{H}^{1/2}_{\mathrm{loc}}(\partial X), \ k \longrightarrow +\infty$$

$$(23b) \qquad\qquad\qquad \partial_{x_n} u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{H}^{-1/2}_{\mathrm{loc}}(\partial X), \ k \longrightarrow +\infty.$$

Note that the standard trace theorems imply conditions (23) with $O(1)$ instead of $o(1)$. All the sequences (u^k) in this work shall satisfy this condition. For the proof of the equivalence between (23a) and (23b) see [2, Lemma 2.6].

Theorem 3.4. *Let u^k be such that (18), (19) and (23) hold. Then there exists a subsequence of (u^k), still denoted by (u^k), and a measure $\mu \in \mathcal{M}^+$, called* **microlocal defect measure** *of (u^k), such that $\mu(\mathcal{E} \cup \mathcal{H}) = 0$ and*

$$(24) \qquad\qquad \forall A_j \in \mathcal{A}^j, \ j \in \{1,2\}, \ \lim_{k \to +\infty} \left(A_2 u^k + A_1 D_{x_n} u^k, u^k \right) = \left\langle \mu, \frac{a_2 + \xi_n a_1}{\tau^2} \right\rangle.$$

In (24), the notation $(.,.)$ stands for the $\boldsymbol{L}^2$-scalar product on M (in local coordinates, it is the scalar product on X using the metric $\mathbf{g}^{1/2} dy\, dt$). We note that $\tau \neq 0$ on $S\widehat{Z} \setminus (\mathcal{E} \cup \mathcal{H})$, and that

$$\forall (x, \xi) \in S\widehat{Z} \cap \{x_n > 0\}, \quad \left| \frac{\xi_n a_1}{\tau^2} \right| \leq \frac{\sqrt{q(x, \xi')} |a_1(x, \xi')|}{\tau^2},$$

so that the function $(x, \xi) \mapsto \frac{\xi_n a_1}{\tau^2}$ extends to a continuous function on $S\widehat{Z} \setminus (\mathcal{E} \cup \mathcal{H})$, which is null on $\mathcal{G}$. Thus the right-hand side of (24) makes sense.

Remark 3.5. The measure $\tilde{\mu} = \mu \mathbb{1}_{\{x_n > 0\}}$ may be seen as the standard microlocal defect measure (cf [10]) of the bounded sequence (u^k) of $\boldsymbol{H}^1_{\mathrm{loc}}(M)$. This interior measure describe the compactness defect of (u^k) in $\boldsymbol{H}^1_{\mathrm{loc}}(M)$ (in particular, it is null when (u^k) converges to 0 in this space), but not in $\boldsymbol{H}^1_{\mathrm{loc}}(\overline{M})$: $\tilde{\mu}$ vanishes when (u^k) concentrates on ∂M, even if it does not converge to 0 in $\boldsymbol{H}^1_{\mathrm{loc}}(\overline{M})$. On the other hand,

$$\varphi \in C_0^\infty(\overline{M}) \implies \int \varphi \left| \nabla_y u^k \right|^2 dx + \int \varphi \left| \partial_t u^k \right|^2 dx \xrightarrow[k \to +\infty]{} \langle \mu, \varphi \rangle.$$

The measure μ thus gives a complete description of the local convergence of (u^k) in $\boldsymbol{H}^1(\overline{M})$. This description is indeed microlocal:

$$\rho \in \operatorname{supp} \mu \iff u^k \xrightarrow[k \to +\infty]{} 0 \text{ in } \begin{cases} \boldsymbol{H}^1_\rho & \text{if } x_n > 0 \\ \boldsymbol{H}^{0,1}_\rho & \text{if } x_n = 0. \end{cases}$$

Theorem 3.4 is a new formulation, using Lemma 2.7 of [2], of Proposition 2.5 of this article. Compared to this proposition, the measure μ of Theorem 3.4 has been renormalized.

3.2.2. *A sufficient condition of nullity for μ.* Let $\tilde{\rho} \in S\widehat{Z}$ an interior point and $A \in \mathcal{A}^2_i$, whose principal symbol is invertible at $\tilde{\rho}$. By elementary symbolic calculus on classical operators, it is easy to show, with formula (24),

$$(25) \qquad\qquad\qquad A u^k \xrightarrow[k \to +\infty]{} 0 \text{ in } \boldsymbol{H}^{-1}_{\tilde{\rho}} \implies \tilde{\rho} \notin \operatorname{supp} \mu.$$

The same statement holds in $\mathcal{G}$:

Proposition 3.6. *Let $\tilde{\rho} \in \mathcal{G}$. Consider an operator of the form*

$$A = A_0 D_{x_n}^2 + A_1 D_{x_n} + A_2, \ A_j \in \mathcal{A}^j, \ a_j := \sigma(A_j),$$

such that

$$(26) \qquad\qquad\qquad A u^k \xrightarrow[k \to +\infty]{} 0 \text{ in } \boldsymbol{H}^{0,-1}_{\tilde{\rho}}.$$

Assume that (u^k) is regular on the boundary and that $a_2(\tilde{\rho})$ is invertible. Then

$$\tilde{\rho} \notin \operatorname{supp} \mu.$$

Remark 3.7. Proposition 3.6 is trivial when $A = A_2 \in \mathcal{A}^2$ (it is essentially the definition of $\boldsymbol{H}_{\tilde{\rho}}^{0,-1}$).

Remark 3.8. Note that according to the appendix, the convergence to 0 of (Au^k) in the space $\boldsymbol{H}^{-1}(X)$ near $\tilde{x}$ would imply (26). Furthermore, the proof of Proposition 3.6 will show that assumption (26) is equivalent to

$$(BAu^k, u^k) \xrightarrow[k\to+\infty]{} 0, \ \forall B \in \mathcal{A}^0, \text{ with support close enough to } \tilde{\rho}.$$

Proof. According to (25), $\mu \mathbb{1}_{\{x_n > 0\}}$ is null near $\tilde{\rho}$. The same property remains to be proved on $\mu \mathbb{1}_{\{x_n = 0\}}$. Let $B \in \mathcal{A}^0$, $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(0) = 1$, $B_\varepsilon := \psi\left(\frac{x_n}{\varepsilon}\right) B$, and $b = \sigma(B)$. In view of (26) and (24),

$$\left\langle \mu, \psi\left(\frac{x_n}{\varepsilon}\right) b \frac{a_0 \xi_n^2 + a_1 \xi_n + a_2}{\tau^2} \right\rangle = \lim_{k\to+\infty} (B_\varepsilon A u^k, u^k) = 0.$$

Letting ε goes to 0, the dominated convergence theorem and the fact that ξ_n is null on the support of $\mu \mathbb{1}_{\{x_n = 0\}}$ give

$$(27) \qquad\qquad\qquad\qquad \left\langle \mu, \mathbb{1}_{\mathcal{G}} \frac{b a_2}{\tau^2} \right\rangle = 0.$$

Let $\phi \in S_b^0$ be scalar, positive, and compactly supported near $\tilde{\rho}$ such that a_2 is invertible on the support of ϕ, and choose B such that

$$b(x, \xi') = \phi(x, \xi') a_2^{-1} \tau^2.$$

The equality (27) then shows that $\langle \mu, \mathbb{1}_{\mathcal{G}} \phi \rangle = 0$, which completes the proof using the positivity of μ. $\qquad\square$

3.3. **The propagation theorem.**

3.3.1. *The generalized bicharacteristic flow.* The characteristic curves of the Hamiltonian flow of p

$$H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$$

define a local flow on T^*X. The symbol p is homogeneous of degree 2 in ξ, so that the flow of H_p does not yield a flow on the quotient space S_b^*X. To get such a flow, we shall replace p by p/τ which is homogeneous of degree 1. Note that on the support of μ (where τ does not vanish), p is null, so that $\frac{1}{\tau} H_p$ and $H_{p/\tau}$ are equal. Furthermore, the integral curves of $\frac{1}{\tau} H_p$ and H_p are the same.

Let Σ be a small conic open subset of $Z = j(\operatorname{Char} P)$. Set $q_0 := q_{\restriction x_n = 0}$, $q_1 := \partial_{x_n} q_{\restriction x_n = 0}$ and

$$\Sigma^0 := \Sigma \cap \{x_n > 0\}$$

$$\Sigma^1 := \mathcal{H} = \Sigma \cap \{x_n = 0, \ q_0 < 0\}$$

$$\Sigma^2 := \Sigma \cap \{x_n = 0, \ q_0 = 0, \ q_1 \neq 0\}$$

$$\Sigma^{k+3} := \Sigma \cap \{x_n = 0, \ q_0 = q_1 = ... = H_{q_0}^k q_1 = 0, \ H_{q_0}^{k+1} q_1 \neq 0\}.$$

Assume that in Σ, there is no contact of infinite order between the bicharacteristic curves of P and the boundary, which means that for a certain finite integer J

$$(28) \qquad\qquad\qquad\qquad \exists J \in \mathbb{N}, \quad \Sigma = \bigcup_{j \leq J} \Sigma^J.$$

Decompose Σ^2 in the disjoint union

$$\Sigma^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}, \quad \mathcal{G}^{2,+} := \Sigma^2 \cap \{q_1 < 0\}, \ \mathcal{G}^{2,-} := \Sigma^2 \cap \{q_1 > 0\}.$$

The set $\mathcal{G}^{2,+}$ is the set of **strictly diffractive** points and $\mathcal{G}^{2,-}$ the set of **strictly gliding** points.

Definition 3.9. Let γ be a map from a real interval I to Σ and

$$\Gamma(s) = j^{-1}(\gamma(s)) \in S\mathrm{Char}\, P$$

which is defined as long as $\gamma(s) \notin \mathcal{H}$. Such a map $\gamma(s) = (x(s), \xi(s))$ is called a **ray**, or a **general bicharacteristic curve** when γ is continuous from I to Σ and for all s_0 in I,

- if $x_n(s_0) > 0$, Γ is differentiable in s_0 and

$$\Gamma'(s_0) = \frac{1}{\tau} H_p \Gamma(s_0);$$

- if $\gamma(s_0) \in \mathcal{H} \cup \mathcal{G}^{2,+}$,

$$\exists \varepsilon > 0, \quad \forall s \in]s_0 - \varepsilon, s_0[\cup]s_0, s_0 + \varepsilon[, \quad x_n(s) > 0;$$

- if $\gamma(s_0) \in \mathcal{G} \backslash \mathcal{G}^{2,+}$, Γ is well defined and differentiable near s_0 and

$$\Gamma'(s_0) = \frac{1}{\tau} H_{q_0} \Gamma(s_0).$$

(Thus, if γ stays in this region, its spatial projection is a geodesic of the boundary.)

Under the assumption (28), R. Melrose and J. Sjöstrand have shown that for any $\rho \in \Sigma$, there exists a unique maximal ray γ taking values in Σ such that $\gamma(0) = \rho$ (cf [17], [11, chap 24.3]). In the sequel, we shall denote by $\phi(s, \rho)$ the resulting flow (satisfying $\phi(0, \rho) = \rho$). The function p/τ being homogeneous of degree 1 in ξ, ϕ also defines a flow on $\Sigma/\mathbb{R}_+^*$.

3.3.2. *The uniform Lopatinsky conditions.*

Notations. Let S_∂^m be the set of symbols $a(x', \xi')$ of pseudo-differential operators on ∂X, with compact support in x', with principal symbol homogeneous of degree m in ξ', and $\mathcal{A}_\partial^m$ the set of corresponding compactly supported pseudo-differential operators (cf §3.1.3 for precise definitions). The Sobolev spaces on ∂X, defined as those on X, shall be denoted by $\boldsymbol{H}_\partial^s$, $\boldsymbol{H}_{\mathrm{loc},\partial}^s$, $\boldsymbol{H}_{\tilde{\rho},\partial}^s$.

An approximate pseudo-differential equation on the traces of u^k is said to satisfy Lopatinsky conditions when it is independent of the equation $Pu^k = 0$. More precisely:

Definition 3.10. Under the assumptions (18) and (19), the sequence (u^k) is said to satisfy **uniform Lopatinsky** boundary conditions at $\tilde{\rho} \in S^*\partial X$ when

- if $\tilde{\rho} \in \mathcal{G}$, $\exists B_{-1} \in \mathcal{A}_\partial^{-1}$ such that

(29)
$$\begin{cases} u^k_{\restriction x_n=0} = B_{-1}\left(D_{x_n} u^k_{\restriction x_n=0} \right) + h^k \\ h^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\tilde{\rho},\partial}^1; \end{cases}$$

- if $\tilde{\rho} \in \mathcal{H}$, $\exists B_0 \in \mathcal{A}_\partial^0$ such that

(29')
$$\begin{cases} D_{x_n} u^k_{\restriction x_n=0} - \Lambda u^k_{\restriction x_n=0} = B_0\left(D_{x_n} u^k_{\restriction x_n=0} + \Lambda u^k_{\restriction x_n=0} \right) + h^k \\ \sigma(B_0)(\tilde{\rho}) \text{ invertible}, \quad h^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}_{\tilde{\rho},\partial}^2 \\ \Lambda \in \mathcal{A}_\partial^1, \quad \sigma(\Lambda) = \sqrt{q_0(x', \xi')} = \sqrt{\nu^2\tau^2 - \|\eta'\|^2} \text{ near } \tilde{\rho}. \end{cases}$$

($\|\eta'\|^2 = {}^t\eta' g'^{-1}\eta'$ is the natural Euclidean norm in the local coordinate system).

