SPECTRAL METHODS IN POLAR COORDINATES FOR THE STOKES PROBLEM. APPLICATION TO COMPUTATION IN UNBOUNDED DOMAINS

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ABSTRACT. We present spectral methods for solving the Stokes problem in a circular domain. Their main feature is the uniform inf-sup condition, which allows for optimal error estimates. We apply them to the resolution of exterior problems by coupling with the transparent boundary condition.

1. INTRODUCTION

When solving a problem in an unbounded domain, it is customary to introduce an artificial boundary, and to prescribe on it a so-called "transparent boundary condition", which replaces the missing part of the domain. This leads to a well-posed problem in a bounded domain, with an integral boundary condition. In [10] a method has been introduced for coupling finite elements and the integral equation for the Laplace equation in an exterior domain. This method has been extended to the Stokes problem in [15] and to the Maxwell equations in [13]. Other numerical methods have been developed, coupling finite elements in the interior and spectral decompositions on the boundary (see for instance [11, 12]).

The finite element method is often preferable when dealing with complicated geometries. Nevertheless, in two dimensions, if the artificial boundary is chosen to be a circle, the transparent boundary condition has a very simple expression in the angular coordinate θ. It seems most natural to take advantage of it to approximate the solution with polynomials in r and trigonometric polynomials in θ. Successful computations using spectral methods have been presented in [4]. A theoretical formalism is the aim of the present paper.

As an interesting illustration we chose the steady Stokes problem in two dimensions. We first consider the problem with homogeneous Dirichlet boundary condition in the disc of center 0 and radius 1. The weak formulation reads (the notations can be found in §2):

find $(\mathbf{u}, p)$ in $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that:

$$\begin{cases} \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), & a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \\ \forall q \in L_0^2(\Omega), & b(\mathbf{u}, q) = 0. \end{cases}$$

Owing to a well-known result of Babuška and Brezzi (see [8]), the most delicate property to fulfill for well-posedness is the so-called inf-sup condition: there exists a positive number $C > 0$ such that

$$\inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} \geq C.$$

We recall in §2 that this condition is fulfilled for this problem in any regular geometry. In our case, the best value C can be given explicitly: $1/\sqrt{2}$ (Theorem 2.1). In most cases we find in the literature (rectangular domains) that the use of spectral methods produces "parasitic modes", which perturb the computation of the pressure (see for instance [2, 17]). This is expressed by the fact that the constant C in the discrete inf-sup condition tends to 0 as the number of modes increases. The spectral methods we suggest here lead to uniform inf-sup conditions.

We start with the Galerkin method (§3). Here, N and K are two integers greater than or equal to 2, $\mathbf{S}_K([0, 2\pi])$ is the space of trigonometric polynomials in θ of degree less than or equal to K, and $\mathbf{P}_N([0, 1])$ the space of polynomials in r of degree less than or equal to N. The approximation is made in $X_N \times M_N$, where $M_N = L_0^2(\Omega) \cap (\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1})$ and $X_N = \mathbf{H}_0^1(\Omega) \cap \{\mathbf{u}$ s.t. div $\mathbf{u}$ and curl $\mathbf{u}$ belongs to $\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}\}$. The approximate problem is:

find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that:

$$\begin{cases} \forall \mathbf{v} \in X_N, & a(\mathbf{u}_N, \mathbf{v}) + b(\mathbf{v}, p_N) = \langle \mathbf{f}, \mathbf{v} \rangle, \\ \forall q \in M_N, & b(\mathbf{u}_h, q) = 0. \end{cases}$$

A convenient decomposition of vector fields on the circle gives the inf-sup condition, and the constant is still equal to $1/\sqrt{2}$. We then give two projection theorems in the weighted Sobolev spaces on $(0, 1)$,

$$H_r^p(0, 1) = \{\psi \in \mathcal{D}'(0, 1), \int_0^1 r|\psi^{(j)}|^2(r)\, dr < +\infty \text{ for } 0 \leq j \leq p\}.$$

The first one, in L_r^2, is classical. The second one, in H_r^1, is more delicate. The technique of the proof is inspired by [1], but the lack of a Hardy inequality requires new partial results. These theorems lead to "optimal" error estimates: if $\mathbf{f}$ belongs to H^p for $p \geq 0$, then $\|\mathbf{u} - \mathbf{u}_N\|_X + \|p - p_N\|_M \leq C N^{-1-p} \|\mathbf{f}\|_p$.

In §4, we present a pseudospectral method. It relies on a Gauss-Lobatto quadrature formula on $[0, 1]$ for the weight r. The constant in the inf-sup condition remains the same. For the error estimates we need results on polynomial interpolation in $H_r^p(0, 1)$. Again, we use the strategy in [1], but some new lemmas are necessary. The error estimates are still "optimal".

With these tools, we are now able to study the problem in an unbounded domain (§5). We first reduce it to a disc by giving the transparent operator, and writing the variational formulation. For the discrete formulation, we introduce the Galerkin method. In both cases, continuous and discrete, the constant in the inf-sup condition is equal to 1, which in turn allows for optimal error estimates.

2. The Stokes problem in a disc

2.1. **Variational formulation in a bounded domain.** Let Ω be a bounded open connected subset of $\mathbf{R}^2$, with smooth boundary Γ. The Stokes problem in Ω with

homogeneous Dirichlet boundary data reads: find $(\mathbf{u}, p)$ such that

(2.1)
$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Here, $\nabla, \nabla\cdot,$ and Δ denote respectively the gradient, divergence, and Laplace operators: $\nabla p = (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}), \nabla\cdot\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$. For any positive integer m, we denote by $H^m(\Omega)$ the Sobolev space of distributions in Ω whose derivatives up to order m belong to $L^2(\Omega)$, furnished with the inner product

$$(v, w)_0 = \int_\Omega v(x)\overline{w}(x)\, dx, \qquad (v, w)_m = \sum_{|k| \le m} (D^k v, D^k w)_0.$$

The corresponding norm is denoted by $\|v\|_m (H^0(\Omega) = L^2(\Omega))$. By $\mathbf{H}^m(\Omega)$ (resp. $\mathbf{L}^2(\Omega)$) we denote the space of vector-valued distributions whose two components belong to $H^m(\Omega)$ (resp. $L^2(\Omega)$), whereas $\|\cdot\|_m$ and $(\cdot, \cdot)_m$ are the norm and scalar product either in $H^m(\Omega)$ or in $\mathbf{H}^m(\Omega)$. Furthermore, $L_0^2(\Omega)$ is the space of distributions in $L^2(\Omega)$ such that $(v, 1)_0 = 0$, $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, or equivalently, $H_0^1(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma\}$, and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle$. In view of the Poincaré-Friedrichs inequality, the seminorm defined by $|v|_1 = \|\nabla v\|_0$ is a norm on $H_0^1(\Omega)$, equivalent to the $\|\cdot\|_1$ norm. Finally, $H^{1/2}(\Gamma)$ is the space of traces on Γ of the elements of $H^1(\Omega)$, and $H^{-1/2}(\Gamma)$ its dual space. The duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ will be denoted by $\langle \cdot, \cdot \rangle_\Gamma$.

According to the following result (cf. [8]), problem (2.1) is well-posed.

Theorem A. *If $\mathbf{f}$ belongs to $\mathbf{H}^{-1}(\Omega)$, there exists a unique solution $(\mathbf{u}, p)$ to (2.1) in $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ and*

$$\|\mathbf{u}\|_1 + \|p\|_0 \le C\|\mathbf{f}\|_{-1}.$$

Moreover, if $\mathbf{f}$ belongs to $\mathbf{H}^m(\Omega)$, then $(\mathbf{u}, p)$ belongs to $\mathbf{H}^{m+2}(\Omega) \times H^{m+1}(\Omega)$ and

$$\|\mathbf{u}\|_{m+2} + \|p\|_{m+1} \le C\|\mathbf{f}\|_m.$$

The existence and uniqueness rely on the following weak formulation: let X be the Hilbert space $\mathbf{H}_0^1(\Omega)$ provided with the $|\cdot|_1$ inner product and M be $L_0^2(\Omega)$ provided with the L^2 scalar product:

$$(\mathbf{u}, \mathbf{v})_X = (\nabla\mathbf{u}, \nabla\mathbf{v})_0 = (\nabla u_1, \nabla v_1)_0 + (\nabla u_2, \nabla v_2)_0; \quad \|\mathbf{u}\|_X = \|\nabla\mathbf{u}\|_0;$$

$$(u, v)_M = (u, v)_0; \quad \|u\|_M = \|u\|_0.$$

We define the bilinear forms a and b, and the linear form L, by

(2.2)
$$\begin{cases} a(\mathbf{u}, \mathbf{v}) = (\nabla\mathbf{u}, \nabla\mathbf{v})_0 = (\mathbf{u}, \mathbf{v})_X, \\ b(\mathbf{v}, q) = -(q, \nabla \cdot \mathbf{v})_0, \\ L(\mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle. \end{cases}$$

The weak formulation reads: find $(\mathbf{u}, p)$ in $X \times M$ such that

(2.3)
$$\begin{cases} \forall \mathbf{v} \in X, a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = L(\mathbf{v}), \\ \forall q \in M, b(\mathbf{u}, q) = 0. \end{cases}$$

We introduce the subspace $V = \{\mathbf{v} \in X, \nabla \cdot \mathbf{v} = 0\}$. Theorem A is a consequence of the following general result (cf. [8]):

Theorem B. *Suppose the following assumptions are fulfilled*:

(i) *a is bilinear continuous on X and there exists a positive constant α such that for any v in V, $a(v,v) \geq \alpha \|v\|_X^2$,*

(ii) *b is bilinear continuous on $X \times M$,*

(iii) *the inf-sup condition of Babuška-Brezzi is satisfied: there exists a real number $C > 0$ such that*

$$\inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} \geq C,$$

(iv) *L is linear continuous on X.*
Then problem (2.3) has a unique solution.

We shall from now on consider the case where $\Omega = D(0,1)$ is the unitary disc with center 0 and radius 1.

2.2. **The inf-sup condition in $D(0,1)$.** In order to construct a "good" approximation, we shall first calculate the constant C.

