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ARTIFICIAL BOUNDARY CONDITIONS FOR INCOMPRESSIBLE VISCOUS FLOWS*

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Abstract. Artificial boundary conditions for the linearized incompressible Navier-Stokes equations are designed by approximating the symbol of the transparent operator. The related initial boundary value problems are well posed in the same spaces as the original Cauchy problem. Furthermore, error estimates for small viscosity are proved.

Key words. incompressible Navier-Stokes equation, artificial boundary conditions, initial boundary value problems, Oseen approximation

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1. Introduction. Many problems arising in fluid mechanics lead to the resolution of a partial differential equation in an unbounded domain. Depending on the applications, various strategies have been developed.

For stationary flows around a body, integral equations are often used (see for instance [6]) or we can also bound the domain and prescribe on the artificial boundary the so-called "transparent" boundary condition, i.e., the boundary condition which simulates the missing part of the domain. This boundary condition is integral on the boundary, and the associated initial boundary value problem is solved numerically using either an eigenvalues expansion on the boundary [2], [22], [10] or a coupling between finite elements in the interior and integral equations on the boundary [18], [26].

For time dependent problems, the "transparent" boundary condition is integral in time and space and thus impractical in general. Tremendous research effort has attempted to design useful boundary conditions for inviscid flows. Most of the studies rely on a linearization of the equation near the boundary (for a nonlinear treatment of the problem see [16] and [29]). Of course in the applications we solve the nonlinear equations in the interior together with the linear boundary conditions.

There are two mathematical frames for these studies. On one hand, Engquist and Majda in [8] and [9] designed absorbing boundary conditions with wave propagation tools. On the other hand, Bayliss and Turkel in [4] and [5] used far field expansions. Both works write sequences of boundary conditions that are local (i.e., differential) in time and space on the boundary. This feature is due to the hyperbolicity or quasi-hyperbolicity of the operators they handle.

The problem becomes less clear when it comes to viscous flows. Some numerical answers have been given for compressible fluids (see Rudy and Strikwerda [25]). On the other hand, calculations have been performed in [11], [12], and [19] on the case of a parabolic equation. Moreover, the case of linear advection-diffusion has been treated in [13]. For incompressible flows it is still, as far as we know, an open mathematical question.

Our previous results were announced in [15].

We are concerned here with the incompressible Navier-Stokes equation,

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for $N=2$ or 3:

$$(1.1)\qquad \begin{cases} u_t+(u\cdot\nabla)u-\nu\Delta u+\nabla p=f & \text{in } \mathbb{R}^N\times]0,T[,\\ \operatorname{div}(u)=0 & \text{in } \mathbb{R}^N\times]0,T[,\\ u(0)=u^0 & \text{in } \mathbb{R}^N,\end{cases}$$

where $u=(u_1,\cdots,u_N)$.

It is important to set the artificial boundaries outside of turbulent regime, sufficiently far for the flow to be considered as constant. We are then allowed to linearize around the constant state and to consider the Oseen approximation as follows.

$$(1.2)\qquad \begin{cases} u_t+(a\cdot\nabla)u-\nu\Delta u+\nabla p=f & \text{in } \mathbb{R}^N\times]0,T[,\\ \operatorname{div}(u)=0 & \text{in } \mathbb{R}^N\times]0,T[,\\ u(0)=u^0 & \text{in } \mathbb{R}^N.\end{cases}$$

The data f and u^0 are supposed to be compactly supported.

All throughout this paper we study the model problem: writing artificial boundary conditions on the hyperplane $\mathbb{R}^{N-1}$, i.e., such that the Oseen equation in the half-space $\mathbb{R}^N_-=\{(x,y)\in\mathbb{R}^N, x<0\}$ with the boundary condition prescribed on the hyperplane $x=0$ is an approximation of the Oseen equation in $\mathbb{R}^N$. This enables us to use the Fourier transform as an essential tool. The problem of designing artificial conditions on a closed artificial boundary will be treated in a forthcoming paper.

In [13], as a first step, the linear advection diffusion equation has been studied. This article concluded with a family of approximate boundary conditions that are local in time and space. In contrast, here the divergence-free condition implies a coupling, which makes the analysis more troublesome. In particular, the symbol of the operator to be approximated contains $|k|$, with k the dual variable of the tangential spatial variable, and it does not seem easy to approximate this symbol by polynomials or rational fractions of low degree in space. Thus, the approximate boundary conditions will be local in time and global in the tangential space variables.

In the course of justifying our calculations precisely, we will prove a number of interesting results on the spaces of divergence-free functions on a half space, without a Dirichlet boundary condition.

The analysis is made in $\mathbb{R}^2$, but is valid in $\mathbb{R}^3$ with slight modifications (see [14]).

In § 2, we define the spaces of divergence-free functions in $\mathbb{R}^2$ and prove trace theorems on $\Gamma=\{(x,y), x=0\}$. We then introduce the Oseen equation, give the formalism necessary for a variational formulation, and state a well-posedness theorem for the Cauchy problem in $\mathbb{R}^2$. We finish by specific trace results for the solution of Stokes equation, which emphasize the regularity of $u_1+\mathcal{H}u_2$ ($\mathcal{H}$ is the Hilbert transform along the boundary), solution of a heat equation.

In § 3 we compute the transparent boundary condition on Γ according to the following principle: Problem (1.2) in $\mathbb{R}^2$ is equivalent (in a sense we will make precise) to the transmission problem in $\Omega_-\times\Omega_+$, where $\Omega_\pm=\{(x,y), \pm x>0\}$ and

$$\begin{aligned} &\frac{\partial u_-}{\partial t}+(a\cdot\nabla)u-\nu\Delta u_-+\nabla p_-=f && \text{in } \Omega_-\times]0,T[,\\ &\operatorname{div}(u_-)=0 && \text{in } \Omega_-\times]0,T[,\\ &\frac{\partial u_+}{\partial t}+(a\cdot\nabla)u_+-\nu\Delta u_++\nabla p_+=0 && \text{in } \Omega_+\times]0,T[,\\ &\operatorname{div}(u_+)=0 && \text{in } \Omega_+\times]0,T[,\end{aligned}$$

with the initial data

$$\begin{cases} u_-(0) = u^0 & \text{in } \Omega_-, \\ u_+(0) = 0 & \text{in } \Omega_+, \end{cases}$$

and the transmission conditions

$$\begin{cases} u_-|_\Gamma = u_+|_\Gamma, \\ \sigma_n(u_-)|_\Gamma = \sigma_n(u_+)|_\Gamma, \end{cases}$$

where σ_n is the normal constraint.

We study the Oseen equation for u_+ in Ω_+, with nonhomogeneous Dirichlet boundary conditions, and prove the well-posedness in spaces where the partial Fourier-Laplace transform in time and tangential space is permissible. We write then a pseudodifferential relation between u_+ and $\sigma_n(u_+)$ on Γ. Thanks to the transmission conditions, it leads to the same pseudo-differential relation between u_- and $\sigma_n(u_-)$. We call it the transparent boundary condition after proving the uniqueness for the related initial boundary value problem in Ω_-.

In § 4 we design a family of approximations to the transparent boundary condition by approximating its symbol. These approximations are local in time and integral along the boundary. Using the tools developed in the previous sections we prove them to be well-posed with the same regularity as the solution of the Cauchy problem in $\mathbb{R}^2$. Even at low order these boundary conditions appear to be good approximations for small viscosity.

Finally in Appendix A, we give some information on Beppo–Levi spaces, and in Appendix B, we show why we cannot use spaces of fast-decreasing functions at infinity in the analysis of the present incompressible problems.

2. Definitions, notations and basic results. In this section, we describe a number of functional spaces which are useful for our study of Stokes problem with an advection term, otherwise called an Oseen system. We need spaces in which we are able to treat nonhomogeneous boundary conditions, and thus transparent and artificial boundary conditions. The typical space is the space of divergence-free functions, which are square integrable, with a square integrable gradient, but without any boundary condition. We give density results relative to these spaces, and corresponding trace results. Then we give rather classical results on the solution of the Oseen problem, in $\mathbb{R}^2$. Finally, if u is the solution of this problem, and if $\mathcal{H}$ is the Hilbert transform in the direction x_2, the function $u_1 + \mathcal{H}u_2$ has a number of special properties which will be useful everywhere in the sequel. In particular, if the support of the data of the Oseen problem is in the region $\{x_1 \leqq -X < 0\}$, the restriction of $u_1 + \mathcal{H}u_2$ to the boundary $\{x_1 = 0\} \times \mathbb{R} \times \mathbb{R}$ is arbitrarily smooth.

2.1. Classical functional spaces. We use the formalism of [24] in many instances. The domain W of $\mathbb{R}^N$ has boundary Γ; the scalar product in $L^2(\Omega)$ is denoted $(\cdot, \cdot)$, with associated norm $\| \ \|$. The classical Sobolev space $H^m(\Omega)$ is the space of square integrable functions, whose derivatives of order at most m are square integrable; the scalar product in $H^m(\Omega)$ is denoted $(\cdot, \cdot)_m$, and the norm $\| \ \|_m$; the scalar product in $L^2(\Omega)$ is denoted $(\cdot, \cdot)_\Omega$ with associated norm $\| \ \|_\Omega$. The Fourier transform is defined on the Schwarz space $\mathcal{S}$ by

$$\hat{v}(k) = \int_{\mathbb{R}^N} v(x) \exp(-ik \cdot x)\, dx, \tag{2.1}$$

where $k \cdot x = k_1 x_1 + k_2 x_2 + \cdots + k_N x_N$; it is extended to $\mathcal{S}'$, and we will often write (2.1) for temperate distributions by an abuse of notation.

For any real s, the Sobolev space of fractional order $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N)=\left\{v\in\mathcal{S}'\Big/\int(1+|k|^2)^s|\hat{v}(k)|^2\,dk<\infty\right\},$$

this space is a Hilbert space, equipped with the norm

$$\|v\|_s=\left[\int(1+|k|^2)^s|\hat{v}(k)|^2\,dk\right]^{1/2}.$$

In particular, we will need the Sobolev spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, when Γ is $\mathbb{R}^N$; the norm of $H^{1/2}(\Gamma)$ will be denoted $\|\cdot\|_{1/2,\Gamma}$, and the norm of $H^{-1/2}(\Gamma)\|\cdot\|_{-1/2,\Gamma}$.

We denote the right-hand side half-space as

$$\Omega_+=\mathbb{R}^N_+=\{x=(x_1,x_2,\cdots,x_N)/x_1>0\},$$

and the left-hand side half-space as

$$\Omega_-=\mathbb{R}^N_-=\{x=(x_1,x_2,\cdots,x_N)/x_1<0\}.$$

A convenient characterization of $H^m(\Omega_+)$ is given by

$$H^m(\Omega_+)=\{u\in L^2(0,\infty;H^m(\mathbb{R}^{N-1}))/\quad\forall k=1,2,\cdots,m,$$
$$\frac{\partial^k u}{\partial x_1^k}\in L^2(0,\infty;H^{m-k}(\mathbb{R}^{N-1}))\}.$$

Then the norm on $H^m(\Omega_+)$ can be written as

$$\|u\|_m^2=\sum\left|\frac{\partial^k u}{\partial x_1^k}\right|^2_{L^2(0,\infty;H^{m-1}(\mathbb{R}^{N-1}))}.$$

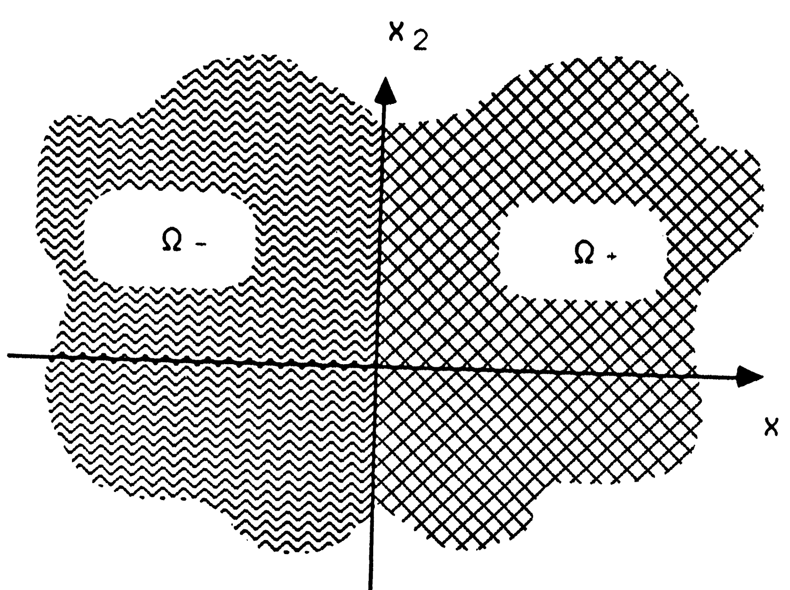

FIG. 1

We can define a partial Fourier transform operating on the tangential variables; for v in $\mathcal{S}'(\Omega_+)$, it is defined formally as

$$\mathcal{F}_{x'\to k'}v(x_1, k') = \int_{\mathbb{R}^{N-1}} v(x_1, x') \exp(-ik \cdot x')\, dx',$$

where $x' = (x_2, \cdots, x_N)$ and $k' = (k_2, \cdots, k_N)$.
The space $H^m(\Omega_+)$ can be characterized too with the help of a partial Fourier transform.

We define the Sobolev spaces of vector-valued functions:

$$\mathbf{H}^s(\Omega) = (H^s(\Omega))^N \text{ and } \mathbf{H}_0^s(\Omega) = (H_0^s(\Omega))^N.$$

The scalar product and the associated norm will be denoted as for the scalar Sobolev spaces.

We denote the gradient by ∇, and the divergence by $\nabla\cdot$.

Finally, we will need the Beppo-Levi spaces, defined as follows.

DEFINITION 2.1. Let Ω be an open domain of $\mathbb{R}^N$. The Beppo-Levi space $BL(H^s(\Omega))$ constructed on $H^s(\Omega)$ is the space of distributions u such that ∇u belongs to $H^s(\Omega)$.

The space $BL(H^s(\Omega))$ is a subspace of $H^{s+1}_{loc}(\Omega) \cap \mathcal{S}'(\Omega)$; more information on the Beppo-Levi spaces is given in Appendix A. Let it suffice here to observe that $BL(H^s(\Omega))$ is a Hilbert space, with Hilbert seminorm $\|\nabla u\|_s$; the kernel of this seminorm is the space of constant functions. Moreover, if Ω is unbounded, $BL(H^s(\Omega))$ contains unbounded elements. Examples of such behavior are given in Appendix *A*.

2.2. Spaces of divergence-free functions. Spaces of divergence-free functions are an essential tool for the study of fluid motion in Eulerian coordinates. One classical set of spaces is defined starting from

$$\mathcal{V}(\Omega) = \{v \in \mathcal{D}(\Omega)^2 / \nabla \cdot v = 0\}.$$

The respective closures of v in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ are denoted $H_0(\Omega)$ and $V(\Omega)$; they are very well adapted to the study of a problem with Dirichlet boundary conditions; they admit the following characterization, when $\Omega = \mathbb{R}^2$.

$$\begin{aligned} V(\mathbb{R}^2) &= \{v \in \mathbf{H}^1(\mathbb{R}^2)/\nabla \cdot v = 0\} \\ H_0(\mathbb{R}^2) &= \{v \in \mathbf{L}^2(\mathbb{R}^2)/\nabla \cdot v = 0\} \end{aligned} \tag{2.2}$$

When Ω is a domain with smooth boundary Γ, the characterization is modified as follows.

$$\begin{aligned} V(\Omega) &= \{v \in \mathbf{H}_0^1(\Omega)/\nabla \cdot v = 0\} \\ H_0(\Omega) &= \{v \in \mathbf{L}^2(\Omega)/\nabla \cdot v = 0 \text{ and } v \cdot n = 0 \text{ on } \Gamma\}, \end{aligned} \tag{2.3}$$

where n is the exterior normal to Γ.
This last characterization makes sense because, on the space

$$E(\Omega) = \{v \in \mathbf{L}^2(\Omega)/\nabla \cdot v \in L^2(\Omega)\},$$

the normal trace $v \cdot n$ is defined, belongs to $\mathbf{H}^{1/2}(\Gamma)$, and the mapping

$$v \to v \cdot n$$

from $E(\Omega)$ to $\mathbf{H}^{1/2}(\Gamma)$ is onto.

A convenient reference for the above classical results is [28]. Nevertheless, we are interested in boundary conditions which are not Dirichlet boundary conditions, and we introduce spaces which are not classical:

$$\mathcal{W}(\Omega_-) = \{v \in \mathcal{D}(\Omega_-)^2 / \nabla \cdot v = 0\}, \tag{2.4}$$

where Ω_- is the closed half-plane. Moreover, we introduce

$$W(\Omega_-)=\{v\in \mathbf{H}^1(\Omega_-)/\nabla\cdot v=0\} \tag{2.5}$$

$$H(\Omega_-)=\{v\in \mathbf{L}^2(\Omega_-)/\nabla\cdot v=0\}. \tag{2.6}$$

The relations between $\mathcal{W}(\Omega_-)$, $W(\Omega_-)$, and $H(\Omega_-)$ as well as the trace properties of these spaces will be stated below. The results presented here differ from the standard ones in two respects: first, we want to allow for nonzero boundary data; second, the open set Ω contains points at an infinite distance from its boundary. The case with zero boundary data can be found in [28]; the case where the points of Ω are at a bounded distance from its boundary can be found in [1].

PROPOSITION 2.2. *The closure of $\mathcal{W}(\Omega_-)$ in $\mathbf{H}^1(\Omega_-)$ is precisely $W(\Omega_-)$.*

Proof. To prove this result, we consider an element orthogonal to the closure of $\mathcal{W}(\Omega_-)$ in $W(\Omega_-)$. The potential of this element u satisfies a homogeneous partial differential equation; as u is in a Sobolev space, it is a temperate distribution. This enables us to deduce estimates on the traces of u on the boundary, and, in the same time, the nullity of u.