Examples. • The Dirichlet boundary condition, $u^k_{\restriction x_n=0} = 0$, or more generally a pseudo-differential boundary condition of the form

(30)
$$u^k_{\restriction x_n=0} = B_{-1} D_{x_n} u^k_{\restriction x_n=0} + h^k, \quad B_{-1} \in \mathcal{A}^{-1}$$
$$h^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\tilde{\rho},\partial}^1,$$

where the eigenvalues of $\sigma(B_{-1})$ are all pure imaginary numbers near $\tilde{\rho}$, is a uniform Lopatinsky boundary condition, whether $\tilde{\rho}$ is glancing or hyperbolic.

- The Neumann condition

$$D_{x_n} u^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}$$

is a uniform Lopatinsky condition near any $\tilde{\rho} \notin \mathcal{G}$, but not at $\tilde{\rho} \in \mathcal{G}$.

In the glancing case, a boundary condition of the form (29) implies that (u^k) is regular near the boundary close to $\tilde{\rho}$ (see Proposition 3.18), so that we can define the microlocal defect measure μ of u^k, and the bicharacteristic flow is defined μ-almost everywhere in a neighbourhood of $\tilde{\rho}$ (since $\mu(\mathcal{E}) = 0$).

The set $\mathcal{G}^{2,+}$ is transverse to the bicharacteristic flow. The following result (proved in [2], §3.4) is needed in the next paragraph to treat the propagation of μ by the flow.

Lemma 3.11. *Suppose that on Σ, u^k satisfies uniform Lopatinsky boundary conditions. Then*

$$\mu\left(\mathcal{G}^{2,+} \cap \Sigma\right) = 0.$$

3.3.3. *The propagation theorem.* When a uniform Lopatinsky condition holds, μ propagates along the integral curves of the bicharacteristic flow. In the hyperbolic region, there is a jump (which depends upon the boundary condition). We shall only state a propagation theorem for the support of μ, without giving a complete description of the propagation of μ.

Theorem 3.12. *Let $\tilde{\rho} \in \mathcal{H} \cap \mathcal{G}$ such that (u^k) satisfies Lopatinsky boundary conditions. Consider a small conic open neighbourhood Σ of $\tilde{\rho}$ in $S\hat{Z}$ such that on Σ, (29) (or (29')) holds. Then the support of μ is, in Σ, invariant by the bicharacteristic flow.*

(cf [2, chap. 3.3, th.1])

In other terms, if $\rho \in \Sigma$ is on the support of μ, so is the entire bicharacteristic passing through ρ in Σ.

Remark 3.13. Inside M, Theorem 3.12 is an easy consequence of the transport equation on μ:

$$(31) \qquad\qquad \langle \mu, \{p/\tau, a\} \rangle, \quad a \in C_0^\infty\left(Z \cap \{x_n > 0\}\right),$$

which may be immediately derived, using symbolic calculus, from the property

$$(32) \qquad\qquad \lim_{k\to+\infty}\left(A_1 P u^k - P A_1 u^k, u^k\right) = 0, \quad A_1 \in \mathcal{A}_i^1,$$

obtained by integration by parts with the equation (22). Near a boundary point, property (32), with $A_1 \in \mathcal{A}_b^1$, still holds with an additional boundary term. Consequently, (31) holds only for a certain class of function $a \in C_0^\infty(SZ)$, satisfying a particular boundary condition on $\{x_n = 0\}$ (condition chosen so that the boundary term vanishes when k goes to ∞). The proof of the propagation theorem, which is fairly technical, uses (31), and near strictly diffractive points, Lemma 3.11. The boundary condition on a gives the exact value of the jump in the hyperbolic region. See [2, par. 3] for details.

3.4. **Estimates on traces.** We now state precise properties of the traces of u^k in the hyperbolic, elliptic and glancing regions, which are one of the main tools of the proofs of the following sections. Those results are fairly classical, and we only shall give a proof (in the appendix) for the glancing case. See [2, appendix] for proofs in the hyperbolic and elliptic cases. In this paragraph, we shall always assume (u^k) satisfies (18) and (19).

3.4.1. *Hyperbolic region.* Near an hyperbolic point, one gains without any boundary condition, half a derivative in comparison with the standard traces theorem.

Proposition 3.14. *Let $\tilde{\rho} \in \mathcal{H}$. Then*

$$u^k_{\restriction x_n=0} \underset{k\to\infty}{=} O(1) \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}, \quad \partial_{x_n} u^k_{\restriction x_n=0} \underset{k\to\infty}{=} O(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

In view of the propagation theorem in the interior of M, the support of μ is, near $\tilde{\rho}$, the union of incoming rays (integral curves of $H_{p/\tau}$ along which $\xi_n < 0$) and outgoing rays (integral curves of $H_{p/\tau}$ along which $\xi_n > 0$). When the sequence satisfies uniform Lopatinsky conditions, Theorem 3.12 is equivalent to the fact that if an incoming (respectively outgoing) ray is in the support of μ, so is the outgoing (respectively incoming) ray passing through the same hyperbolic point. In the opposite case where the support of μ contains, locally, only incoming (or only outgoing) rays, one gets a boundary condition which is orthogonal to the uniform Lopatinsky conditions:

Proposition 3.15. *Assume that near $\tilde{\rho} \in \mathcal{H}$, on the support of μ, $\xi_n > 0$. Then*

$$D_{x_n} u^k_{\upharpoonright x_n=0} + \Lambda u^k_{\upharpoonright x_n=0} = o(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}, \quad \Lambda \in \mathcal{A}^1_\partial, \quad \sigma_1(\Lambda) = \sqrt{\nu^2 \tau^2 - \|\eta'\|^2}$$

On the other hand, if near $\tilde{\rho} \in \mathcal{H}$, on the support of μ, $\xi_n < 0$, then

$$D_{x_n} u^k_{\upharpoonright x_n=0} - \Lambda u^k_{\upharpoonright x_n=0} = o(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

In particular, if μ is null near $\tilde{\rho}$,

$$D_{x_n} u^k_{\upharpoonright x_n=0} \underset{k \to +\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}, \quad u^k_{\upharpoonright x_n=0} \underset{k \to +\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}.$$

3.4.2. *In the elliptic region.* In $\mathcal{E}$, the equation (22) implies a pseudo-differential traces equation on u^k:

Proposition 3.16. *Let $\tilde{\rho} \in \mathcal{E}$ and (u^k) satisfy the assumptions of Theorem 3.4. Let $M > 0$. Then*

(33)
$$D_{x_n} u^k_{\upharpoonright x_n=0} + \Xi u^k_{\upharpoonright x_n=0} \underset{k \to +\infty}{\longrightarrow} 0 \text{ in } H^M_{\tilde{\rho}}$$

$$\Xi \in \mathcal{A}^1, \quad \sigma_1(\Xi) = i\sqrt{q_0} = i\sqrt{\|\eta'\|^2 - \nu^2 \tau^2} \text{ near } \tilde{\rho}.$$

In particular, if a boundary condition independent of (33) holds on u^k near $\tilde{\rho}$ (such a condition is called as in the glancing and hyperbolic cases a uniform Lopatinsky condition), the traces of u^k converge to 0 in $H^M_{\tilde{\rho}}$. Proposition 3.16 still holds in a much more general case, for example if P is replaced by a non-scalar operator $\mathbf{P}$. The principal symbol of Ξ depends again upon the principal symbol of $\mathbf{P}$. In the next proposition, we only state a consequence of this fact when (u^k) satisfies Dirichlet boundary conditions (which are of uniform Lopatinsky type).

Proposition 3.17. *Let $\mathbf{P} := D^2_{x_n} + \mathbf{Q}_1 D_{x_n} + \mathbf{Q}_2$, where each $\mathbf{Q_j}$ is a matrix pseudo-differential operator of degree j, with principal symbol $\mathbf{q}_j$. Let $\mathbf{p}(x,\xi) := \xi_n^2 + \mathbf{q}_2 + \xi_n \mathbf{q}_1$ be the principal symbol of $\mathbf{P}$, and $\tilde{\rho} = (\tilde{x}', \tilde{\xi}')$ be a point of $S^*\partial X$ such that the matrix $\mathbf{p}(\tilde{x}', 0, \tilde{\xi}', \xi_n)$ is invertible for any real number ξ_n. Consider a sequence (u^k), weakly converging to 0 in $\boldsymbol{H}^1_{\mathrm{loc}}(\overline{X})$ and satisfying*

$$\mathbf{P}u^k = 0, \quad u^k_{\upharpoonright x_n=0} = 0.$$

Then for all M,

$$\partial_{x_n} u^k_{\upharpoonright x_n=0} \underset{k \to +\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^M_{\tilde{\rho},\partial}.$$

We shall later apply the preceding proposition in the elliptic zone ($\mathcal{E}_T \cap \mathcal{E}_L$ with notations of §3.5.2) of the Lamé operator $\partial_t^2 - \Delta_e$.

3.4.3. *In the glancing region.* The strong results of the two preceding paragraphs do not hold in the neighbourhood of a glancing point. In this case, one needs boundary conditions to get better estimates than the standard traces theorem with loss of one half-derivative. In the case of Lopatinsky boundary conditions, the results are similar to those of the hyperbolic region.

Proposition 3.18. a) *Let $\tilde{\rho} \in \mathcal{G}$. Assume that (u^k) satisfies Lopatinsky uniform boundary conditions near $\tilde{\rho}$. Then*

(34)
$$u^k_{\upharpoonright x_n=0} = O(1) \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}, \quad \partial_{x_n} u^k_{\upharpoonright x_n=0} = O(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

Furthermore, if μ vanishes near $\tilde{\rho}$, then

$$(35) \qquad u^k_{\upharpoonright x_n=0} = o(1) \ in \ \boldsymbol{H}^1_{\tilde{\rho},\partial}, \quad \partial_{x_n} u^k_{\upharpoonright x_n=0} = o(1) \ in \ \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

b) *Assume*

$$(36) \qquad u^k_{\upharpoonright x_n=0} = o(1) \ in \ \boldsymbol{H}^1_{\tilde{\rho},\partial}, \quad \partial_{x_n} u^k_{\upharpoonright x_n=0} = o(1) \ in \ \boldsymbol{L}^2_{\tilde{\rho},\partial},$$

and that every $\tilde{\rho} \in \mathcal{G}$ is not diffractive, in the sense that at least one off the two half-bicharacteristic passing through $\tilde{\rho}$ stays in $\partial\Omega$ near $\tilde{\rho}$. Then $\mu = 0$ near $\tilde{\rho}$.

Remark 3.19. As seen in Propositions 3.14 and 3.15, point **a)** holds in the hyperbolic case, where no boundary condition is required.

The proof of Proposition 3.18 is given in the appendix.

3.5. **The Lamé system.** This subsection is devoted to the Lamé system with Dirichlet boundary conditions on an open bounded subset Ω of $\mathbb{R}^3$:

$$(37) \qquad \begin{cases} \partial_t^2 u - \Delta_e u = 0, \ (t,y) \in \mathbb{R} \times \Omega \\ u_{\upharpoonright \partial\Omega} = 0 \\ u_{\upharpoonright t=0} = u_0 \in \boldsymbol{H}^1_0, \ \partial_t u_{\upharpoonright t=0} = u_1 \in \boldsymbol{L}^2. \end{cases}$$

In §3.5.1, (37) is decomposed into two wave equations. In §3.5.2, we shall introduce the defect measures associated to these equations. Next paragraphs are devoted to a few elementary properties of these measures.

We will often work in local coordinates. As before (but with $n = 3$) we will only use geodesic normal coordinates near the boundary, choosing the coordinate x_3 to be the distance to the boundary $\{x_3 = 0\}$.

3.5.1. *Transversal and longitudinal waves.* The natural energy

$$E(t) := \frac{1}{2} \int_\Omega \left(|\partial_t u|^2 + \varsigma |\nabla u|^2 + (\lambda + \varsigma)|\operatorname{div} u|^2 \right) \, \mathrm{d}y$$

is time-invariant. Let E_0 be its constant value. The next classical proposition is proved, for example, in [2]. Recall that $M := \mathbb{R} \times \Omega$.

Proposition 3.20 (Decomposition of the Lamé system)**.** *There exists a constant $C > 0$ such that for every solution u of (37), there exists*

$$u_T \in \boldsymbol{H}^1_{\mathrm{loc}}\left(\overline{M}\right), \ u_L \in \boldsymbol{H}^1_{\mathrm{loc}}\left(\overline{M}\right),$$

such that

(1) $u = u_T + u_L$, $\operatorname{div} u_T = 0$, $\operatorname{curl} u_L = 0$.
(2) $(\partial_t^2 - c_T^2 \Delta)u_T = 0$, *where* $c_T^2 := \varsigma$.
(3) $(\partial_t^2 - c_L^2 \Delta)u_L = 0$, *where* $c_L^2 := \lambda + 2\varsigma$.
(4) *For every bounded interval I of $\mathbb{R}$, of length $|I|$,*

$$\|u_L\|^2_{\boldsymbol{H}^1(I\times\Omega)} + \|u_T\|^2_{\boldsymbol{H}^1(I\times\Omega)} \le C|I|E_0.$$

(5) *If u_0 and u_1 are linear combinations of a finite number of eigenfunctions of Δ_e, then*

$$u_T \in C^\infty(\overline{M}), \quad u_L \in C^\infty(\overline{M}).$$

Definition 3.21. The function u_T is called the **transversal wave**, and u_L the **longitudinal wave**.