Theorem 2.1. *For $\Omega = D(0,1)$, one has*

$$\inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = \frac{1}{\sqrt{2}}.$$

Proof. Following [8] or [17], we write

$$\inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = \inf_{q \in M} \frac{\|\mathbf{w}(q)\|_X}{\|q\|_M},$$

where $\mathbf{w}(q)$ is the unique solution to the problem

(2.4)
$$\mathbf{w} \in X, \forall \mathbf{v} \in X, \quad a(\mathbf{w}, \mathbf{v}) = b(\mathbf{v}, q),$$

which can be rewritten as

(2.5)
$$\begin{cases} \mathbf{w} \in H_0^1(\Omega), \\ \Delta \mathbf{w} + \nabla q = \mathbf{0}. \end{cases}$$

We shall write $\mathbf{w}$ as a function of q. This can easily be done in polar coordinates. A basis in $L^2(\Gamma)$ is given by the sequence $H_m(\theta) = \frac{1}{\sqrt{2\pi}} \exp(im\theta)$ for $m \in \mathbf{Z}$. A basis in $\mathbf{L}^2(\Gamma)$ is given by the two sequences $\mathbf{V}_m(\theta)$ and $\mathbf{W}_m(\theta)$ for $m \in \mathbf{Z}$, with

(2.6)
$$\begin{cases} 2\mathbf{V}_m(\theta) = H_m(\theta)(\mathbf{e}_1 + i\mathbf{e}_2) = H_{m+1}(\theta)(\mathbf{e}_r + i\mathbf{e}_\theta), \\ 2\mathbf{W}_m(\theta) = H_m(\theta)(\mathbf{e}_1 - i\mathbf{e}_2) = H_{m-1}(\theta)(\mathbf{e}_r - i\mathbf{e}_\theta). \end{cases}$$

Here, $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the usual basis in $\mathbf{R}^2$, $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ the moving basis. Note that $\mathbf{V}_m(\theta) = \overline{\mathbf{W}_{-m}(\theta)}$. The sequence H_m is orthonormal in $L^2(\Gamma)$ (the norm is 1), the sequence $\{\mathbf{V}_m, \mathbf{W}_m\}$ is orthonormal in $\mathbf{L}^2(\Gamma)$ (the norm is $\frac{1}{\sqrt{2}}$).

Let us write q and $\mathbf{w}$ in separate variables:

(2.7)
$$q(r, \theta) = \sum_{m \in \mathbf{Z}} q_m(r) H_m(\theta),$$

(2.8)
$$\mathbf{w}(r, \theta) = \sum_{m \in \mathbf{Z}} v_m(r) \mathbf{V}_m(\theta) + \sum_{m \in \mathbf{Z}} w_m(r) \mathbf{W}_m(\theta).$$

Since $\mathbf{w}$ and q are real functions, we have, for any m, $q_m = \overline{q}_{-m}$ and $v_m = \overline{w}_{-m}$.

We introduce $L_r^2(0,1) = \{\psi \in \mathcal{D}'(0,1), \int_0^1 r|\psi|^2(r)\,dr < +\infty\}$, furnished with the natural norm $\|\psi\|_{L_r^2}^2 = \int_0^1 r|\psi|^2(r)\,dr$, and the corresponding inner product $(\cdot,\cdot)_r$. The norms of q in $L^2(\Omega)$ and of $\mathbf{w}$ in X are given by

$$\|q\|_0^2 = \sum_{m \in \mathbf{Z}} \|q_m\|_{L_r^2}^2 \quad \text{and} \quad \|\mathbf{w}\|_X^2 = \sum_{m \in \mathbf{Z}} \left(\left\|\frac{dv_m}{dr}\right\|_{L_r^2}^2 + \left\|\frac{m}{r}v_m\right\|_{L_r^2}^2 \right).$$

According to [14], for any $m \neq 0$, we have $v_m(0) = 0$. Define, for any m, the operator D_m by

$$(2.9) \qquad\qquad D_m\varphi = \frac{d\varphi}{dr} - \frac{m}{r}\varphi = r^m \frac{d}{dr}(r^{-m}\varphi).$$

The norm of $\mathbf{w}$ in X is given by

$$(2.10) \qquad\qquad \|\mathbf{w}\|_X^2 = \sum_{m \in \mathbf{Z}} \|D_m v_m\|_{L_r^2}^2.$$

Remark 2.1. Since q belongs to $L_0^2(\Omega)$, there holds $\int_0^1 rq_0\,dr = 0$.

We can expand ∇q and $\Delta \mathbf{w}$ in the basis $(\mathbf{V}_m, \mathbf{W}_m)$:

$$(2.11) \quad \nabla q = \sum_{m \in \mathbf{Z}} \left(q'_{m+1} + \frac{m+1}{r}q_{m+1} \right) \mathbf{V}_m + \sum_{m \in \mathbf{Z}} \left(q'_{m-1} - \frac{m-1}{r}q_{m-1} \right) \mathbf{W}_m,$$

$$\Delta \mathbf{w} = \sum_{m \in \mathbf{Z}} \left(\Delta_r v_m - \frac{m^2}{r^2}v_m \right) \mathbf{V}_m + \sum_{m \in \mathbf{Z}} \left(\Delta_r w_m - \frac{m^2}{r^2}w_m \right) \mathbf{W}_m.$$

Noting that

$$\Delta_r - \frac{m^2}{r^2} = \left(\frac{d}{dr} + \frac{m+1}{r} \right)\left(\frac{d}{dr} - \frac{m}{r} \right) = \left(\frac{d}{dr} - \frac{m-1}{r} \right)\left(\frac{d}{dr} + \frac{m}{r} \right),$$

we see that $\Delta \mathbf{w} + \nabla q = \mathbf{0}$ is equivalent to

$$(2.12) \qquad\qquad \forall m \in \mathbf{Z}, \quad \left(\frac{d}{dr} + \frac{m+1}{r} \right)(q_{m+1} + D_m v_m) = 0.$$

For any m in $\mathbf{Z}$, (2.12) can be rewritten as $q_{m+1} + D_m v_m = c_m r^{-(m+1)}$. If m is positive, $r^{-(m+1)}$ does not belong to L_r^2, which contradicts the fact that $\mathbf{w}$ belongs to $\mathbf{H}^1(\Omega)$ and q to $L^2(\Omega)$. Hence, c_m vanishes. For $m < 0$, the constant is determined by the boundary conditions, and we finally get

$$(2.13\text{a}) \qquad\qquad m \geq 0, \quad D_m v_m = -q_{m+1};$$

$$(2.13\text{b}) \qquad m < 0, \quad D_m v_m = -q_{m+1} - 2mr^{-(m+1)} \int_0^1 \rho^{-m} q_{m+1}(\rho)\,d\rho.$$

This can be solved in the correct spaces by

$$(2.14\text{a}) \qquad\qquad m \geq 0, \quad v_m = r^m \int_r^1 \rho^{-m} q_{m+1}(\rho)\,d\rho;$$

$$(2.14\text{b}) \qquad m < 0, \quad v_m = -r^m \int_0^r \rho^{-m} q_{m+1}(\rho)\,d\rho + r^{-m} \int_0^1 \rho^{-m} q_{m+1}(\rho)\,d\rho.$$

We now express the norms:

For $m \le -2$, $\quad \|q_{m+1}\|_{L_r^2}^2 - \|D_m v_m\|_{L_r^2}^2 = -2m \left| \int_0^1 \rho^{-m} q_{m+1}(\rho) \, d\rho \right|^2$

$$= -2m \left| \int_0^1 \rho^{-m} q_{-m-1}(\rho) \, d\rho \right|^2$$

$$= -2m \left| \int_0^1 \rho^{-m} D_{-m-2} v_{-m-2}(\rho) \, d\rho \right|^2.$$

By the Cauchy-Schwarz inequality we get

for $m \le -2$, $\quad \|q_{m+1}\|_{L_r^2}^2 \le \|D_m v_m\|_{L_r^2}^2 + \|D_{-m-2} v_{-m-2}\|_{L_r^2}^2;$

for $m \ge -1$, $\quad \|q_{m+1}\|_{L_r^2}^2 = \|D_m v_m\|_{L_r^2}^2,$

which gives

$$\|q\|_0^2 \le 2\|\mathbf{w}\|_X^2.$$

If q is given by $q_m = a_m r^{|m|}$; $q_0 = 0$, then $D_m v_m$ vanishes for $m < 0$, which gives equality.

3. A Galerkin method for the Stokes problem in a disc

Let N and K be two integers greater than or equal to 2. Let $\mathbf{S}_K([0, 2\pi])$ be the set of trigonometric polynomials in θ of degree less than or equal to K, and $\mathbf{P}_N([0, 1])$ the set of polynomials in r of degree less than or equal to N. Before introducing the discrete spaces, let us write precisely the bilinear forms a and b. If u and v are expanded in the $(\mathbf{V}_m, \mathbf{W}_m)$ basis with coefficients (v_m, w_m) and $(\tilde{v}_m, \tilde{w}_m)$, and if the coefficients of q in the $\{H_m\}$ are denoted by q_m, we have

$$a(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \sum_{m \in \mathbf{Z}} [(D_{m-1} v_{m-1}, D_{m-1} \tilde{v}_{m-1})_r + (D_{-m-1} w_{m+1}, D_{-m-1} \tilde{w}_{m+1})_r],$$

$$b(\mathbf{v}, q) = \frac{1}{2} \sum_{m \in \mathbf{Z}} (q_m, D_{m-1} \tilde{v}_{m-1} + D_{-m-1} \tilde{w}_{m+1})_r.$$

This suggests to choose q in $\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}$, and $\mathbf{v}$ in such a space that $D_{m-1} \tilde{v}_{m-1}$ vanishes for $|m-1| \ge N-1$, and belongs to $\mathbf{P}_{N-1}$. This introduces a term in $\ln r$, which cannot be avoided.

The discrete space M_N is defined by

(3.1) $$M_N = M \cap (\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}),$$

where $M = L_0^2(\Omega)$. Any q in $\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}$ is expanded in separate variables as

(3.2) $$q = \sum_{|m| \le N-1} q_m H_m; \quad q_m \in \mathbf{P}_{N-1},$$

and the discrete space X_N is defined by

(3.3) $$X_N = X \cap H_N,$$

where $X = \mathbf{H}_0^1(\Omega)$ and H_N is the space of real functions $\mathbf{v}$ such that

(3.4) $$\mathbf{v} = \sum_{m=-N}^{N-2} v_m \mathbf{V}_m + \sum_{m=-(N-2)}^{N} w_m \mathbf{W}_m,$$

where v_m belongs to $\mathbf{P}_N$ for $-N \leq m \leq 0$, and to $\mathbf{P}_N \oplus \mathbb{Q}_m$ for $1 \leq m \leq N-2$, where $\mathbb{Q}_m = \{g(r) : g(r) = cr^m \ln r, \ c \in \mathbb{C}\}$. We provide X_N and M_N with the inner products of X and M. The discrete problem is the following: find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that

$$(3.5) \qquad \begin{cases} \forall \mathbf{v} \in X_N, & a(\mathbf{u}_N, \mathbf{v}) + b(\mathbf{v}, p_N) = L(\mathbf{v}), \\ \forall q \in M_N, & b(\mathbf{u}_N, q) = 0. \end{cases}$$

In order to analyze this problem, we need some classical results on Jacobi polynomials. The definitions and results can be found in [1] or [6].