Let $W^{\#}$ be the closure of $\mathcal{W}(\Omega_-)$ in $\mathbf{H}^1(\Omega_-)$; clearly, $W^{\#}$ is included in $W(\Omega_-)$. To prove that $W^{\#}$ is precisely $W(\Omega_-)$, let u be an element of $W(\Omega_-)$ that is orthogonal to $W^{\#}$, or equivalently to $\mathcal{W}(\Omega_-)$; then, for any v in $\mathcal{W}(\Omega_-)$,

$$(u_1,v_1)+(u_2,v_2)+\left(\frac{\partial u_1}{\partial x_1},\frac{\partial v_1}{\partial x_1}\right)+\left(\frac{\partial u_1}{\partial x_2},\frac{\partial v_1}{\partial x_2}\right)+\left(\frac{\partial u_2}{\partial x_1},\frac{\partial v_2}{\partial x_1}\right)+\left(\frac{\partial u_2}{\partial x_2},\frac{\partial v_2}{\partial x_2}\right)=0 \tag{2.7}$$

As v belongs to $\mathcal{W}(\Omega_-)$, there is a C^∞ function φ such that

$$v_1=\frac{\partial\varphi}{\partial x_2},\qquad v_2=-\frac{\partial\varphi}{\partial x_1}, \tag{2.8}$$

v vanishes outside of a compact set of Ω_-. Conversely, any φ in $\mathcal{D}(\Omega_-)$ defines a v in $\mathcal{W}(\Omega_-)$, with the help of (2.4). Similarly, the theory of Beppo-Levi spaces shows that there exists ψ in $L^2_{\text{loc}}(\Omega_-)\subset\mathcal{S}'(\Omega_-)$ such that

$$u_1=\frac{\partial\psi}{\partial x_2},\qquad u_2=-\frac{\partial\psi}{\partial x_1}. \tag{2.9}$$

If we substitute u and v in (2.7) with their expressions in terms of φ and ψ, we obtain:

$$\begin{aligned}&\left(\frac{\partial\psi}{\partial x_2},\frac{\partial\varphi}{\partial x_2}\right)+\left(\frac{\partial\psi}{\partial x_1},\frac{\partial\varphi}{\partial x_1}\right)+2\left(\frac{\partial^2\psi}{\partial x_1\,\partial x_2},\frac{\partial^2\varphi}{\partial x_1\,\partial x_2}\right)+\left(\frac{\partial^2\psi}{\partial x_2^2},\frac{\partial^2\varphi}{\partial x_2^2}\right)+\left(\frac{\partial^2\psi}{\partial x_1^2},\frac{\partial^2\varphi}{\partial x_1^2}\right)\\&\quad=0,\quad\forall\varphi\in\mathcal{D}(\Omega_-).\end{aligned} \tag{2.10}$$

Therefore, in the sense of distributions, ψ satisfies the equation

$$-\Delta\psi+\Delta^2\psi=0 \text{ in } \Omega_-. \tag{2.11}$$

If we perform a Fourier transform in the tangential variable x_2, which is sent to k, it is not difficult to see that the general solution of (2.11) is of the form

$$\begin{aligned}\hat\psi(x_1,k)&=\alpha\exp(|k|x_1)+\beta\exp((1+|k|^2)^{1/2}x_1)+\gamma\exp(-|k|x_1)\\&\quad+\delta\exp(-(1+|k|^2)^{1/2}x_1).\end{aligned}$$

For $\hat\psi$ to be a temperate distribution, the coefficients γ and δ must vanish, because the corresponding modes increase exponentially at $x_1=-\infty$. Therefore, $\hat\psi$ is of the form

$$\hat\psi(x_1,k)=\alpha\exp(|k|x_1)+\beta\exp(x_1\sqrt{1+|k|^2}). \tag{2.12}$$

We will prove that α and β vanish identically. For this purpose, we need some regularity. We integrate $|k\hat{\psi}(x_1, k)|^2 = |\hat{u}_1|^2$ on Ω_-, and we obtain an estimate on α and β.

$$\int |k|^2\left\{\frac{|\alpha(k)|^2}{|k|}+\frac{|\beta(k)|^2}{\sqrt{1+|k|^2}}+4\,\mathrm{Re}\,\frac{\alpha(k)\overline{\beta(k)}}{|k|+\sqrt{1+|k|^2}}\right\}dk<+\infty. \tag{2.13}$$

We compute the smallest eigenvalue of the quadratic form that appears in (2.13). This shows that there is a positive weight $h(k)$ such that

$$\int |\hat{u}_1|^2\,dk\,dx_1 \geqq \int |k|^2 h(k)[|\alpha|^2+|\beta|^2](k)\,dk,$$

the weight h satisfies the estimates:

$$h(k) \simeq \tfrac{1}{2} \quad \text{in a neighborhood of } k=0$$

$$h(k) \simeq \frac{1}{16|k|^5} \quad \text{in a neighborhood of } |k|=+\infty.$$

Therefore,

$$ik\alpha \text{ and } ik\beta \text{ belong to the space } H^{-5/2}. \tag{2.14}$$

If we return to formulation (2.10), written in Fourier variables, we obtain the variational problem

$$-(|k|^2\hat{\psi}, \hat{\bar{\varphi}})+\left(\frac{\partial\hat{\psi}}{\partial x_1}, \frac{\partial\hat{\bar{\varphi}}}{\partial x_1}\right)+(|k|^4\hat{\psi}, \hat{\bar{\varphi}})+\left(|k|^2\frac{\partial\hat{\psi}}{\partial x_1}, \frac{\partial\hat{\bar{\varphi}}}{\partial x_1}\right)+\left(\frac{\partial^2\hat{\psi}}{\partial x_1^2}, \frac{\partial^2\hat{\bar{\varphi}}}{\partial x_1^2}\right)=0 \quad \forall\varphi\in\mathscr{D}(\Omega_-). \tag{2.15}$$

According to (2.14) and (2.12), $\partial\hat{\psi}/\partial x_1$, $\partial^2\hat{\psi}/\partial x_1^2$ and $\partial^3\hat{\psi}/\partial x_1^3$ have a trace on Γ, even if it is in a very weak sense. Accordingly, we can perform several integrations by parts on (2.15), and we obtain the relations

$$\left.\left[\frac{\partial\hat{\psi}}{\partial x_1}+|k|^2\frac{\partial\hat{\psi}}{\partial x_1}-\frac{\partial^3\hat{\psi}}{\partial x_1^3}\right]\right|_\Gamma=0$$

$$\left.\left[\frac{\partial^2\hat{\psi}}{\partial x_1^2}\right]\right|_\Gamma=0,$$

substituting (2.12), we obtain a linear system:

$$|k|(1+|k|^2)\alpha+|k|^2\sqrt{1+|k|^2}\beta=0$$

$$|k|^2\alpha+(1+|k|^2)\beta=0.$$

This system is not singular, thus α and β vanish identically, and so does the potential ψ thanks to (2.12). This proves that u is zero, and that the orthogonal of the closure of $\mathscr{W}(\Omega_-)$ in $W(\Omega_-)$ is zero. □

We have an analogous result for the space $H(\Omega_-)$.

PROPOSITION 2.3. *The closure of* $\mathscr{W}(\Omega_-)$ *in* $\mathbf{L}^2(\Omega_-)$ *is precisely* $H(\Omega_-)$.

Proof. The technique of proof is absolutely identical to the technique employed for the previous proposition. It is in fact easier; the main steps begins with an analogue of equation (2.11):

$$\Delta\psi=0 \quad \text{in } \Omega_-.$$

Then the Fourier transform of the potential y is of the form

$$\hat{\psi}(x_1, k)=\alpha\exp(|k|x_1),$$

and it is very easy to prove that $\alpha\sqrt{\gamma|k|}$ is square integrable; the remainder of the proof is left to the reader. □

Let us consider now the trace spaces of $W(\Omega_-)$ and $H(\Omega_-)$. The main result we need is as follows.

PROPOSITION 2.4. *For any* $u=(u_1, u_2)^T$ *in* $W(\Omega_-)$, *the trace of* u_1 *on* Γ *is in* $H^{1/2}(\Gamma)$ *and satisfies*

$$\int_{\mathbb{R}} |\hat{u}_1(0, k)|^2 \sqrt{|k|^2+\frac{1}{|k|^2}}\, dk<+\infty. \tag{2.16}$$

If $W^{1/2}$ *is the space of functions that satisfy* (2.16), *then the trace mapping* $u \to u_1|_\Gamma$ *from* $W(\Omega_-)$ *to* $W^{1/2}$ *is onto.*

Proof. Let u belong to $\mathcal{W}(\Omega_-)$; we may write

$$\frac{\partial}{\partial x_1}\left(\frac{|\hat{u}_1(x_1, k)|^2}{|k|}\right)=\frac{2}{|k|}\operatorname{Re}\left(u_1(x_1, k)\,\overline{\frac{\partial \hat{u}_1(x_1, k)}{\partial x_1}}\right)$$

$$=\frac{2}{|k|}\operatorname{Re}\,(u_1(x_1, k)\; ik\,\overline{\hat{u}_2(x_1, k)}),$$

since u is divergence free. The function u vanishes for x_1 small enough, because the support of u is compact. Thus we obtain

$$\int \frac{|\hat{u}_1(x_1, k)|^2}{|k|}\, dk \leqq \|u\|^2. \tag{2.17}$$

The trace space of $W(\Omega_-)$ is included in $\mathbf{H}^{1/2}(\Gamma)$; together with (2.17), we obtain the first statement of the proposition, because $\mathcal{W}(\Omega_-)$ is dense in $W(\Omega_-)$. To see that the trace mapping is onto, take the orthogonal of the image of $W(\Omega_-)$ in $W^{1/2}$ by the normal trace mapping. For all φ in $\mathcal{D}(\Omega_-)$, we have

$$\operatorname{Re}\left(\int_{\mathbb{R}}\left(|k|^2+\frac{1}{|k|^2}\right)^{1/2} \hat{u}_1 ik\,\bar{\hat{\varphi}}\, dk\right)=0,$$

and this shows immediately that the mapping is onto. □

From the proof of Proposition 2.4, we deduce

COROLLARY 2.5. *For any* u *in* $H(\Omega_-)$, *the trace of* u *on* Γ *exists and satisfies*

$$\int_{\mathbb{R}} \frac{|\hat{u}_1(x_1, k)|^2}{|k|}\, dk<+\infty.$$

Moreover, the expression

$$s(u, v)=(u, v)+\int_{\mathbb{R}} \frac{\hat{u}_1(x_1, k)\overline{\hat{v}_1(x_1, k)}}{|k|}\, dk \tag{2.18}$$

is a scalar product on $H(\Omega_-)$, *which is equivalent to the scalar product* $(\cdot, \cdot)$ *induced by* $L^2(\Omega_-)$.

Proof. It is clear that

$$s(u, u) \geqq (\|u\|_0)^2;$$

the inequality

$$s(u, u) \leqq 2(\|u\|_0)^2$$

follows from (2.17). □

The dual space of $W^{1/2}$ will be denoted $W^{-1/2}$ and is the space of functions which satisfy

(2.19) $$u \in W^{-1/2} \Leftrightarrow \int_{\mathbb{R}} |\hat{u}_1(0,k)|^2 \left(|k|^2 + \frac{1}{|k|^2}\right)^{1/2} dk < +\infty.$$

Remark 2.6. The space $W^{1/2}$ is included in $H^{1/2}$ because the weight $(|k|^2 + 1/|k|^2)^{1/2}$ satisfies the inequality

$$\left(|k|^2 + \frac{1}{|k|^2}\right)^{1/2} \geqq \frac{1+|k|^2}{2} \quad \forall k \in \mathbb{R}^*.$$

Dually, $W^{-1/2}$ contains $H^{-1/2}$.
This completes our review of divergence-free functional spaces.

2.3. Oseen system in full space. Consider the Navier-Stokes system in $\mathbb{R}^2 \times \mathbb{R}^+$:

(2.20) $$u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0; \ \nabla \cdot u = 0.$$

Here ∇ is the gradient operator, Δ is the Laplacian operator; $v \cdot \nabla$ denotes the differential operator $v_1\, \partial/\partial x_1 + v_2\, \partial/\partial x_2$.

We linearize this system around a constant state $a = (a_1, a_2)$, with a_1 positive, and we obtain a Stokes system with an advection term or Oseen system:

(2.21) $$u_t + (a \cdot \nabla)u - \nu \Delta u + \nabla p = 0;$$

(2.22) $$\nabla \cdot u = 0.$$

A differential operator $\mathscr{A}$ is defined by

(2.23) $$\mathscr{A}(u,p) = u_t + (a \cdot \nabla)u - \nu \Delta u + \nabla p.$$

For functional analysis reasons, it will be convenient to study the differential operator $\mathscr{A}_\mu$ defined, for $\mu > 0$, by

$(2.23)_\mu$ $$\mathscr{A}_\mu(u,p) = u_t + \mu u + (a \cdot \nabla)u - \nu \Delta u + \nabla p.$$

We define a bilinear form $\mathbf{a}$ on $\mathbf{H}^1(\Omega)$ by

(2.24) $$a(u,v) = ((\mathbf{a} \cdot \nabla)u, v) + \nu(\nabla u, \nabla v),$$

where

$$(\nabla u, \nabla v) = (\nabla u_1, \nabla v_1) + (\nabla u_2, \nabla v_2).$$

Clearly, $\mathbf{a}$ is continuous on $\mathbf{H}^1(\Omega)$; moreover, we have

$$((a \cdot \nabla)u, u) = \frac{1}{2} \int_\Omega a \cdot \nabla(|u|^2)\, dx = \frac{1}{2} \int_\Gamma a \cdot n |u|^2\, d\Gamma.$$

If $\Omega = \mathbb{R}^2$, this expression vanishes; if $\Omega = \Omega_-$, this expression is greater than or equal to zero. Therefore,

$$\mathbf{a}(u,u) \geqq \nu(\|\nabla u\|_0)^2;$$

there exists a positive constant α such that

(2.25) $$(\|u\|_0)^2 + \mathbf{a}(u,u) \geqq \alpha(\|u\|_1)^2, \quad \forall u \text{ in } \mathbf{H}^1(\Omega).$$

We will need a partially antisymmetrized form of $\mathbf{a}$, in order to uncouple the boundary part of Green's formula; let

(2.26) $$\tilde{\mathbf{a}}(u,v) = \tfrac{1}{2}[((a \cdot \nabla)u, v) - ((a \cdot \nabla)v, u)] + \nu(\nabla u, \nabla v).$$

As a is constant, $\mathbf{a}$ and $\tilde{\mathbf{a}}$ can differ only by a boundary term; if $\Omega=\mathbb{R}^2$, there is no such term; if $\Omega=\Omega_-$,

$$\tilde{\mathbf{a}}(u,v)-\mathbf{a}(u,v)=-\frac{1}{2}\int_\Gamma a_1 u\cdot v\,d\Gamma. \tag{2.27}$$

In particular, in the full space case,

$$\mathbf{a}(u,u)=\tilde{\mathbf{a}}(u,u)=\nu\int|\nabla u|^2\,dx. \tag{2.28}$$

Consider now the linearized Navier–Stokes system in the plane:

$$\mathcal{A}_\mu(u,p)=f \quad \text{in } \mathbb{R}^2\times\mathbb{R}^+; \tag{2.29}$$

$$\nabla\cdot u=0 \quad \text{in } \mathbb{R}^2\times\mathbb{R}^+; \tag{2.30}$$

$$u(\cdot,0)=u^0 \quad \text{in } \mathbb{R}^2. \tag{2.31}$$

This system is well posed, if we take u^0 and f in adequate functional spaces.

PROPOSITION 2.7. *For all u^0 in $H(\mathbb{R}^2)$, and all f in $L^2(0,\infty;\,L^2(\mathbb{R}^2))$, and for all nonnegative μ, there exists a unique u and a p unique up to an additive constant such that* (2.29)–(2.31) *hold and*

$$u\in L^\infty_{\text{loc}}([0,\infty);\,H(\mathbb{R}^2)),$$
$$\nabla u\in L^2_{\text{loc}}([0,\infty);\,\mathbf{L}^2(\mathbb{R}^2)),$$
$$u_t\in L^2_{\text{loc}}([0,\infty);\,V'(\mathbb{R}^2)),$$
$$p=\frac{\partial P}{\partial t},\ P\in L^2_{\text{loc}}([0,\infty);\,BL(H^{-1}(\mathbb{R}^2))).$$

Here $V'(\mathbb{R}^2)$ is the dual of $V(\mathbb{R}^2)$ and $BL(H^1(\mathbb{R}^2))$ is the Beppo-Levi space of functions whose gradient is in $H^1(\mathbb{R})$. The function u belongs to $L^p_{\text{loc}}([0,\infty);X)$ if any restriction of u to a finite time interval $[0,T]$ belongs to $L^p([0,T];X)$.

Moreover, for any strictly positive μ, the following global estimates hold:

$$u\in L^\infty([0,\infty);\,H(\mathbb{R}^2)),$$
$$\nabla u\in L^2([0,\infty);\,L^2(\mathbb{R}^2)),$$
$$u_t\in L^2([0,\infty);\,V'(\mathbb{R}^2)),$$
$$p=\frac{\partial P}{\partial t},\ P\in L^2([0,\infty);\,BL(H^{-1}(\mathbb{R}^2))).$$

Remark 2.8. If Ω were a bounded open set with smooth enough boundary, it would be an exercise to extend the existence proof in [28] for the Stokes problem to the present situation. Though the forthcoming proof is quite classical, it does not appear to be written with all the necessary details in the literature of which we know.

Proof. We know from [24] that for every positive and finite T, there exists a unique u in $L^2(0,T;\,V(\mathbb{R}^2))$ with $\partial u/\partial t$ in $L^2(0,T;\,V'(\mathbb{R}^2))$ such that

$$(u_t,v)+\mathbf{a}(u,v)+\mu(u,v)=(f,v)\quad\forall v\in V(\mathbb{R}^2); \tag{2.32$_\mu$}$$

$$u(0)=u^0. \tag{2.33}$$

Let us first consider the case $\mu=0$. Relation (2.28) implies an energy estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2+\nu\|\nabla u\|_0^2\leqq\|f(t)\|_0\|u\|_0.$$

From this inequality we make a classical Gronwall estimate and deduce the estimates on u and ∇u, using the coerciveness; the variational formulation $(2.32)_\mu$ gives the estimate on u_t. Precisely, we get the following estimates

$$\|u(t)\|_0 \leqq \|u^0\|_0 + \sqrt{t}\|f\|_{L^2([0,t]\times\mathbb{R}^2)},$$

$$\int_0^t (\|\nabla u\|_0)^2\, ds \leqq C(\|f\|_{L^2([0,t]\times\mathbb{R}^2)})(\|u^0\|_0 + \|f\|_{L^2([0,t]\times\mathbb{R}^2)})(1+t).$$

These two estimates show polynomial growth in time. In order to have an estimate on p, we proceed as in [28]. Let

$$U(t) = \int_0^t u(s)\, ds, \qquad F(t) = \int_0^t f(s)\, ds.$$

Then ∇U belongs to $L^2(0, T; L^2(\mathbb{R}^2))$. If we integrate (2.21) in time, we obtain

$$(u(t) - u^0 + (a\cdot\nabla)U - \nu\Delta U, v) = (F, v) \quad \forall v \text{ in } V(\mathbb{R}^2).$$

The expression $u(t) - u^0 + (a\cdot\nabla)U - \nu\Delta U - F$ is orthogonal to divergence-free vectors, and belongs to $L^\infty(0, T; \mathbf{H}(\mathbb{R}^2)) + L^2(0, T; \mathbf{H}^{-1}(\mathbb{R}^2))$, which is included in $L^2(0, T; \mathbf{H}^{-1}(\mathbb{R}^2))$. Therefore, there exists P in $L^2(0, T; BL(H^{-1}(\mathbb{R}^2))$ such that

$$u(t) - u^0 + (a\cdot\nabla)U - \nu\Delta U + \nabla P = F.$$

If we define $p = \partial P/\partial t$, we obtain the announced estimate.

For the last statement, we observe that (u, p) is a solution of $(2.32)_0$ if and only if $(w, q) = (ue^{-\mu t}, pe^{-\mu t})$ is solution of $(2.32)_\mu$.