Remark 3.22. In the sequel we shall often reduce the longitudinal wave to a scalar function, writing $u_L = \nabla\varphi$, with

$$\varphi \in \boldsymbol{H}^2_{\mathrm{loc}}\left(\overline{M}\right), \quad \|\varphi\|_{\boldsymbol{H}^2(I\times\Omega)} \le C|I|E_0, \quad (\partial_t^2 - c_L^2 \Delta)\varphi = 0.$$

3.5.2. *Measures.* Let (u^k) be a sequence of solutions of the Lamé system with

$$(u_0^k, u_1^k) \underset{k \to +\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega).$$

In view of Proposition 3.20, both sequences (u_T^k) and (u_L^k) weakly converge to 0 in $\boldsymbol{H}_{\mathrm{loc}}^1(\mathbb{R} \times \overline{\Omega})$. Likewise, the sequence $(\partial_t \varphi^k)$ weakly converges to 0 in $H_{\mathrm{loc}}^1(\mathbb{R} \times \overline{\Omega})$ (we consider a derivative of φ in order to work in a H^1 space since we have defined the defect measures in such spaces).

Lemma 3.23. *The sequences (u_T^k), (u_L^k), $(\partial_t \varphi^k)$ are regular on the boundary.*

(cf [2, lemme 4.2])

Notations. Let

- μ_T, μ_L and μ be the defect measures respectively associated (up to a subsequence) to (u_T^k), (u_L^k), and $(\partial_t \varphi^k)$ by Theorem 3.4;
- $\mathcal{H}_T$, $\mathcal{H}_L$, $\mathcal{G}_T$, $\mathcal{G}_L$, $\mathcal{E}_T$, $\mathcal{E}_L$ the hyperbolic, glancing and elliptic region of the transversal and longitudinal waves.

All the calculation shall be carried out in one of the $J+1$ local coordinate systems chosen in the beginning of this section. We shall make a distinction between the spaces of scalar operators $\mathcal{A}^m$, $\mathcal{A}_\partial^m$ (defined in §3.1.3, §3.1.4, §3.1.8, with $N = 1$), and the spaces of matrix operators $\boldsymbol{\mathcal{A}}^m$, $\boldsymbol{\mathcal{A}}_\partial^m$ (with $N = 3$).

The notation $x = (t, y)$ always refers to local coordinates. When a distinction is necessary, we shall write coordinates on Ω before the change of variables $z = (z_1, z_2, z_3)$. This global system of coordinates has been chosen so that the magnetic field $\boldsymbol{B}$ has coordinates $(B, 0, 0)$.

Remark 3.24. The condition $c_L^2 \neq c_T^2$ means that the intersection of $\mathcal{G}_T$ and $\mathcal{G}_L$ is empty.

3.5.3. *Link between μ and μ_L.* In a fixed coordinate patch, let χ be the local diffeomorphism from global spatial coordinates (z_1, z_2, z_3) to local coordinates:

$$z = \chi(y), \quad {}^t\chi'(y)\zeta = \eta.$$

Proposition 3.25.

$$\forall a \in C_0^0 \left(S_b^*(\mathbb{R} \times \overline{\Omega}), M^3(\mathbb{C}) \right), \ \langle \mu, \frac{{}^t\zeta a\zeta}{\tau^2} \rangle = \langle \mu_L, a \rangle.$$

In particular, the measures μ and μ_L have the same supports.

Proof. Let $A_j \in \boldsymbol{\mathcal{A}}^j$, $j = 1, 2$, $\quad A := A_{-1}D_{x_3} + A_0$.
Set $I_k := (A\partial_t u_L^k, \partial_t u_L^k)$.
On one hand,

$$I_k = -(\partial_t A \partial_t u_L^k, u_L^k) \underset{k \to +\infty}{\longrightarrow} \langle \mu_L, a \rangle.$$

On the other hand,

$$I_k = -(\mathrm{div}\, A\nabla \partial_t \varphi^k, \partial_t \varphi^k) + o(1) \underset{k \to +\infty}{\longrightarrow} \left\langle \mu, \frac{{}^t\zeta a\zeta}{\tau^2} \right\rangle.$$

The boundary terms of this preceding integration by parts converge to 0 according to Lemma 3.23. This implies Proposition 3.25 when a is of the form $\xi_3 a^{-1} + a_0$, and then by a density argument for any a. $\quad\square$

3.5.4. *Polarization of μ_T and μ_L.* Let π bet the orthogonal projection in $\mathbb{C}^3$ on the line generated by ζ, and $\pi_\perp$ the orthogonal projection on the plane normal to ζ.

(38) $$\pi(V) := |\zeta|^{-2} ({}^t\zeta.V)\zeta, \quad \pi_\perp := \mathrm{Id}_{\mathbb{C}^3} - \pi.$$

The projectors π and $\pi_\perp$ are defined by the formulas (38) on $S^*\overline{X}$.

Proposition 3.26. *The measure μ_L is polarized along the direction of propagation, and μ_T orthogonally to this direction:*

$$\mu_L = \pi \mu_L \pi, \quad \mu_T = \pi_\perp \mu_T \pi_\perp.$$

Proof. The statement on μ_L is an immediate consequence of Proposition 3.25. To show the statement on μ_T, take $A_0 \in \mathcal{A}_0$. The nullity of div u_T^k implies

$$0 = (A_0 \nabla \operatorname{div} u_T^k, u_T^k) \underset{k \to +\infty}{\longrightarrow} \langle \mu_T, \frac{a_0 \zeta \, {}^t\zeta}{\tau^2} \rangle.$$

Thus $< \mu_T, a\pi >= 0$ i.e. $< \mu_T, a\pi_\perp >=< \mu_T, a >$. The symmetry of μ_T completes the proof. $\qquad\square$

Remark 3.27. To get more intrinsic formulations of the preceding results, i.e. statements where the two coordinate system do not mix, one should have considered u_L^k and u_T^k as section of the tangent space $T\Omega$, and defined measures with values endomorphism of $T\Omega$ (instead of endomorphism of $\mathbb{C}^3$).

3.5.5. *A decoupling lemma.* The next result, converting an approximate differential equation on u^k into two equations on u_L^k and u_T^k, is of crucial importance in the sequel. As before, $(.,.)$ stands for the $\boldsymbol{L}^2$ scalar product on $\mathbb{R} \times \Omega$.

Lemma 3.28 (decoupling lemma)**.** *Let A be a pseudo-differential operator of order 2 of the following form*

$$(39) \qquad A = \sum_{j=-M}^{2} A_j \partial_{x_3}^{2-j} + A_i, \quad A_i \in \mathcal{A}_i^2, \quad A_j \in \mathcal{A}^j.$$

Then

$$(40) \qquad \lim_{k \to +\infty} (Au_T^k, u_L^k) = \lim_{k \to +\infty} (Au_L^k, u_T^k) = 0.$$

Let A be a $(1, 3)$ matrix of pseudo-differential operators, with coefficients of the form (39), but with **scalar** *operators. Then*

$$(41) \qquad \lim_{k \to +\infty} (Au_T^k, \partial_t \varphi^k) = 0.$$

Proof. We shall only prove the convergence to 0 of (Au_T^k, u_L^k). The proof of rest of the lemma is very much the same. We may obviously assume that the operators A_j have compact support in one of the local coordinate system introduced in §3.1.8. In view of the equations

$$-\mathrm{g}^{-1/2}\partial_{x_3}\mathrm{g}^{1/2}\partial_{x_3}u_T^k + Q_T u_T^k = 0, \quad -\mathrm{g}^{-1/2}\partial_{x_3}\mathrm{g}^{1/2}\partial_{x_3}u_L^k + Q_L u_L^k = 0,$$

where Q_T and Q_L are tangential, it is sufficient to prove the lemma in the cases $j = 1, 2$ and in the interior case.

First case: $A \in \mathcal{A}^i$.

Since $\nu_L \neq \nu_T$ it is easy to construct two operators $\Psi_T, \Psi_L \in \mathcal{A}_i^0$ such that

$$(\Psi_T + \Psi_L)_{\restriction U} = \mathrm{Id}, \quad \operatorname{supp}\sigma_0(\Psi_T) \cap \mathcal{Z}_L = \operatorname{supp}\sigma_0(\Psi_L) \cap \mathcal{Z}_T = \emptyset,$$

where U is an open subset of Ω such that there exists a function $\varphi \in C_0^\infty(U)$ satisfying $\varphi A \varphi = A$. Writing

$$A = A\varphi = A\Psi_L \varphi + A\Psi_T \varphi,$$

we may assume that the principal symbol of A does not intersect $\mathcal{Z}_T$ (or does not intersect $\mathcal{Z}_L$). For such operators, (40) holds has a direct consequence of Proposition 3.2. For example, in the first case we have

$$Au_T^k = O(1) \text{ in } \boldsymbol{L}^2.$$

Second case: $A \in \mathcal{A}_\partial^2$.

We know that the support of $\mu_T \mathbb{1}_{\{x_3=0\}}$ is included in $\mathcal{G}_T$ and the support of $\mu_L \mathbb{1}_{\{x_3=0\}}$ in $\mathcal{G}_L$. As a consequence, we may write $A = A(\Theta_T + \Theta_L)$ where Θ_T and Θ_L are tangential operators of degree 0 such that

$$\operatorname{supp}\sigma_0(\Theta_T) \cap \mathcal{G}_L = \operatorname{supp}\sigma(\Theta_L) \cap \mathcal{G}_T = \emptyset.$$

We may thus assume that the support of the principal symbol a_2 of A is disjoint from one of the two glancing sets, say $\mathcal{G}_T$. We have

$$\left(Au_T^k, u_L^k\right) = \left(\chi(x_3/\varepsilon)Au_T^k, u_L^k\right) + \left((1-\chi(x_3/\varepsilon))Au_T^k, u_L^k\right).$$

where χ is a compactly supported function in $\mathbb{R}$ equal to 1 near the origin. We first fix ε and let k tend to ∞. The second term of the sum tends to 0 in view of the preceding case. As for the first term, we have, u_L^k being bounded in $\boldsymbol{H}^1$,

$$\left|\left(\chi(x_3/\varepsilon)Au_T^k, u_L^k\right)\right| \leq C\|\chi(x_3/\varepsilon)Au_T^k\|_{L^2(0,l,\boldsymbol{H}^{-1}(X'))}\|u_L^k\|_{L^2(0,l,\boldsymbol{H}^1(X'))}$$
$$\leq C\|\Lambda'_{-1}\chi(x_3/\varepsilon)Au_T^k\|_{\boldsymbol{L}^2} + o(1), \ k \to +\infty,$$

where $\Lambda'_{-1} \in \mathcal{A}_\partial^{-1}$, with principal symbol equal to $\|\xi'\|^{-1}$ near the support of A. We have

$$\|\Lambda'_{-1}\chi(x_3/\varepsilon)A\, u_T^k\|_{\boldsymbol{L}^2}^2 \underset{k\to+\infty}{\longrightarrow} \left\langle \mu_T, \frac{(\chi(x_3/\varepsilon))^2\, |a_2|^2}{\tau^2\|\xi'\|^2}\right\rangle$$
$$\limsup_{k\to+\infty} \left|\left(Au_T^k, u_L^k\right)\right| \leq \left\langle \mu_T, \frac{(\chi(x_3/\varepsilon))^2\, |a_2|^2}{\tau^2\|\xi'\|^2}\right\rangle.$$

When ε goes to 0, the right side of this inequality converges (by the dominated convergence theorem) to

$$\left\langle \mu_T, \mathbb{1}_{\{x_3=0\}}|a_2|^2\tau^{-2}\|\xi'\|^{-2}\right\rangle,$$

which is null, because $\mathcal{G}_T$ and the support of a_2 are disjoint.

Third case: $A = A_1 D_{x_3}$, A_1 tangential.

As in the preceding case, we may assume that $\sigma_1(A_1)$ is disjoint from one of the two glancing sets, say $\mathcal{G}_T$. Then

$$\left(Au_T^k, u_L^k\right) = \left(\chi(x_3/\varepsilon)Au_T^k, u_L^k\right) + \left((1-\chi(x_3/\varepsilon))Au_T^k, u_L^k\right)$$

The second term converges to zero when $k\to\infty$ for the same reasons as in the case $A \in \mathcal{A}_i^2$. The first term may be written

$$\left(\chi(x_3/\varepsilon)A_1 D_{x_3}u_T^k, u_L^k\right) = \left(D_{x_3}(\chi(x_3/\varepsilon)A_1 u_T^k), u_L^k\right) + \left(R_\varepsilon u_T^k, u_L^k\right), \ R_\varepsilon \in \boldsymbol{\mathcal{A}}^1$$
$$= \left((\chi(x_3/\varepsilon)A_1 u_T^k), D_{x_3}u_L^k\right) + \left(R_\varepsilon u_T^k, u_L^k\right) + \{boundary\ terms\}$$

The boundary terms tend to zero when k tends to infinity because (u_T^k) and (u_L^k) are regular on the boundary. The proof may be completed as in the preceding case, letting k go to infinity then ε go to zero. $\qquad\square$

4. Sufficient condition

If γ is a $\boldsymbol{B}$-resistant ray defined on a real interval $]a, b[$, we shall call life-length the positive quantity $|t(b) - t(a)|$. In this section we use the tools introduced in the preceding section to prove the following:

Proposition 4.1. *Let $T > 0$. Assume that every $\boldsymbol{B}$-resistant ray in Ω is of life-length strictly less than T. Then there exists $C > 0$ such that for every solution of the Lamé system (37)*

$$(42) \qquad \|u_0\|_{\boldsymbol{H}_0^1}^2 + \|u_1\|_{\boldsymbol{L}^2}^2 \leq C\left(\|\mathrm{curl}\,(\partial_t u \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 + \|u_0\|_{\boldsymbol{L}^2}^2 + \|u_1\|_{\boldsymbol{H}^{-1}}^2\right).$$

Inequality (42) is the sufficient condition (11) for uniform decay stated in point a) of Proposition 2.5. Proposition 4.1 thus completes the proof of the sufficient condition of Theorem 1, namely that the non-existence of arbitrarily large $\boldsymbol{B}$-resistant rays on Ω implies the uniform decay of the energy for solutions of the system of magnetoelasticity.