3.1. **Jacobi polynomials on** $[0,1]$. Let ω be a positive function on $[0,1]$ such that, for any $k \geq 0$, ωr^k is integrable. Define

$$L^2_\omega(0,1) = \left\{ v \in \mathcal{D}'(0,1), \int_0^1 \omega(r)|v|^2(r)\, dr < +\infty \right\}$$

and provide this with the natural scalar product $(v, w)_\omega = \int_0^1 \omega(r) v(r) \overline{w}(r)\, dr$ and the corresponding norm $\|\cdot\|_{L^2_\omega}$. For any given weight ω, there exists a sequence of orthogonal polynomials in $L^2_\omega(0,1)$. If α, β are two integers, and $\omega = \omega_{\alpha,\beta} = (1-r)^\alpha r^\beta$, they are the sequence of Jacobi polynomials $J_n^{\alpha,\beta}$, normalized by $J_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}$. Their norm is given by

$$(3.6) \qquad \|J_n^{\alpha,\beta}\|^2_{L^2_{\omega_{\alpha,\beta}}} = \frac{(n+\alpha)!(n+\beta)!}{(n+\alpha+\beta)!(2n+\alpha+\beta+1)n!}.$$

The unbounded operator $L_{\alpha,\beta}$ on $L^2_{\omega_{\alpha,\beta}}(0,1)$ is defined by

$$(3.7) \qquad L_{\alpha,\beta} = -\frac{1}{\omega_{\alpha,\beta}} \frac{d}{dr} [\omega_{\alpha+1,\beta+1} \frac{d}{dr}].$$

It is selfadjoint positive on $L^2_{\omega_{\alpha,\beta}}(0,1)$, the eigenfunctions are $J_n^{\alpha,\beta}$ with eigenvalues $\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$. The polynomials $J_n^{\alpha,\beta}$ satisfy the differential equation

$$(3.8) \qquad r(1-r)(J_n^{\alpha,\beta})'' + (\beta+1-(\alpha+\beta+2)r)(J_n^{\alpha,\beta})' + \lambda_n^{\alpha,\beta} J_n^{\alpha,\beta} = 0.$$

Moreover, the Jacobi polynomials are given by the recursion formula

$$(3.9)$$

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)J_{n+1}^{\alpha,\beta}$$
$$= (2n+\alpha+\beta+1)[\alpha^2-\beta^2+(2n+\alpha+\beta+2)(2n+\alpha+\beta)(2r-1)]J_n^{\alpha,\beta}$$
$$\qquad - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)J_{n-1}^{\alpha,\beta},$$

$$J_0^{\alpha,\beta} = 1; \quad J_1^{\alpha,\beta} = (\alpha+\beta+2)r - (\beta+1).$$

We shall use the formula relating $J_n^{\alpha,\beta}$ and $J_{n-1}^{\alpha+1,\beta+1}$:

$$(3.10) \qquad \frac{d}{dr} J_n^{\alpha,\beta} = (n+\alpha+\beta+1)J_{n-1}^{\alpha+1,\beta+1},$$

and several easy results on $J_n^{0,1}$ and $J_n^{1,1}$. The sequence $J_n^{0,1}$ is orthogonal for the weight r. One has

$$(3.11) \qquad J_n^{0,1}(0) = (-1)^n(n+1),$$

and (by integration of (3.8) on $[0, 1]$)

$$(3.12) \qquad \int_0^1 J_n^{0,1}(r) \, dr = 2(-1)^n \|J_n^{0,1}\|_{L^2_{\omega_{0,1}}}^2 .$$

The polynomials $J_n^{0,1}$ and $J_n^{0,0}$ (the nth Legendre polynomial) are related by

$$(3.13) \qquad J_n^{0,1} = \frac{J_n^{0,0} + J_{n+1}^{0,0}}{2r}.$$

3.2. **Existence and uniqueness: the discrete inf-sup condition.** Since X_N and M_N are subspaces of X and M, properties (i), (ii) and (iv) in Theorem B are satisfied. For existence and uniqueness, we merely need to prove (iii). We shall prove the constant in (iii) to be the same as in the continuous case.

Theorem 3.1. *On $X_N \times M_N$ one has the uniform inf-sup condition*

$$(3.14) \qquad \inf_{q \in M_N} \sup_{\mathbf{v} \in X_N} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = \frac{1}{\sqrt{2}}.$$

Proof. Again, we have

$$(3.15) \qquad \inf_{q \in M_N} \sup_{\mathbf{v} \in X_N} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = \inf_{q \in M_N} \frac{\|\mathbf{w}\|_X}{\|q\|_M},$$

where $\mathbf{w}$ is the unique solution of

$$(3.16) \qquad \forall \mathbf{v} \in X_N, \quad (\Delta \mathbf{w} + \nabla q, \mathbf{v})_0 = 0.$$

Using formula (2.14a), we can easily see that if q belongs to M_N, then $\mathbf{w} = -(\Delta)^{-1} \nabla q$ belongs to X_N. The constant is thus greater than or equal to $\frac{1}{\sqrt{2}}$. The choice $q_0 = 0$, $q_m = r^m$, $1 \le m \le N - 1$, gives equality. $\qquad \square$

This result, together with Theorem A, leads to the conclusion:

Theorem 3.2. *For any $\mathbf{f}$ in $H^{-1}(D(0,1))$, problem (3.5) has a unique solution $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$, and*

$$\|\mathbf{u}_N\|_X^2 + \|p_N\|_0^2 \le C\|\mathbf{f}\|_{-1}^2.$$

3.3. **Projection in weighted spaces on $[0, 1]$.** In order to obtain error estimates, we need one-dimensional projection results in weighted Sobolev spaces on $[0, 1]$. Results of the same type have been obtained in [1] for the weights $\omega_{\alpha,\alpha}$; our proofs rely in a large part on their methods. The additional difficulties come from the fact that we cannot use any Hardy inequality.

For any positive integer m, we denote $H_r^p(0, 1) = \{\psi \in \mathcal{D}'(0, 1), \psi^{(j)} \in L_r^2(0, 1)$ for any $j, 0 \le j \le p\}$ and furnish it with the norm $\|\psi\|_{H_r^p}^2 = \sum_{0 \le j \le p} \|\psi^{(j)}\|_{L_r^2}^2$.

Theorem 3.3. (i) *For any q in L_r^2, there exists a unique polynomial $\Pi_N q$ in $\mathbf{P}_N$ such that*

$$(3.17) \qquad \forall Q \in \mathbf{P}_N, \quad \|q - \Pi_N q\|_{L_r^2} \le \|q - Q\|_{L_r^2}.$$

(ii) *For any positive integer p, one has*

$$(3.18) \qquad \forall q \in H_r^p, \quad \|q - \Pi_N q\|_{L_r^2} \le C N^{-p} \|q\|_{H_r^p}.$$

Proof. Expand q in the polynomials $Q_n = J_n^{0,1}$:

$$(3.19) \qquad q = \sum_{n=0}^{\infty} q_n Q_n; \qquad q_n = \frac{(q, Q_n)_r}{\|Q_n\|_{L_r^2}^2}.$$

Then $\Pi_N q$ is given by

$$(3.20) \qquad \Pi_N q = \sum_{n=0}^{N} q_n Q_n,$$

and

$$(3.21) \qquad \|q - \Pi_N q\|_{L_r^2}^2 = \sum_{n=N+1}^{\infty} |q_n|^2 \|Q_n\|_{L_r^2}^2.$$

Here, Q_n is an eigenfunction of the selfadjoint operator $L = L_{0,1}$, corresponding to the eigenvalue $\lambda_n = \lambda_n^{0,1} = n(n+2)$. Then, for any integer s,

$$(3.22) \qquad (q, Q_n)_r = \frac{1}{\lambda_n^s} (L^s q, Q_n)_r,$$

$$(3.23) \qquad \|q - \Pi_N q\|_{L_r^2}^2 = \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n^{2s}} \frac{[(L^s q, Q_n)_r]^2}{\|Q_n\|_{L_r^2}^2},$$

which gives, for q sufficiently smooth, the bounds

$$(3.24) \quad \|q - \Pi_N q\|_{L_r^2}^2 \le \frac{C}{N^{4s}} \|L^s q\|_{L_r^2}^2, \qquad \|q - \Pi_N q\|_{L_r^2}^2 \le \frac{C}{N^{4s+2}} |(L^s q, L^{s+1} q)_r|.$$

In order to estimate $\|L^s q\|_{L_r^2}^2$ and $(L^s q, L^{s+1} q)_r$, we introduce, for any integer k and any function q defined on $[0, 1]$, the quantities

$$|||q|||_k^2 = \sum_{j=0}^{k} \int_0^1 r^{j+1} (1 - r)^j |q^{(j)}|^2 \, dr.$$

If q belongs to H_r^k, then $|||q|||_k^2$ is well defined and $|||q|||_k^2 \le \|q\|_{H_r^k}^2$. It is easy to see by induction that for any positive integer s, one has the following bounds:

$$(3.25a) \qquad \forall q \in H_r^{2s}, \quad \|L^s q\|_{L_r^2} \le |||L^s q|||_0 \le C |||q|||_{2s};$$

$$(3.25b) \qquad \forall q \in H_r^{2s+1}, \quad (L^s q, L^{s+1} q)_r \le |||L^s q|||_1 \le C |||q|||_{2s+1}.$$

This ends the proof of the theorem. $\qquad \qquad \qquad \qquad \qquad \square$

The results in H_r^1 are less classical. Let $\widetilde{\mathbf{P}}_N([0,1])$ be the subspace of $\mathbf{P}_N([0,1])$ of polynomials vanishing at $r = 1$.