Thus, the polynomial growth estimates for u ensure the global estimates for w. □

There is a regularity result which will be useful in what follows; denote

$$H^\infty(\mathbb{R}^2) = \bigcap_{m\geqq 0} H^m(\mathbb{R}^2); \tag{2.34}$$

this is a Frechet space, with an obvious topology. The regularity result is as follows:

LEMMA 2.9. *Let u^0 belong to $H^\infty(\mathbb{R}^2)$, and f to $C^\infty(\mathbb{R}^+; H^\infty(\mathbb{R}^2))$. Then, the solution (u, p) of* (2.29)-(2.31) *belongs to $C^\infty(\mathbb{R}^+; H^\infty(\mathbb{R}^2))$.*

Proof. The spatial derivatives of u are divergence free, and satisfy (2.24); if we apply the estimates of Proposition 2.7 to the differentiated equation, we obtain the desired estimates. In order to differentiate in time, we check that the time derivative $u_t(., t)$ is in $H(\mathbb{R}^2)$; from (2.29), $u_t(., t)$ is the projection of $\nu\Delta u - (a\cdot\nabla)u$ onto $H(\mathbb{R}^2)$; u_t satisfies (2.29), (2.30), and thus Proposition 2.6 is applicable. By induction,

$$u \in H^m(0, T; H^m(\mathbb{R}^2)), \quad \forall\ m \text{ in } \mathbb{N}.$$

Leaving all details to the reader, this ends the proof. □

Remark 2.10. For μ strictly positive, it is possible to prove that if u^0 belongs to $H^\infty(\mathbb{R}^2)$, and f belongs to $\mathcal{S}([0,\infty); H^\infty(\mathbb{R}^2))$, then u belongs to $\mathcal{S}([0,\infty); H^\infty(\mathbb{R}^2))$. This result will be proved indeed for the case of the half-plane in the next section. The reader is referred to Proposition 3.1, whose proof can be completely copied to obtain this result. In any case, it is a question of estimating solutions of linear equations with a nice exponentially decreasing behavior, and the proof is an easy application of semigroup theory.

Remark 2.11. The solution of (2.29)-(2.31) does not decrease fast at infinity in space, in general; the trouble is with the pressure. Assume that f vanishes for large x; by taking the divergence of (2.29), in the smooth case, one can see that $\Delta p = \nabla\cdot f$, so that the pressure is harmonic; if p decreased rapidly at infinity in space to a constant, it would be identically equal to this constant (see Appendix B for a proof of this result); this is not a general situation. If ∇p decays like some power of x, at infinity, then, once ∇p is known, u is essentially a solution of the heat equation with source $\nabla p+f$, and it cannot, in general, decrease rapidly at infinity in space.

2.4. On specific trace results for the solution of Oseen system. Consider the solution u of (2.29)-(2.31). Extend u and p by 0 for $t \leqq 0$, and denote the extended functions still by u and p. Then,

$$\mathcal{A}_\mu(u, p) = u^0 \otimes \delta^t + f \text{ in } \mathbb{R}^2 \times \mathbb{R}; \ \nabla \cdot u = 0. \tag{2.35}$$

We cannot expect that u restricted to $\sum = \{(0, x_1, t)/(x_1, t) \in \mathbb{R}^2\}$ will be smooth in time, even if we assume that

(2.36) The support of u^0 is compact and included in Ω_-,

(2.37) The support of f is in the product of a compact subset of Ω_- with $\mathbb{R}_+$.

Remark 2.12. We explain why smoothness in time cannot be expected in general, and give sufficient conditions to have it. Denote by Π the projection in $\mathbf{L}^2(\mathbb{R}^2)$ onto $H(\mathbb{R}^2)$; assume that u^0 is smooth enough for the foregoing computations. Then, interpreting the pressure as a Lagrange multiplier, we can take a limit as t decreases to zero:

$$u_t(\cdot, 0^+) = \Pi f + u^0 \otimes \delta^t - (a \cdot \nabla) u^0 + \nu \Delta u^0 - \mu u^0.$$

In this relation all the terms containing u^0 vanish on $\sum \cap \{t=0\}$, thanks to assumption (2.36). Nevertheless, there is no reason why Πf should vanish there, since Π is not a local operator. Thus, in order to have some smoothness in time, we should ask that f, and a number of its time derivatives vanish at time 0. This assumption is not reasonable, and we shall not make it because it turns out that we are interested in less than the regularity of u.

More precisely, let $\sigma = \operatorname{sgn}(k)$ and let $\mathcal{H}$ denote the Hilbert transform on Γ [27, Chap. V and VI]:

$$(\mathcal{H} u)^\wedge(k) = -i\sigma \hat{u}(k). \tag{2.38}$$

In physical variables, the Hilbert transform is defined as the convolution with the principal value *v.p.* $(1/\pi x_2)$. We shall see later that we are interested only in the trace of $u_1 + \mathcal{H} u_2$ on Σ. It turns out that this trace is very regular in t and x_2, under the support conditions (2.36), (2.37).

A sequence of lemmas will describe the precise regularity of the trace of $u_1 + \mathcal{H} u_2$.

LEMMA 2.13. *Let u_0 belong to $\mathbf{H}(\mathbb{R}^2)$, and f to $L^2(0, \infty; \mathbf{H}(\mathbb{R}^2))$, and let u be the solution of* (2.29)-(2.31). *Assume* (2.36) *and* (2.37). *Let*

$$z = u_1 + \mathcal{H} u_2. \tag{2.39}$$

Then z belongs to $L^2_{\text{loc}}([0, \infty); H^1(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}([0, \infty); L^2(\mathbb{R}^2))$ and satisfies a heat equation of the form

$$z_t + \mu u + a \cdot \nabla u - \nu \Delta u = g, \tag{2.40}$$

with an initial condition

$$z(\cdot, 0) = u_1(\cdot, 0) + \mathcal{H} u_2(\cdot, 0) \tag{2.41}$$

belonging to $L^2(\mathbb{R}^2)$. Here, g has support in $(-\infty, -X) \times \mathbb{R} \times \mathbb{R}^+$, and X is some strictly positive number.

Proof. The function z is well defined as a function of x_1, x_2, and t, because $\mathscr{H}$ is an isometry from $L^2(\mathbb{R})$ to itself. Moreover, z belongs to the spaces mentioned in the lemma, because $\mathscr{H}$ commutes with the differentiations, so that if w is in $H^1(\mathbb{R}^2)$, so is $\mathscr{H}w$. If we compute the right-hand side of (2.40) in the sense of distributions, we obtain

$$g = f_1 + \mathscr{H}f_2 - \frac{\partial p}{\partial x_1} - \mathscr{H}\frac{\partial p}{\partial x_2}.$$

It remains to show that $\partial p/\partial x_1 + \mathscr{H}(\partial p/\partial x_2)$ has its support in $\{x_1 \leqq -X\}$, because assumption (2.37) shows that f has its support in this set. If we take the divergence of (2.29) in the sense of distributions, we have

$$\Delta p = \nabla \cdot f.$$

Therefore, p is harmonic in the region $\{x_1 \geqq -X\} \times \mathbb{R} \times \mathbb{R}^+$. By partial Fourier transform in x_2,

$$\frac{\partial^2 \hat{p}}{\partial x_1^2} - |k|^2 \hat{p} = 0.$$

As p is temperate in x_1, x_2, p is necessarily of the form, for $x_1 \geqq -X$,

$$\hat{p} = \hat{p}(0, k, .)\, e^{-|k|x_1},$$

and therefore,

$$\frac{\partial \hat{p}}{\partial x_1} + \widehat{\left(\mathscr{H}\frac{\partial p}{\partial x_2}\right)} = (-|k| + ik(-i\sigma))\hat{p} = 0 \text{ for } x_1 \geqq -X.$$

This proves that the support of g is indeed in the region $\{x_1 \leqq -X\}$. □

LEMMA 2.14. *Let w be a solution of*

$$w_t + a_2 \partial w/\partial x_2 - \nu \Delta w = g \quad \text{for } x \in \mathbb{R}^2, \quad t \in \mathbb{R},$$
$$w = 0 \quad \text{for } t \leqq T_1,$$

where the support of the distribution g is included in $(-\infty, -X] \times \mathbb{R} \times \mathbb{R}^+$. If g belongs to $H^s(\mathbb{R}^2 \times \mathbb{R})$, then the trace of w on $\Sigma = \{0\} \times \mathbb{R} \times \mathbb{R}$ belongs to $H^\infty(\mathbb{R} \times [0, T])$, for all T.

Proof. Among the many possible ways of proving this result, we choose the one which is the closest to the spirit of this article; namely, we consider the problem with respect to the variable x_1, instead of a problem in time. We will perform a Fourier transform in time and in the partial space variable x_2 and perform a Fourier analysis of the ordinary differential equation obtained in this fashion. For a correct argument, we multiply w by $e^{-\mu t}$, where μ is a strictly positive number. Then $v = w\, e^{-\mu t}$ satisfies the equation

$$v_t + \mu v + a_2 \frac{\partial v}{\partial x_2} - \Delta v = g\, e^{-\mu t} = h, \tag{2.42}$$

where h has the same support properties and smoothness properties as g. This amounts to performing a Fourier–Laplace transform in time instead of a Fourier transform. If ω denotes the dual variable of t, and k the dual variable of x_2, the partial Fourier transform of (2.42) is

$$(i\omega + \mu + ia_2 k + \nu|k|^2)\hat{v} - \nu \frac{\partial^2 \hat{v}}{\partial x_1^2} = \hat{h}. \tag{2.43}$$

Let ρ be the root of $\nu\rho^2-(i(\omega+a_2k)+\mu+\nu|k|^2)=0$ which has positive real part. An elementary computation shows that the unique solution of (2.43) which is temperate is given by

$$\hat{v}(x_1,k,\omega)=-\frac{1}{2\rho\nu}\left[2\int_{-\infty}^{x_1}\operatorname{sh}(\rho(y-x_1))\hat{h}(y,k,\omega)\,dy\right]$$
$$+\int_{-\infty}^{\infty}\exp(\rho(x_1-y))\hat{h}(y,k,\omega)\,dy.$$

Thus, we can write

$$\hat{v}(0,k,\omega)=-\frac{1}{2\rho\nu}\left[2\int_{-\infty}^{-X}\operatorname{sh}(\rho y)\hat{h}(y,k,\omega)\,dy+\int_{-\infty}^{-X}\exp(-\rho y)\hat{h}(y,k,\omega)\,dy\right]$$
$$=\frac{1}{2\rho\nu}\int_{-\infty}^{X}e^{\rho y}\hat{h}(y,k,\omega)\,dy.$$

We can see that

$$\operatorname{Re}(\rho)\geqq C(1+|k|+\sqrt{|\omega|}),\tag{2.44}$$

for some constant C strictly positive, and thus, it is possible to estimate

$$\int(1+|k|^2+|\omega|^2)^m|\hat{v}(0,k,\omega)^2\,dk\,d\omega$$
$$\leqq\int(1+|k|^2+|\omega|^2)^m\int_{-\infty}^{-X}|e^{\rho y}\hat{h}(x_1,k,\omega)\,dy|^2\,dk\,d\omega$$
$$\leqq\int(1+|k|^2+|\omega|^2)^m\left\{\int_{-\infty}^{X}|e^{2\rho y}|\,dy\right\}\left\{\int|\hat{h}(x,k,\omega)|^2\,dx\right\}dk\,d\omega.$$

If we first strengthen somewhat the hypotheses by assuming that, for some n,

$$\int(1+|k|^2+|\omega|^2)^n|\hat{h}(x,k,\omega)|^2\,dx\,dk\,d\omega<\infty,$$

then, the result we look for can be easily deduced:

$$\int_{-\infty}^{-X}|e^{2\rho y}|\,dy=\frac{e^{-2X\operatorname{Re}(\rho)}}{2\operatorname{Re}(\rho)},$$

and thanks to (2.24), this quantity is dominated by all powers of k and ω, and the result is proved. To get rid of the extra assumption we made, we observe that an h in $H^s(\mathbb{R}^2\times\mathbb{R})$ is a finite sum of x_1 derivatives of some functions h_j, and the h_j can be taken with the same support property as h, and each h_j satisfies

$$\int_{-\infty}^{-X}(1+|k|^2+|\omega|^2)^n|\hat{h}_j(x,k,\omega)|^2\,dx\,dk\,d\omega<\infty.$$

We treat the terms

$$\int_{-\infty}^{-X}e^{\rho y}\frac{\partial^j\hat{h}_j(y,k,\omega)}{\partial y^j}\,dy$$

by performing an integration by parts, which will amount to a multiplication by some extra powers of ρ; these powers will be nevertheless dominated by exp $(2X \text{ Re }(\rho))$, and

$$\int (1+|k|^2+|\omega|^2)^m \left| \int_{-\infty}^{-X} e^{\rho y} \frac{\partial^j \hat{h}_j(x_1, k, \omega)}{\partial y^j} dy \right|^2 dk\, d\omega$$

is finite for all real m.

Going back to w, we obtain the required result. □

To conclude this sequence of results, we can state now the

LEMMA 2.15. *Let z be as in Lemma* 2.13. *Then, the trace of z on Σ belongs to $H^\infty(\Sigma)$.*

Proof. Consider the new variables

$$t'=t,\ x_1'=x_1-a_1t,\ x_2'=x_2.$$

In these new variables, the equation satisfied by z becomes

$$z_{t'}+a_2\frac{\partial z}{\partial x_2}-\nu\Delta' z=g,$$

and the support of g is included in the set

$$\{(x', t')/t'\geqq 0 \text{ and } x_1'+a_1t'\leqq -X\}.$$

As a_1 is strictly positive, a_1t' is less than or equal to zero, and we are in the case of Lemma 2.14. □

3. Analysis of the transparent boundary condition.

3.1. Introduction. Let (u, p) be the solution of (2.29)-(2.31) with initial data satisfying (2.36)-(2.37). The normal constraint σ_n is defined by

$$\sigma_n=(\sigma_{11}, \sigma_{12});\quad \sigma_{11}=\nu\frac{\partial u_1}{\partial x_1}-p;\quad \sigma_{12}=\nu\frac{\partial u_2}{\partial x_1}. \tag{3.1}$$

We will show in this section that the restrictions to $\Sigma=\{0\}\times\mathbb{R}\times\mathbb{R}^+$ of u and the normal constraint σ_n satisfy a linear pseudo-differential relation. This relation will be computed by Fourier techniques, working on the problem

$$\begin{cases} a(u,p)=0 & \text{in } \Omega_+\times\mathbb{R}; \\ \nabla\cdot u=0 & \text{in } \Omega_+\times\mathbb{R}; \\ u(0,x_2,t)=g(x_2,t) & \text{in } \mathbb{R}\times\mathbb{R}. \end{cases} \tag{3.2}$$

We assume here that g is given in $\mathcal{S}(\mathbb{R}; \mathbf{H}^\infty(\mathbb{R}))$. The choice of $\mathbf{H}^\infty(\mathbb{R})$ is justified by Remark 2.11. On the other hand, there is no such constraint in time, and it is permissible to have a solution with values in $\mathbf{H}^m(\mathbb{R})$ which decreases fast in time for all nonnegative m. Moreover, we ask that

$$k\to\frac{1}{\sqrt{|k|}}\hat{g}(k,\omega)\in\mathbf{L}^2(\mathbb{R}),\quad \forall\ \omega.$$

Under these assumptions, we will show at Proposition 3.1 that for all such g, (3.2) admits a unique solution u in $C^\infty(\Omega_+\times\mathbb{R})$.

The mapping

$$\mathscr{E}: g\to\sigma_n$$

can be completely described in Fourier variables:

$$(\mathscr{E}g)\hat{}\,(k,\omega)=E(k,\omega)\hat{g}(k,\omega), \tag{3.3}$$

where E is a two-by-two matrix that will be given explicitly in terms of k and ω at Corollary 3.3.

From this operator $\mathscr{E}$, we will define another operator $\mathscr{L}$, which is a nice pseudo-differential operator, such that the restriction of the solution of Oseen system to the left half-plane Ω_- satisfies the variational equation

$$s(u_t, v) + \tilde{\mathbf{a}}(u, v) + \langle \mathscr{L}g, v|\Sigma\rangle = 0, \tag{3.4}$$

where s is the scalar product defined at (2.18) and $\mathscr{L}$ is defined with the help of an explicit matrix L by

$$(\mathscr{L}g)^\wedge(k, \omega) = L(k, \omega)\hat{g}(k, \omega).$$

The next step is to study the properties of $\mathscr{L}$. This operator is causal, which means that if g vanishes for $t \leqq 0$, so does $\mathscr{L}g$. This property plus some estimates will enable us to extend $\mathscr{L}$ to much larger spaces.

In view of well-posedness results, we shall prove that $\mathscr{L}$ has some useful positivity properties; in particular, the symmetrized matrix $(L + L^*)/2$ is positive semidefinite.

With this detailed study of $\mathscr{L}$, we prove an existence and uniqueness result for the solution of

$$\mathscr{A}(u, p) = 0 \quad \text{in } \Omega_- \times \mathbb{R}, \tag{3.5}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega_- \times \mathbb{R}, \tag{3.6}$$

$$u(x_1, x_2, 0) = u^0(x_1, x_2) \quad \text{in } \Omega_-, \tag{3.7}$$

$$\sigma_n|_\Sigma = \mathscr{E}(u|_\Sigma). \tag{3.8}$$

This problem is written in variational form, and, with very smooth data, it admits a solution that is simply the restriction of the full-space problem with initial data extended by zero in Ω_+; the uniqueness will be a consequence of the positivity of the operators. Once we have the uniqueness, we can extend the class of solutions for which we have a solution, and, then, conditions (3.8) may be called transparent.

Most of the time, it will be convenient to replace (3.5) by $(3.5)_\mu$ where

$$\mathscr{A}_\mu(u, p) = \mathscr{A}(u, p) + \mu u; \tag*{$(3.5)_\mu$}$$

here, μ is a positive number. This amounts to considering the system solved by $(u, p)e^{-\mu t}$, or, in other words, to extend the frequency ω to the half plane $Im(\omega) < 0$. This is permissible because we work with causal operators, and we can apply the Paley-Wiener-Schwarz theorem.

3.2. The boundary problem for Oseen system. In this section, we consider the problem

$$\begin{cases} \mathscr{A}_\mu(u, p) = 0 & \text{in } \Omega_+ \times \mathbb{R}, \\ \nabla \cdot u = 0 & \text{in } \Omega_+ \times \mathbb{R}, \\ u(0, x_2, t) = g(x_2, t) & \text{in } \mathbb{R} \times \mathbb{R}, \end{cases} \tag{3.9}$$

where

$$\mathscr{A}_\mu(u, p) = \partial u/\partial t + \mu u + (a \cdot \nabla)u - \nu \Delta u + \nabla p \tag{3.10}$$

and

$$\mu > 0.$$

If g vanishes for $t \leqq t_0$, and if (u, p) is a solution of (3.9) for $\mu > 0$, then $(ue^{\mu t}, pe^{\mu t})$ is a solution of (3.9) for $\mu = 0$, with data $g_\mu(x_2, t) = g(x_2, t)\, e^{\mu t}$.

We first prove a result of existence and regularity.