To show (42), we shall argue by contradiction, considering the defect measures $\mu_{T,L}$ of Subsection 3.5.2 associated to a sequence (u^k) which contradicts (42) (cf Subsection 4.1). The bound on $\mathrm{curl}\,(\partial_t u^k \wedge \boldsymbol{B})$ given by the negation of (42) implies a strong condition on the supports of these measures (see Subsection 4.2). In Subsection 4.3, we make use of this condition, together with propagation arguments near the

boundary of Ω. Subsection 4.4 completes the proof, using the assumption of non-existence of $\boldsymbol{B}$-resistant rays of life-length larger than T.

4.1. **Introduction of measures.** Assume that (42) does not hold. Then there exists a sequence (u^k) of solutions of the Lamé system such that

$$(43) \qquad 1 = \|u_0^k\|_{\boldsymbol{H}_0^1}^2 + \|u_1^k\|_{\boldsymbol{L}^2}^2 > k \left(\|\mathrm{curl}\,(\partial_t u^k \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 + \|u_0^k\|_{\boldsymbol{L}^2}^2 + \|u_1^k\|_{\boldsymbol{H}^{-1}}^2 \right).$$

Up to the extraction of a subsequence, one may assume that (u_0^k, u_1^k) weakly converges in $\boldsymbol{H}_0^1 \times \boldsymbol{L}^2$. Inequality (43) implies that its weak limit is 0. We may thus introduce the defect measures μ_T, μ_L and μ of §3.5.2, associated to the sequences (u_T^k), (u_L^k) and $(\partial_t \varphi^k)$. To contradict (43), we need to show that these measures are null. Note that (43) implies

$$(44) \qquad \mathrm{curl}\,(\partial_t u^k \wedge \boldsymbol{B}) \xrightarrow[k\to+\infty]{} 0 \text{ in } H^{-1}((0,T)\times\Omega).$$

Remark 4.2. By a density argument, it suffices to show (42) with (u_0, u_1) generated by a finite number of eigenfunctions of $\mathcal{L}$. We may thus assume, that u_L^k and u_T^k are C^∞.

Remark 4.3. We will indeed show a more precise statement than Proposition 4.1 namely that if (u^k) is a sequence of solutions of the Lamé system converging weakly to 0 in the energy space and satisfying (44) then the set $(\mathrm{supp}\,\mu_T \cup \mathrm{supp}\,\mu_L) \cap \{t \in (0,T)\}$ a a union of $\boldsymbol{B}$-resistant rays of length T.

4.2. **Condition on the supports.** We may see $\boldsymbol{B}$ as a vector field on Ω, i.e. a section of $T\Omega$. To avoid confusions, the magnetic field considered as a vector field shall be referred as $\overrightarrow{B}$. In a local coordinate system, if

$$\chi : U \subset \overline{\Omega} \longrightarrow \overline{\mathbb{R}}_+^3$$

is the change of coordinates, and χ' its differential, $\overrightarrow{B}$ is equal to $\chi'\boldsymbol{B}$. Notation $\boldsymbol{B}$ shall always refer to the vector of $\mathbb{R}^3$ of coordinates $(B,0,0)$. As before, (z_1, z_2, z_3) refers to the global spatial coordinates on Ω, before the change of variable.

Lemma 4.4. *Assume (44). Then, on the interval* $(0,T)$,

$$(45) \qquad \mu_T \mathbb{1}_{(0,T)} = \mu_T \mathbb{1}_{(0,T)} \mathbb{1}_{\overrightarrow{B}^\perp}, \quad \overrightarrow{B}^\perp := \{(t,y,\tau,\eta);\ {}^t\overrightarrow{B}\eta = 0\}$$

$$(46) \qquad \mu_L \mathbb{1}_{(0,T)} = \mu_L \mathbb{1}_{(0,T)} \mathbb{1}_{\overrightarrow{B}^{//}}, \quad \overrightarrow{B}^{//} := \{(t,y,\tau,\eta);\ \eta \in vect\,(g\overrightarrow{B})\}.$$

Proof. Set

$$(47) \qquad Ru := \mathrm{curl}\,(\partial_t u \wedge \overrightarrow{B}) = B\partial_t \begin{pmatrix} -\partial_{z_2} u_2 - \partial_{z_3} u_3 \\ \partial_{z_1} u_2 \\ \partial_{z_1} u_3 \end{pmatrix}.$$

<u>Transversal measure</u>. The measure μ_T does not charge neither $\mathcal{H}_T$ nor $\mathcal{E}_T$. Thus, it suffices to check

$$\mathrm{supp}\,\mu_T \mathbb{1}_{(0,T)} \mathbb{1}_{\{x_3>0\}\cup\mathcal{G}_T} \subset \overrightarrow{B}^\perp.$$

Near the boundary, by Proposition A.1 of the appendix,

$$Ru^k \xrightarrow[k\to+\infty]{} 0 \text{ in } L^2([0,l[, \boldsymbol{H}_{\mathrm{loc}}^{-1}(X')).$$

Thus, according to the decoupling lemma,

$$(48) \qquad \forall A_0 \in \mathcal{A}^0,\ \mathrm{supp}\,A_0 \subset \{t \in (0,T)\} \implies (A_0 Ru_T^k, u_T^k) \xrightarrow[k\to+\infty]{} 0.$$

Because $\mathrm{div}\,u_T^k = 0$, the definition (47) of Ru_T implies $Ru_T^k = \partial_t \partial_{z_1} u_T^k$. By §3.2.2,

$$\mu_T \Big(\{t \in (0,T)\} \cap \{\sigma_2(\partial_t \partial_{z_1}) \neq 0\} \Big) = 0.$$

This clearly shows the announced result near an interior point. When $\tilde{\rho} \in \mathcal{G}_T$ one may write $\partial_{z_1} = f_0 \partial_{x_3} + F_1$, where f_0 is a function and F_1 a first order tangential differential operator, and the following basic fact completes the proof:

$$\sigma(F_1)(\tilde{\rho}) = 0 \iff \tilde{\eta}' \perp \overrightarrow{B}.$$

<u>Longitudinal measure.</u> The first coordinate of Ru_L^k is $-\partial_t(\partial_{z_2}^2 + \partial_{z_3}^2)\varphi^k$. Its scalar product with $\partial_t \varphi^k$ gives

$$\forall A_0 \in \mathcal{A}^0,\ \text{supp}\,(A_0) \subset \{t \in (0,T)\} \implies \lim_{k \to +\infty} \left(A_0(\partial_{z_2}^2 + \partial_{z_3}^2)\partial_t \varphi^k, \partial_t \varphi^k\right) = 0.$$

This implies (again by §3.2.2), that μ (thus μ_L) vanishes, in $(0,T)$, on the set of all ρ such that

$$\rho \in \{x_3 > 0\} \cup \mathcal{G}_T, \quad \sigma_2(\partial_{z_2}^2 + \partial_{z_3}^2)(\rho) \neq 0.$$

Hence (46). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.3. **Support of the measure near the boundary.** For any symbol q_0 with support in $\{x_3 > 0\}$, we have

$$\langle \mu_T, H_{p_T/\tau} q_0 \rangle = 0 \qquad \langle \mu_L, H_{p_L/\tau} q_0 \rangle = 0,$$

which shows, in the interior of Ω, the invariance of each measure by the associated Hamiltonian flow. Unfortunately, the condition

$$(49) \qquad\qquad\qquad\qquad u_{T\restriction\partial\Omega}^k + u_{L\restriction\partial\Omega}^k = 0$$

is not always sufficient to extend such a property in the neighbourhood of $\partial\Omega$. Indeed, μ_L and μ_T are not deterministic for general sequences of solutions of the Lamé system, in the sense that the value of the two measures for time $t > t_0$ is not uniquely determined by their value for time $t \leq t_0$. In our case, this convenient property holds thanks to the strong conditions on the support of μ_T and μ_L. As announced before, we shall only describe the propagation of the supports of the measure.

Lemma 4.5. *Let μ_T and μ_L be the defect measures associated to a sequence of solutions of the Lamé system satisfying (44). Let*

$$\tilde{\rho} = (\tilde{x}', \tilde{\xi}') = (\tilde{t}, \tilde{y}', \tilde{\tau}, \tilde{\eta}') \in S^*\partial X$$

and $\mathbf{n}$ the unitary exterior normal vector to $\partial\Omega$ at $\tilde{y}'$. Them μ_T and μ_L both vanish near $\tilde{\rho}$ except possibly in the following cases (cf figure 2):

(1) *μ_L is null. The support of μ_T propagates along the transversal flow and*
- *$\underline{(\mathcal{H}_{T[1]})\ case}$: $\tilde{\rho} \in \mathcal{H}_T$, $\tilde{\eta}' = 0$ and $\overrightarrow{B}$ is orthogonal to $\mathbf{n}$;*
- *$\underline{(\mathcal{H}_{T[2]})\ case}$: $\tilde{\rho} \in \mathcal{H}_T$, $\tilde{\eta}' \neq 0$ and $\overrightarrow{B}$ is normal to the reflection plane;*
- *$\underline{(\mathcal{G}_{T[1]})\ case}$: $\tilde{\rho}$ is diffractive for the transversal wave (i.e. $\tilde{\rho} \in \mathcal{G}_T$ and the bicharacteristic ray passing through $\tilde{\rho}$ intersects the boundary only at $\tilde{\rho}$), and $\overrightarrow{B}$ is orthogonal to $\tilde{\eta}'$;*
- *$(\mathcal{G}_{T[2]})\ case$: $\tilde{\rho} \in \mathcal{G}_T$ is not diffractive for the transversal wave, and $\overrightarrow{B}$ is normal to the reflection plane.*

(2) *μ_T is null, the support of μ_L propagates along the longitudinal flow and*
- *$\underline{(\mathcal{H}_L)\ case}$: $\tilde{\rho} \in \mathcal{H}_L$, $\tilde{\eta}' = 0$ and $\overrightarrow{B}$ is parallel to $\mathbf{n}$;*
- *$\underline{(\mathcal{G}_L)\ case}$: $\tilde{\rho}$ is a diffractive point for the longitudinal wave and $\overrightarrow{B}$ is parallel to $\tilde{\eta}'$.*

(3) *Both measures μ_T and μ_L are non null, $\tilde{\rho} \in \mathcal{H}_T \cap \mathcal{H}_L$ and*
- *$\underline{(T \to L)\ case}$: $\overrightarrow{B}$ is orthogonal to the transversal ray coming in, and parallel to the longitudinal ray going out of $\tilde{\rho}$. The support of μ_T is a union of incoming transversal rays. The support of μ_L is the union of all outgoing longitudinal rays going out of points of $\mathcal{H}_T \cap \mathcal{H}_L$ where the transversal rays of the support of μ_T come in;*
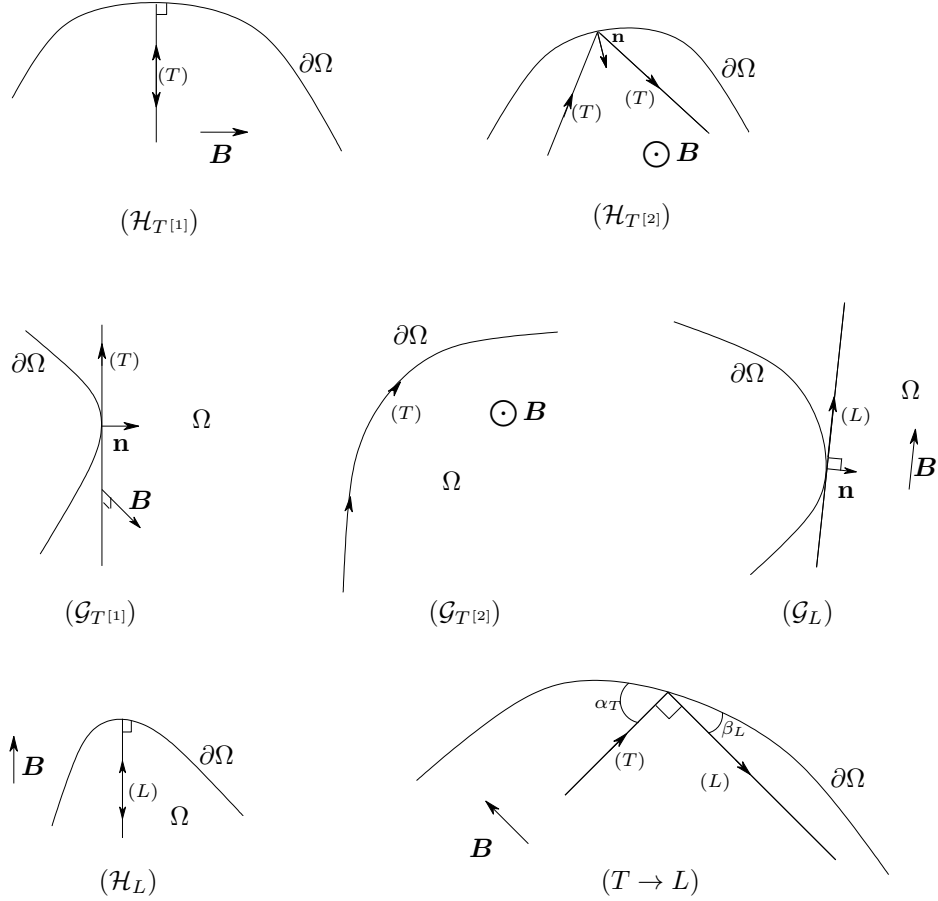
FIGURE 2. Cases arising in Lemma 4.5

- $(L \to T)$ *case*: $\overrightarrow{B}$ *is orthogonal to the transversal ray going out of and parallel to the longitudinal ray coming in* $\tilde{\rho}$. *The support of* μ_L *is a union of incoming longitudinal rays. The support of* μ_T *is the union of all outgoing transversal rays going out of points of* $\mathcal{H}_T \cap \mathcal{H}_L$ *where the longitudinal rays of the support of* μ_L *come in.*

All the assertions of Lemma 4.5 should be understood in a neighbourhood of $\tilde{\rho}$. The reflection plane at a boundary point $\rho = (t, y', \tau, \eta')$, defined as long as $\eta' \neq 0$, is the plane passing through y' and generated by $\mathbf{n}$ and η', thus containing the bicharacteristic ray passing through ρ. The statement "$\overrightarrow{B}$ is parallel to η'" must be understood as "the vector $g\overrightarrow{B}$ of the cotangent bundle of Ω is parallel to η'".