Theorem 3.4. (i) *For any q in $H_r^1 \cap \mathcal{C}^0([0,1])$, there exists a unique polynomial $\Pi_N^1 q$ in $\mathbf{P}_N$ such that*

$$(3.26) \qquad \Pi_N^1 q(1) = q(1); \quad \forall Q \in \widetilde{\mathbf{P}}_N, \quad ((q - \Pi_N^1 q)', Q')_r = 0.$$

(ii) *For any q in H_r^p with $p \ge 2$,*

$$(3.27) \qquad \|(q - \Pi_N^1 q)'\|_{L_r^2} \le C N^{-(p-1)} \|q\|_{H_r^p},$$

$$(3.28) \qquad |(q - \Pi_N^1 q)(0)| \le C N^{-(p-1)} \|q\|_{H_r^p}.$$

Proof. (i) Since q' belongs to L_r^2, its projection $\Pi_{N-1}q'$ on $\mathbf{P}_{N-1}$ is well defined. Then $\Pi_N^1 q$ is uniquely determined by

$$(3.29) \qquad\qquad \Pi_N^1 q = q(1) - \int_r^1 \Pi_{N-1}q'(\rho)\,d\rho.$$

(ii) Assume now that q belongs to H_r^p for $p \geq 2$. According to [14], q belongs to $\mathcal{C}^0([0,1])$, so $\Pi_N^1 q$ is well defined, and

$$\|(q - \Pi_N^1 q)'\|_{L_r^2} = \|q' - \Pi_{N-1}q'\|_{L_r^2},$$

which, together with (3.18) proves (3.27).

In order to prove (3.28), we write

$$(q - \Pi_N^1 q)(0) = \int_0^1 (q' - \Pi_{N-1}q')(\rho)\,d\rho$$

and proceed as in Theorem 3.3: expand q' in the polynomials $Q_n : q' = \sum_{n=0}^{\infty} q_n Q_n$, and

$$\int_0^1 (q' - \Pi_{N-1}q')(\rho)\,d\rho = \sum_{n=N}^{+\infty} q_n \int_0^1 Q_n(\rho)\,d\rho = 2\sum_{n=N}^{+\infty} (-1)^n q_n \|Q_n\|_{L_r^2}^2$$

(using (3.12)). Then, for any $s \geq 0$,

$$\int_0^1 (q' - \Pi_{N-1}q')(\rho)\,d\rho = 2\sum_{n=N}^{+\infty} \frac{(-1)^n}{\lambda_n^s}(L^s q', Q_n)_r,$$

and by the Cauchy-Schwarz inequality,

$$|(q - \Pi_N^1 q)(0)|^2 \leq 4\left(\sum_{n=N}^{+\infty} \frac{[(L^s q', Q_n)_r]^2}{\|Q_n\|_{L_r^2}^2}\right)\left(\sum_{n=N}^{+\infty} \frac{\|Q_n\|_{L_r^2}^2}{\lambda_n^{2s}}\right).$$

The first term has been estimated in Theorem 3.3. As for the second, we have

$$2\sum_{n=N}^{+\infty} \frac{\|Q_n\|_{L_r^2}^2}{\lambda_n^{2s}} \leq \sum_{n=N}^{+\infty} \frac{1}{n^{4s+1}},$$

and for $s > 0$ (cf. [7])

$$\sum_{n=N}^{+\infty} \frac{1}{n^{4s+1}} \sim \int_N^{+\infty} x^{-(4s+1)}\,dx = \frac{1}{4s}N^{-4s}.$$

This, together with (3.25a), gives the successive bounds:

$$\forall s > 0, \quad |(q - \Pi_N^1 q)(0)| \leq CN^{-2s}\|L^s q'\|_{L_r^2} \leq CN^{-2s}\|q'\|_{H_r^{2s}} \leq CN^{-2s}\|q\|_{H_r^{2s+1}}.$$

In the same way, for any $s \geq 0$, one has

$$|(q - \Pi_N^1 q)(0)|^2 \leq CN^{-4s-2}|(L^s q', L^{s+1} q')_r|$$
$$\leq CN^{-4s-2}\|q'\|_{H_r^{2s+1}}^2 \leq CN^{-4s-2}\|q\|_{H_r^{2s+2}}^2,$$

which gives (3.28) for any integer $p \geq 2$. $\square$

We also need to estimate the norm of $q - \Pi_N^1 q$ in L_r^2.

Theorem 3.5. *For any integer $p \geq 2$, for any q in H_r^p, one has*

$$(3.30) \qquad\qquad \|q - \Pi_N^1 q\|_{L_r^2} \leq CN^{-p}\|q\|_{H_r^p}.$$

Proof. It goes by a duality argument:

$$(3.31) \qquad \|q - \Pi_N^1 q\|_{L_r^2} = \sup_{g \in L_r^2} \frac{(q - \Pi_N^1 q, g)_r}{\|g\|_{L_r^2}}.$$

We shall use a bidimensional result: if Ω is smooth enough, we know (see for example [3]) that for any f in $L^2(\Omega)$, there exists a unique u in $H_0^1(\Omega)$ such that $-\Delta u = f$. Furthermore u belongs to $H^2(\Omega)$ and $\|u\|_2 \leq C\|f\|_0$. Here, Ω is $D(0,1)$, and we choose $f = g(r)$ in L_r^2. Then $u = \psi(r)$ belongs to H_r^2 and is such that

$$(3.32) \qquad -\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = g,$$

$$(3.33) \qquad \|\psi\|_{H_r^2} \leq C\|g\|_{L_r^2}.$$

Integration by parts, using the boundary data, gives

$$(q - \Pi_N^1 q, g)_r = ((q - \Pi_N^1 q)', \psi')_r.$$

Moreover, since $\psi(1) = 0$, we have that $\Pi_N^1 \psi$ belongs to $\widetilde{\mathbf{P}}_N$ and

$$((q - \Pi_N^1 q)', (\Pi_N^1 \psi)')_r = 0.$$

Hence,

$$(q - \Pi_N^1 q, g)_r = ((q - \Pi_N^1 q)', (\psi - \Pi_N^1 \psi)')_r \leq \|(q - \Pi_N^1 q)'\|_{L_r^2}\|(\psi - \Pi_N^1 \psi)'\|_{L_r^2}.$$

Using (3.27) and (3.33), we have

$$\|(\psi - \Pi_N^1 \psi)'\|_{L_r^2} \leq CN^{-1}\|\psi\|_{H_r^2} \leq C\|g\|_{L_r^2},$$

$$\|(q - \Pi_N^1 q)'\|_{L_r^2} \leq CN^{-(p-1)}\|q\|_{H_r^p},$$

and for any g in L_r^2,

$$(3.34) \qquad (q - \Pi_N^1 q, g)_r \leq CN^{-p}\|q\|_{H_r^p}\|g\|_{L_r^2}.$$

We now plug (3.34) in (3.31) and get (3.30). $\qquad\qquad\qquad\qquad\qquad\square$

3.4. **Approximation results.** Because of the ellipticity of the bilinear form a and the uniform inf-sup condition, Theorem 1.1 in [8] gives a first approximation result: if $(\mathbf{u}, p)$ and $(\mathbf{u}_N, p_N)$ are the solutions to (2.3) and (3.5), respectively, there exists a positive constant C such that

$$(3.35) \quad \|\mathbf{u} - \mathbf{u}_N\|_X + \|p - p_N\|_0 \leq C\left\{\inf_{\mathbf{v}_N \in V_N} \|\mathbf{u} - \mathbf{v}_N\|_X + \inf_{q_N \in M_N} \|p - q_N\|_0\right\},$$

where V_N is the discrete space corresponding to V:

$$(3.36) \qquad V_N = \{\mathbf{v}_N \in X_N, \forall q_N \in M_N, b(\mathbf{v}_N, q_N) = 0\}.$$

It remains to estimate the expressions in the right-hand side of (3.35).

Theorem 3.6. *Let p be an integer greater than or equal to zero. For any q in $M \cap H^p(\Omega)$, its projection $\tilde{q} = \Pi_{N,K}q$ on $S_K \otimes \mathbf{P}_N$ belongs to M and satisfies the following estimate:*

$$(3.37) \qquad \|q - \tilde{q}\|_0 \leq C\min\left\{\frac{1}{N^p}, \frac{1}{K^p}\right\}\|q\|_p.$$

On the other hand, $\mathbf{u}$ belongs to V. It can be approximated in $V \cap X_N$:

Theorem 3.7. *Let p be a positive integer. For any $\mathbf{v}$ in $V \cap \mathbf{H}^p(\Omega)$, there exists $\mathbf{w}$ in $V \cap X_N$ such that*

$$(3.38) \qquad\qquad \|\mathbf{v} - \mathbf{w}\|_X \leq CN^{1-p}\|\mathbf{v}\|_p.$$

These two theorems give the final optimal estimate:

Theorem 3.8. *Let p be an integer greater than or equal to zero. If $\mathbf{f}$ belongs to $\mathbf{H}^p(\Omega)$, the solutions $(\mathbf{u}, p)$ and $(\mathbf{u}_N, p_N)$ to (2.3) and (3.5) satisfy the following estimate:*

$$(3.39) \qquad\qquad \|\mathbf{u} - \mathbf{u}_N\|_X + \|p - p_N\|_0 \leq CN^{-1-p}\|\mathbf{f}\|_p.$$

Proof of Theorem 3.6. We expand q in $\{H_k\}$ as $q = \sum_{k\in\mathbf{Z}} q_k(r)H_k$. Its projection $\widetilde{\Pi}_K$ on $S_K \otimes L_r^2$ is given by $\widetilde{\Pi}_K q = \sum_{|k|\leq K} q_k(r)H_k$, and the operator $\Pi_{N,K}$ is defined by

$$\tilde{q} = \Pi_{N,K}q = \sum_{|k|\leq K} \Pi_N q_k(r)H_k.$$

In particular, if q belongs to M, then $\tilde{q}$ belongs to M, and

$$(3.40) \qquad\qquad \|q - \tilde{q}\|_0 \leq \|q - \widetilde{\Pi}_K q\|_0 + \|\widetilde{\Pi}_K q - \tilde{q}\|_0.$$

The first term is estimated through the one-dimensional result in [5]:

$$\|q - \widetilde{\Pi}_K q\|_0 \leq CK^{-p}\left\|\frac{\partial^p q}{\partial\theta^p}\right\|_0,$$

which gives

$$(3.41) \qquad\qquad \|q - \widetilde{\Pi}_K q\|_0 \leq CK^{-p}\|q\|_p.$$

As for the second term, we have

$$\|\widetilde{\Pi}_K q - \tilde{q}\|_0^2 = \sum_{|k|\leq K} \|q_k - \Pi_N q_k\|_{L_r^2}^2,$$

and using Theorem 3.3, we get

$$\|\widetilde{\Pi}_K q - \tilde{q}\|_0^2 \leq CN^{-2p} \sum_{|k|\leq K} \|q_k\|_{H_r^p}^2,$$

$$(3.42) \qquad\qquad \|\widetilde{\Pi}_K q - \tilde{q}\|_0 \leq CN^{-p}\|q\|_p.$$