PROPOSITION 3.1. *Let Z be the subspace of $\mathcal{S}'(\mathbb{R})$ defined by*

$$g \in Z \text{ iff } g \in H^\infty(\mathbb{R}) \text{ and } \mathcal{F}^{-1}\left(\frac{\hat{g}_1(k)}{\sqrt{|k|}}\right) \in H^\infty(\mathbb{R}). \tag{3.11}$$

Assume that

$$g \in \mathcal{S}(\mathbb{R}; Z). \tag{3.12}$$

Then (3.9) *possesses a unique solution* (u, p) *such that,*

$$u \in L^\infty(\mathbb{R}; H(\Omega_+)), \tag{3.13}$$

$$\nabla u \in L^2(\mathbb{R}; \mathbf{L}^2(\Omega_+)), \tag{3.14}$$

$$\frac{\partial u}{\partial t} \in L^2(\mathbb{R}; W'(\Omega_+)), \tag{3.15}$$

$$\nabla p \in L^\infty(\mathbb{R}; \mathbf{L}^2(\Omega_+)) + L^2(\mathbb{R}; \mathbf{H}^{-1}(\Omega_+)). \tag{3.16}$$

Moreover, u is infinitely differentiable, and if $\mu > 0$,

$$u, p \in \mathcal{S}(\mathbb{R}; \mathbf{H}^\infty(\Omega_+)). \tag{3.17}$$

Proof. Let us first construct a function z such that

$$\begin{cases} z \in \mathcal{S}(\mathbb{R}; \mathbf{H}^\infty(\Omega_+)), \\ \nabla \cdot z = 0, \\ z|_\Sigma = g. \end{cases} \tag{3.18}$$

the function z will be a sum of two functions defined by different means. We first extend g_1; let ζ be defined by

$$\hat{\zeta}_1(x_1, k, \omega) = \hat{g}_1(k, \omega) \exp(-|k|x_1),$$
$$\hat{\zeta}_2(x_1, k, \omega) = -i\sigma \hat{g}_1(k, \omega) \exp(-|k|x_1),$$

where

$$\sigma = \text{sign}(k).$$

We have used the Hilbert transform, of symbol $-i\sigma$, mentioned in § 2, and defined at (2.38):

$$\zeta_2(x_1, \cdot, t) = \mathcal{H}\zeta_1(x_1, \cdot, t).$$

Moreover,

$$\nabla \cdot \zeta = 0; \ \zeta_1(0, x_2, t) = g_1(x_2, t).$$

Now, in order to compensate for the bad boundary condition of the second component, we define a function h by

$$h(x_1, x_2, t) = \psi(x_1)[g_2(x_2, t) - (\mathcal{H}g_1)(x_2, t)],$$

where ψ belongs to $\mathcal{D}(\Omega_+)$, $\psi(0) = 0$, $\psi'(0) = 1$, and finally, we let

$$z_1 = \zeta_1 - \frac{\partial h}{\partial x_2}; \qquad z_2 = \zeta_2 + \frac{\partial h}{\partial x_1}.$$

Clearly h satisfies the boundary conditions. If (3.12) holds, then an integration in x_1 shows that

$$\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|k|^2)^m |\hat{\zeta}_1(x_1, k, \omega)|^2 dx_1 dk\, d\omega$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|k|^2)^m (2|k|)^{-1} |\hat{g}_1(k, \omega)|^2\, dk\, d\omega < \infty.$$

Similarly,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \left| \frac{\partial^m \hat{\zeta}_1(x_1, k, \omega)}{\partial x_1^m} \right|^2 dx_1\, dk\, d\omega = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty |k|^{2m} |\hat{\zeta}_1(x_1, k, \omega)|^2\, dx_1\, dk\, d\omega$$
$$= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |k|^{2(m-1/2)} |\hat{g}_1(k, \omega)|^2\, dk\, d\omega < \infty.$$

This shows that ζ_1 belongs to $L^2(\mathbb{R}; H^m(\Omega_+))$, for all m. We look at time derivatives multiplied by polynomials in t, and we show by induction that ζ_1 belongs to $\mathcal{S}(\mathbb{R}; H^\infty(\Omega_+))$. A similar argument shows that ζ_2 belongs to $\mathcal{S}(\mathbb{R}; H^\infty(\Omega_+))$. By construction, g_1 and g_2 belong to $\mathcal{S}(\mathbb{R}; H^\infty(\mathbb{R}))$; the Hilbert transform in the variable x_2 leaves this space invariant; therefore, $\mathcal{H} g_1$ belongs to it, and so h and its gradient belong to $\mathcal{S}(\mathbb{R}; H^\infty(\Omega_+))$. The function z we obtain finally satisfies (3.18).

Let now

$$u = v + z;$$

then (u, p) solves (3.9) if and only if (v, p) solves

$$\begin{cases} \mathcal{A}_\mu(v, p) = -\mathcal{A}_\mu(z, 0) & \text{in } \Omega_+ \times \mathbb{R}, \\ \nabla \cdot v = 0 & \text{in } \Omega_+ \times \mathbb{R}, \\ v(0, x_2, t) = 0 & \text{in } \mathbb{R} \times \mathbb{R}. \end{cases} \tag{3.19}$$

The existence of a solution of (3.19) is obtained by many classical methods; for instance, define the generator A_0 of the Stokes semigroup (without advection) by

$$D(A_0) = \{u \in H_0(\Omega_+) / v \to \int \nabla u \cdot \nabla v\, dx \text{ is continuous}\}$$

and

$$(A_0 u, v)_{H_0} = \nu \int \nabla u \cdot \nabla v\, dx. \tag{3.20}$$

The operator A_0 is defined starting from a quadratic form on the Hilbert space H_0; therefore, according to Phillips' theorem, it generates a contraction semigroup in H_0. Its domain can be computed explicitly and is equal to

$$D(A_0) = \left\{ u \in H_0(\Omega_+) \cap \mathbf{H}^2(\Omega_+) \Big/ \frac{\partial u_2}{\partial x_1}(0, x_2) = 0 \right\}.$$

Let A be defined by

$$D(A) = D(A_0);$$

$$(Au, v)_{H_0} = \int [(a \cdot \nabla) u \cdot v + \nu \nabla u \cdot \nabla v]\, dx. \tag{3.21}$$

Then A is obtained from A_0 by adding a strongly relatively bounded perturbation to A_0; this is because we have the inequality for all ε

$$\|\nabla u\|_0 \leqq \varepsilon \|\Delta u\|_0 + C(\varepsilon)\|u\|_0.$$

Therefore, A generates a strongly continuous semigroup $\mathcal{T}(t)$; this semigroup turns out to satisfy the estimate,

$$\|\mathcal{T}(t)\| \leqq e^{-\mu t}, \text{ for all } t \geqq 0,$$

because, for all u in $D(A)$, $(Au, u) \geqq \mu(\|u\|_0)^2$. The solution of (3.20) is given by

$$v(\cdot, t) = \int_{-\infty}^{t} \mathcal{T}(t-s)(-\mathcal{A}_\mu(z, 0)(s))\, ds,$$

and it is clear that $v(\cdot, t)$ is bounded uniformly on $\mathbb{R}$, with values in H_0, and that, if μ is positive, it decreases fast to zero as t tends to $\pm\infty$. All the other estimates are easy to obtain by differentiation and semigroups estimates, and they are global on $\mathbb{R}$. Details are left to the reader. □

We now perform the Fourier analysis of (3.9). For simplicity of notation, the frequency variable ω can be real, or complex with negative imaginary part; if needed, it will be denoted $\tau = \omega - i\mu$ when this negative imaginary part is present. We need a few notations; the differential operator C is given by

$$Cf = -\nu \frac{\partial^2 f}{\partial x_1^2} + a_1 \frac{\partial f}{\partial x_1} + (\nu|k|^2 + i\tau + a_2 ik)f; \tag{3.22}$$

its associated characteristic polynomial is given by

$$P(k, \tau; \lambda) = -\nu\lambda^2 + a_1\lambda + \nu|k|^2 + i\tau + a_2 ik. \tag{3.23}$$

The discriminant of P is given by:

$$a_1^2 + 4\nu(i(\tau + a_2 k) + \nu|k|^2). \tag{3.24}$$

The real part of (3.24) is positive for all ω and k, and for all $\mu \geqq 0$, because we assumed $a_1 > 0$. We denote by ρ the determination of the square root of (3.24) with positive real part:

$$\rho^2 = a_1^2 + 4\nu(i(\tau + a_2 k) + \nu|k|^2), \text{ Re } \rho > 0. \tag{3.25}$$

Then, the roots of $P(k, \tau, \cdot)$ are given by

$$\lambda = \frac{a_1 - \rho}{2\nu}; \lambda' = \frac{a_1 + \rho}{2\nu},$$

and it is not difficult to check that

$$\text{Re } \lambda < 0 \text{ except if } k = \tau = 0. \tag{3.26}$$

Moreover, we have immediately the following important estimate; there exists a strictly positive constant γ such that

$$\frac{1}{\gamma}(1 + |k| + \sqrt{|\tau|}) \leqq |\rho| \leqq \gamma(1 + |k| + \sqrt{|\tau|}). \tag{3.27}$$

The first result pertaining to the Fourier analysis of (3.9) is as follows.

PROPOSITION 3.2. *Let (u, p) be the solution of* (3.9). *Then, there exist locally integrable functions α and β such that*

$$\hat{u}(x_1, k, \tau) = \exp(-|k|x_1)\alpha(k, \tau) + \exp(\lambda x_1)\beta(\alpha, \tau), \tag{3.28}$$

where

(3.29)
$$\begin{cases} \alpha_1 = \dfrac{\lambda \hat{g}_1 + ik\hat{g}_2}{\lambda + |k|}, \ \beta_1 = k\dfrac{\sigma\hat{g}_1 - i\hat{g}_2}{\lambda + |k|}, \\ \alpha_2 = -i\sigma\dfrac{\lambda \hat{g}_1 + ik\hat{g}_2}{\lambda + |k|}, \ \beta_2 = i\lambda\dfrac{\sigma\hat{g}_1 - i\hat{g}_2}{\lambda + |k|}. \end{cases}$$

Moreover, any couple of distributions (α', β') *which satisfies* (3.27) *is equal to* (α, β), *up to the addition of* $(1, -1)S$, *where S is an arbitrary distribution with values in* $\mathbb{R}^2$ *and support in* $\{0\}_{k,\omega}$.

Proof. We perform a partial Fourier transform on u and p; the transformed quantities are denoted $\hat{u} = (\hat{u}_1, \hat{u}_2)$ and $\hat{p}$; they satisfy a system of ordinary differential equations, with respect to the variable x_1, with k and τ as parameters; this system can be written

(3.30)
$$C\hat{u}_1 + \frac{\partial \hat{p}}{\partial x_1} = 0,$$

(3.31)
$$C\hat{u}_2 + ik\hat{p} = 0,$$

(3.32)
$$\frac{\partial \hat{u}_1}{\partial x_1} + ik\hat{u}_2 = 0.$$

If we eliminate the pressure from this system, by multiplying (3.30) by $-ik$, differentiating (3.31) with respect to x_1, and adding the two resulting inequalities, we obtain

(3.33)
$$C\left(-ik\hat{u}_1 + \frac{\partial \hat{u}_2}{\partial x_1}\right) = 0.$$

With the help of (3.32), we eliminate $\hat{u}_2$; then $\hat{u}_1$ satisfies

$$C\left(\frac{\partial^2 \hat{u}_1}{\partial x_1^2} - |k|^2 \hat{u}_1\right) = 0.$$

This is an ordinary differential equation of the fourth order in x_1 parameterized by k and τ; its general solution is of the form

$$\alpha_1(k, \tau) \exp(-|k|x_1) + \beta_1(k, \tau) \exp(\lambda x_1) + \gamma_1(k, \tau) \exp(|k|x_1) + \delta_1(k, \tau) \exp(\lambda' x_1).$$

From Proposition 3.1, u_1 belongs to $L^2(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$ if $\mu > 0$, and so does $\hat{u}_1$; thus, for almost every k and τ, $\hat{u}_1(\cdot, k, \tau)$ is square integrable. Therefore, γ_1 and δ_1 vanish for almost every k and τ; thus,

(3.34)
$$\hat{u}_1 = \alpha_1(k, \tau) \exp(-|k|x_1) + \beta_1(k, \tau) \exp(\lambda x_1).$$

Similarly, eliminating $\hat{u}_1$ from (3.33) and (3.34), we obtain

$$C\left(\frac{\partial^2 \hat{u}_2}{\partial x_1^2} - |k|^2 \hat{u}_2\right) = 0.$$

With the same argument as above,

(3.35)
$$\hat{u}_2 = \alpha_2(k, \tau) \exp(-|k|x_1) + \beta_2(k, \tau) \exp(\lambda x_1).$$

The divergence-free relation (3.32) implies immediately that

(3.36)
$$-|k|\alpha_1 + ik\alpha_2 = 0, \quad \lambda\beta_1 + ik\beta_2 = 0.$$

If we take into account the boundary conditions,

$$\hat{g}_1 = \alpha_1 + \beta_1, \quad \hat{g}_2 = \alpha_2 + \beta_2, \tag{3.37}$$

we will now express α_1, α_2, β_1, and β_2 in terms of $\hat{g}_1$ and $\hat{g}_2$. To obtain α_1 and β_1. we have to divide by $\lambda + |k|$, which vanishes only for $k = \tau = 0$. In order to obtain α_2 and β_2, we have to divide moreover by k. In order to obtain locally integrable functions α and β, we have to show that $\lambda / \lambda + |k|$ is bounded in a neighborhood of zero. But,

$$\lambda = -2 \frac{i(\tau + a_2 k) + |k|^2}{a_1 + \rho}$$

and

$$\lambda + |k| = -2 \frac{\tau + a_2 k}{a_1 + \rho + 2\nu |k|},$$

which shows immediately the desired estimate. Eliminating between (3.36) and (3.37), we can write

$$\alpha_1 = \frac{\lambda \hat{g}_1 + ik\hat{g}_2}{\lambda + |k|} = \frac{\lambda(\hat{g}_1 - i\sigma\hat{g}_2) + i\sigma(\lambda + |k|)}{\lambda + |k|},$$

and there is a locally integrable function α_1 which is almost everywhere equal to the expression given by the first of formulae (3.29). The second of these formulae gives a locally integrable β_1, because $\beta_1 = \hat{g}_1 - \alpha_1$. From the first formula of (3.36) divided by k, α_2 is locally integrable. From the second formula of (3.36) and the expression of β_1, a division by k gives the formula for β_2. If α' and β' are distributions which satisfy (3.28), they differ from a and β by distributions S and T with support in $\mathbb{R}_k \times \{0\}_\omega$. We must have

$$S \exp(-|k|x_1) + T \exp(\lambda x_1) = 0, \text{ for all } x_1.$$

This is possible only if S and T have their support in $\{0\}_{k,\omega}$, and $T + S = 0$. Therefore, we can say that α and β are respectively determined up to the addition of S and $-S$, with S a distribution with support in $\{0\}_{k,\omega}$. □

The most important consequence of the previous proposition is the following result on the operator, which assigns to the boundary value of u the normal constraint at the boundary.

COROLLARY 3.3. *Let (u, p) be the solution of* (3.9). *Let E be the two-by-two matrix given by*

$$E(k, \tau) = -\begin{pmatrix} (i\tau/|k|) + ia_2\sigma + \nu(|k| - \lambda) & i\sigma(\nu\lambda - a_1) \\ -i\sigma\nu\lambda & \nu(|k| - \lambda) \end{pmatrix}. \tag{3.38}$$

Then the normal constraint $(-p + \nu\, \partial u_1/\partial x_1,\ \nu\, \partial u_2/\partial x_1)|_\Sigma = \sigma_n$ is given by

$$\sigma_n = \mathscr{E} g,$$

or equivalently

$$\hat{\sigma}_n(k, \tau) = E(k, \tau)\hat{g}(k, \tau).$$

Proof. From (3.29), we compute $\partial \hat{u}_1/\partial x_1$ and $\partial \hat{u}_2/\partial x_1$ on the boundary:

$$\frac{\partial \hat{u}}{\partial x_1} = -|k|\alpha + \lambda\beta = -|k|\alpha + \lambda(\hat{g} - \alpha) = \lambda\hat{g} - \begin{pmatrix} \lambda\hat{g}_1 + ik\hat{g}_2 \\ -i\sigma(\lambda\hat{g}_1 + ik\hat{g}_2) \end{pmatrix}.$$

We observe that the distribution S does not contribute, because $-(|k|+\lambda)S=0$. The above formula proves that

$$\frac{\partial \hat{u}_1}{\partial x_1}=-ik\hat{g}_2, \tag{3.39}$$

$$\frac{\partial \hat{u}_2}{\partial x_1}=i\lambda\sigma\hat{g}_1+(\lambda-|k|)\hat{g}_2. \tag{3.40}$$

To obtain an expression for the pressure, we observe that, from (3.31), we can write

$$ik\hat{p}=-C\hat{u}_2=-P(k,\tau;-|k|)\alpha_2\exp(-|k|x_1)-P(k,\tau;\lambda)\beta_2\exp(\lambda x_1).$$

The second term of this expression vanishes by definition of λ; in the first term,

$$P(k,\tau;|k|)=-a_1|k|+i(\tau+a_2k).$$

Thus,

$$ik\hat{p}=\frac{i\sigma(\lambda\hat{g}_1+ik\hat{g}_2)(-a_1|k|+i(\tau+a_2k))}{\lambda+|k|}.$$

This expression can be simplified by algebraic manipulations: let $y=\lambda+|k|$; then

$$P(k,\tau,y-|k|)=0$$

so that

$$-\nu y^2+2\nu|k|y+a_1y-a_1|k|+i(\tau+a_2k)=0.$$

Hence,

$$y(-\nu y+2\nu|k|+a_1)=a_1|k|-i(\tau+a_2k),$$

that is

$$\frac{a_1|k|-i(\tau+a_2k)}{\lambda+|k|}=a_1+\nu|k|-\nu\lambda.$$

If we substitute this in the above expression of $ik\hat{p}$, we obtain

$$ik\hat{p}=-i\sigma(a_1+\nu|k|-\nu\lambda)(\lambda\hat{g}_1+|k|\hat{g}_2).$$

Using once again the equation which defines λ, we have

$$a_1\lambda+\nu|k|\lambda-\nu\lambda^2=\nu|k|\lambda-\nu|k|^2-i(\tau+a_2k),$$

and from here, we obtain

$$ik\hat{p}=ik\left[\hat{g}_1\left\{\nu(|k|-\lambda)+i\left(a_2\sigma+\frac{\tau}{|k|}\right)\right\}+\hat{g}_2i\sigma\{\nu(\lambda-|k|)-a_1\}\right].$$

If we divide both sides of this equation by k, we obtain, with the help of (3.39) the unique locally integrable function equal to $\hat{p}$; thus

$$-\nu\frac{\partial \hat{u}_1}{\partial x_1}+\hat{p}=\hat{g}_1\left\{(\nu(|k|-\lambda)+i\left(a_2\sigma+\frac{\tau}{|k|}\right)\right\}+\hat{g}_2i\sigma(\nu\lambda-a_1).$$

Putting together this last expression and (3.40), we obtain the matrix E. A distribution $\hat{p}$ which is solution of our equations is equal to the expression $\hat{g}_1\ \{\nu(|k|-\lambda)+i(a_2\sigma+|k|^{-1}\tau)\}+\hat{g}_2i\sigma\{\nu(\lambda-|k|)-a_1\}$ up to the addition of a distribution with support in $\{k=0\}$. This corresponds to the addition of a space independent constant to p, which is natural because p is defined only via its gradient. But, as our problem was obtained through linearization, around a constant velocity field and a constant pressure, this constant is equal to zero. □

3.2. Variational formulation for the problem with transparent condition. To obtain a tractable formulation for the Oseen problem and "transparent" boundary conditions, we will write a variational formulation. The existence of a solution of the variational problem will be ensured by simply taking the restriction of a full-space problem. A functional problem has to be settled in order to justify this formulation; a certain pseudo-differential operator $\mathscr{L}$ acts on the trace of u on Σ. Originally $\mathscr{L}$ is defined only on very smooth functions, and it has to be extended to a larger functional class. After this is done, we have a clean presentation of the problem with "transparent" condition. The quotes will be removed only after uniqueness is proved.