Notation. Let ρ be an hyperbolic point for the transversal (respectively longitudinal) wave. We shall denote by ξ_T^-, ξ_T^+ (respectively ξ_L^-, ξ_L^+) the incoming and outgoing vectors through ρ:

$$\xi_T^+ := \left(\begin{array}{c} \xi' \\ \xi_{T3} = \sqrt{\nu_T^2 \tau^2 - \|\eta'\|^2} \end{array} \right) \quad \xi_T^- = \left(\begin{array}{c} \xi' \\ -\xi_{T3} \end{array} \right)$$

(respectively with "L" instead of "T").

We shall write $\eta_T^{\pm}$, $\eta_L^{\pm}$ the spatial components of this vectors. For example,

$$\eta_T^+ = \left(\begin{array}{c} \eta' \\ \sqrt{\nu_T^2 \tau^2 - \|\eta'\|^2} \end{array} \right).$$

Let us postpone the proof of Lemma 4.3 to show, as stated in the introduction of this article, that in the $(T \to L)$ and $(L \to T)$ cases, the angles of refraction and incidence have a fixed value, determined by the quotient c_T/c_L. Consider for example the $(T \to L)$ case. Let α_T be the angle of incidence of the transversal wave, β_L the angle of refraction of the longitudinal wave, a_T and a_L the following numbers:

$$a_T := \tan \alpha_T = \frac{\tilde{\xi}_{T3}}{\|\tilde{\eta}'\|}, \quad a_L := \tan \beta_L = \frac{\tilde{\xi}_{L3}}{\|\tilde{\eta}'\|}.$$

The incident and refracted waves are orthogonal, so that

(50) $$\|\tilde{\eta}'\|^2 - \tilde{\xi}_{T3}\tilde{\xi}_{L3} = 0, \text{ i.e. } a_T a_L = 1.$$

Furthermore, the definitions of $\tilde{\eta}_{L3}^-$ and $\tilde{\eta}_{T3}^+$ yield

$$c_T^2 \|\tilde{\eta}_T^+\|^2 - \tau^2 = 0, \quad c_L^2 \|\tilde{\eta}_L^-\|^2 - \tau^2 = 0,$$

which gives the equation

(51) $$c_T^2(1 + a_T^2) = c_L^2(1 + a_L^2).$$

Equations (50) and (51) imply the formula announced in the introduction:

$$\alpha_T = \arctan \frac{c_L}{c_T}, \ \beta_L = \arctan \frac{c_T}{c_L}.$$

By a similar calculation, one gets, in the $(T \to L)$ case, $\alpha_L = \arctan \frac{c_T}{c_L}$, $\beta_T = \arctan \frac{c_L}{c_T}$.

Proof of Lemma 4.5. We start with case (3): $\mu_T \neq 0$, $\mu_L \neq 0$ near $\tilde{\rho}$. In this case, $\tilde{\rho} \notin \mathcal{E}_T \cup \mathcal{E}_L$. It is also easy to show that $\tilde{\rho} \notin \mathcal{G}_T \cup \mathcal{G}_L$. Indeed, if $\tilde{\rho} \in \mathcal{G}_T$ then it also belongs to $\mathcal{H}_L$ (it cannot be a point of $\mathcal{E}_L$, and $\mathcal{G}_L$ and $\mathcal{G}_T$ are disjoint). But μ_T being non-null near $\tilde{\rho}$, $\tilde{\eta}$ is orthogonal to $\overrightarrow{B}$ (by Lemma 4.4) so neither $\tilde{\eta}_L^+$ nor $\tilde{\eta}_L^-$ are parallel to $\overrightarrow{B}$, which implies (again by Lemma 4.4) that $\mu_L = 0$ near $\tilde{\rho}$, contradicting our assumptions. Likewise, if $\tilde{\rho} \in \mathcal{G}_L$, $\tilde{\eta}'$ must be parallel to $\overrightarrow{B}$ and μ_T null near $\tilde{\rho}$. Thus $\tilde{\rho} \in \mathcal{H}_T \cap \mathcal{H}_L$. The support of measures μ_T and μ_L is, near $\tilde{\rho}$, a union of incoming and outgoing maximal rays.

Let us first assume that the support of μ_L contains the ray going out of $\tilde{\rho}$. Then

$$\tilde{\eta}_L^+ /\!/ \overrightarrow{B}$$

so that $\tilde{\eta}_T^+$ is not orthogonal to $\overrightarrow{B}$. As a consequence, the support of μ_T is only made of incoming rays, and the fact that $\mu_T \neq 0$ implies

$$\tilde{\eta}_T^- \perp \overrightarrow{B}.$$

Thus $\tilde{\eta}_L^-$ is not parallel to $\overrightarrow{B}$. This is the $(T \to L)$ case, and it remains to show the statement of the lemma about the transfer from transversal incoming waves to longitudinal outgoing waves, which may be formulated as follow: for any $\check{\rho} \in \mathcal{H}_T \cap \mathcal{H}_L$ near $\tilde{\rho}$ the following equivalence holds:

(52) $$\check{\rho} \in \operatorname{supp} \mu_T \iff \check{\rho} \in \operatorname{supp} \mu_L.$$

Let us assume for example $\check{\rho} \notin \operatorname{supp} \mu_T$. Then μ_T is null near rays coming in and going out of $\check{\rho}$ and by the hyperbolic theory (see Proposition 3.15),

$$u_{L \restriction x_3=0}^k = -u_{T \restriction x_3=0}^k \xrightarrow[k \to +\infty]{} 0 \text{ in } \boldsymbol{L}_{\tilde{\rho},\partial}^2.$$

By the propagation theorem, the support of μ_L propagates near $\check{\rho}$. But this support is a union of outgoing rays. Consequently, it is empty near $\check{\rho}$ and μ_L is null near $\check{\rho}$. The implication $\check{\rho} \notin \operatorname{supp} \mu_L \implies \check{\rho} \notin \operatorname{supp} \mu_T$ may be shown in the same manner.

If the support of μ_L contains no outgoing ray near $\tilde{\rho}$, it must contain incoming rays. This corresponds to the $(L \to T)$ case, which may be treated as the $(T \to L)$ case.

The study of the other cases is based on the transfer of the boundary conditions stated in Lemma 4.6 below: roughly, a boundary condition on the longitudinal wave implies one on the transversal wave and vice versa.

Notation. Let $v_{T,L}^k$ be the functions $u_{T,L}^k$ considered as vector fields on Ω. In local coordinates, if χ denotes the change of coordinates, we have

$$v_T^k = \chi'(y)u_T^k = \begin{pmatrix} v_{T1}^k \\ v_{T2}^k \\ v_{T3}^k \end{pmatrix} \quad v_L^k = \chi'(y)u_L^k = \begin{pmatrix} v_{L1}^k \\ v_{L2}^k \\ v_{L3}^k \end{pmatrix}.$$

Lemma 4.6. *Let $\tilde{\rho} \in S^*\partial X$, and (u^k) be any sequence of solutions of the Lamé system weakly converging to 0 in $\boldsymbol{H}_{\mathrm{loc}}^1(\mathbb{R} \times \overline{\Omega})$.*

- *Assume the following approximate equation for some $A_1 \in \mathcal{A}^1$:*

(53) $$\partial_{x_3} v_{T3\restriction x_3=0}^k = A_1 v_{T3\restriction x_3=0}^k + o(1) \text{ in } L_{\tilde{\rho},\partial}^2.$$

  Then

(53') $$\Delta_{y'} \varphi_{\restriction x_3=0}^k = -A_1 \partial_{x_3} \varphi_{\restriction x_3=0}^k + o(1) \text{ in } L_{\tilde{\rho},\partial}^2.$$

- *Conversely, if, for some $A_{-1} \in \mathcal{A}^{-1}$ the following equation holds:*

(54) $$\varphi_{\restriction x_3=0}^k = A_{-1} \partial_{x_3} \varphi_{\restriction x_3=0}^k + o(1) \text{ in } H_{\tilde{\rho},\partial}^2.$$

  Then

(54') $$\partial_{x_3} v_{T3\restriction x_3=0}^k = -\Delta_{y'} A_{-1} v_{T3\restriction x_3=0}^k + o(1) \text{ in } L_{\tilde{\rho},\partial}^2.$$

- *Moreover if, in addition to (54), $\tilde{\eta}' \neq 0$ and $\sigma(A_{-1})(\tilde{\rho}) \neq 0$, then*

(54") $$v_{T\restriction x_3=0}^k = \mathbf{Z}_{-1} \partial_{x_3} v_{T\restriction x_3=0}^k + o(1) \text{ in } \boldsymbol{H}_{\tilde{\rho},\partial}^1$$

$$\mathbf{Z}_{-1} \in \boldsymbol{\mathcal{A}}^{-1}, \ \sigma_{-1}(\mathbf{Z}_{-1}) = \|\eta'\|^{-2} \begin{pmatrix} 0 & 0 & ig'^{-1} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \\ 0 & 0 & \sigma(A_{-1})^{-1} \end{pmatrix}.$$

Proof. First note that the Dirichlet condition on u^k implies

(55) $$\partial_{x_3} v_{T3\restriction x_3=0}^k = \Delta_{y'} \varphi_{\restriction x_3=0}^k + O(1) \text{ in } H_{\mathrm{loc}}^{1/2}(\partial X),$$

Indeed, the equation $\operatorname{div} u_T^k = 0$ implies, recalling that $\operatorname{div} v_T^k = \frac{1}{\sqrt{g}} \sum_{j=1}^3 \partial_{y_j}(\sqrt{g} v_{Tj}^k)$.

$$\partial_{x_3} v_{T3}^k + \partial_{y_1} v_{T1}^k + \partial_{y_2} v_{T2}^k = O(1) \text{ in } H_{\mathrm{loc}}^1(X).$$

Using $v_{3\restriction x_3=0}^k = 0$, we obtain

$$\partial_{x_3} v_{T3\restriction x_3=0}^k = \partial_{y_1} v_{L1\restriction x_3=0}^k + \partial_{y_2} v_{L2\restriction x_3=0}^k + O(1) \text{ in } H_{\mathrm{loc}}^{1/2}(\partial X),$$

yielding (55) by the definition of φ.

Assume (53). By (55) and the nullity of $v_{3\restriction x_3=0}^k$,

$$\Delta_{y'} \varphi_{\restriction x_3=0}^k = -A_1 v_{L3\restriction x_3=0}^k + o(1) = -A_1 \partial_{x_3} \varphi_{\restriction x_3=0}^k + o(1) \text{ in } \boldsymbol{L}_{\tilde{\rho},\partial}^2.$$

Now assume (54). Then $\Delta_{y'} \varphi_{\restriction x_3=0}^k = \Delta_{y'} A_{-1} \partial_{x_3} \varphi_{\restriction x_3=0}^k + o(1)$ in $L_{\tilde{\rho},\partial}^2$, which implies (54') using (55) on the left-hand side of the equation, and the Dirichlet condition on v_3^k on its right-hand side.

If, in addition to the assumption (54), $\tilde{\eta}'$ and $\sigma(A_{-1})(\tilde{\rho}')$ are nonzero, both operators A_{-1} and $\Delta_{y'}$ are elliptic at $\tilde{\rho}$, and equations (54') and (55) may be rewritten

$$(56) \qquad v_{T3|x_3=0}^k = Y_{-1}\partial_{x_3}v_{T3|x_3=0}^k + o(1) \text{ in } H_{\tilde{\rho},\partial}^1 \quad \sigma(Y_{-1}) = \|\eta'\|^{-2}\sigma(A_{-1})^{-1},$$

$$\varphi_{|x_3=0}^k = E_{-2}\partial_{x_3}v_{T3|x_3=0}^k + O(1) \text{ in } H_{\tilde{\rho},\partial}^{5/2}, \quad \sigma(E_{-2}) = -\|\eta'\|^{-2} \text{ near } \tilde{\rho}$$

Hence

$$(57) \qquad \begin{pmatrix} v_{T1|x_3=0}^k \\ v_{T2|x_3=0}^k \end{pmatrix} = g'^{-1}\begin{pmatrix} \partial_{y_1}\varphi_{|x_3=0}^k \\ \partial_{y_2}\varphi_{|x_3=0}^k \end{pmatrix} = Z_{-1}\partial_{x_3}v_{T3|x_3=0}^k + O(1) \text{ in } H_{\tilde{\rho},x_3=0}^{1/2}$$

$$\sigma(Z_{-1}) = i\|\eta'\|^{-2}g'^{-1}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \text{ near } \tilde{\rho}.$$

Assertion (54") is an easy consequence of (56) and (57). $\qquad\square$

We may now study cases (1) and (2) of Lemma 4.5.