Plugging (3.41) and (3.42) in (3.40) gives the desired estimate in the theorem. $\square$

Proof of Theorem 3.7. Recall that for $\mathbf{v}$ in $H^p(\Omega) \cap H_0^1(\Omega)$, its curl $\nabla \wedge \mathbf{v}$ belongs to $H^{p-1}(\Omega)$, and

$$\|\nabla \wedge \mathbf{v}\|_{p-1} \leq 2\|\mathbf{v}\|_p.$$

If moreover $\mathbf{v}$ belongs to V, then $\|\mathbf{v}\|_X = \|\nabla \wedge \mathbf{v}\|_0$. Here, $\nabla \wedge \mathbf{v}$ can be projected on $\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}$ in the following way. We expand $\nabla \wedge \mathbf{v}$ and $\nabla \cdot \mathbf{v}$ in $\{H_m(\theta)\}$:

$$\nabla \cdot \mathbf{v} = \sum_{m\in\mathbb{Z}} D_m v_m H_{m+1} + \sum_{m\in\mathbb{Z}} D_{-m} w_m H_{m-1},$$

$$\nabla \wedge \mathbf{v} = i\left(\sum_{m\in\mathbb{Z}} D_m v_m H_{m+1} - \sum_{m\in\mathbb{Z}} D_{-m} w_m H_{m-1}\right).$$

If $\mathbf{v}$ belongs to V, the function φ defined by $\varphi = -\frac{i}{2}\nabla \wedge \mathbf{v}$ belongs to $L^2(\Omega)$, and by Theorem 3.6, one has the following error estimate on its projection on $\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}, \chi = \sum_{m=-N}^{N-2} \chi_m H_{m+1}$,

$$\|\varphi - \chi\|_0 \leq cN^{1-p}\|\varphi\|_{p-1}.$$

We can define $\mathbf{w}$ in $V \cap X_N$ such that $\nabla \wedge \mathbf{w} = 2i\chi$. It is given by

$$\mathbf{w} = \sum_{m=-N}^{N-2} \tilde{v}_m \mathbf{V}_m + \sum_{m=-(N-2)}^{N} \tilde{w}_m \mathbf{W}_m,$$

the coefficients being defined for $-(N-2) \leq m \leq N$ by $\tilde{w}_m = \bar{\tilde{v}}_m$, and

$$* - N \leq m \leq -1, \quad \tilde{v}_m = r^m \int_0^r \rho^{-m}\chi_m(\rho)\, d\rho,$$

$$*\tilde{v}_0 = -\int_r^1 \chi_0(\rho)\, d\rho,$$

$$*1 \leq m \leq N-2, \quad \tilde{v}_m = -r^m \int_r^1 \rho^{-m}\chi_m(\rho)\, d\rho.$$

The assumptions we made force $\mathbf{w}$ to belong to $X_N \cap V$. Moreover,

$$\|\mathbf{v} - \mathbf{w}\|_X = \|\nabla \wedge \mathbf{v} - \nabla \wedge \mathbf{w}\|_0 = 2\|\varphi - \chi\|_0 \leq CN^{1-p}\|\varphi\|_{p-1},$$

$$\|\mathbf{v} - \mathbf{w}\|_X \leq CN^{1-p}\|\mathbf{v}\|_p. \qquad \square$$

There is also an L^2-estimate on the velocity:

Theorem 3.9. *Let p be a positive integer. If $\mathbf{f}$ belongs to $\mathbf{H}^p(\Omega)$, there exists a positive constant C such that*

(3.43) $$\|\mathbf{u} - \mathbf{u}_N\|_0 \leq CN^{-2-p}\|\mathbf{f}\|_p.$$

The proof is classical and will be omitted (cf. [8]).

4. Pseudospectral method

We start with a description of the quadrature formula we shall use in the r-variable.

4.1. **Discrete formulation.** The fully discrete formulation relies on the Gauss-Lobatto quadrature formula for the weight r on $[0,1]$ (cf. [6]).

Theorem 4.1. *Let N be an integer ≥ 2, $r_0 = 0$, $r_N = 1$. There exists a unique set of $N-1$ points r_j in $(0,1)$ and $N-1$ positive weights ρ_j such that*

(4.1) $$\forall g \in \mathbf{P}_{2N-1}, \quad \int_0^1 rg(r)\, dr = \sum_{j=0}^N \rho_j g(r_j).$$

The points r_j are the zeros of $(J_N^{0,1})' = J_{N-1}^{1,2}$, the weights ρ_j are given by

(4.2) $$1 \leq j \leq N, \quad \rho_j = \frac{1}{N(N+2)[J_N^{0,1}(r_j)]^2}, \quad \rho_0 = \frac{2}{N(N+2)(N+1)^2}.$$

We now introduce on $C^0([0,1])$ the discrete inner product

$$(4.3) \qquad (f,g)_N = \sum_{j=0}^{N} \rho_j f(r_j) \overline{g}(r_j).$$

The corresponding Hermitian form is denoted by $|\cdot|_N$.

Theorem 4.2. *For any integer $N \geq 2$, $|\cdot|_N$ is a norm on $\mathbf{P}_N([0,1])$, which is equivalent to $\|\cdot\|_{L_r^2}$. More precisely, one has*

$$(4.4) \qquad \forall \varphi \in \mathbf{P}_N([0,1]), \quad \|\varphi\|_{L_r^2} \leq |\varphi|_N \leq \sqrt{3}\|\varphi\|_{L_r^2}.$$

Proof. Expand φ in $\mathbf{P}_N([0,1])$ in the $J_k^{0,1}$, $\varphi = \sum_0^N \lambda_k J_k^{0,1}$, express $\|\varphi\|_{L_r^2}^2$ and $|\varphi|_N^2$, use the fact that the continuous and discrete integration formulae agree on $\mathbf{P}_{2N-1}$, and formula (4.2) (for details see [1] for the method and [9] in this case). $\qquad\square$

The discrete inner product is now defined in $L^2(\Omega)$ by

$$(4.5) \qquad (\varphi, \chi)_{N,N} = \int_0^{2\pi} (\varphi(\cdot,\theta), \chi(\cdot,\theta))_N \, d\theta.$$

Since we use trigonometric polynomials, we do not discretize the tangential integral. The discrete bilinear forms a_N and b_N are given by

$$(4.6) \qquad \begin{cases} a_N(\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u}, \nabla\mathbf{v})_{N,N}, \\ b_N(\mathbf{v},q) = -(q, \nabla\cdot\mathbf{v})_{N,N}, \end{cases}$$

which can be rewritten by expanding u and v in the $(\mathbf{V}_m, W_m)$ basis with coefficients (v_m, w_m) and $(\tilde{v}_m, \tilde{w}_m)$, and q in the $\{H_m\}$ with coefficients q_m,

$$a_N(\mathbf{u},\mathbf{v}) = \frac{1}{2} \sum_{m=-(N-1)}^{N-1} [(D_{m-1}v_{m-1}, D_{m-1}\tilde{v}_{m-1})_N$$

$$+ (D_{-m-1}w_{m+1}, D_{-m-1}\tilde{w}_{m+1})_N],$$

$$b_N(\mathbf{v},q) = \frac{1}{2} \sum_{m=-(N-1)}^{N-1} (q_m, D_{m-1}\tilde{v}_{m-1} + D_{-m-1}\tilde{w}_{m+1})_N.$$

The special form of our discrete spaces allows the following pleasant result:

Lemma 4.1. *The discrete forms a_N and b_N are exact on $X_N \times M_N$, i.e.,*

$$\forall(\mathbf{u},\mathbf{v}) \in X_N \times X_N, \quad a_N(\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}),$$

$$\forall(\mathbf{v},q) \in X_N \times M_N, \quad b_N(\mathbf{v},q) = b(\mathbf{v},q).$$

The proof is straightforward and will be omitted.

The discrete problem now reads: find $(\tilde{\mathbf{u}}_N, \tilde{p}_N)$ in $X_N \times M_N$ such that

$$(4.7) \qquad \begin{cases} \forall\mathbf{v} \in X_N, \quad a_N(\tilde{\mathbf{u}}_N, \mathbf{v}) + b_N(\mathbf{v}, \tilde{p}_N) = L_N(\mathbf{v}), \\ \forall q \in M_N, \quad b_N(\tilde{\mathbf{u}}_N, q) = 0, \end{cases}$$

where L_N is defined in the following way: $\mathbf{f}$ and $\mathbf{v}$ are expanded as

$$\mathbf{f} = \sum_{m\in\mathbf{Z}} f_m \mathbf{V}_m + \sum_{m\in\mathbf{Z}} g_m \mathbf{W}_m, \quad \mathbf{v} = \sum_{m=-N}^{N-2} v_m \mathbf{V}_m + \sum_{m=-(N-2)}^{N} w_m \mathbf{W}_m;$$

then

$$2(\mathbf{f}, \mathbf{v}) = \sum_{m=-N}^{N-2} (f_m, v_m)_r + \sum_{m=-(N-2)}^{N} (g_m, w_m)_r.$$

For $1 \leq m \leq N - 2$, define an operator R_m by

$$(4.8) \qquad R_m h = -r^{-m-1} \int_0^r \rho^{m+1} h(\rho) \, d\rho.$$

If h belongs to $L_r^2(0, 1)$, then $R_m h$ belongs to $L_r^2(0, 1)$, and

$$(4.9) \qquad \|R_m h\|_{L_r^2}^2 \leq \frac{1}{4(m+1)} \|h\|_{L_r^2}^2.$$

Using integration by parts, we can write

$$2(\mathbf{f}, \mathbf{v}) = \sum_{m=-N}^{0} (f_m, v_m)_r + \sum_{m=1}^{N-2} (R_m f_m, D_m v_m)_r$$
$$+ \sum_{m=0}^{N} (g_m, w_m)_r + \sum_{m=-(N-2)}^{-1} (R_{-m} g_m, D_{-m} w_m)_r.$$

We can now define the discrete linear operator by

$$(4.10) \qquad 2L_N(\mathbf{v}) = \sum_{m=-N}^{0} (f_m, v_m)_N + \sum_{m=1}^{N-2} (R_m f_m, D_m v_m)_N$$
$$+ \sum_{m=0}^{N} (g_m, w_m)_N + \sum_{m=-(N-2)}^{-1} (R_{-m} g_m, D_{-m} w_m)_N.$$