Let μ be strictly positive, let f satisfy the assumptions of Lemma 2.9, and let u^0 equal zero. According to Lemma 2.9, the solution (u, p) of (2.29)-(2.31) belongs to $C^\infty(\mathbb{R}^+; H^\infty(\mathbb{R}^2))$. Assume that f satisfies (2.37). We assume moreover that:

$$f \in \mathscr{S}((0, \infty); \mathbf{H}^\infty(\mathbb{R}^2)). \tag{3.41}$$

In particular, f vanishes of infinite order at $t=0$. Then, the solution of (2.29)-(2.31) extended by 0 for $t \leqq 0$ belongs to $C^\infty(\mathbb{R}; \mathbf{H}^\infty(\mathbb{R}^2))$. Arguing as in Proposition 3.1, u belongs to $\mathscr{S}(\mathbb{R}; \mathbf{H}^\infty(\mathbb{R}^2))$, and its trace on Σ belongs to $\mathscr{S}(\mathbb{R}; \mathbf{H}^\infty(\mathbb{R}))$. In the region $\Omega_+ \times \mathbb{R}$, u satisfies (3.9), with

$$g = u|_\Sigma.$$

If we still denote u the restriction of u to $\Omega_- \times \mathbb{R}$, and if we multiply the equation satisfied by u on $\Omega_- \times \mathbb{R}$ by an arbitrary v in $W(\Omega_-)$, and integrate, we obtain, thanks to Green's formula

$$(u_t, v) + \mu(u, v) + \tilde{\mathbf{a}}(u, v) + \int_{\mathbb{R}} \left(\frac{a_1}{2} g \cdot v|_\Sigma - \sigma_n \cdot v|_\Sigma \right) dx_2 = (f, v).$$

From Corollary 3.3, this relation can be written

$$(u_t, v) + \mu(u, v) + \tilde{\mathbf{a}}(u, v) + \int_{\mathbb{R}} \left[\left\{ \frac{a_1}{2} g - \mathscr{E} g \right\} \cdot v|_\Sigma \right] dx_2 = (f, v). \tag{3.42}$$

Let us write down the matrix $a_1(I/2) + E$:

$$\frac{a_1 I}{2} - E(k, \tau) = \begin{pmatrix} (a_1/2) + (i\tau/|k|) + ia_2\sigma + \nu(|k| - \lambda) & i\sigma(\nu\lambda - a_1) \\ -i\sigma\nu\lambda & (a_1/2) + \nu|k| - \lambda \end{pmatrix}. \tag{3.43}$$

We can decompose this matrix into the sum of two matrices:

$$\frac{a_1 I}{2} - E = K_1 + L,$$

where K_1 is given by

$$K_1 = \begin{pmatrix} 1/|k| & 0 \\ 0 & 0 \end{pmatrix} i\tau. \tag{3.44}$$

Let $\mathscr{K}_1$ be the operator of symbol K_1. From the definition (2.18) of the scalar product s, we can see that

$$(u_t, v) + \mu(u, v) + \langle \mathscr{K}_1 u, v \rangle = s(u_t, v) + \mu s(u, v). \tag{3.45}$$

The matrix L can be written as follows, recalling that $(a_1/2)-\nu\lambda=\rho/2$:

$$L=\begin{pmatrix} ia_2\sigma+\nu|k|+\frac{\rho}{2} & \frac{-i\sigma(a_1+\rho)}{2} \\ \frac{i\sigma(\rho-a_1)}{2} & \nu|k|+\frac{\rho}{2}\end{pmatrix}. \tag{3.46}$$

It is convenient to write L as a sum of two matrices, one of which has ρ as a factor:

$$L=M+N_1,\ N_1=\rho\frac{N}{2}. \tag{3.47}$$

The expressions of M and N are given by

$$M=\begin{pmatrix} ia_2\sigma+\nu|k| & -i\sigma a_1/2 \\ -i\sigma a_1/2 & \nu|k|\end{pmatrix}, \tag{3.48}$$

$$N=\begin{pmatrix} 1 & -i\sigma \\ i\sigma & 1\end{pmatrix}. \tag{3.49}$$

The matrix N is Hermitian positive semidefinite; we have

$$\mathcal{N}u=(u_1+\mathcal{H}u_2,\ \mathcal{H}(u_1+\mathcal{H}u_2)). \tag{3.50}$$

This relation explains why, as we mentioned in § 2, the trace of $u_1+\mathcal{H}u_2$ on Σ plays a particular role in the analysis of transparent boundary conditions. As $\mathrm{Re}(\rho)$ is strictly positive, we have

$$\mathrm{Re}\,(N\hat{u},\hat{u})=(\mathrm{Re}\,\rho)|\hat{u}_1-i\sigma\hat{u}_2|^2\geqq 0.$$

The matrix M is not Hermitian, but the symmetrized M, $(M+M^*)/2$ is equal to

$$\frac{M+M^*}{2}=\begin{pmatrix} \nu|k| & 0 \\ 0 & \nu|k|\end{pmatrix}. \tag{3.51}$$

This is a Hermitian positive definite matrix, for all nonzero k. Therefore L satisfies

$$\mathrm{Re}\,(L\hat{u},\hat{u})\geqq 0\quad \forall\hat{u}\text{ in }\mathbb{C}^2. \tag{3.52}$$

In particular, $L+I$ is invertible, for all values of the parameters τ and k, and in Hermitian norm on $\mathbb{C}^2$, we have

$$|(L+I)^{-1}\hat{u}|\leqq|\hat{u}|\quad \forall\hat{u}\text{ in }\mathbb{C}^2. \tag{3.53}$$

We have thus shown the following result.

PROPOSITION 3.4. *For any f satisfying* (2.37) *and* (3.41), *for $u^0=0$, and for any positive μ, let (u,p) be the solution of* (2.29)-(2.31). *If we denote still by u the function u restricted to $\Omega_-\times\mathbb{R}^+$ and extended by 0 for $t\leqq 0$, then u satisfies the variational inequality*

$$s(u_t,v)+\mu s(u,v)+\tilde{\mathbf{a}}(u,v)+\langle\mathcal{L}u,v\rangle=(f,v)\quad \forall v\in W(\Omega_-), \tag{3.54}$$

where the pseudo-differential operator $\mathcal{L}$ is defined by its symbol L, given by (3.46).

3.3. Functional analysis of the operator $\mathcal{L}$. The class of data u^0 and f for which we have a solution of (3.54) is much too restricted. Thus, we extend it in several steps, by giving first a suitable definition of the domain of $\mathcal{L}$. We rely, of course, on the positivity of the matrix L observed in (3.47) to (3.52). Then, we shall give a dense subset of the domain of $\mathcal{L}$, and prove that $\mathcal{L}$ is a causal operator. Finally, the expression $\int_0^T\langle(\mathcal{L}u)(t),u(t)\rangle\,dt$ is greater than or equal to zero, if u is in the domain of $\mathcal{L}$, and vanishes for $t\leqq 0$.

The operator $\mathscr{L}$ is a pseudo-differential operator that belongs to the class $S^1_{1,0}$ of [17], [20], and [30]; it belongs in fact to a certain anisotropic class, which could be defined. As we do not seek the highest possible generality, we will be content with results specific to our variational problem:

DEFINITION 3.5. The domain $D(\mathscr{L})$ of $\mathscr{L}$ is the space of functions g such that there exists an h in $L^2(\mathbb{R}; W^{-1/2}(\mathbb{R})\times H^{-1/2}(\mathbb{R}))$ such that

$$\hat{g}=(I+L)^{-1}\hat{h}. \tag{3.55}$$

Then, on $D(\mathscr{L})$, $\mathscr{L}$ is defined by

$$\mathscr{L}g=h-g. \tag{3.56}$$

An obvious consequence of the definition and of estimate (3.53) is that $D(\mathscr{L})$ is a subspace of $L^2(\mathbb{R}; W^{1/2}\times H^{-1/2})$. A sufficient condition for g to belong to the domain of $\mathscr{L}$ is given by the following lemma.

LEMMA 3.6. *The set Y of functions g on Σ such that*

$$g\in L^2(\mathbb{R}; \mathbf{H}^{1/2}(\mathbb{R})) \quad \text{and} \quad g_1+\mathscr{H}g_2\in H^{1/2}(\mathbb{R}; H^{-1/2}(\mathbb{R})) \tag{3.57}$$

is a subset of $D(\mathscr{L})$; moreover, if Z is the space defined at (3.11), $\mathscr{S}(\mathbb{R}; Z)$ is dense in $L^2(\mathbb{R}; W^{1/2}\times H^{-1/2})$ in the following sense; for every g in $D(\mathscr{L})$, there exists a sequence of elements g_n of $\mathscr{S}(\mathbb{Q}; Z)$ such that g_n converges to g and $\mathscr{L}g_n$ converges to $\mathscr{L}g$ in $L^2(\mathbb{R}; W^{-1/2}\times H^{-1/2})$.

Proof. According to estimate (3.27), there exists a decomposition

$$\rho=\rho_1+\rho_2,$$

such that

$$|\rho_1|\leqq \gamma\sqrt{1+|k|^2},\quad |\rho_2|\leqq \gamma\sqrt[4]{1+|\tau|^2}.$$

We decompose L as a sum

$$L=\left(M+\frac{\rho_1 N}{2}\right)+\left(\frac{\rho_2 N}{2}\right).$$

The first functional assumption on g implies that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\sqrt{1+|k|^2}|\hat{g}|^2\,dk\,d\omega<+\infty.$$

The coefficients of M are bounded by $\gamma'(1+|k|)$, and thus,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\sqrt{1+|k|^2}}\left|\left(M+\frac{\rho_1 N}{2}\right)\hat{g}\right|^2 dk\,d\omega\leqq(\gamma+\gamma')\int_{\mathbb{R}}\int_{\mathbb{R}}\sqrt{1+|k|^2}|\hat{g}|^2\,dk\,d\omega<+\infty.$$

The second functional assumption means that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\sqrt{\frac{1+|\tau|^2}{1+|k|^2}}|\hat{g}_1-i\sigma\hat{g}_2|^2\,dk\,d\omega<+\infty.$$

The second piece of $L\hat{g}$ is estimated by

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\rho_2 N\hat{g}|^2}{\sqrt[4]{1+|k|^2}}\,dk\,d\omega\leqq\gamma\int_{\mathbb{R}}\int_{\mathbb{R}}\sqrt{\frac{1+|\tau|^2}{1+|k|^2}}|\hat{g}_1-i\sigma\hat{g}_2|^2\,dk\,d\omega<+\infty.$$

This proves that $\mathscr{L}g$ belongs to $L^2(\mathbb{R}, W^{-1/2}(\mathbb{R})\times H^{-1/2}(\mathbb{R}))$, since $H^{-1/2}$ is a subspace of $W^{-1/2}$. On the other hand, g belongs to $L^2(\mathbb{R}; W^{-1/2}(\mathbb{R})\times H^{-1/2}(\mathbb{R}))$, and thus, g belongs to $D(\mathscr{L})$.

To build a sequence g_n with the required properties, let g belong to $D(\mathscr{L})$, and let $h=\mathscr{L}g$; let φ_n, ψ_n and χ_n be elements of $\mathscr{S}(\mathbb{R})$ such that

$$\hat{\varphi}_n(k)=0 \text{ if } |k|\leqq\frac{1}{n} \text{ or if } |k|\geqq n;\ \hat{\varphi}_n=1 \quad \text{if } \frac{2}{n}\leqq|k|\leqq n-1,$$

$$0\leqq\hat{\varphi}_n\leqq 1 \text{ everywhere};$$

$$\psi_n(t)=n\psi(nt-1) \text{ with } \psi \text{ belonging to } \mathscr{D}(\mathbb{R}),\ \int_{\mathbb{R}}\psi\,dx=1,$$

$$\chi_m(t)=\exp\left(-\frac{t^2}{2m}\right)\Rightarrow\hat{\chi}_m(\omega)=\sqrt{\frac{m}{2\pi}}\exp\left(-\frac{m\omega^2}{2}\right).$$

We define

$$h_n=h*(\varphi_n\psi_n).$$

Clearly, h_n belongs to $\mathbf{H}^\infty(\mathbb{R},Z)$, because the choice of φ_n disposes of all problems at $k=0$. Moreover, for any given ε, there exists an $n(\varepsilon)$ such that for $n>n(\varepsilon)$,

$$\|h-h_n\|_{L^2(\mathbb{R};W^{-1/2}(\mathbb{R})\times H^{-1/2}(\mathbb{R}))}\leqq\frac{\varepsilon}{2}.$$

Fix n such that this inequality holds. We define now

$$g_n=(\mathscr{L}+1)^{-1}h_n;\ g_{mn}=\chi_m g_n.$$

From estimate (3.53),

$$\|g-g_n\|_{L^2(\mathbb{R};\ W^{-1/2}(\mathbb{R})\times H^{-1/2}(\mathbb{R}))}\leqq\frac{\varepsilon}{2}.$$

Let

$$h_{mn}=(\mathscr{L}+1)g_{mn},$$

which is well defined thanks to the first part of the lemma. In the Fourier variable,

$$\hat{h}_{mn}(k,\tau)-\hat{h}_n(k,\tau)=(L(k,\tau)+1)\{(\hat{\chi}_m(\cdot)-\delta^\omega)*\hat{g}_n(k,\cdot)\}(\tau).$$

As m tends to infinity, $\hat{\chi}_m*\hat{g}_n$ converges to $\hat{g}_n$ in $\mathscr{F}\mathbf{H}^m(\Sigma)$, for all m. As the multiplication by $L(k,\tau)$ maps $\mathscr{F}\mathbf{H}^1(\Sigma)$ continuously into $\mathscr{F}\mathbf{L}^2(\Sigma)$, there exists an m such that

$$\|h_{mn}-h_n\|_{L^2(\mathbb{R};\ W^{-1/2}(\mathbb{R})\times H^{-1/2}(\mathbb{R}))}\leqq\frac{\varepsilon}{2}.$$

this proves the density of $\mathscr{S}(\mathbb{R};Z)$ in $D(\mathscr{L})$. □

Now, we are able to prove that $\mathscr{L}$ is a causal operator.

LEMMA 3.7. *If g belongs to $D(\mathscr{L})$, and vanishes for $t<0$, then $\mathscr{L}g$ vanishes for $t<0$.*

Proof. Let g be an element of $D(\mathscr{L})$, and let $h=(\mathscr{L}+1)g$; assume that g vanishes for $t\leqq 0$. Let φ_n, ψ_n and g_n, h_n be as in the previous lemma. Then, as we can see in the Fourier variable,

$$g_n=g*(\varphi_n\psi_n).$$

The choice we made of ψ_n implies that

$$g_n = 0 \text{ for } t > 0.$$

The choice of φ_n implies that

$$\hat{g}_n(k, \omega) = 0 \text{ for } |k| \geqq n.$$

If θ is a square integrable function on $\mathbb{R}$ that vanishes on $\mathbb{R}_-$, we can write

$$\hat{\theta}(\omega - i\mu) = \int_0^\infty \theta(t) \exp(-it(\omega - i\mu))\, dt,$$

for $\mu = 0$, with the estimate

$$|\hat{\theta}(\omega - i\mu)| \leqq |\theta|_{L^2(\mathbb{R})} \sqrt{\int_0^\infty e^{-2t\mu}\, dt} \leqq |\theta|_{L^2(\mathbb{R})} \frac{1}{\sqrt{2\mu}}.$$

Therefore, for g_n, we have the estimate

$$|\hat{g}_n(k, \omega - i\mu)| \leqq |\mathscr{F}_{x_2 \to k} g_n(k, \cdot)|_{L^2(\mathbb{R}^t)} \frac{1}{\sqrt{2\mu_0}} \text{ if } \mu \geqq \mu_0 > 0,$$

where $L^2(\mathbb{R}^t)$ is the space of square integrable functions of time.
We define an operator $\mathscr{L}_k$ by its symbol:

$$\mathscr{L}_k v = (\mathscr{F}_{t \to \omega})^{-1}\{L(k, \cdot)\tilde{v}(\cdot)\},$$

where the tilda $\tilde{}$ denotes the Fourier transform with respect to the time variable. If v belongs to $L^2(\mathbb{R})$, then $\mathscr{L}_k v$ belongs to $H^{-1/2}(\mathbb{R})$. The root ρ can be extended to the half-plane $\tau = \omega - i\mu$, $\mu \geqq 0$, as an analytic function of ω, with the estimate

$$|\rho(k, \omega)| \leqq C(1 + |k| + \sqrt{|\omega|} + \sqrt{|\mu|}).$$

We can apply the Paley-Wiener-Schwarz theorem to $\mathscr{L}_k \mathscr{F}_{x_2 \to k} g_n(k, \cdot)$; from the previous considerations, for almost every k, $L(k, \tau)\hat{g}_n(k, \tau)$ is analytic in τ for $\mu = -\mathrm{Im}(\tau) > 0$, and satisfies an estimate of the form

$$|L(k, \tau)\hat{g}_n(k, \tau)| \leqq C(k, \mu_0)(1 + |\tau| + |\mu|) \quad \forall \mu \geqq \mu_0.$$

Thus, for almost every k, $\omega \to L(k, \omega - i\mu)\hat{g}_n(k, \omega - i\mu)$ is the Fourier-Laplace transform of a distribution with support in $[0, \infty)$. On the other hand, $L(k, \omega - i\mu)\hat{g}_n(k, \omega - i\mu)$ is square integrable with respect to ω, and therefore,

$$(\mathscr{F}_{t \to \omega})^{-1}(L(k, \cdot)\hat{g}_n(k, \cdot)) = 0 \quad \forall\, t \leqq 0, \text{ and a.e. } k,$$

and finally,

$$\mathscr{L} g_n = 0 \quad \forall\, t \leqq 0, \text{ and for all } x_2.$$

By density, the same result holds for g. □

An immediate consequence of Lemma 3.7 is the following.

LEMMA 3.8. *Let u^0 belong to* $\mathbf{H}(\mathbb{R}^2)$, *and f to* $L^2(0, \infty; \mathbf{L}^2(\mathbb{R}^2))$. *Assume that u^0 and f satisfy respectively the support conditions* (2.36) *and* (2.37). *If μ is strictly positive, and if (u, p) is the solution of* (2.29)–(2.31), *extended by zero for negative time, then the trace of u on Σ belongs to $D(\mathscr{L})$.*

Proof. According to Proposition 2.7, u belongs to $L^2(0, \infty, W(\Omega_-))$, and according to Proposition 2.4, the trace of u on Σ belongs to $L^2(0, \infty, W^{1/2} \times H^{1/2})$. On the other hand, the support condition enables us to apply Lemma 2.14, and therefore, $(u_1 + \mathscr{H} u_2)|_\Sigma$ belongs to $H^\infty(\Sigma)$. □

We first obtain a result of existence, under the support condition on u^0 and f, with some functional analysis on $\mathscr{L}$.

PROPOSITION 3.9. *For all u^0 in $H(\Omega_-)$ and all f in $L^2(0, \infty, L^2(\Omega_-))$ that satisfy the support conditions* (2.36), (2.37), *and for all strictly positive μ, there exists a function u satisfying the variational equality* (3.54) *with initial data u^0, and such that*

$$u \in L^\infty([0, \infty); H(\Omega_-)),$$
$$\nabla u \in L^2([0, \infty); L^2(\Omega_-)),$$
$$u_t \in L^2([0, \infty); W'(\Omega_-)).$$

Proof. We have to approximate a right-hand side $f + u^0 \otimes \delta'$ by a smooth right-hand side f_n belonging to $\mathcal{S}([0, \infty); H^\infty(\mathbb{R}^2))$. This is clearly possible by truncation and regularization. The passage to the limit in the variational inequality is easy.

We will obtain a much better result of existence, not only for the sake of exhaustivity, but also to have a convenient frame for the proof of uniqueness. We write (3.52) as a problem with time in the full real line, extending u by 0 for $t \geqq 0$; after an integration by parts in time, we have for all v in $L^2(\mathbb{R}; W(\Omega_-))$ such that v_t belongs to $L^2(\mathbb{R}; H(\Omega_-))$

$$(3.58)\quad \int_{\mathbb{R}} \{-s(u, v_t) + \mu s(u, v) + \tilde{\mathbf{a}}(u, v)\}\, dt + \langle u, \mathcal{L}^* v\rangle_\Sigma = s(u^0, v(0)) + \int_{\mathbb{R}} (f, v)\, dt.$$

Here, of course, $\mathcal{L}^*$ has L^* for symbol, the domain of $\mathcal{L}^*$ is defined similarly to the domain of $\mathcal{L}$, and the trace of a test function v on Σ belongs to $D(\mathcal{L}^*)$ thanks to Lemma 3.6 and simple interpolation. We need to define three more operators to prove existence and uniqueness.