Case (1): assume $\mu_L = 0$, $\mu_T \neq 0$ near $\tilde{\rho}$. There are three possibilities:

- If $\tilde{\rho} \in \mathcal{H}_L$, the nullity of μ_L implies, by standard hyperbolic theory (Proposition 3.15),

$$u_{T|x_3=0}^k = -u_{L|x_3=0}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\tilde{\rho},\partial}^1,$$

so that near $\tilde{\rho}$, the support of μ_T propagates. It is easy to see that condition (45) on the support of μ_T implies, if μ_T does not vanish, that this is one of the four cases described in Lemma 4.5, (1) .

- Assume $\tilde{\rho} \in \mathcal{E}_L$. The standard elliptic theory (Proposition 3.16) implies

$$\partial_{x_3}\varphi_{|x_3=0}^k + i\Xi\varphi_{|x_3=0}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\tilde{\rho},|x_3=0}^1,$$

where $\sigma_1(\Xi) = i\sqrt{\|\eta'\|^2 - \nu_L^2\tau^2}$. With Lemma 4.6, this yields the following equation on the traces of u_T^k:

$$(58) \qquad u_{T|x_3=0}^k = i\widetilde{\mathbf{Z}}_{-1}D_{x_3}u_{T|x_3=0}^k + o(1) \text{ in } \boldsymbol{H}_{\tilde{\rho},\partial}^1, \quad \widetilde{\mathbf{Z}}_{-1} = \chi'^{-1}\mathbf{Z}_{-1}\chi'$$

where the principal symbol of the operator $\mathbf{Z}_{-1} \in \boldsymbol{\mathcal{A}}^{-1}$ is given by (54"), with $\sigma_{-1}(A_{-1})^{-1} = -i\sigma_1(\Xi)$. Notice that the eigenvalues of $\sigma_{-1}(i\mathbf{Z}_{-1})$, thus those of $\sigma_{-1}(i\widetilde{\mathbf{Z}}_{-1})$ are pure imaginary numbers. As a consequence, the boundary condition (58) is a uniform Lopatinsky boundary condition near $\tilde{\rho}$ (see the example following Definition 3.10), which shows again the propagation of μ_T. As in the case where $\tilde{\rho} \in \mathcal{H}_L$, it is easy to see that this is one of the four cases of Lemma 4.5.

- The case $\tilde{\rho} \in \mathcal{G}_L$ is the most difficult. If μ_T is non-null near $\tilde{\rho}$, then $\tilde{\rho}$ must be in $\mathcal{H}_T$. We use a contradiction argument to prove the propagation of the support of μ_T. Let $\breve{\rho} \in \mathcal{H}_T$ such that the ray coming in $\breve{\rho}$ is in the support of μ_T, but not the ray going out of $\breve{\rho}$. According to the standard hyperbolic theory (Proposition 3.15),

$$D_{x_3}u_{T|x_3=0}^k - \Lambda_T u_{T|x_3=0}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}_{\tilde{\rho},\partial}^2, \quad \sigma_1(\Lambda_T) = \sqrt{\nu_T\tau^2 - \|\eta'\|^2}.$$

This implies, by Lemma 4.6, a boundary equation on $\partial_t\varphi^k$, of the following form:

$$\partial_t\varphi_{|x_3=0}^k = Y_{-1}D_{x_3}\partial_t\varphi_{|x_3=0}^k + o(1) \text{ in } H_{\tilde{\rho},\partial}^1,$$

which is a uniform Lopatinsky condition near $\breve{\rho}$ because $\tilde{\rho} \in \mathcal{G}_L$. In view of Proposition 3.18 on traces in the glancing region, such an equation implies, with the nullity of μ near $\breve{\rho}$ the following conditions:

$$\partial_t\varphi_{|x_3=0}^k \to 0 \text{ in } H_{\breve{\rho},\partial}^1, \quad \partial_n\partial_t\varphi_{|x_3=0}^k \to 0 \text{ in } L_{\breve{\rho},\partial}^2.$$

The operator ∂_t being elliptic at $\breve{\rho}$, this shows that u_L^k tends to 0 in $\boldsymbol{H}^1_{\breve{\rho},\partial}$, and thus

$$u^k_{T\restriction x_3=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\breve{\rho},\partial}.$$

Hence the propagation of the support of μ_T near $\breve{\rho}$, which contradicts the assumption on rays coming in and going out of $\breve{\rho}$.

Similar arguments show that if the ray going out of $\breve{\rho}$ is in the support of μ_T, so is the ray coming in $\breve{\rho}$. This proves that the support of μ_T propagates near $\tilde{\rho}$. Notice that this is necessarily the $(\mathcal{H}_{T^{[2]}})$ case.

Case 2: we assume now that $\mu_T = 0$ and $\mu_L \neq 0$. We argue in a similar way, considering three possibilities:

- If $\tilde{\rho} \in \mathcal{H}_T$, the standard hyperbolic theory gives an approximate Dirichlet boundary equation on u_T^k, which implies

$$u^k_{L\restriction x_3=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}.$$

  As a consequence, the support of μ_L propagates. On this support, η is parallel to $\overrightarrow{B}$, which shows, as stated in Lemma 4.5, that $\tilde{\eta}' = 0$ and $\tilde{\rho} \in \mathcal{H}_L$, or $\tilde{\eta}'/\!/\overrightarrow{B}$ and $\tilde{\rho} \in \mathcal{G}_L$.

- If $\tilde{\rho} \in \mathcal{E}_T$, we write (as in the similar situation when $\tilde{\rho} \in \mathcal{E}_L$), the boundary equation of the elliptic region,

$$D_{x_3} u^k_{T\restriction x_3=0} + \Xi_T u^k_{T\restriction x_3=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

  This implies in view of Lemma 4.6 a uniform Lopatinsky boundary equation on $\partial_t \varphi^k$, thus the propagation of the support of μ, which is the same as that of μ_L. The fact that $\eta' \neq 0$ shows that $\tilde{\rho}$ cannot be hyperbolic for the longitudinal wave (in this case outgoing and incoming directions are not parallel, thus at least one is not parallel to $\overrightarrow{B}$). Consequently, $\tilde{\rho} \in \mathcal{G}_L$. More precisely, it is a diffractive point: the bicharacteristic passing through $\tilde{\rho}$ must stay parallel to $\overrightarrow{B}$, thus its direction is constant which is not possible for gliding rays because Ω has no contact of infinite order with its tangents. We are in the $(\mathcal{G}_L)$ case of Lemma 4.5.

- If $\tilde{\rho} \in \mathcal{G}_T$, then $\tilde{\rho} \in \mathcal{H}_L$. The fact that $\tilde{\rho} \in \mathcal{G}_T$ implies that $\tilde{\eta}' \neq 0$, so directions $\tilde{\eta}_L^+$ and $\tilde{\eta}_L^-$ cannot be both parallel to $\overrightarrow{B}$. Consequently, the support of μ_L is a union of only incoming rays (or only outgoing rays). This gives a boundary equation of the following form:

$$\partial_{x_3} \partial_t \varphi^k_{\restriction x_3=0} + \iota \Lambda_L \partial_t \varphi^k_{\restriction x_3=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial},$$

  where $\iota \in \{+1, -1\}$. Notice that ∂_t is elliptic at $\tilde{\rho}$, so that we may rewrite this last property taking out all the ∂_t and with $\boldsymbol{H}^1$ instead of $\boldsymbol{L}^2$. This yields, in view of Lemma 4.6, a uniform Lopatinsky boundary condition on u_T^k. The nullity of μ_T gives as before (by Proposition 3.18)

$$u_L^k = -u_T^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde{\rho}},$$

  so that μ_L propagates, and in view of the particular form of its support, vanishes near $\tilde{\rho}$. This shows that this particular situation ($\tilde{\rho} \in \mathcal{G}_T$ and $\mu_L \neq 0$) is impossible, and completes the proof of Lemma 4.5.

$$\square$$

Definition 4.7. We shall call $\boldsymbol{B}$-admissible points the points of the boundary of $S_b^* M$ which are of one of the eight types described in Lemma 4.5.

4.4. **Conclusion of the proof.** Let $S := \operatorname{supp} \mu_L \cup \operatorname{supp} \mu_T$, and BR the subset of $S_b^* M$, of all points ρ satisfying one of the following properties:

- $x_3 > 0$, $\rho \in S\widehat{Z}_L$ and $\eta /\!/ \overrightarrow{B}$;
- $x_3 > 0$, $\rho \in S\widehat{Z}_T$ and $\eta \perp \overrightarrow{B}$;
- $x_3 = 0$ and ρ is $\boldsymbol{B}$-admissible.

Let $\Phi_T(\rho, s)$ and $\Phi_L(\rho, s)$ be the bicharacteristic flows for the transversal and longitudinal waves. We shall define a local continuous flow on BR, denoted by

$$\Phi(\rho, s) = \left(\Phi_{x'}, \Phi_{x_3}, \Phi_{\xi'}, \Phi_{\xi_3}\right),$$

in the following way:

- if $\Phi_{x_3}(\rho, s) > 0$ and $\Phi(\rho, s) \in S\widehat{Z}_T$, or if $\Phi(\rho, s)$ is a $\boldsymbol{B}$-admissible boundary point of the form (1) of Lemma 4.5, Φ is near (ρ, s) the restriction to BR of the transversal bicharacteristic flow;
- if $\Phi_{x_3}(\rho, s) > 0$ and $\Phi(\rho, s) \in S\widehat{Z}_L$, or if $\Phi(\rho, s)$ is a $\boldsymbol{B}$-admissible boundary point of the form (2) of Lemma 4.5, Φ is near (ρ, s) the restriction to BR of the longitudinal bicharacteristic flow;
- if $\Phi(\rho, s)$ is $\boldsymbol{B}$-admissible of type $(T \to L)$, then

$$\Phi(\rho, r) = \Phi_T(\rho, r), \text{ if } r < s$$
$$\Phi(\rho, r) = \Phi_L(\rho, r), \text{ if } r > s;$$

- if $\Phi(\rho, s)$ is $\boldsymbol{B}$-admissible of type $(L \to T)$, then

$$\Phi(\rho, r) = \Phi_L(\rho, r), \text{ if } r < s$$
$$\Phi(\rho, r) = \Phi_T(\rho, r), \text{ if } r > s.$$

In view of Lemma 4.3, S (which a subset of BR) is stable under the flow Φ on $(0, T)$. Furthermore, if for some $\rho \in \mathsf{BR}$,

$$\Phi(\rho, s) \xrightarrow[s \to \tilde{s}]{} \tilde{\rho} \notin \mathsf{BR}, \ \tilde{s} \in (0, T),$$

(thus $\tilde{\rho}$ is a boundary point which is not $\boldsymbol{B}$-admissible), then ρ is not in S. Consequently, S is a union of $\boldsymbol{B}$-resistant rays of life-length T. The assumption of non-existence of such rays made in Proposition 4.1 shows that S is empty, which completes the proof.

5. NECESSARY CONDITION

Proposition 5.1. *Assume that for all $T > 0$, there exists a $\boldsymbol{B}$-resistant ray of life-length T. Then for all $T > 0$, there exists a sequence (u^k) of solutions of the Lamé system such that*

$$\text{(59)} \qquad \qquad \|\partial_t u^k_{\restriction t=0}\|^2_{\boldsymbol{L}^2} + \|u^k_{\restriction t=0}\|^2_{\boldsymbol{H}^1_0} \xrightarrow[k \to +\infty]{} 1$$

$$\text{(60)} \qquad \qquad \|u^k \wedge \boldsymbol{B}\|_{\boldsymbol{H}^1((0,T) \times \Omega)} \xrightarrow[k \to +\infty]{} 0$$

Corollary 5.2. *Under the assumption of Proposition 5.1, the energy of the solutions of the magnetoelasticity equations does not decay uniformly. In other terms, the necessary condition of Theorem 1 holds.*

Corollary 5.2 is a direct consequence of point b) of Proposition 2.5.

Proof of Proposition 5.1. This proof is inspired by the one of Theorem 4 of [2]. Denote by Z one of the indices T or L and set $P_Z := \Delta - \nu_Z^2 \partial_t^2$. We start by an elementary remark:

Remark 5.3. Let E be a vector subspace of $\mathbb{C}^3$ and π_E the orthogonal projection on E. The defect measure of $\pi_E u^k_Z$ is $\pi_E^* \mu_Z \pi_E$. Furthermore, $P_Z \pi_E u^k_Z = 0$, so that the theorem of propagation 3.12 holds for the measure $\pi_E^* \mu_Z \pi_E$ if a uniform Lopatinsky boundary condition holds on $\pi_E u^k_Z$. Notice that any **scalar** uniform Lopatinsky condition on u^k_Z yields such a condition on $\pi_E u^k_Z$. If $\pi_E^* \mu_Z \pi_E = \mu_Z$, the measure μ_Z will be said to be **polarized along** E. If E is the line generated by a vector H of $\mathbb{C}^3$ we shall also use the phrase "polarized along H". If both measures $\mu_T \mathbb{1}_{]-\varepsilon, T+\varepsilon[}$ and $\mu_L \mathbb{1}_{]-\varepsilon, T+\varepsilon[}$ are polarized along $\boldsymbol{B}$, then condition (60) is fulfilled.