Theorem 4.3. *If f belongs to $C^0(\Omega)$, problem (4.7) has a unique solution.*

Proof. By Lemma 4.1, we only need to check that the mapping $\mathbf{v} \to L_N(\mathbf{v})$ is continuous on X_N. By Theorem 4.2 and the Cauchy-Schwarz inequality,

$$|L_N(\mathbf{v})|^2 \leq C \left[\sum_{m=-N}^{0} |f_m|_N^2 + \sum_{m=1}^{N-2} |R_m f_m|_N^2 \right] \|\mathbf{v}\|_X^2.$$

The first sum is bounded by a constant times $\|\mathbf{f}\|_\infty^2$. As for the second, we have for any i,

$$|R_m f_m(r_i)|^2 \leq \frac{1}{2m+2} \|f_m\|_{L_r^2}^2$$

and

$$|L_N(\mathbf{v})| \leq C \|\mathbf{f}\|_\infty \|\mathbf{v}\|_X. \qquad \square$$

4.2. Interpolation formula on $[0, 1]$. Let h be an element of $H_r^p(0, 1)$, for $p \geq 2$. By [14], h is continuous on $[0, 1]$. Then define $I_N h$ as the polynomial interpolating h at the Gauss-Lobatto points i.e.,

$$(4.11) \qquad I_N h \in \mathbf{P}_N; \quad \forall j, 1 \leq j \leq N, \quad I_N h(r_j) = h(r_j).$$

The aim of this section is to prove the following result:

Theorem 4.4. *Let p be an integer ≥ 2. If h belongs to H_r^p, one has*

$$(4.12) \qquad \|h - I_N h\|_{L_r^2} \leq C N^{-p} \|h\|_{H_r^p}.$$

The proof goes along the lines in [1]:

(i) Estimate zeros and weights in (4.1).

(ii) If $h(1) = 0$, estimate $\|I_N h\|_{L_r^2}$ by Theorem 4.3.

(iii) Estimate $\|h - I_N h\|_{L_r^2}$ by the projection theorems of §3.2.

(i) *Estimation of the weights.* From ([16, p. 353]), one has

$$(4.13) \qquad 1 \le j \le N-1, \quad \rho_j \sim \frac{\pi}{N} r_j^{3/2} (1-r_j)^{1/2}.$$

Location of the zeros.

Lemma 4.2. *For any integer $N \ge 2$, the zeros $r_j = \cos^2 \frac{\theta_j}{2}$ of $J_{N-1}^{1,2}$ are such that $\theta_j \in K_j$, where the intervals K_j are defined by*

$$(4.14) \qquad 1 \le j \le [\frac{N}{2}] - 1; \quad K_j = \left(\frac{j - \frac{1}{4}}{N + \frac{3}{2}} \pi, \frac{j+1}{N+1} \pi \right),$$

$$[\frac{N}{2}] \le j \le [\frac{N}{2}] + 1; \quad K_j = \left(\frac{j - \frac{1}{4}}{N + \frac{3}{2}} \pi, \frac{j + \frac{3}{4}}{N + \frac{1}{2}} \pi \right),$$

$$[\frac{N}{2}] + 1 \le j \le N - 1; \quad K_j = \left(\frac{j}{N+2} \pi, \frac{j + \frac{3}{4}}{N + \frac{1}{2}} \pi \right).$$

Proof. In ([16, p. 138]) we find the location of the zeros of the Legendre polynomials $J_N^{0,0}$. We now use (3.10) and (3.13) to get (4.14). □

(ii) *Estimation of $\|I_N h\|_{L_r^2}$.*

Lemma 4.3. *Let p be an integer ≥ 2. If h belongs to H_r^p and $h(1) = 0$, then*

$$(4.15) \qquad \|I_N h\|_{L_r^2}^2 \le C(\|h\|_{L_r^2}^2 + N^{-2} \|h'\|_{L_r^2}^2 + N^{-4} |h(0)|^2).$$

Proof. From Theorem 4.3, we have

$$\|I_N h\|_{L_r^2}^2 \le |I_N h|_N^2 = \sum_{j=0}^{N} \rho_j |h|^2(r_j).$$

If $h(1) = 0$, then from (4.2),

$$\|I_N h\|_{L_r^2}^2 \le 2N^{-4} |h(0)|^2 + \sum_{j=1}^{N-1} \rho_j |h|^2(r_j),$$

and, from (4.13),

$$(4.16) \qquad \|I_N h\|_{L_r^2}^2 \le C \left(N^{-4} |h(0)|^2 + N^{-1} \sum_{j=1}^{N-1} r_j^{3/2} (1-r_j)^{1/2} |h|^2(r_j) \right).$$

Using the function G defined by

$$(4.17)$$

$$g(r) = r^{3/2} (1-r)^{1/2} f(r), \quad F(\theta) = h(r), \quad G(\theta) = g(r), \quad \text{with } r = \cos^2 \frac{\theta}{2},$$

we can rewrite (4.13) as

$$(4.18) \qquad \|I_N h\|_{L_r^2}^2 \le C \left(N^{-4} |h(0)|^2 + N^{-1} \sum_{j=1}^{N-1} \sup_{\theta \in K_j} |G(\theta)|^2 \right).$$

Assume for the time being the two following results.

Lemma 4.4. *Let p be an integer ≥ 2. If ϕ belongs to H_r^p and $\phi(1) = 0$, then*

$$(4.19) \qquad \int_0^1 \frac{|\phi(r)|^2}{1-r}\, dr \leq \int_0^1 r|\phi'(r)|^2\, dr.$$

Lemma 4.5. *Let p be an integer ≥ 2. If h belongs to H_r^p, the function $G(\theta)$ defined in (4.17) belongs to $H^1(0,\pi)$ and is such that*

$$(4.20) \qquad \|G\|_{L^2(0,\pi)} = \|h\|_{L_r^2}, \quad \|G'\|_{L^2(0,\pi)} \leq C\|h'\|_{L_r^2}.$$

Lemma 1.4 in [1] reads

$$(4.21) \qquad \sup_{\theta \in K_j} |G(\theta)|^2 \leq C\left(\frac{1}{|K_j|}\|G\|_{L^2(K_j)}^2 + |K_j|\|G'\|_{L^2(K_j)}^2\right),$$

where $|K|$ is the length of the interval K. Note that, for any j, K_j and K_{j+3} are disjoined. Thus, the union of intervals K_j covers at most 3 times $(0,\pi)$. Moreover, there exists a strictly positive number C such that $|K_j| \sim \frac{C}{N}$. These two remarks, together with (4.20) and (4.21), give (4.15). $\qquad\square$

Proof of Lemma 4.4. Since ϕ belongs to $H^1(0,1)$, we write $\phi(r) = -\int_r^1 \phi'(s)\, ds$, and by the Cauchy-Schwarz inequality,

$$|\phi(r)|^2 \leq (1-r)\int_r^1 |\phi'(s)|^2\, ds,$$

which proves that $\frac{|\phi(r)|^2}{1-r}$ belongs to $L^1(0,1)$ and

$$\int_0^1 \frac{|\phi(r)|^2}{1-r}\, dr \leq \int_0^1 \int_r^1 |\phi'(s)|^2\, ds\, dr = \int_0^1 r|\phi'(r)|^2\, dr.$$

Proof of Lemma 4.5. A mere change of variables in the integral shows at once that $\|G\|_{L^2(0,\pi)} = \|h\|_{L_r^2}$. As for the derivative, we have

$$\|G'\|_{L^2(0,\pi)} = \int_0^1 r^2(1-r)|h'(r)|^2\, dr + \frac{1}{16}\int_0^1 \frac{20r - 16r^2 - 3}{1-r}|h(r)|^2\, dr.$$

On $[0,1]$, the numerator in the second integral is bounded by a strictly positive constant, and Lemma 4.4 allows us to conclude.

(iii) *Estimation of $\|h - I_N h\|_{L_r^2}$.* For any h in H_r^p, we introduce $\Pi_N^1 h$ as in Theorem 3.4. Since it belongs to $\mathbf{P}_N$, we can write

$$\|h - I_N h\|_{L_r^2}^2 \leq 2(\|h - \Pi_N^1 h\|_{L_r^2}^2 + \|I_N(\Pi_N^1 h - h)\|_{L_r^2}^2).$$

Using Lemma 4.3 for $\Pi_N^1 h - h$, we get

$$\|h - I_N h\|_{L_r^2}^2 \leq C(\|\Pi_N^1 h - h\|_{L_r^2}^2 + N^{-2}\|(\Pi_N^1 h - h)'\|_{L_r^2}^2 + N^{-4}|(\Pi_N^1 h - h)(0)|^2).$$

Theorems 3.4 and 3.5 give (4.12) in Theorem 4.4.

4.3. **Approximation results.** We start with the same result as in §3.4. The solutions $(\mathbf{u}, p)$ and $(\tilde{\mathbf{u}}_N, \tilde{p}_N)$ to (2.3) and (4.7) satisfy a first estimate:

$$(4.22) \qquad \|\mathbf{u} - \tilde{\mathbf{u}}_N\|_X + \|p - \tilde{p}_N\|_0$$

$$\leq C \left\{ \inf_{\mathbf{v}_N \in V_N} \|\mathbf{u} - \mathbf{v}_N\|_X + \inf_{q_N \in M_N} \|p - q_N\|_0 + \|L - L_N\| \right\},$$

where $V_N = \{\mathbf{v}_N \in X_N, \forall q_N \in M_N, b_N(\mathbf{v}_N, q_N) = 0\}$, and

$$(4.23) \qquad \|L - L_N\| = \sup_{\mathbf{v}_N \in V_N} \frac{|L(\mathbf{v}_N) - L_N(\mathbf{v}_N)|}{\|\mathbf{v}_N\|_X}.$$

Using the results in §§3.3 and 4.3, we shall prove the following result.