DEFINITION 3.10. The operator A is an operator from $W(\Omega_-)$ to $W'(\Omega_-)$ defined by

$$(3.59)\qquad s(Au, v) = \tilde{\mathbf{a}}(u, v) \quad \forall v \in W(\Omega_-).$$

The operator $B(\tau)$ is an operator from $W(\Omega_-)$ to $W'(\Omega_-)$ defined by

$$(3.60)\qquad s(B(\tau)u, v) = \int_{\mathbb{R}} L(k, \tau)\hat{u}(0, k)\bar{\hat{v}}(k, \tau)\, dk, \quad \forall v \in W(\Omega_-).$$

The operator S is an operator from $L^2(\Omega_-)$ to $H(\Omega_-)$ defined by

$$(3.61)\qquad s(Sf, v) = (f, v) \quad \forall v \in H(\Omega_-).$$

The operator A is well defined; this is a classical result. The mapping which assigns to a pair $(u; v)$ belonging to $W(\Omega_-) \times W(\Omega_-)$ the expression

$$\int_{\mathbb{R}} L(k, \tau)\hat{u}(0, k)\bar{\hat{v}}(0, k)\, dk$$

is clearly a sesquilinear continuous mapping, and thus $B(\tau)$ is well defined. Finally, the mapping

$$v \to (f, v)$$

is linear continuous on $H(\Omega_-)$, and Sf is thus well defined.

The existence and uniqueness theorem reads.

THEOREM 3.11. *For all u^0 in $H(\Omega_-)$ and all f in $L^2(0, \infty; \mathbf{L}^2(\Omega_-))$, and for all strictly positive μ, there exists a unique u in $L^\infty([0, \infty); H(\Omega_-)) \cap L^2([0, \infty); W(\Omega_-))$ such that u_t belongs to $L^2([0, \infty); W(\Omega_-))$ and*

$$(3.62)\quad \int_{\mathbb{R}} \{-s(u, v_t) + \mu s(u, v) + \tilde{\mathbf{a}}(u, v)\}\, dt + \langle u, \mathcal{L}^* v\rangle)_\Sigma = s(u^0, v(0)) + \int_{\mathbb{R}} (f, v)\, dt.$$

for all v in $L^2(\mathbb{R}; W(\Omega_-))$ such that v_t belongs to $L^2(\mathbb{R}; H(\Omega_-))$.

In order to prove this theorem, we need a positivity result which will be useful for the existence part.

LEMMA 3.12. *Let g belong to the space Y defined at* (3.57). *Assume that u vanishes for negative time. Then, for all positive T, we have*

$$\int_0^T (\mathscr{L}g, g)\, dx_2\, dt \geqq 0.$$

Proof. Assume first that g belongs to $\mathscr{S}(\mathbb{R}; Z)$, and vanishes for negative time. Let u be the solution of (3.9). Then, if we multiply the equation $\mathscr{A}_\mu(u, p) = 0$ by u and integrate over Ω_+, we obtain

$$\int_0^T (\mathscr{L}g, g)\, dt = \|u(\cdot, T)\|^2 + \mu \int_0^T \{\tilde{\mathbf{a}}(u, u) + \|u(\cdot, t)\|^2\}\, dt - (\mathscr{K}_1 g_1(\cdot, T), g_1(\cdot, T))_\Gamma$$
$$- \mu \int_0^T (\mathscr{K}_1 g_1(\cdot, t), g_1(\cdot, t))_\Gamma\, dt,$$

where $\mathscr{K}$ is the boundary operator of symbol $|k|^{-1}$. We check that $\|u\|^2 - (\mathscr{K}g, g)_\Gamma$ is greater than or equal to zero. By an argument analogous to the one we used at proposition 2.4, we have the identity

$$(\mathscr{K}g_1, g_1)_\Gamma = -2(u_1, \mathscr{K}u_2),$$

and

$$\|u\|^2 - (\mathscr{K}u_1, u_1) = \|u_1 - \mathscr{K}u_2\|^2 \geqq 0.$$

Thus the positivity holds for smooth data. By density and causality, it will hold as stated in the statement of the lemma. □

Proof of the theorem. We first prove uniqueness; denote $\tilde{u}$ the partial Fourier transform of u in time. If u is the solution of (3.62), it satisfies

$$i\omega\tilde{u}(\tau) + \mu\tilde{u}(\tau) + A\tilde{u}(\tau) + B(\tau)\tilde{u}(\tau) = u^0 + S\tilde{f}(\tau). \tag{3.63}$$

If the data vanish, we have

$$i\omega\tilde{u}(\tau) + \mu\tilde{u}(\tau) + A\tilde{u}(\tau) + B(\tau)\tilde{u}(\tau) = 0.$$

For almost every ω, $\tilde{u}(\tau)$ belongs to $W(\Omega_-)$. If we multiply the above equation scalarly by $\tilde{u}(\tau)$ and take the real part, we obtain

$$\operatorname{Re}\{\mu s(\tilde{u}(\tau), \bar{\tilde{u}}(\tau)) + \tilde{\mathbf{a}}(\tilde{u}(\tau), \bar{\tilde{u}}(\tau)) + s(B(\tau)\tilde{u}(\tau), \bar{\tilde{u}}(\tau))\} = 0. \tag{3.64}$$

The definition of $\tilde{\mathbf{a}}$ and $B(\tau)$ implies that the corresponding terms in (3.64) are nonnegative; there remains

$$\operatorname{Re}(\mu s(\tilde{u}(\tau), \bar{\tilde{u}}(\tau))) \leqq 0,$$

and this implies that $\tilde{u}$ vanishes almost everywhere. We can conclude the uniqueness of the solution of (3.62).

Let us prove the existence under the assumptions of our theorem; if u and f satisfy our support condition, we know that there exists a unique solution to (3.62). If this support condition is not satisfied, extend u^0 and f by 0 in Ω_+, and approximate these extended u^0 and f by translated data u_n^0 and f_n that satisfy the support condition. For the solution u_n the positivity of $\mathscr{L}$ implies the estimate

$$s(u_n(T), u_n(T)) + \int_0^T \{\mu s(u_n(t), u_n(t)) + \tilde{\mathbf{a}}(u_n(t), u_n(t))\}\, dt \leqq s(u_n^0, u_n^0) \leqq s(u^0, u^0).$$

An easy passage to the limit gives the result. □

4. Absorbing boundary conditions. This section is dedicated to the approximation of the pseudo-differential operator $\mathscr{L}$ by more manageable operators. We have seen that the symbol L of $\mathscr{L}$ is a two-by-two matrix which is algebraic in τ, k and $\sigma = \operatorname{sgn}(k)$. We would like our approximation of $\mathscr{L}$ to be local in space and time, which would mean that its symbol is polynomial or rational in τ and k. Unfortunately, we do not know how to approximate σ or $|k|$ by rational fractions of low degree; thus, we will be content with an approximation which is rational in τ, k and σ. The idea is to approximate the root ρ by a sequence of r_n which keeps the essential property $\operatorname{Re}(r_n) \geqq 0$. We need a number of technical results that precisely describes the properties of the sequence of approximations. These properties enable us to prove a rather weak existence and uniqueness theorem for the problem with absorbing boundary conditions. Then, we estimate the difference between the solution of the problem with absorbing boundary conditions and the solution of the problem with transparent conditions. From this estimate, we deduce a better existence theorem. If ν is small, the difference between the two solutions is small with a power of ν; when u^0 and f satisfy the support conditions (2.36), (2.37), the difference is estimated in a space of smooth functions.

4.1. Approximation of the symbol L. We approximate the transparent boundary conditions in constraint formulation. If we approximated the symbol of the operator $g \to (\partial u/\partial x_1, p)$, we could run into trouble and obtain an ill-posed problem.

We recall that we obtained at (3.47) a decomposition of the matrix $L(k, \tau)$, which we write now with an explicit dependance on the viscosity ν.

$$L(k, \tau, \nu) = M(k, \nu) + \rho(k, \tau, \nu)\frac{N(k)}{2}. \tag{4.1}$$

The matrix M is of degree 1 in ν; $N(k)$ and the Hermitian symmetrization of $M(k, \nu)$ are both positive semidefinite matrices.

Now, we have to approximate $\rho(k, \tau, \nu)$ so that the successive approximations, denoted r_n, will satisfy the essential property

$$\operatorname{Re}(r_n(k, \tau, \nu)) \geqq 0 \quad \forall k, \omega, \nu. \tag{4.2}$$

This problem has been solved in [13] but with very few proofs. The principle of the approximation has been obtained by observing that for all complex number d we have

$$\lambda = \frac{\lambda d - |k|^2 - i\omega'\alpha}{-\lambda + \alpha + d}, \tag{4.3}$$

where $\alpha = a_1/\nu$ and $\omega' = (\tau + a_2 k)/a_1$.

The choice of $d = i\omega'$ and of the initialization $\lambda_1 = -i\omega'$ (because λ_0 might be a special case) defines a sequence recursively by:

$$\lambda_{n+1} = \frac{\lambda_n d - |k|^2 - i\omega'\alpha}{-\lambda_n + \alpha + d},$$

which has the three important properties:

(i) Each of the λ_n is rational in k, τ and σ;

(ii) The associated initial boundary value problem is, at least formally, well-posed because $\operatorname{Re}(\lambda_n) \leqq 0$, and we will prove that it is actually well-posed;

(iii) λ_n is the $[n-1, n-1]$ Padé approximant of λ around $\nu = 0$ (see [13]).

More precisely, we have the

DEFINITION 4.1. Denote

$$\omega' = \frac{\tau + a_2 k}{a_1}, \tag{4.4}$$

$$\alpha = \frac{a_1}{\nu}. \tag{4.5}$$

We define:

$$\lambda_0 = 0, \tag{4.6}$$

$$\lambda_1 = -i\omega', \tag{4.7}$$

$$\lambda_{n+1} = \frac{-i\omega'\lambda_n + i\alpha\omega' + k^2}{\lambda_n - \alpha - i\omega'}, \tag{4.8}$$

$$r_n = a_1 - 2\nu\lambda_n. \tag{4.9}$$

We will give now a sequence of technical lemmas on r_n and λ_n.

LEMMA 4.2. *For all $n \geqq 0$, $k \in \mathbb{R}$, $\omega \in \mathbb{R}$, $\nu > 0$, r_n is well defined and*

$$\operatorname{Re}(\lambda_n) \leqq 0. \tag{4.10}$$

Proof. For $n = 0$ or 1, the result is obvious. Assume that $\operatorname{Re}(\lambda_n) \leqq 0$; then, the expression $\lambda_n - \alpha + i\omega'$ has a strictly negative real part; moreover,

$$\lambda_{n+1} = \frac{(-i\omega'\lambda_n + i\alpha\omega' + k^2)(\bar{\lambda}_n - \alpha + i\omega')}{|\lambda_n - \alpha + i\omega'|^2}$$

so that,

$$\operatorname{Re}(\lambda_{n+1}) = \frac{[k^2 + \omega'^2] \operatorname{Re}(\lambda_n - \alpha)}{|\lambda_n - \alpha + i\omega'|^2},$$

and by induction, the results holds true. □

LEMMA 4.3. *Let ρ be as in* (3.24) *and $\lambda = (a_1 - \rho)/2\nu$; then, the following relation holds*:

$$\lambda_{n+1} - \lambda = -\frac{(\lambda_n - \lambda)[k^2 + \omega'^2]}{[\lambda_n - \alpha - i\omega'][\lambda - \alpha - i\omega']}. \tag{4.11}$$

Proof. We substitute $d = -i\omega'$ in (4.2)

$$\lambda = \frac{-i\omega'\lambda + i\alpha\omega' + k^2}{\lambda - \alpha - i\omega'}.$$

Therefore, subtracting this expression λ from the expression (4.7), we obtain immediately (4.10). □

The recursive definition of λ_n by a sequence of homographic transformations shows that λ_n is a rational fraction in k and ω. To obtain more information on the sequence λ_n, we study the particular case $\alpha = 1$. Then (4.8) becomes

$$\lambda_{n+1} = \frac{-i\omega'\lambda_n + i\omega' + k^2}{\lambda_n - i\omega' - 1}; \tag{4.12}$$

we define P_n and Q_n by

$$P_1 = -i\omega' \tag{4.13}$$

$$Q_1 = 1 \tag{4.14}$$

and

$$P_{n+1} = -i\omega' P_n + (i\omega' + k^2) Q_n \tag{4.15}$$

$$Q_{n+1} = P_n - (i\omega' + 1) Q_n. \tag{4.16}$$

Then, we have

$$\lambda_n = \frac{P_n}{Q_n}. \tag{4.17}$$

An obvious induction shows that P_n is at most of degree n, and that Q_n is at most of degree $n-1$, globally in k and ω'. Let P_n and Q_n be decomposed as a sum of globally homogeneous polynomials in k and ω', of decreasing degree:

$$P_n = P_n^n + P_n^{n-1} + \cdots + P_n^0, \tag{4.18}$$

$$Q_n = Q_n^{n-1} + Q_n^{n-2} + \cdots + Q_n^0; \tag{4.19}$$

we now define the polynomial Z_n by

$$Z_n = P_n + kQ_n. \tag{4.20}$$

Let

$$z = k - i\omega'. \tag{4.21}$$

Then Z_n satisfies the following relation:

$$Z_{n+1} = z(Z_n - Q_n), \tag{4.22}$$

and if we decompose Z_n into a sum of homogeneous polynomials

$$Z_n = Z_n^n + Z_n^{n-1} + \cdots + Z_n^1, \tag{4.23}$$

then we can deduce Q_n and P_n from Z_n by

$$P_n = i \operatorname{Im}(Z_n^1) + \operatorname{Re}(Z_n^2) + i \operatorname{Im}(Z_n^3) + \cdots \tag{4.24}$$

$$Q_n = \frac{\operatorname{Re}(Z_n^1) + i \operatorname{Im}(Z_n^2) + \operatorname{Re}(Z_n^3) + \cdots}{k}. \tag{4.25}$$

Of course, these expressions terminate differently according to the parity of n. □

LEMMA 4.4. *Assume that $\alpha = 1$. Then, there exists a polynomial R_n of degree $n-1$ in one variable such that Q_n^{n-1} can be written as*

$$Q_n^{n-1}(k, \omega') = k^{n-1} R_n\left(\frac{\omega'}{k}\right).$$

The zeros ζ_p of R_n are real and simple and except for $k=0$, the next term $Q_n^{n-2}(k, k\zeta_p)$ does not vanish.

Proof. A simple induction shows that P_n^n and Q_n^{n-1} are given for n odd by

$$\begin{cases} P_n^n = i \operatorname{Im}(z^n), \\ Q_n^{n-1} = \operatorname{Re}(z^n)/k. \end{cases} \tag{4.26}$$

When n is even, they are given by

$$\begin{cases} P_n^n = \operatorname{Re}(z^n), \\ Q_n^{n-1} = [i \operatorname{Im}(z^n)]/k. \end{cases} \tag{4.27}$$

The polynomial $R_n(X)$ is given by

$$R_n(X) = i \operatorname{Im}(1 - iX)^i \quad \text{if } n \text{ is even},$$

$$R_n(X) = \operatorname{Re}(1 - iX)^n \quad \text{if } n \text{ is odd}.$$

From these formulae, R_n is of degree $n-1$ and the roots of R_n are real and simple. Geometrically, these roots are the ordinates of the intersection of the straight lines $\zeta = 1 + r\,e^{i\pi k/n}$ with the imaginary axis, when n is even; when n is odd, they are the abscissae of the intersection of these same lines with the real axis. There are exactly n of these lines. When n is even, one of these lines is parallel to the imaginary axis and another is the real axis; this yields the announced $n-1$ real solutions, one of which is zero. If n is odd, one of the lines is the real axis, which we discard, and there remain $n-1$ real distinct nonzero solutions. For the last assertion, we consider first the case of n even:

$$Q_n^{n-2} = \frac{\operatorname{Re}(Z_n^{n-1})}{k}.$$

Relation (4.22) implies that

$$Z_n = z^n - \sum_{j=1}^{n-1} z^j Q_{n-j}.$$

With the help of (4.26) and (4.25), we obtain

$$\begin{aligned}
\operatorname{Re}(Z_n^{n-1}) &= -\operatorname{Re}\left(\sum_{j=1}^{n-1} z^j Q_{n-j}^{n-j-1}\right) \\
&= -\operatorname{Re}\left(\sum_{\substack{j=1\\ j \text{ even}}}^{n-1} z^j \frac{i \operatorname{Im} z^{n-j}}{k} + \sum_{\substack{j=1\\ j \text{ odd}}}^{n-1} z^j \frac{\operatorname{Re} z^{n-j}}{k}\right) \\
&= \frac{1}{k}\left(\sum_{\substack{j=1\\ j \text{ even}}}^{n-1} \operatorname{Im}(z^j)\operatorname{Im}(z^{n-j}) - \sum_{\substack{j=1\\ j \text{ odd}}}^{n-1} \operatorname{Re}(z^j)\operatorname{Re}(z^{n-j})\right).
\end{aligned}$$

Assume now that z is a root of Q_n^{n-1} which does not vanish. Then, it is real and the above formula reduces to

$$\operatorname{Re}(Z_n^{n-1}) = -\frac{n}{2}\frac{z^n}{k},$$

which does not vanish.

The proof in the case of n odd is analogous and left to the reader. □

In the next lemma we give an estimate on $\lambda_n + (i\omega'/n)$, which will enable us to work on the variational problem for absorbing boundary conditions.

LEMMA 4.5. Assume that $\alpha = 1$. Then, for each n, there is a constant C_n, which depends neither on k nor on ω', such that

$$\left|\lambda_n + \frac{i\omega'}{n}\right| \leqq C_n(1+|k|)^2. \tag{4.28}$$

Proof. We observe that in

$$\lambda_n + \frac{i\omega'}{n} = \frac{nP_n + i\omega' Q_n}{nQ_n};$$

the term of highest degree in ω' of P_n is $(-i\omega')^n$, and the term of highest degree in ω' of Q_n is $(-i\omega')^{n-1}$, so that $nP_n + i\omega' Q_n$ does not have a term of degree n in ω'. Thus, we have the estimate

$$|nP_n + i\omega' Q_n| \leqq C_{1n}\{1 + |z|^{n-1} + |k|\,|z|^{n-1}\}. \tag{4.29}$$

To estimate from below the denominator Q_n, we consider first the case when n is even. Let $(\zeta_p)_{1\leqq p\leqq n-1}$ denote the zeros of R_n, and let ξ_q denote the zeros of $X\to(Q_n^{n-2})(1, X)$. We make the convention that

$$\zeta_1=0.$$

Lemma 4.4 implies that there exists a β such that

$$\min_{\substack{1\leqq q\leqq n-2\\ 1\leqq p\leqq n-1}} |\xi_q-\zeta_p|=\beta>0.$$

Therefore, this suggests an estimate on three different regions:

$$\mathcal{R}_1\colon \{k, \omega')/|k|^2+|\omega'|^2<r^2\},$$

$$\mathcal{R}_2\colon \{(k, \omega')/|k|^2+|\omega'|^2\geqq r^2 \text{ and } \min_p |\zeta_p-\omega'/k|\leqq\gamma\},$$

$$\mathcal{R}_3\colon \{(k, \omega')/|k|^2+|\omega'|^2\geqq r^2 \text{ and } \min_p |\zeta_p-\omega'/k|>\gamma\}.$$

For any positive r, Q_n is bounded away from zero on $\mathcal{R}_1$; this is a consequence of Lemma 4.2. The precise choice of r and γ will be made below.