Let $T' > T$. Consider a $\boldsymbol{B}$-resistant ray defined on an open interval I of length T':

$$\gamma(s) = (t_\gamma(s), y_\gamma(s), \tau_\gamma(s), \eta_\gamma(s)) = (x_\gamma(s), \xi_\gamma(s)).$$

If T' is large enough, then one of the two following assertions holds:

   a) $\gamma(I)$ contains an interior point;
   b) $y_\gamma(I) = \Gamma \subset \partial\Omega$ where Γ is a closed curved, contained in a plane P which is normal to $\boldsymbol{B}$, boundary of a convex subset of P, and such that on Γ, $\mathbf{n}$ is orthogonal to $\boldsymbol{B}$.

Case b) occurs when there exists an infinite boundary $\boldsymbol{B}$-resistant ray. This case reduces to case a), choosing a transversal ray contained in P which only meets the boundary at hyperbolic points.

Thus, we may assume that $\gamma(I)$ has an interior point. We may also assume, possibly translating the origin, that this interior point is $\gamma(0)$, and that $(t_\gamma(0), y_\gamma(0)) = (0,0)$. Recall that the magnetic field is vertical: $\boldsymbol{B} = (B, 0, 0)$. We shall denote by $-T^-$ and T^+ the extremal points of I: $I = (-T^-, T^+)$.

If $\eta_\gamma(0)$ is parallel to $\boldsymbol{B}$ (i.e. if $\gamma(0)$ is in the longitudinal characteristic set), choose a non-zero function $\varphi \in C_0^\infty(\Omega)$, and set

$$\varphi^k(y) = K^{-1} k^{-5/4} e^{iky_1} \varphi(\sqrt{k}y), \quad u_0^k = \nabla\varphi^k, \quad u_1^k = ikc_L u_0^k.$$

Where u^k is the solution of the Lamé system with initial data $(u^k, \partial_t u^k)_{\upharpoonright t=0} = (u_0^k, u_1^k)$. Then

$$\|u_0^k\|_{\boldsymbol{H}^1} \underset{k\to+\infty}{\longrightarrow} K^{-1}\|\varphi\|_{\boldsymbol{L}^2}, \quad \|u_1^k\|_{\boldsymbol{L}^2} \underset{k\to+\infty}{\longrightarrow} K^{-1}\|\varphi\|_{\boldsymbol{L}^2}.$$

Thus, condition (59) is fulfilled with an appropriate choice of K.

For small t, by finite speed of propagation for the wave equation, u_L^k has compact support in Ω. Thus $u_T^k = 0$ and $u_L^k = u^k$. As a consequence, for small t,

   (1) $\mu_T = 0$;
   (2) μ_L is polarized along $\boldsymbol{B}$;
   (3) the projection of the support of μ_L on $\mathbb{R}_t \times \overline{\Omega}$ is contained in $x_\gamma(I)$.

If $\eta_\gamma(0)$ is orthogonal to $\boldsymbol{B}$, we construct a sequence of solution of the Lamé system, with the following initial data (cf [2]):

$$\psi^k = K^{-1} k^{-5/4} e^{iky_1} \psi(\sqrt{k}y), \quad u_0^k = \mathrm{curl}\,(0, 0, -\psi^k), \quad u_1^k = ikc_T u_0^k.$$

In this case, condition (59) is fullfilled for an appropriate K and the defect measures satisfy the following properties for small t:

   (1) $\mu_L = 0$;
   (2) μ_T is polarized along $\boldsymbol{B}$;
   (3) the projection of the support of μ_T on $\mathbb{R}_t \times \overline{\Omega}$ is contained in $x_\gamma(I)$.

To show (60), we shall prove that both measures μ_T and μ_L, are, for $t \in I$, polarized along $\boldsymbol{B}$. For $t > 0$, we shall denote by $\mathcal{P}(t)$ the following property: **in a neighbourhood of** $[0,t]$**, both measures** μ_L **and** μ_T **are polarized along** $\boldsymbol{B}$ **and the projections of their support on** $\mathbb{R}_t \times \overline{\Omega}$ **are contained in** $x_\gamma(I)$.

Let $\mathcal{T}$ be the set of t in $[0, T^+)$ such that $\mathcal{P}(t)$ holds. By its definition, $\mathcal{T}$ is an open subset of $[0, T^+)$. We have just shown that 0 is in $\mathcal{T}$. We shall now prove that $\mathcal{T}$ is closed. We will use the following lemma:

Lemma 5.4. *Let* $\tilde\rho = (\tilde{t}, \tilde{y}, \tilde\tau, \tilde\eta) \in S_b^* M$. *If* $\mu_T \mathbb{1}_{t<\tilde{t}}$ *and* $\mu_L \mathbb{1}_{t<\tilde{t}}$ *vanish in a neighbourhood of* $\tilde\rho$, *the measures* μ_T *and* μ_L *also vanish in a neighbourhood of* $\tilde\rho$.

This is a trivial assertion in the interior of Ω by the propagation of both measures. Near a point of the boundary of Ω, one may show Lemma 5.4 using the Dirichlet boundary condition on u^k and the theorem of propagation 3.12, together with the same type of arguments as in Lemma 4.5.

Let $s_0 > 0$ such that $\mathcal{P}(s_0)$ holds for $s < s_0$. We must check that $\mathcal{P}(s_0)$ holds. Three cases arise, depending on the nature of $\rho := \gamma(s_0)$.

   i) ρ is an interior point.

$\mathcal{P}(s_0)$ is obvious in view of the propagation of both measures in the interior of Ω.

ii) ρ is of the type (1) of Lemma 4.5.

This case, where μ_L vanishes for time $t < t_\gamma(s_0)$ near $t_\gamma(s_0)$, was studied in [2]. The authors show that μ_L remains null for times greater than $t_\gamma(s_0)$ and that μ_T propagates near $\gamma(s_0)$, in such a way that in our case, its polarization along $\boldsymbol{B}$ is preserved. In particular property $\mathcal{P}(s_0)$ holds.

iii) ρ is of the type $(\mathcal{H}_L)$ of Lemma 4.5: $y'_\gamma(s_0) \in \partial\Omega$, $\eta'_\gamma(s_0) = 0$ and $\mathbf{n}//\boldsymbol{B}$.

In view of Lemma 5.4, the support of the measure μ_L is contained, near ρ in the union of the longitudinal ray coming in ρ and the longitudinal ray going out of ρ. The support of μ_T, if not empty, if the transversal ray going out of ρ. Let E be the plane orthogonal to $\boldsymbol{B}$ in $\mathbb{C}^3$. The polarization of μ_L along $\boldsymbol{B}$ shows that $\pi_E^* \mu_L \pi_E = 0$ and thus, by Remark 5.3 and the standard hyperbolic theory of Proposition 3.14,

$$\pi_E u_{T\restriction x_3=0}^k = -\pi_E u_{L\restriction x_3=0}^k \xrightarrow[k\to\infty]{} 0 \text{ in } \boldsymbol{H}_{\rho,\partial}^1$$

which implies, using again Remark 5.3 that $\pi_E^* \mu_T \pi_E$ propagates along the transversal flow near $\gamma(s_0)$. Thus $\pi_E^* \mu_T \pi_E$ vanishes near ρ. But μ_T is polarized orthogonally to its direction of propagation which is exactly $\boldsymbol{B}$ on the support of μ_T near ρ. This show that μ_T vanishes near ρ, completing the proof of $\mathcal{P}(s_0)$.

iv) ρ is of the type $(\mathcal{G}_L)$ of Lemma 4.5: ρ is a diffractive point for the longitudinal wave, and $\eta'_\gamma(s_0)$ is parallel to $\boldsymbol{B}$.

Then $\rho \in \mathcal{H}_T \cup \mathcal{E}_T$. Furthermore, if $\rho \in \mathcal{H}_T$, $\mu_T \mathbb{1}_{t<t_\gamma(s_0)}$ vanishes near ρ. Thus, according to standard elliptic or hyperbolic theory (cf Propositions 3.15 and 3.16), u_T^k satisfies a boundary condition of the following form:

$$D_{x_3} u_{T\restriction x_3=0}^k = A u_{T\restriction x_3=0}^k + o(1) \text{ in } \boldsymbol{L}_{\rho,\partial}^2, \quad A \in \boldsymbol{\mathcal{A}}_\partial^1,$$

where $\sigma_1(A) = -i\sqrt{\|\eta'\|^2 - \nu_T^2 \tau^2}$ in the elliptic case and $\sigma_1(A) = -\sqrt{\nu_T^2 \tau^2 - \|\eta'\|^2}$ in the hyperbolic case. This yields, in view of Lemma 4.6, an uniform Lopatinsky boundary condition on φ^k:

$$(61) \qquad\qquad \varphi_{\restriction x_3=0}^k = B_{-1} D_{x_3} \varphi_{\restriction x_3=0}^k + o(1) \text{ in } \boldsymbol{H}_{\rho,\partial}^1.$$

As a consequence, the support of μ (and that of μ_L) propagates near ρ. The polarization of μ_L along $\boldsymbol{B}$ is immediate. The nullity of μ_T near ρ remains to be checked. This is a general property in the elliptic case $\rho \in \mathcal{E}_T$. In the hyperbolic case, first note that the propagation theorem of Burq and Lebeau [2, Theorem 1] implies with the boundary condition (61) that μ is invariant by the longitudinal flow near diffractive points. So the total mass of μ_L is preserved by time, for t close enough to $t_\gamma(s_0)$. The next lemma, which is a measure version of the conservation of energy for the Lamé system, completes the proof of $\mathcal{P}(s_0)$:

Lemma 5.5. *Let* $\varphi \in C_0^\infty(\mathbb{R})$. *Then*

$$\langle \mu_T + \mu_L, \varphi'(t) \rangle = 0.$$

Thus the total mass of the measure $(\mu_T + \mu_L)_{\restriction t=s}$ *is well defined, and does not depend on s.*

Proof. In this proof, we denote by $x = (t,z) \in \mathbb{R} \times \Omega$ the global space-time variable. By the equation $(\partial_t^2 - \Delta_e)u^k = 0$,

$$\text{Re} \int \partial_t^2 u^k \partial_t \overline{u}^k \varphi(t) dx - \text{Re} \int \Delta_e u^k \partial_t \overline{u}^k \varphi(t) dx = 0.$$

Set $\nabla_e u := (\varsigma \nabla u, (\lambda + \varsigma) \text{div } u) \in \mathbb{C}^4$. A simple integration by parts yields

$$\int \varphi'(t) \left| \partial_t u^k \right|^2 dx + \int \varphi'(t) \left| \nabla_e u^k \right|^2 dx = 0.$$

Using another integration by parts, and then the decoupling Lemma 3.28,

$$-\int \partial_t^2 u^k \overline{u}^k \varphi'(t) dx - \int \Delta_e u^k \overline{u}^k \varphi'(t) dx = o(1) \text{ as } k \to +\infty$$

$$\int (\partial_t^2 + c_T^2 \Delta) u_T^k \overline{u}_T^k \varphi'(t) dx + \int (\partial_t^2 + c_L^2 \Delta) u_L^k \overline{u}_L^k \varphi'(t) dx = o(1) \text{ as } k \to +\infty,$$

and thus, by the definition of μ_T and μ_L,

$$\left\langle \mu_T, \frac{\tau^2 + c_T^2 \|\eta\|^2}{2\tau^2} \varphi'(t) \right\rangle + \left\langle \mu_L, \frac{\tau^2 + c_L^2 \|\eta\|^2}{2\tau^2} \varphi'(t) \right\rangle = 0.$$

This completes the proof, noting that on the support of μ_T (respectively μ_L), $c_T \|\eta\|$ (respectively $c_L \|\eta\|$) is equal to τ. □

Lemma 5.5 and the mass conservation of μ_L imply that the mass of μ_T is also preserved near ρ, which shows that μ_T vanishes in a neighbourhood of ρ.

v) ρ is of the type $(L \to T)$ of Lemma 4.5.

In view of Lemma 5.4 and of the assumption $P(s)$ for $s < s_0$, the support of μ_L is contained in the two longitudinal half-rays passing through ρ, and that of μ_T is only contained in the ray going out of ρ. To prove $P(s_0)$, it remains to show that $\mu_L = 0$ along the longitudinal ray going out of ρ. We shall do so by a simple polarization argument. Let H (respectively J) be a unitary vector of $\mathbb{C}^3$ parallel to the direction of the transversal (respectively longitudinal) ray going out of ρ. The polarization of μ_T shows that $\pi_H^* \mu_T \pi_H$ is null, so that $\pi_H^* \mu_L \pi_H$ propagates near ρ. Now, H is orthogonal to the direction of the longitudinal ray coming in ρ, so that $\pi_H^* \mu_L \pi_H$ vanishes, along incoming rays but also, in view of the propagation, along outgoing rays. Furthermore $\mu_L \mathbb{1}_{t > t_\gamma(s_0)}$ is polarized along J. It is easy to see that this last measure vanishes. Indeed, the polarization of μ_L implies

$$\mathbb{1}_{t > t_\gamma(s_0)} \pi_J^* \pi_H^* \pi_J^* \mu_L \pi_J \pi_H \pi_J = \mathbb{1}_{t > t_\gamma(s_0)} \pi_J^* \pi_H^* \mu_L \pi_H \pi_J = 0$$

But

$$\pi_J \pi_H \pi_J = \frac{<H, J>^2}{|H|^2 |J|^2} \pi_J$$

Noting that H and J are not orthogonal this yields the nullity of $\mu_L \mathbb{1}_{t > t_\gamma(s_0)}$ in a neighbourhood of ρ.

vi) ρ is of the type $(T \to L)$ of Lemma 4.5 .