Theorem 4.5. *Let p be an integer ≥ 0. If f belongs to $H^p(\Omega)$, the solutions $(\mathbf{u}, p)$ and $(\tilde{\mathbf{u}}_N, \tilde{p}_N)$ to (2.3) and (4.7) satisfy the following estimates:*

$$(4.24) \qquad \|\mathbf{u} - \tilde{\mathbf{u}}_N\|_X + \|p - \tilde{p}_N\|_0 \leq CN^{-p}\|\mathbf{f}\|_p.$$

Proof. From Theorem 3.6, since p belongs to $H^{p+1}(\Omega)$, we have

$$(4.25) \qquad \|p - \Pi_{N-1,N-1} p\|_0 \leq CN^{-(p+1)}\|p\|_{p+1} \leq CN^{-(p+1)}\|\mathbf{f}\|_p,$$

and since $\mathbf{u}$ belongs to $V \cap \mathbf{H}^{p+2}(\Omega)$, there exists $\mathbf{v}_N$ in $V \cap X_N$ such that

$$(4.26) \qquad \|\mathbf{u} - \mathbf{v}_N\|_X \leq CN^{1-(p+2)}\|\mathbf{u}\|_{p+2} \leq CN^{-(p+1)}\|\mathbf{f}\|_p.$$

It remains to estimate $\|L - L_N\|$. For any $\mathbf{v}$ in X_N, we have

$$|L(\mathbf{v}) - L_N(\mathbf{v})| \leq \sum_{m=-N}^{0} |(f_m, v_m)_N - (f_m, v_m)_r|$$

$$+ \sum_{m=1}^{N-2} |(R_m f_m, D_m v_m)_N - (R_m f_m, D_m v_m)_r|.$$

Let us estimate the first sum. For $-N \leq m \leq 0$, since the quadrature formula is exact on $\mathbf{P}_{2N-1}$,

$$(f_m, v_m)_N - (f_m, v_m)_r = (I_N f_m - \Pi_{N-1} f_m, v_m)_N - (f_m - \Pi_{N-1} f_m, v_m)_r$$

and by Theorem 4.2,

$$|(I_N f_m - \Pi_{N-1} f_m, v_m)_N| \leq 3\|I_N f_m - \Pi_{N-1} f_m\|_{L_r^2}\|v_m\|_{L_r^2},$$

so that

$$|(f_m, v_m)_N - (f_m, v_m)_r| \leq C[\|f_m - \Pi_{N-1} f_m\|_{L_r^2} + \|f_m - I_N f_m\|_{L_r^2}]\|v_m\|_{L_r^2}.$$

By Theorems 3.3 and 4.4, we conclude that

$$(4.27) \qquad \sum_{m=-N}^{0} |(f_m, v_m)_N - (f_m, v_m)_r| \leq CN^{-p}\|\mathbf{f}\|_p\|\mathbf{v}\|_0.$$

In order to estimate the second sum, we need a lemma.

Lemma 4.6. *For any $p \geq 0$, for any m, $-1 \leq m \leq N - 2$, for any φ in H_r^p, $R_m \varphi$ belongs to H_r^p and*

$$(4.28) \qquad \|R_m \varphi\|_{H_r^p} \leq 2\|\varphi\|_{H_r^p}.$$

Let us assume the lemma. For $1 \le m \le N - 2$, $I_N(R_m f_m)$ belongs to $\mathbf{P}_N$ and $D_m v_m$ belongs to $\mathbf{P}_{N-1}$, so

$$(R_m f_m, D_m v_m)_N - (R_m f_m, D_m v_m)_r = (I_N(R_m f_m) - R_m f_m, D_m v_m)_r$$

and by Theorem 4.4,

(4.29) $$\sum_{m=1}^{N-2} |(R_m f_m, D_m v_m)_N - (R_m f_m, D_m v_m)_r| \le CN^{-p}\|\mathbf{f}\|_p\|\mathbf{v}\|_X.$$

Adding (4.27) and (4.29) gives

(4.30) $$|L(\mathbf{v}) - L_N(\mathbf{v})| \le CN^{-p}\|\mathbf{f}\|_p\|\mathbf{v}\|_X.$$

Plugging (4.25), (4.26) and (4.30) in (4.22), we obtain (4.24). $\qquad\square$

Proof of Lemma 4.6. It is easy to see by induction that, for any $k \ge 0$,

$$(R_m\varphi)^{(k)} = -\frac{1}{m+k+1}\{k\varphi^{(k-1)} + (m+1)R_{m+k+1}\varphi^{(k)}\}.$$

We now use (4.9) to get an upper bound on $\|(R_m\varphi)^{(k)}\|_{L_r^2}$:

$$\|(R_m\varphi)^{(k)}\|_{L_r^2}^2 \le 2\{\|\varphi^{(k-1)}\|_{L_r^2}^2 + \frac{1}{4}\|\varphi^{(k)}\|_{L_r^2}^2\}.$$

Summing for $1 \le k \le p$ gives (4.28).

5. COUPLING SPECTRAL METHOD AND TRANSPARENT BOUNDARY CONDITION

We consider the Stokes problem in the whole plane:

(5.1) $$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \mathbf{R}^2, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbf{R}^2. \end{cases}$$

We shall assume $\mathbf{f}$ to be compactly supported in the disc $D(0, R)$ centered at point 0 and of radius R. If Ω is an unbounded domain, $W^1(\Omega)$ is defined by

(5.2) $$W^1(\Omega) = \{v \in \mathcal{D}'(\Omega), \frac{v}{(1+r^2)^{1/2}(1+\ln(1+r^2))} \in L^2(\Omega), \nabla v \in L^2(\Omega)\},$$

furnished with the natural inner product and norm

$$\|\mathbf{v}\|_{W^1(\Omega)}^2 = \left\|\frac{\mathbf{v}}{(1+r^2)^{1/2}(1+\ln(1+r^2))}\right\|_0^2 + \|\nabla v\|_0^2.$$

Note that $\mathbf{R} \subset W^1(\Omega)$. A result in [15] asserts that if $\mathbf{f}$ belongs to $(L_0^2(\mathbf{R}^2))^2$, this problem has a unique solution $(\mathbf{u}, p)$ in $(W^1(\mathbf{R}^2)/\mathbf{R})^2 \times L^2(\mathbf{R}^2)$. In order to compute $(\mathbf{u}, p)$, we shall introduce a fictitious boundary, the circle centered at point 0 and of radius R, and solve the Stokes problem in $D(0, R)$ with the so-called transparent boundary condition. This boundary condition represents the solution outside the disc.

5.1. **Transparent boundary condition and reduction to a bounded domain.** We shall denote $\Omega = D(0, R)$, $\Omega' = \mathbf{R}^2 - \Omega$, Γ their common boundary, $\Gamma = C(0, R)$. The normal vector to Γ is oriented toward the exterior of Ω; we shall call it $\mathbf{n}$ (it is $\mathbf{e}_r$ with the notations of §2). Problem (5.1) is equivalent to the coupling

$$\begin{cases} -\Delta\mathbf{u}_1 + \nabla p_1 = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_1 = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\Delta\mathbf{u}_2 + \nabla p_2 = \mathbf{0} & \text{in } \Omega', \\ \nabla \cdot \mathbf{u}_2 = 0 & \text{in } \Omega', \end{cases}$$

with the transmission conditions

$$\begin{cases} \mathbf{u}_1 = \mathbf{u}_2 & \text{on } \Gamma, \\ \sigma_n(\mathbf{u}_1) = \sigma_n(\mathbf{u}_2) & \text{on } \Gamma, \end{cases}$$

where σ_n is the normal strain, i.e.,

$$(5.3) \qquad \qquad \sigma_n(\mathbf{u}) = \frac{\partial\mathbf{u}}{\partial n} - p\mathbf{n}.$$

Consider the problem

$$(5.4) \qquad \begin{cases} -\Delta\mathbf{w} + \nabla q = \mathbf{0} & \text{in } \Omega', \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega', \\ \mathbf{w} = \mathbf{g} & \text{on } \Gamma. \end{cases}$$

According to [15] again, if $\mathbf{g}$ belongs to $\mathbf{H}^{1/2}(\Gamma)$, this problem has a unique solution $(\mathbf{w}, q)$ in $(W^1(\Omega'))^2 \times L^2(\Omega')$ and

$$\|\mathbf{v}\|_{(W^1(\Omega'))^2} \leq C\|\mathbf{g}\|_{H^{1/2}(\Gamma)}.$$

Denote by K the linear operator from $\mathbf{H}^{1/2}(\Gamma)$ to $\mathbf{H}^{-1/2}(\Gamma)$ defined by $K\mathbf{w} = -\sigma_n(\mathbf{w})$. Owing to the transmission conditions on Γ, problem (5.1) is equivalent to the following boundary value problem in Ω:

$$(5.5) \qquad \begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \sigma_n(\mathbf{u}) + K\mathbf{u} = 0 & \text{on } \Gamma. \end{cases}$$

5.2. **Basic properties and expression of the transparent operator K.** General results valid in any sufficiently smooth geometry assert (see [15] or [9]):

Theorem 5.1. *The linear operator K from $\mathbf{H}^{1/2}(\Gamma)$ to $\mathbf{H}^{-1/2}(\Gamma)$ is continuous, symmetric, positive*: *for any $\mathbf{g}$ in $\mathbf{H}^{1/2}(\Gamma)$, $\langle K\mathbf{g}, \mathbf{g}\rangle_\Gamma \geq 0$.*

We shall now give the expression of K in polar coordinates. We shall use the notations in §2, the singularity being here at infinity.

We decompose $\mathbf{g}$ in

$$(5.6) \qquad \begin{cases} \mathbf{g} = \mathbf{g}^{(1)} + \mathbf{g}^{(2)}, \\ \mathbf{g}^{(1)}(\theta) = \sum_{m\in\mathbf{Z}} g_m \mathbf{V}_m(\theta), \quad \mathbf{g}^{(2)}(\theta) = \overline{\mathbf{g}}^{(1)}(\theta) = \sum_{m\in\mathbf{Z}} h_m \mathbf{W}_m(\theta). \end{cases}$$

The function (Euclidean) orthogonal to $\mathbf{g}$ is given by $\mathbf{g}^\perp = i(-\mathbf{g}^{(1)} + \mathbf{g}^{(2)})$. The Hilbert operator is defined by

$$(5.7) \qquad \mathcal{H}\mathbf{g} = \sum_{m \in \mathbf{Z}} i\, \text{sign}(m) g_m \mathbf{V}_m(\theta) + \sum_{m \in \mathbf{Z}} i\, \text{sign}(m) h_m \mathbf{W}_m(\theta)$$

(with the convention, $\text{sign}(m) = 0$ if $m = 0$).