We can see that Q_n^{n-1} is the product with k^{n-1} of an imaginary polynomial R_n whose roots are all real. On the other hand, Q_n^{n-2} is real. Therefore,

$$|Q_n|\geqq\min(|Q_n^{n-1}|, |Q_n^{n-2}|)-\sum_{j=1}^{n-3}|Q_n^j|$$

$$\geqq\min(|Q_n^{n-1}|, |Q_n^{n-2}|)-\sum_{j=1}^{n-3}) C_j|z|^j,$$

where the C_j are positive numbers depending only on n. We have the relations

$$Q_n^{n-2}=K'k^{n-2}\prod_{q=1}^{n-2}\left(\frac{\omega'}{k}-\xi_q\right)$$

and

$$Q_n^{n-1}=Kk^{n-1}\prod_{p=1}^{n-1}\left(\frac{\omega'}{k}-\zeta_p\right).$$

In the region $\mathcal{R}_2$, $|(\omega'/k)-\xi_p|\geqq\beta-\gamma$. Therefore, if we choose γ so that

$$\gamma\leqq\frac{\beta}{2}, \tag{4.30}$$

we obtain the estimate

$$|Q_n^{n-2}|\geqq|K'||k|^{n-2}\left(\frac{\beta}{2}\right)^{n-2}.$$

In the region $\mathcal{R}_2$, $|\omega'|\leqq|k|(\gamma+\max_p|\zeta_p|)$, and therefore, there exists a constant C_{2n} such that

$$|Q_n^{n-2}|\geqq C_{2n}|z|^{n-2}. \tag{4.31}$$

We argue similarly in the region $\mathcal{R}_3$; the homogeneous term Q_n^{n-1} satisfies

$$|Q_n^{n-1}|\geqq|K||k|^{n-1}\prod_{p=1}^{n-1}\left|\frac{\omega'}{k}-\zeta_p\right|=|K||\omega'|^{n-1}\prod_{p=1}^{n-1}\left|\frac{k\zeta_p}{\omega'}-1\right|.$$

If $|k/\omega'|\leqq\varepsilon$, where ε is so chosen that $\varepsilon\max_p|\zeta_p|\leqq\frac{1}{2}$, then

$$|Q_n^{n-1}|\geqq|K||\omega'|^{n-1}\frac{1}{2^{n-1}\varepsilon^{n-1}}\geqq C_{3n}|z|^{n-1}.$$

If the converse inequality holds, then

$$|Q_n^{n-1}|\geqq|K||\gamma k|^{n-1}\geqq C_{4n}|z|^{n-1}.$$

Finally there exists a constant C_{5n} such that

$$|Q_n^{n-1}|\geqq C_{5n}|z|^{n-1}\quad\forall\, z \text{ in } \mathscr{R}_3. \tag{4.32}$$

The number r must be chosen so that

$$|z|\geqq r\Rightarrow\min(C_{2n}|z|^{n-2}, C_{5n}|z|^{n-1})\geqq 2\sum_{j=1}^{n-3}C_j|z|^j.$$

Then, there exists a constant C_{6n} such that

$$|Q_n|\geqq C_{6n}(1+|z|^{n-2})\qquad\text{on } \mathscr{R}_2,$$

$$|Q_n|\geqq C_{6n}(1+|z|^{n-1})\qquad\text{on } \mathscr{R}_3.$$

Finally,

$$\left|\lambda_n+\frac{i\omega'}{n}\right|\leqq\frac{C_{1n}\{1+|z|^{n-1}+|k||z|^{n-1}\}}{C_{6n}(1+|z|^{n-2})}\leqq C_{8n}(1+|z|+|k||z|)\text{ on } \mathscr{R}_2;$$

the inequality of the statement of the lemma is satisfied, because $|\omega'|\leqq$ constant $|k|$ on $\mathscr{R}_2$. On the other region,

$$\left|\lambda_n+\frac{i\omega'}{n}\right|\leqq\frac{C_{1n}(1+|z|^{n-1}+|k||z|^{n-1})}{C_{7n}(1+|z|^{n-1})}\leqq C_{9n}(1+|k|)\text{ on } \mathscr{R}_3.$$

Here, the inequality stated is clearly satisfied. No difficulty can come from region $\mathscr{R}_3$. The proof when n is odd is analogous and left to the reader. □

We can now state an estimate in the general case.

PROPOSITION 4.6. *There exists a constant c_n depending only on a_1 and a_2, such that*

$$\left|\lambda_n+\frac{i\omega'}{n}\right|\leqq c_n\frac{(1+\nu|k|)^2}{\nu}. \tag{4.33}$$

Proof. In the general case,

$$\lambda_{n+1}=\frac{-i\omega'\lambda_n+i\alpha\omega'+k^2}{\lambda_n-\alpha-i\omega'},$$

then,

$$\lambda_n(k,\omega',\alpha)=\frac{P_n(k,\omega',\alpha)}{Q_n(k,\omega',\alpha)},$$

where

$$P_n(k,\omega',\alpha)=P_n^n+\alpha P_n^{n-1}+\cdots+\alpha^nP_n^0,$$

$$Q_n(k,\omega',\alpha)=Q_n^{n-1}+\alpha Q_n^{n-2}+\cdots+\alpha^{n-1}Q_n^0,$$

or, in other terms,

$$P_n(k, \omega', \alpha) = \alpha^n P_n\left(\frac{k}{\alpha}, \frac{\omega'}{\alpha}, 1\right),$$

$$Q_n(k, \omega', \alpha) = \alpha^{n-1} Q_n\left(\frac{k}{\alpha}, \frac{\omega'}{\alpha}, 1\right).$$

Now, we can use (4.26):

$$\left|\lambda_n(k, \omega', \alpha) + \frac{i\omega'}{n}\right| = \left|\alpha\left\{\lambda_n\left(\frac{k}{\alpha}, \frac{\omega'}{\alpha}, 1\right) + \frac{i\omega'}{\alpha n}\right\}\right| \leqq C_n \alpha(1+|k/\alpha|)^2.$$

This proves the proposition. □

4.2. Variational formulation: existence, uniqueness, error estimates. In this section, we shall state a variational formulation which is suitable for the analysis of absorbing conditions. Given data u^0 and f, we obtain by the Fourier method a unique solution of the problem with absorbing boundary conditions of any order n. This is not enough to obtain existence in nice spaces. One expects that the larger n, the closer the solution with artificial boundary conditions to the full space solution. This result is obtained by the analysis of the error, if the data satisfy the support conditions (2.36) and (2.37). It turns out that the error is in a better space than the solution u_n; using causality and positivity, we obtain the existence and uniqueness in convenient spaces.

DEFINITION 4.7. Let, for $n = 0$

$$L_0(k, \tau, \nu) = M(k, \tau, 0) + \frac{a_1}{2} N(k) \tag{4.34}$$

and

$$L_n(k, \tau, \nu) = M(k, \tau, \nu) + r_n(k, \tau, \nu) N(k)/2. \tag{4.35}$$

The operator $\mathscr{L}_n$ is defined by its symbol L_n:

$$\mathscr{L}_n(u) = \mathscr{F}^{-1}(L_n(\cdot, \cdot, \nu)\hat{u}(0, \cdot, \cdot)). \tag{4.36}$$

The variational formulation (3.58) is approximated by:

$$\begin{gathered}\int_{\mathbb{R}} \{-s(u, v_t) + \mu s(u, v) + \tilde{\mathbf{a}}(u, v)\}\, dt + \langle u, \mathscr{L}_n^* v\rangle_\Sigma = s(u^0, v(0)) + \int_{\mathbb{R}} (f, v)\, dt, \\ \forall\, v \in L^2(\mathbb{R}; W(\Omega_-)) \text{ such that } v_t \in L^2(\mathbb{R}; H(\Omega_-)).\end{gathered} \tag{4.37}$$

We have a first result of existence and uniqueness, as follows.

PROPOSITION 4.8. *Let u^0 belong to H, and let f belong to $L^2(\mathbb{R}; H(\Omega_-))$. Then, for $n \geqq 1$, there exists a unique u in $H^{-\varepsilon}(\mathbb{R}; H)$, such that, for all v in $H^1(\mathbb{R}; W(\Omega_-))$, (4.37) holds.*

Proof. Define an operator $B_n(\omega)$ by

$$s(B_n(\tau)u, v) = \int L_n(k, \tau)\hat{u}(0, k) \cdot \bar{\hat{v}}(0, k)\, dk \quad \forall\, v \in W. \tag{4.38}$$

We solve (4.37) by Fourier transform in time:

$$i\omega\tilde{u}_n(\tau) + A\tilde{u}_n(\tau) + \mu\tilde{u}_n(\tau) + B_n(\tau)\tilde{u}_n(\tau) = u^0 + S\tilde{f}(\tau). \tag{4.39}$$

Here A and S are defined respectively at (3.59) and (3.61).

We estimate the solution of

$$i\omega\tilde{v}_n(\tau)+A\tilde{v}_n(\tau)+\mu\tilde{v}_n(\tau)+B_n(\tau)\tilde{v}_n(\tau)=v^0 \tag{4.40}$$

in terms of $\|v^0\|_H$ and $\|v^0\|_W$. If we multiply (4.40) scalarly by the conjugate of $\tilde{v}_n(t)$ and take the real part of the result, we obtain the estimate

$$\|\tilde{v}_n(\tau)\|_W \leqq C\|v^0\|_{W'}, \tag{4.41}$$

from the positivity of the operator B_n.

In order to obtain the $H^{-\varepsilon}$ estimate, we take the imaginary part of (4.40) scalarly multiplied by the conjugate of $\tilde{v}_n(\omega)$, and we obtain, for $n\geqq 1$,

$$\begin{aligned}\omega s(\tilde{v}_n(\tau), \bar{\tilde{v}}_n(\tau))&+\left(a_1+\frac{2\tau\nu}{na_1}\right)\int N\hat{v}_n(0,k,\tau)\cdot\bar{\tilde{v}}_n(0,k,\omega)\,dk\\ &=\operatorname{Im}\Big\{s(v^0,\bar{\tilde{v}}_n(\tau))-s(A\bar{\tilde{v}}_n(\tau),\tilde{v}_n(\tau))\\ &\quad-\int M\hat{v}_n(0,k,\tau)\cdot\bar{\tilde{v}}_n(0,k,\tau)\,dk\\ &\quad+\int 2\nu\left[\lambda_n+\frac{i\omega'}{n}-i\frac{a_1}{na_2}\right]N\hat{v}_n(0,k,\tau)\cdot\bar{\tilde{v}}_n(0,k,\tau)\,dk\Big\}.\end{aligned} \tag{4.42}$$

The most interesting term in the right-hand side of the above relation is the integral term which has $(\lambda_n+i\omega'/n)$ as a factor of the integrand; to estimate this term, we use Proposition 4.6. We will have a result if we are able to estimate $k\hat{v}_n(0,\cdot,\omega)$. This will depend on additional regularity on $\tilde{v}_n(\tau)$. Let D denote the differentiation with respect to x_2; if we apply D to (4.40), we can write, because $B_n(\tau)$ commutes obviously with D,

$$i\omega D\tilde{v}_n(\tau)+\mu\tilde{v}_n(\tau)+AD\tilde{v}_n(\tau)+B_n(\tau)D\tilde{v}_n(\tau)=Dv^0. \tag{4.43}$$

From (4.41), we have

$$\|D\tilde{v}_n(\tau)\|_W\leqq C\|Dv^0\|_{W'}\leqq C\|v^0\|_H.$$

Now, this inequality enables us to conclude that, for $\omega\neq 0$,

$$|\omega| s(\tilde{v}_n(\tau),\tilde{v}_n(\tau))\leqq C(\|v^0\|_H)^2. \tag{4.44}$$

This relation concludes the proof of the existence. The uniqueness is as in the proof of Theorem 3.12. □

There is an analogous result for $n=0$; as it is easier, its proof is left to the reader.

PROPOSITION 4.9. *Let u^0 belong to W'; then, there exists a unique u in $H^{-\varepsilon}(\mathbb{R}; H)$ such that, for all v in $H^1(\mathbb{R}; W)$,* (4.34) *holds.*

These $H^{-\varepsilon}$ estimates are of course very bad, but we will obtain better after we make error estimates. Let us denote

$$e_n=u_n-u. \tag{4.45}$$

THEOREM 4.10. *Assume that u^0 and f satisfy the support conditions* (2.36) *and* (2.37); *then, for $n\geqq 1$, and for all positive p, the error e_n satisfies the estimate*

$$\|e_n\|_{H^p(\mathbb{R};\,W)}\leqq C'_{p,n}\nu^{2n}(\|u^0\|_H+\|f\|_{L^2(\mathbb{R};\,H)}). \tag{4.46}$$

For $n=0$, the exponent $2n$ has to be replaced by 1 in relation (4.46).

Proof. We subtract (3.63) from (4.39), and we obtain

$$i\omega\tilde{e}_n+A\tilde{e}_n+\mu\tilde{e}_n+B_n(\tau)\tilde{e}_n=(B-B_n)(\tau)\tilde{u}. \tag{4.47}$$

The point now is to estimate the norm of $(B-B_n)(\tau)\tilde{u}(\tau)$ in $W'(\Omega_-)$, so as to utilize (4.38). For $n \geqq 1$, we deduce from (4.8) and (4.32) that

$$s((B-B_n)(\tau)\tilde{u}(\tau), v) = \int \nu(\lambda_n - \lambda) N\hat{u}(0, \cdot, \tau) \cdot \hat{v}(0, \cdot, \tau)\, dk.$$

We already know from (4.10) that

$$|\lambda_n - \lambda| \leqq C^{n-1} \nu^{2n-2} |\lambda_1 - \lambda| (|k|^2 + |\tau|^2)^{n-1}.$$

An easy algebraic computation shows that there exists a constant c' such that

$$|\lambda_1 - \lambda| \leqq c' \nu (|k|^2 + |\tau|^2).$$

On the other hand, under the support conditions, $\mathcal{N}u|_\Sigma$ belongs to $H^\infty(\mathbb{R}^2)$. Thus, for all p, there exists a constant c_p such that

$$|N\hat{u}(0, k, \tau)| \leqq c_p (1 + |k|^2 + |\tau|^2)^{-p}.$$

These considerations imply the estimate

$$|s((B-B_n)(\tau)\tilde{u}(\tau), v)| \leqq C^{n-1} \nu^{2n} c_p c' \int (|k|^2 + |\tau|^2)^n (1 + |k|^2 + |\tau|^2)^{-p} |\hat{v}(k)|\, dk.$$

If v is taken in W, then $\hat{v}$ is integrable on Σ, and there is a constant $C_{p,n}$ such that

$$|s((B-B_n)(\tau)\tilde{u}(\tau), v)| \leqq C_{p,n} \nu^{2n} \|v\|_W (1 + |\tau|^2)^{p-n}.$$

Thus, we obtain

$$\|(B-B_n)(\tau)\tilde{u}(\tau)\|_{W'} \leqq C_{p,n} \nu^{2n} (1 + |\tau|^2)^{p-n}.$$

From here, the conclusion of the theorem is immediate. The case $n = 0$ is completely analogous. □

Remark 4.11. The constant $C_{p,n}$ which appears in (4.46) increases with n; moreover, the sequence λ_n converges to λ only for bounded values of λ and ω. If one wanted to prove that the limit of the sequence u_n is u, one would have to estimate an integral involving arbitrary powers of $(\omega'^2 + k^2)$. Therefore, it is likely that a convergence theorem would need very strong conditions over the data.

COROLLARY 4.12. Under the assumptions of Theorem 4.10, u_n is an element of $L^\infty(\mathbb{R}^+, H(\Omega_-)) \cap L^2(\mathbb{R}^+; W(\Omega_-))$.

Proof. From Theorem 3.12, we know that u is an element of $L^\infty(\mathbb{R}^+, H(\Omega_-)) \cap L^2(\mathbb{R}^+; W(\Omega_-))$; as the error e_n belongs to the same space, the corollary holds.

In order to get rid of the support conditions, we prove some positivity results for $\mathcal{L}_n$.

LEMMA 4.13. The operator $\mathcal{L}_n$ is causal; if g is an element of $H^\infty(\mathbb{R}^2)$, which vanishes for t lesser than or equal to zero, then

$$\text{Re} \int_0^T \langle \mathcal{L}_n g, g \rangle_\Gamma \, dt \geqq 0. \tag{4.48}$$

Proof. The causality of $\mathcal{L}_n$ comes from the inductive construction of λ_n; it suffices to observe that λ_1 admits an extension to the half-plane $\text{Im}(\tau) < 0$, and that this extension has polynomial growth. By induction, the same holds for the λ_n. Details of this proof are left to the reader. The second part of the lemma relies on the decomposition

$$\mathcal{L}_n = \mathcal{R}_n + \frac{1}{n} \frac{d}{dt}, \tag{4.49}$$

where the symbol of $\mathcal{R}_n$ is bounded with respect to τ. This decomposition is an immediate consequence of Lemma 4.6. Arguing as in the proof of Lemma 3.7, and with the help of a technique used in [21], we define $\mathcal{L}_n^{\#}(k,\cdot)$ by

$$\mathcal{L}_n^{\#}(k,\cdot)u(t)=\mathcal{F}_{\omega\to t}\{L_n(k,\tau)\hat{u}(\tau)\}$$

and similarly $N^{\#}(k,\cdot)$, recalling the definition (3.50) of $\mathcal{N}$.

If we let

$$u_\varepsilon(t)=\begin{cases} u(t) \text{ if } 0\leqq t\leqq T,\\ u(T)(T-t+\varepsilon)\varepsilon^{-1} \quad \text{if } T\leqq t\leqq T+\varepsilon,\\ 0 \text{ if } T+\varepsilon\leqq t,\end{cases}$$

then

$$\operatorname{Re}\int \mathcal{L}_n^{\#}(k,\cdot)u_\varepsilon(t)\cdot u_\varepsilon(t)\,dt=\operatorname{Re}\int L_n(k,\tau)\hat{u}_\varepsilon(\tau)\cdot\bar{\hat{u}}_\varepsilon(\tau)\,d\omega\geqq 0.$$

On the other hand, by causality,

$$\operatorname{Re}\int_0^T \mathcal{L}_n^{\#}(k,\cdot)u(t)\cdot u(t)\,dt=\operatorname{Re}\int \mathcal{L}_n^{\#}(k,\cdot)u_\varepsilon(t)\cdot u_\varepsilon(t)\,dt$$
$$-\operatorname{Re}\int_T^{T+\varepsilon} \mathcal{L}_n^{\#}(k,\cdot)u_\varepsilon(t)\cdot u_\varepsilon(t)\,dt.$$

The first term is greater than or equal to zero; we estimate the second one as ε tends to zero; with the help of decomposition (4.49),

$$\lim_{\varepsilon\to 0}\operatorname{Re}\int_T^{T+\varepsilon} \mathcal{R}_n^{\#}(k,\cdot)u_\varepsilon(t)\cdot u_\varepsilon(t)\,dt=0,$$

because $\mathcal{R}_n$ is bounded with respect to τ;

$$\lim_{\varepsilon\to 0}\operatorname{Re}\int_T^{T+\varepsilon} \mathcal{N}^{\#}(k,.)\frac{du_\varepsilon(t)}{dt}\cdot u_\varepsilon(t)\,dt=-\frac{1}{2}\mathcal{N}^{\#}(k,.)u(T)\cdot u(T),$$

as is shown by a straightforward computation. This shows the lemma. □

Now we can give a much more precise estimate under the assumptions of Theorem 4.10.