One may argue as before, showing that for every vector K orthogonal to J, $\pi_K^* \mu_T \pi_K = 0$ near ρ, which implies the nullity of $\mu_T \mathbb{1}_{t > t_\gamma(s_0)}$ near ρ.

The proof is completed by reversing time, which yields $\mathcal{P}(s)$ for $-T^- < s < 0$. □

APPENDIX A. REGULARITY OF BOUNDARY VALUE PROBLEMS

The following proposition concerns solutions of a partial differential equation which is transverse to the boundary of an open set. It says that for such functions, the control of derivatives which are tangential to the boundary suffices to control all the derivatives. As in Section 3, we shall work in an open subset $\overline{X}$ of $\mathbb{R}_+^{n+1}$, of the form $X' \times [0, l[$, where X' is an open subset of $\mathbb{R}^n$. Let P be a differential operator of degree r on $\overline{X}$, of the following form:

$$P = \sum_{j=0..r} Q_{r-j} \partial_{x_n}^j,$$

where the Q_j's are $N \times N$ matrices of tangential differential operators, with $C^\infty(\overline{X})$ coefficients, and Q_0 is the identity of $\mathbb{C}^N$. To simplify the following statements, we suppose $u \in C^\infty(\overline{X})$.

Proposition A.1. *Let $s \geq 0$, $j \in \mathbb{N}$, and suppose that $Pu = 0$ on $\overline{X}$. Then*

$$\forall \varphi \in C_0^\infty(\overline{X}), \quad \exists \tilde{\varphi} \in C_0^\infty(\overline{X}), \quad \|\varphi \partial_{x_n}^j u\|_{\boldsymbol{H}^{s-j}} \leq C \|\tilde{\varphi} u\|_{L^2(0, l; \boldsymbol{H}^s(X'))},$$
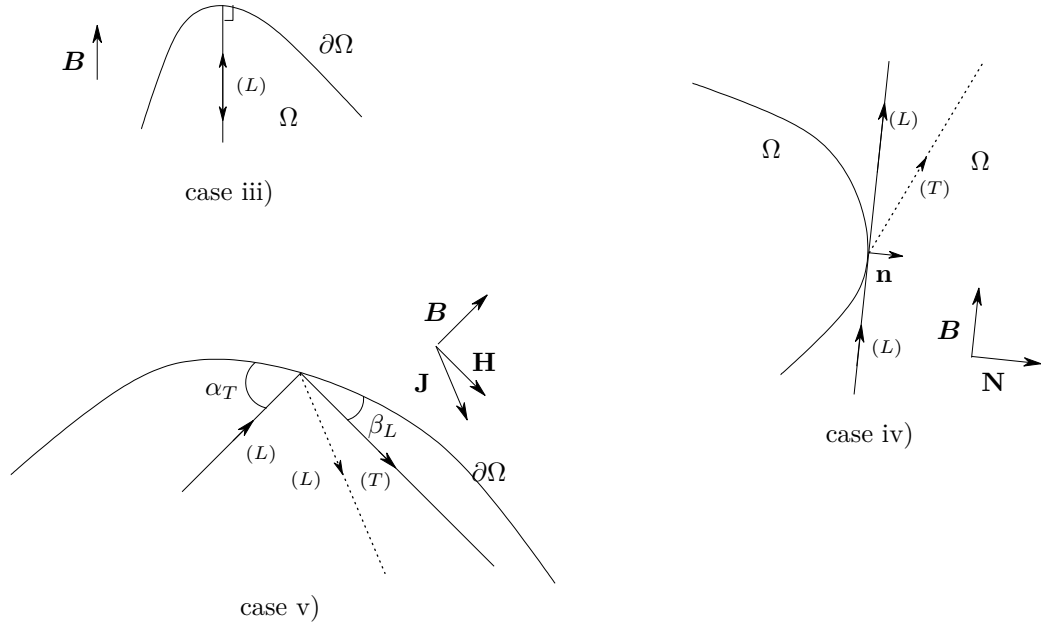
FIGURE 3. Cases iii), iv) and v)

where C does not depend on u. Likewise (assuming again $Pu = 0$),

$$\forall \varphi \in C_0^\infty\left(\overline{X}\right), \quad \exists \tilde{\varphi} \in C_0^\infty\left(\overline{X}\right), \quad \|\varphi u\|_{L^2(0,l;\boldsymbol{H}^{-s}(X'))} \leq C\|\tilde{\varphi} u\|_{\boldsymbol{H}^{-s}(X)}.$$

APPENDIX B. PROOF OF PROPOSITION 3.18

As mentioned in the introduction of Section 3, we shall assume that each u^k is smooth enough, so that all the quantities appearing in the following calculation are well defined and finite. Let

$$A = A_0 D_{x_n}, \ A_0 \in \mathcal{A}^0, \ \mathcal{C}^k := \left([P,A]u^k, u^k\right).$$

Take the support of A_0 in a small enough neighbourhood of $\tilde{\rho}$. The operator P is formally self-adjoint. A simple integration by parts yields

$$\mathcal{C}^k = \underbrace{-(APu^k, u^k) + (Au^k, Pu^k)}_{0} - (Au^k_{\restriction x_n = 0}, iD_{x_n}u^k_{\restriction x_n = 0})_\partial + (iD_{x_n}Au^k_{\restriction x_n = 0}, u^k_{\restriction x_n = 0})_\partial,$$

where $(.,.)_\partial$ is the $\boldsymbol{L}^2$ scalar product on $\{x_n = 0\}$, with respect to the measure $\sqrt{g_{\restriction x_n = 0}}dx'dt$. We have

$$D_{x_n}A_0 D_{x_n}u^k = A_0 D_{x_n}^2 u^k + [D_{x_n}, A_0]D_{x_n}u^k = -A_0 Qu^k + R_0 D_{x_n}u^k, \quad R_0 \in \mathcal{A}^0$$

$$\mathcal{C}^k = i(A_0 D_{x_n}u^k, D_{x_n}u^k)_\partial - i(A_0 Qu^k, u^k)_\partial + (R_0 D_{x_n}u^k, u^k)_\partial.$$

So, using condition (29),

$$\mathcal{C}^k = i(A_0 D_{x_n}u^k, D_{x_n}u^k)_\partial - i(A_0 QB_{-1}D_{x_n}u^k, B_{-1}D_{x_n}u^k)_\partial - i(A_0 Qh^k, B_{-1}D_{x_n}u^k)_\partial$$
$$- i(A_0 QB_{-1}D_{x_n}u^k, h^k)_\partial + (R_0 D_{x_n}u^k, B_{-1}D_{x_n}u^k)_\partial + o(1), \ k \to +\infty$$

Take A_0 of the form $T_0^* T_0$, where $T_0 \in \mathcal{A}_0$ is scalar, elliptic at $\tilde{\rho}$ and has support in a small neighbourhood of $\tilde{\rho}$. Then

$$\mathcal{C}^k = i(E_0 D_{x_n} u^k, D_{x_n} u^k)_\partial - i(T_0 Q h^k, B_{-1} T_0 D_{x_n} u^k) - i(Q B_{-1} T_0 D_{x_n} u^k, T_0 h^k) + o(1), \ k \to +\infty$$

$$E_0 \in \mathcal{A}^0, \ E_0 = T_0^* T_0 - B_{-1}^* T_0^* T_0 Q B_{-1} + B_{-1}^* R_0.$$

Denoting by t_0 the principal symbol (which is scalar) of T_0, we have

$$\sigma_0(E_0) = |t_0|^2 \left(1 - b_{-1}^* b_{-1} q_2 \right).$$

Since q_2 vanishes at $\tilde{\rho}$, we may choose t_0 with support in a small enough neighbourhood of $\tilde{\rho}$ such that

$$\sigma(E_0) \geq 1/2 |t_0|^2$$

(in the sense of quadratic positive Hermitian forms). The weak Gårding inequality, applied to the operator $E_0 - 1/2 T_0^* T_0$, thus yields

$$\liminf_{k \to +\infty} \mathrm{Re}\, (E_0 D_{x_n} u^k, D_{x_n} u^k) - \frac{1}{2} \| T_0 D_{x_n} u^k \|_{\boldsymbol{L}_\partial^2}^2 \geq 0.$$

This implies, using the convergence to 0 of h^k in $\boldsymbol{H}_{\tilde{\rho}}^1$

$$\liminf_{k \to +\infty} \mathrm{Im}\, \mathcal{C}^k \geq \frac{1}{4} \| T_0 D_{x_n} u^k \|_{\boldsymbol{L}_\partial^2}^2.$$

Thus $D_{x_n} u^k_{\upharpoonright x_n = 0}$ is bounded in $\boldsymbol{L}_{\tilde{\rho}, \partial}^2$, which yields, with the boundary condition, that $u^k_{\upharpoonright x_n = 0}$ is bounded in $\boldsymbol{H}_{\tilde{\rho}, \partial}^1$. The proof of (34) is complete.

When μ is null near $\tilde{\rho}$, we have

$$\lim_{k \to +\infty} \mathcal{C}^k = \left\langle \mu, \frac{\{ p, a_1 \xi_n \}}{\tau^2} \right\rangle = 0,$$

which yields (35).

Point b) of Proposition 3.18 may be seen as a consequence of the propagation theorem of [2]. The assumptions (36) imply that **any** uniform Lopatinsky boundary condition holds on the traces of u^k, which shows that the measure μ propagates near $\tilde{\rho}$ with any smooth multiplicative factor, which is impossible unless μ is null.

References

[1] E. Andreou and G. Dassios. Dissipation of energy for magnetoelastic wavec in a conductive medium, *Quart. Appl. Math.*, 55 (1997) 23-39.

[2] N. Burq and G. Lebeau. Mesures de défaut de compacité, application au système de Lamé. *Annales Scientifiques de l'École Normale Supérieure (4)* 34 (2001), no. 6, 817–870. Mesures de défaut de compacité, application au système de Lamé. *A paraître aux Annales de L'Ecole Normale Supérieure*, 1999.

[3] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *Siam Journal of Control and Optimization*, 305:1024–1065, 1992.

[4] N. Burq, P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. (French) *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. 325 (1997), no. 7, 749–752.

[5] R. C. Charão, J. C. Oliveira, G. Perla Menzala. Energy decay rates of magnetoelastic waves in a bounded conductive medium. *Discrete and Continuous Dynamical Systems* 25 (2009), no. 3, 797-821.

[6] R. C. Charão, J. C. Oliveira and G. Perla Menzala. Decay rates of magnetoelastic waves in an unbounded conductive medium *Electron. J. Differ. Equ.* 2011, paper no. 127, 14 p.

[7] C. R. da Luz, J. C. Oliveira. Asymptotic behavior of solutions for the magneto-thermo-elastic system in $\mathbb{R}^3$. J. Math. Anal. Appl. 432, No. 2, 1200–1215 (2015).

[8] T. Duyckaerts. Stabilization of the Linear System of Magnetoelasticity. Preprint arXiv 0407257 (2004).

[9] M. de Lima Santos and J. E. Muñoz Rivera. Polynomial stability to three dimensional magnetoelasticity. *Acta Appl. Math.* 76 (2003), 265-281.

[10] P. Gérard. Microlocal Defect Measures. *Communication in Partial Differential Equations*, 16, 1761-1794, 1991.

[11] L. Hörmander. The analysis of linear partial differential operators. III, vol. 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994.

[12] G. Lebeau. Équation des ondes amorties. In *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, vol. 19 of *Math. Phys. Stud.* Kluwer Acad. Publ., Dordrecht, 1996, 73-109.

[13] G. Lebeau and L. Robbiano. Stabilisation de l'équation des ondes par le bord. *Duke Mathematical Journal* 86 (1997), no. 3, 465-491

[14] G. Lebeau and E. Zuazua. Null-controllability of a system of linear thermoelasticity. *Archive for Rational Mechanics and Analysis* 141 (1998), No. 4, 297-329.

[15] G. Lebeau and E. Zuazua. Decay rates for the three-dimensional linear system of thermoelasticity. *Archive for Rational Mechanics and Analysis* 148 (1999), no. 3, 179-231.

[16] T. F. Ma, J. M. Rivera, H. P. Oquendo and F. M. S. Suárez. Polynomial stabilization of magnetoelastic plates. *IMA J. Appl. Math.* 79 (2014), no. 2, 241-253.

[17] R. Melrose and J. Sjöstrand. Singularities of boundary value problems I. *Communications in Pure and Applied Mathematics*, 16, 1761-1794, 1991.

[18] J. E. Muñoz Rivera and R. Racke. Polynomial stability in two-dimensional magnetoelasticity. *IMA J. Appl. Math.* 66 (2001), no. 3, 269–283.

[19] J. E. Muñoz Rivera and R. Racke. Magneto-thermo-elasticity. Large-time behavior for linear systems. *Adv. Differ. Equ.* 6 (2001), no. 3, 359-384.

[20] G. Perla Menzala and E. Zuazua. Energy decay of magnetoelastic waves in a bounded conductive medium. *Asymptotic Analysis*, 18 (1998), 349-362.

[21] R. Quintanilla and R. Racke. Stability in thermoelasticity of type III. *Discrete and Continuous Dynamical Systems, Ser. B* 3 (2003), no. 3, 383-400.

[22] L. Tartar. H-Measures, A New Approach for Studying Homogenisation, Oscillations and Concentration Effects in Partial Differential Equations. *Proceedings of the Royal Society, Edinburgh*, 115-A 193-230, 1990

[23] R. Temam. Navier-Stokes equations. *Studies in mathematics and its applications, North-Holland 1979*

[24] X. Zhang, and E. Zuazua. Decay of solutions of the system of thermoelasticity of type III. Commun. Contemp. Math. 5 (2003), no. 1, 25-83.