Theorem 5.2. *The operator K is given by*

$$(5.8) \qquad\qquad K\mathbf{g} = \frac{1}{R}\frac{\partial}{\partial \theta}(\mathbf{g}^\perp + \mathcal{H}\mathbf{g}),$$

or in extended form,

$$(5.9) \qquad K\mathbf{g} = \frac{1}{R}\left[\sum_{m \le -1} -m g_m \mathbf{V}_m(\theta) + \sum_{m \ge 1} 3 m g_m \mathbf{V}_m(\theta)\right.$$

$$\left. + \sum_{m \le -1} -3 m h_m \mathbf{W}_m(\theta) + \sum_{m \ge 1} m h_m \mathbf{W}_m(\theta)\right].$$

Proof. Since $\mathbf{g}$ is given by (5.6), we solve (5.4) in polar coordinates. The first step is to notice that q is harmonic in Ω' and belongs to $L_r^2(1, +\infty)$. Thus, $q = \sum_{m \in \mathbf{Z}} q_m H_m$, with

$$(5.10) \qquad\qquad q_m = \begin{cases} \alpha_m r^{-|m|}, & |m| \ge 2, \\ 0, & |m| \le 1. \end{cases}$$

With the notations given in (2.9), the first equation gives

$$(5.11) \qquad\qquad \forall m \in \mathbf{Z}, \quad D_m v_m - q_{m+1} = c_m r^{-(m+1)}.$$

For any $m \le 0$, $r^{-(m+1)}$ does not belong to $L_r^2(1, +\infty)$; therefore, $c_m = 0$:

$$\forall m \le 0, \quad D_m v_m - q_{m+1} = 0.$$

For any $m \ge 1$, integrate (5.11) from r to $+\infty$ and use (5.10). Then

$$v_m = -\frac{1}{2m}(c_m + \alpha_{m+1})r^{-m}, \quad \text{and } \forall m \ge 1, \quad D_m v_m = -2m\frac{v_m}{r}.$$

The divergence-free condition reads, $\forall m \in \mathbf{Z}$, $D_{m-1}v_{m-1} + D_{-m-1}w_{m+1} = 0$; thus,

$$\forall m \ge 1, \quad D_m v_m = -2m\frac{v_m}{r} = -q_{m+1}.$$

The operator K is now defined by

$$K\mathbf{g} = \sum_{m \in \mathbf{Z}}(q_{m+1} - D_m v_m - m\frac{v_m}{r})(R)\mathbf{V}_m$$

$$+ \sum_{m \in \mathbf{Z}}(q_{m-1} - D_{-m}w_m + m\frac{w_m}{r})(R)\mathbf{W}_m,$$

which gives (5.9). The compact formulation (5.8) comes in a straightforward way from (5.9). ∎

5.3. **Weak formulation of** (5.4). For simplicity, we shall assume from now on that $R = 1$. We define on $H^1(\Omega)$ a bilinear form $\tilde{a}$ by

$$(5.12) \qquad \tilde{a}(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + \langle K\mathbf{u}, \mathbf{v}\rangle_\Gamma.$$

Lemma 5.1. *The bilinear form $\tilde{a}$ defines on $(H^1(\Omega)/\mathbf{R})^2$ a scalar product. The corresponding norm is equivalent to the natural norm in $\mathbf{H}^1(\Omega)$.*

The proof is straightforward, since K is positive, and the L^2-norm of the gradient is equivalent to the H^1-norm in $H^1(\Omega)/\mathbf{R}$.

The Hilbert space X is $(H^1(\Omega)/\mathbf{R})^2$, furnished with the bilinear form $\tilde{a}$, $M = \mathbf{L}^2(\Omega)$. The variational formulation of problem (5.5) reads:

$$\text{find } (\mathbf{u}, p) \text{ in } X \times M \text{ such that}$$

$$(5.13) \qquad \begin{cases} \forall \mathbf{v} \in X, & \tilde{a}(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v}\rangle, \\ \forall q \in M, & b(\mathbf{u}, q) = 0. \end{cases}$$

Theorem 5.3. *Problem* (5.13) *has a unique solution. In particular, one has the Babuška-Brezzi condition*

$$(5.14) \qquad \inf_{q\in M} \sup_{\mathbf{v}\in X} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = 1.$$

Proof. By Theorem B, it is enough to prove (5.14). We proceed as in (2.5):

$$\inf_{q\in M} \sup_{\mathbf{v}\in X} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = \inf_{q\in M} \frac{\|\mathbf{w}\|_X}{\|q\|_M},$$

where $\mathbf{w}$ is the unique solution in X to

$$(5.15) \qquad \begin{cases} \mathbf{w} \in H^1(\Omega), \\ \Delta\mathbf{w} + \nabla q = \mathbf{0} & \text{in } \Omega, \\ \frac{\partial\mathbf{w}}{\partial r} + q\mathbf{e}_r + K\mathbf{w} = 0 & \text{on } \Gamma. \end{cases}$$

We shall express $\mathbf{w}$ in X in terms of q in M. Here, q and $\mathbf{w}$ are given by (2.7), (2.8), and the norms are easily computed:

$$a(\mathbf{w}, \mathbf{w}) = \sum_{m\in\mathbf{Z}} \|D_m v_m\|_{L_r^2}^2 + \sum_{m\in\mathbf{Z}} m|v_m(1)|^2,$$

$$\langle K\mathbf{w}, \mathbf{w}\rangle_\Gamma = 3\sum_{m\geq 1} m|v_m(1)|^2 - \sum_{m\leq -1} m|v_m(1)|^2,$$

$$(5.16) \qquad \tilde{a}(\mathbf{w}, \mathbf{w}) = \sum_{m\in\mathbf{Z}} \|D_m v_m\|_{L_r^2}^2 + 4\sum_{m\geq 1} m|v_m(1)|^2.$$

The same arguments as in (2.13a) prove

$$(5.17) \qquad \forall m \in \mathbf{Z}, \quad \left(\frac{d}{dr} + \frac{m+1}{r}\right)(q_{m+1} + D_m v_m) = 0,$$

$$(5.18) \qquad \forall m \geq 0, \quad q_{m+1} + D_m v_m = 0.$$

The boundary condition, expanded in the basis $\{\mathbf{V}_m, \mathbf{W}_m\}$, gives

(5.19)
$$\begin{cases} \text{for } m \leq -1, \quad (q_{m+1} + D_m v_m)(1) = 0, \\ \text{for } m \geq 0, \quad (q_{m+1} + D_m v_m + 4m v_m)(1) = 0. \end{cases}$$

From (5.17), (5.18) and (5.19), we get, for $m \geq 1$, $v_m(1) = 0$, and for any m in $\mathbf{Z}$, $\frac{1}{2}q_{m+1} + D_m v_m = 0$. This can be solved explicitly (modulo a constant in v_0):

(5.20a)
$$m \geq 1, \quad v_m = \frac{1}{2} r^m \int_r^1 \rho^{-m} q_{m+1}(\rho) \, d\rho,$$

(5.20b)
$$v_0 = \alpha - \frac{1}{2} \int_r^1 q_{m+1}(\rho) \, d\rho,$$

(5.20c)
$$m < 0, \quad v_m = -\frac{1}{2} r^m \int_0^r \rho^{-m} q_{m+1}(\rho) \, d\rho,$$

and

$$\|q\|_0^2 = \sum_{m \in \mathbf{Z}} \|q_m\|_{L_r^2}^2 = \sum_{m \in \mathbf{Z}} \|D_m v_m\|_{L_r^2}^2 = a(\mathbf{w}, \mathbf{w}).$$

This completes the proof of the theorem. □

5.4. **The Galerkin method.** The discrete spaces are the same as in §3, i.e.,

(5.21)
$$\begin{cases} M_N = M \cap (\mathbf{S}_{N-1} \otimes \mathbf{P}_{N-1}), \\ X_N = X \cap H_N, \end{cases}$$

and the discrete problem reads: find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that

(5.22)
$$\begin{cases} \forall \mathbf{v} \in X_N, a(\mathbf{u}_N, \mathbf{v}) + b(\mathbf{v}, p_N) = L(\mathbf{v}), \\ \forall q \in M_N, b(\mathbf{u}_N, q) = 0. \end{cases}$$

Theorem 5.4. *On $X_N \times M_N$ one has the uniform inf-sup condition*

(5.23)
$$\inf_{q \in M_N} \sup_{\mathbf{v} \in X_N} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = 1.$$

Proof. We write again

(5.24)
$$\inf_{q \in M_N} \sup_{\mathbf{v} \in X_N} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_X \|q\|_M} = \inf_{q \in M_N} \frac{\|\mathbf{w}\|_X}{\|q\|_M},$$

where $\mathbf{w}$ is the unique solution to

(5.25)
$$\forall \mathbf{v} \in X_N, \quad -(\Delta \mathbf{w} + \nabla q, \mathbf{v})_0 + \langle K\mathbf{w} + \frac{\partial \mathbf{w}}{\partial r} + q \mathbf{e}_r, v \rangle_\Gamma = 0.$$

Using formula (5.20a), we can easily see that if q belongs to M_N, then $\mathbf{w} = -(\Delta)^{-1} \nabla q$ belongs to X_N. The constant is thus greater than or equal to 1. The choice $q_0 = 0$, $q_m = r^m$, $1 \leq m \leq N-1$, gives equality. □

Theorem B gives the conclusion:

Theorem 5.5. *For any $\mathbf{f}$ in $H^{-1}(\Omega)$, the problem (5.21) has a unique solution $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ and*

$$\|\mathbf{u}_N\|_X + \|p_N\|_0 \leq C\|\mathbf{f}\|_{-1}.$$

Slight modifications to the proofs in §3 give the optimal error estimates:

Theorem 5.6. *For any integer $p \geq 0$, if $\mathbf{f}$ belongs to $\mathbf{H}^p(\Omega)$, the solutions $(\mathbf{u}, p)$ and $(\mathbf{u}_N, p_N)$ to (5.13) and (5.22) satisfy the optimal error estimates*

$$(5.26) \qquad \|\mathbf{u} - \mathbf{u}_N\|_X + \|p - p_N\|_0 \leq CN^{-1-p}\|\mathbf{f}\|_p,$$

$$(5.27) \qquad \|\mathbf{u} - \mathbf{u}_N\|_0 \leq CN^{-2-p}\|\mathbf{f}\|_p.$$

Remark 5.1. By scaling we can solve the problem in $D(0, R)$ with a Galerkin method. This leads to the following error estimates:

$$\|\mathbf{u} - \mathbf{u}_N\|_X + \|p - p_N\|_0 \leq C(\frac{N}{R})^{-1-p}\|\mathbf{f}\|_p,$$

$$\|\mathbf{u} - \mathbf{u}_N\|_0 \leq C(\frac{N}{R})^{-2-p}\|\mathbf{f}\|_p.$$

6. Conclusion

This is a first step toward the solution of exterior problems by spectral methods in a bounded domain. The second step should be to deal with operators with nonconstant coefficients, and the third step the three-dimensional case, with the use of spherical harmonic functions. This will be of great use, for instance in meteorology.

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