COROLLARY 4.14. *Under the assumptions of Theorem* 4.10, *the solution u_n with absorbing boundary conditions satisfies the estimate*

$$\frac{1}{2}s(u_n(t),u_n(t))+\int_0^T\{\mu s(u_n(t),u_n(t))+\tilde{\mathbf{a}}(u_n(t),u_n(t))\}\,dt$$
$$\leqq\frac{1}{2}s(u^0,u^0)+\int_0^T(f(t),u_n(t))\,dt. \tag{4.50}$$

Proof. This result is an immediate consequence of Lemma 4.13; one has only to substitute v by u_n in the variational equality (4.37). We can do this for smooth enough data; if u^0 and f are smooth enough, u_t belongs to $L^2(\mathbb{R}^+;H(\Omega_-))$. By error estimate (4.43), u_n belongs to the same space. Therefore, using the integration over $[0,T]$, the positivity and a density argument, one obtains the desired result. □

Finally, we obtain the most general existence and uniqueness theorem.

THEOREM 4.15. *For all u^0 in $H(\Omega_-)$ and all f in $L^2(0,\infty;\mathbf{L}^2(\Omega_-))$, and for all strictly positive μ there exists a unique u in $L^\infty(\mathbb{R}^+;H(\Omega_-))\cap L^2(\mathbb{R}^+;W(\Omega_-))$ such that u_t belongs to $L^2(\mathbb{R};W'(\Omega_-))$ and the variational equality* (4.37) *and the energy inequality* (4.50) *hold.*

Proof. It is enough to approximate any initial data by initial data satisfying the support condition. Then, the energy estimate gives the existence by standard procedure of extraction of subsequences. The uniqueness depends only on the positivity and still holds. □

4.3. Explicit formulations. We write problem (4.37) explicitly for $n=0$ and $n=1$. The variational form will be convenient for computations. We give the associated boundary conditions; for $n=2$, we use an auxiliary unknown.

Let us recall first that a product in Fourier space corresponds to a convolution in physical space. One of the important operators is the convolution by the inverse Fourier transform of $\mathrm{pf}(1/|k|)$; it is defined by

$$\mathcal{K}u(y)=\bar{\mathcal{F}}(u(k)pf(1/|k|)).$$

The kernel K of $\mathcal{K}$ is given by

$$K(x)=\frac{1}{\pi}(\gamma-\mathrm{Log}\,|x|), \tag{4.51}$$

where $\gamma\simeq 0.57721\ldots$ is the Euler constant. Then, the scalar product s admits the expression

$$\begin{cases} s(u,v)=\displaystyle\int_{\Omega_-} u(x_1,x_2)v(x_1,x_2)\,dx_1\,dx_2 \\ +\displaystyle\frac{1}{\pi}\left(\gamma\int_\Gamma u_1(0,x_2)v_1(0,x_2)\,dx_2-\fint_{\Gamma\times\Gamma}\mathrm{Log}\,|x_2-y_2|u_1(0,x_2)v_1(0,y_2)\,dx_2\,dy_2\right)\end{cases} \tag{4.52}$$

where the barred integral means that a principal value has to be taken.

Another important kernel is the kernel of the Hilbert transform, which is equal to $vp(1/\pi x)$. This distribution is not locally square integrable, but from the relation

$$\langle\mathcal{H}\varphi,\psi\rangle=-\frac{1}{2\pi}\int_{\mathbb{R}} i\sigma\hat{\varphi}\bar{\hat{\psi}}\,dk=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{|k|}\hat{\varphi}\overline{ik}\,\bar{\hat{\psi}}\,dk.$$

We deduce

$$\langle\mathcal{H}\varphi,\psi\rangle=\left\langle\mathcal{K}\varphi,\frac{\partial\psi}{\partial x_2}\right\rangle, \tag{4.53}$$

and the kernel K of $\mathcal{K}$ is locally integrable.

According to Definition 4.7, the symbol of $\mathcal{L}_0$ is the matrix L_0 given by

$$L_0(k,\tau,\nu)=M(k,\tau,0)+\frac{r_0}{2}N(k).$$

the value of r_0 is given by (4.6) and (4.9); $r_0=a_1$. Therefore,

$$L_0(k,\tau,\nu)=\begin{pmatrix} ia_2\sigma+(a_1/2) & -i\sigma a_1 \\ 0 & a_1/2\end{pmatrix}. \tag{4.54}$$

Therefore,

$$(L_0\hat{u},\hat{v})=\frac{a_1}{2}(\hat{u},\hat{v})_\Gamma+(-i\sigma(a_1\hat{u}_2-a_2\hat{u}_1),\hat{v}_1)_\Gamma.$$

If the vector product is denoted by the symbol $\times$, this last expression can be rewritten as

$$(L_0\hat{u}, \hat{v}) = \frac{a_1}{2}(\hat{u}, \hat{v})_\Gamma + (-i\sigma\, a \times \hat{u}, \hat{v}_1)_\Gamma.$$

We obtain the following expression of $(\mathcal{L}_0 u, v)$:

$$(\mathcal{L}_0 u, v) = \frac{a_1}{2}(u, v)_\Gamma + (a \times \mathcal{H} u, v_1)_\Gamma,$$

or, with the help of (4.53),

$$(\mathcal{L}_0 u, v) = \frac{a_1}{2}(u, v)_\Gamma + \left(a \times \mathcal{H} u, \frac{\partial v_1}{\partial x_2}\right)_\Gamma. \tag{4.55}$$

Finally, the variational formulation for 0-artificial conditions is stated in the following proposition.

PROPOSITION 4.16. *For $n = 0$, the variational formulation* (4.37) *is equivalent to*

$$s(u, v) = (u, v)_{\Omega_-} + (\mathcal{H} u, v)_\Gamma; \tag{4.56}$$

$$s(u_t, v) + \mu s(u, v) + \tilde{\mathbf{a}}(u, v) + \frac{a_1}{2}(u, v)_\Gamma + \left(a \times \mathcal{H} u, \frac{\partial v_1}{\partial x_2}\right)_\Gamma = (f, v) \qquad \forall v \in W(\Omega_-). \tag{4.57}$$

Proof. The above dictionary of kernels proves that the following variational equality holds:

$$\int_{\mathbb{R}} \left[-s(u, v_t) + \mu s(u, v) + \tilde{\mathbf{a}}(u, v) + \frac{a_1}{2}(u, v)_\Sigma + \left(a \times \mathcal{H} u, \frac{\partial v_1}{\partial x_2}\right)_\Sigma\right] dt$$
$$= \int_{\mathbb{R}} (f, v)\, dt + s(u^0, v(0)).$$

for all test functions v in $L^2(\mathbb{R}; W(\Omega_-))$ such that v_t belongs to $L^2(\mathbb{R}; H(\Omega_-))$. Observe that the second integral along Γ makes sense because of the characterization of the trace on Γ. Using a test function of the form

$$v = W \otimes \varphi,$$

where v belongs to $W(\Omega_-)$ and φ is a smooth function, we obtain (4.57). □

In the case $n = 1$, L_1 is more complicated:

$$L_1(k, \tau, \nu) = M(k, \tau, \nu) + \frac{r_1}{2} N(k),$$

r_1 is given by

$$r_1 = a_1 + \frac{2\nu i}{a_1}(\tau + a_2 k).$$

Therefore,

$$L_1(k, \tau, \nu) = L_0(k, \tau, \nu) + \nu|k|\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\nu i}{a_1}(\tau + a_2 k) N(k). \tag{4.58}$$

Therefore, we have only to compute the extra terms which did not appear before:

$$\frac{1}{2\pi}\int_{\mathbb{R}} |k|\hat{u} \cdot \bar{\hat{v}}\, dk = \left(\mathcal{H}\frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2}\right)_\Gamma.$$

In the same fashion,

$$\frac{1}{2\pi}\int_{\mathbb{R}} ik\,N(k)\hat{u}\cdot\bar{\hat{v}}\,dk=\frac{1}{2\pi}\int_{\mathbb{R}} ik\hat{u}\cdot\bar{\hat{v}}\,dk+\frac{1}{2\pi}\int_{\mathbb{R}}|k|\bar{\hat{v}}\times\hat{u}\,dk$$
$$=\left(u,\frac{\partial v}{\partial x_2}\right)_\Gamma+\left(\mathcal{K}\frac{\partial v}{\partial x_2}\times\frac{\partial u}{\partial x_2},1\right)_\Gamma,$$
$$\frac{1}{2\pi}\int_{\mathbb{R}} N(k)\hat{u}\cdot\bar{\hat{v}}\,dk=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{u}\cdot\bar{\hat{v}}\,dk+\frac{1}{2\pi}\int_{\mathbb{R}}(-i\sigma)\bar{\hat{v}}\times\hat{u}\,dk$$
$$=(u,v)_\Gamma+(\mathcal{H}v\times u,1)_\Gamma.$$

We can summarize this calculation:

$$(\mathcal{L}_1u,v)=(\mathcal{L}_0u,v)+\nu\left(\mathcal{K}\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_\Gamma+\frac{\nu a_2}{a_1}\left(u,\frac{\partial v}{\partial x_2}\right)_\Gamma$$
$$+\left(\mathcal{K}\frac{\partial v}{\partial x_2}\times\frac{\partial u}{\partial x_2},1\right)_\Gamma+\frac{\nu}{a_1}(u_t+\mu u,v)_\Gamma+(\mathcal{H}v\times(u_t+\mu u),1)_\Gamma.$$

Finally, we have the following result:

PROPOSITION 4.17. *For $n=1$, the variational formulation* (4.37) *is equivalent to*

$$s(u_t,v)+\frac{\nu}{a_1}(u_t,v_t)_\Gamma+(\mathcal{H}v\times u_t,1)_\Gamma+\mu s(u,v)+\tilde{\mathbf{a}}(u,v)+\frac{a_1}{2}(u,v)_\Gamma$$
$$+\left(a\times\mathcal{K}u,\frac{\partial v_1}{\partial x_2}\right)_\Gamma+\nu\left(\mathcal{K}\frac{au}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_\Gamma+\frac{\nu a_2}{a_1}\left(u,\frac{\partial v}{\partial x_2}\right)_\Gamma+\left(\mathcal{K}\frac{\partial v}{\partial x_2}\times\frac{\partial u}{\partial x_2},1\right)_\Gamma \tag{4.59}$$
$$=(f,v)\quad\forall\, v \text{ in } W(\Omega_-).$$

Proof. This is simply a consequence of the previous computation. □

In order to obtain the strong formulation of the boundary conditions, we observe that if the matrix E_m is defined by

$$E_m=\frac{a_1I}{2}-K_1-L_m, \tag{4.60}$$

where K_1 is given by (3.44), then the associated operator $\mathcal{E}_m$ gives the boundary condition

$$\sigma_n=\mathcal{E}_mu|_\Sigma. \tag{4.61}$$

The matrix E_0 is given by

$$E_0=\begin{pmatrix} ia_2\sigma-(i\tau/|k|) & ia_1\sigma\\ 0 & 0\end{pmatrix}.$$

Therefore, the corresponding boundary condition is given by

$$\sigma_{11}=-\mathcal{K}\frac{\partial u_1}{\partial t}+\mathcal{H}u\times a, \tag{4.62}$$

$$\sigma_{12}=0. \tag{4.63}$$

The same kind of computation gives the result for $n=1$:

$$\sigma_{11}=-\mathcal{K}\frac{\partial u_1}{\partial t}+\mathcal{H}u\times a+\nu\mathcal{H}\frac{\partial u_1}{\partial x_2}-\frac{\nu}{a_1}\left(\frac{\partial}{\partial t}+a_2\frac{\partial}{\partial x_2}\right)(u_1+\mathcal{H}u_2), \tag{4.64}$$

$$\sigma_{21}=\nu\mathcal{H}\frac{\partial u_2}{\partial x_2}+\frac{\nu}{a_1}\left(\frac{\partial}{\partial t}+a_2\frac{\partial}{\partial x_2}\right)(-\mathcal{H}u_1+u_2). \tag{4.65}$$

In [13], the family λ_n defined by (4.4)–(4.8) had been introduced to design absorbing boundary conditions for the advection-diffusion equation $u_t + a\nabla u - \nu\Delta u = 0$. It turned out that with this special choice, the boundary conditions assumed a very special form, namely

$$\left(\frac{\partial}{\partial t} + a\nabla\right)^n u = 0.$$

The analysis is much more intricate here, and we will merely outline a possible strategy for the case $n=2$. It relies on the introduction of an unknown auxiliary, defined on the boundary. This technique had been initiated in [23] and proved to be very useful [3]. Using the symbols and the expression of λ_2 and λ_1, we get a formulation of L_2 as

$$L_2 = L_1 + \nu(\lambda_1 - \lambda_2)N, \quad \text{or} \quad L_2 = L_1 - \nu^2 \frac{k^2 + \omega'^2}{1 + 2i\omega'\nu} N.$$

Let us introduce the auxiliary function φ defined on the boundary through its Fourier transform by the expression

$$-\nu \frac{k - i\omega'}{1 + 2i\omega'} N\hat{u} = \hat{\varphi}, \tag{4.66}$$

or equivalently, using the fact that $(Nu)_1 - \mathcal{H}(Nu)_2 = 0$, $\varphi_1 + \mathcal{H}\varphi_2 = 0$;

$$\left(1 + \frac{2\nu}{a_2}\left(\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x_2}\right)\right)(\varphi_1 + \mathcal{H}\varphi_2) + 2\nu\mathcal{H}\left(i\frac{\partial}{\partial x_2} + \frac{1}{a}\left(\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x_2}\right)\right)(u_1 + \mathcal{H}u_2) = 0. \tag{4.67}$$

then $L_2\hat{u}$ is given by

$$L_2\hat{u} = L_1\hat{u} + \nu(k + i\omega')\hat{\varphi}. \tag{4.68}$$

We conclude that the boundary condition associated with the second approximation $\mathcal{L}_2$ reads

$$\begin{aligned} \sigma_{11} = &-\mathcal{K}\frac{\partial u_1}{\partial t} + \mathcal{H}u \times a + \nu\mathcal{H}\frac{\partial u_1}{\partial x_2} \\ &-\frac{\nu}{a_1}\left(\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x_2}\right)(u_1 + \mathcal{H}u_2 + \varphi_1) + i\nu\frac{\partial \varphi_1}{\partial x_2}, \end{aligned} \tag{4.69}$$

$$\sigma_{21} = \nu\mathcal{H}\frac{\partial u_2}{\partial x_2} - \frac{\nu}{a_1}\left(\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x_2}\right)(u_2 - \mathcal{H}u_1 + \varphi_2) + i\nu\frac{\partial \varphi_2}{\partial x_2} = 0. \tag{4.70}$$

These formulae have to be supplemented with (4.67).

Remark 4.12. The first boundary condition (4.62), (4.63) is actually local and can be written in the form:

$$\frac{\partial u_1}{\partial t} + (a \cdot \nabla)u_1 = 0 \tag{4.71}$$

$$\frac{\partial u_2}{\partial x_1} = 0. \tag{4.72}$$

Appendix A. The standard spaces of Beppo-Levi functions are studied in [7]; the distribution spaces described by Definition 2.1 are of the very same nature. The common property of the spaces $BL(H^s(\Omega))$ is that their local properties are the same as the local properties of $H^{s+1}(\Omega)$, but their properties in the large are quite different. In particular, LEMMA A.1. *For any n, for any unbounded open set Ω of $\mathbb{R}^n$, and for any real s, there exists an unbounded function in $BL(H^s(\Omega))$.*

Proof. If is enough to exhibit examples; outside of a compact subset, we require u to be equal to

$$r^{1/4} \text{ if } n=1;$$

$$\text{Log Log } r \text{ if } n=2;$$

$$r^{-1/4} \text{ if } n \geqq 3.$$

Checking that these functions give the answer is an exercise left to the reader.

Though the elements of $BL(H^s(\Omega))$ may be unbounded, they are nonetheless elements of $\mathscr{S}'$. This is a consequence of the fact that $vp(1/ik)$ in dimension 1, and proper generalizations of this in dimension greater than or equal to 2 admit a Fourier transform, which is known explicitly. Moreover, the growth of the elements of $BL(H^s(\Omega))$ is polynomial at most and can be estimated precisely.

Finally, the Beppo-Levi spaces are natural spaces on an unbounded domain, where the only estimate is an energy estimate that involves only the gradient.

Appendix B. In this appendix, we prove a result on the behavior at infinity of harmonic functions. This result is probably part of the folklore of the subject, but we know no source where to find it in simple form.

LEMMA B.1. *Let p be a function on $\mathbb{R}^N$ such that Δp (computed in the sense of distributions) has compact support. If p decreases fast at infinity to zero, then p is constant outside of a compact set of $\mathbb{R}^N$.*

Proof. Let ρ be a C^∞ test function that is radial and has support in the ball of center 0 and radius 1. Denote $\rho_\varepsilon(x)=\varepsilon^{-N}\rho(x/\varepsilon)$. If p is harmonic for all x in $E_{R-2}=\{x/|x|\geqq R-2\}$, then it has the mean property in this region, and, whenever $|x|\geqq R-2+\varepsilon$, we have

$$u(x)=u*\rho_\varepsilon(x).$$

Therefore, u has derivatives of all orders in int (E_{R-2}), and all of its derivatives decrease rapidly at infinity. Let φ be an infinitely differentiable function on $\mathbb{R}^N$ such that

$$\varphi=0 \text{ for } |x|\leqq R-1$$

$$\varphi=1 \text{ for } |x|\geqq R.$$

Then $q=p\varphi$ is in $\mathscr{S}(\mathbb{R}^N)$, and

$\psi=\Delta q$ has support in the ball of center 0 and radius R.

We perform a Fourier transform on the equation $\Delta q=\psi$, and we obtain

$$|\xi|^2\hat{q}(\xi)=\hat{\psi}(\xi). \tag{B.1}$$

Thanks to the Paley-Wiener theorem, $\hat{\psi}(\xi)$ can be extended to all of $\mathbb{C}^N$ and is an entire function of ξ; moreover, for all n, there is a constant C_n such that

$$|\hat{\psi}(\xi)|\leqq C_n(1+|\xi|)^n e^{R|\mathrm{Im}(\xi)|}.$$

From (B.1), we can see that $\hat{q}$ can be extended to all $\mathbb{C}^N$ as a meromorphic function of ξ, with possibly a pole at zero. But, since q is in $\mathscr{S}(\mathbb{R}^N)$, $\hat{q}$ is in $\mathscr{S}(\mathbb{R}^N)$, too, and there is no pole of $\hat{q}$ at zero. Thus $\hat{q}$ satisfies the estimate

$$|\hat{q}(\xi)|\leqq C_0 \quad \text{for } |\xi|\leqq 1,$$
$$|\hat{q}(\xi)|\leqq C_n(1+|\xi|)^n e^{R|\mathrm{Im}(\xi)|}, \quad \text{for } |\xi|\geqq 1.$$

Thus q has compact support in the ball $|x| \leqq R$, and therefore p vanishes for $|x| \geqq R$. $\square$

If p is known in (2.29)-(2.31), then u is the solution of an advection diffusion with right-hand side $f - \nabla p$; therefore, if p does not decrease rapidly to zero at infinity, and if f has compact support, for instance, u cannot generally decrease rapidly to zero at infinity. Therefore, a smooth solution u is in a space of function with polynomial estimates at infinity, and the dual of this space is a strict subspace of $\mathcal{S}'$.

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