HYPERBOLIC BOUNDARY VALUE PROBLEMS WITH TRIHEDRAL CORNERS

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It is a pleasure to contribute this paper to celebrate the 90th *birthday of Peter Lax. Forty eight years ago Peter suggested the study of mixed initial boundary value problems for hyperbolic equations as a thesis topic for JBR. This article returns to this rich area. We thank Peter for his friendship, teaching, and inspiration. We offer our best wishes on this landmark birthday.*

Abstract. Existence and uniqueness theorems are proved for boundary value problems with trihedral corners and distinct boundary conditions on the faces. Part I treats strictly dissipative boundary conditions for symmetric hyperbolic systems with elliptic or hidden elliptic generators. Part II treats the Bérenger split Maxwell equations in three dimensions with possibly discontinuous absorptions. The discontinuity set of the absorptions or their derivatives has trihedral corners. Surprisingly, there is almost no loss of derivatives for the Bérenger split problem. Both problems have their origins in numerical methods with artificial boundaries.

1. Introduction.

1.1. **Overview.** This paper analyses mixed initial boundary value problems in domains with corners that arise when one computes approximate solutions of hyperbolic equations on unbounded or large domains by simulations on a smaller computational domain. The computational domain is very often a ball or a rectangle. The latter is the most common and has corners as in the figure 1 below. At the external boundaries absorbing boundary conditions are imposed. The boundary conditions on adjacent faces are usually different, so the initial boundary value problem is of mixed type because of the change in boundary condition.

In spatial dimension $d = 3$ the external corner is a meeting point of three orthogonal faces making a trihedral angle. The study of hyperbolic problems in such regions is very little developed. For nontrivial absorbing conditions we know of no previous work asserting existence and uniqueness with trihedral angles. It is often
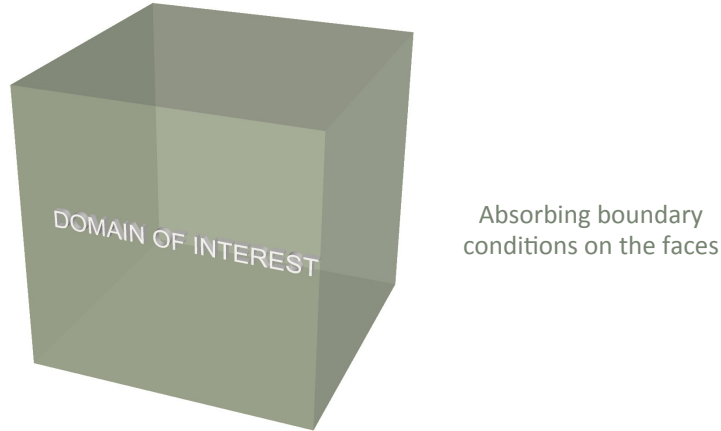
FIGURE 1. Artificial boundary

easy to prove existence of fairly weak solutions and uniqueness of fairly regular ones. Closing this gap for these **external corners** is the subject of Part I.

A second set of problems leading to domains with trihedral angles is the use of the perfectly matched layers of Bérenger. The geometry for this method in dimension $d = 2$ is a rectangular domain including in its interior the domain of interest, surrounded by absorbing layers where Bérenger split equations are satisfied with transmission conditions on all the solid horizontal and vertical lines in the Figure 2. On the dotted lines absorbing boundary conditions are prescribed. Note
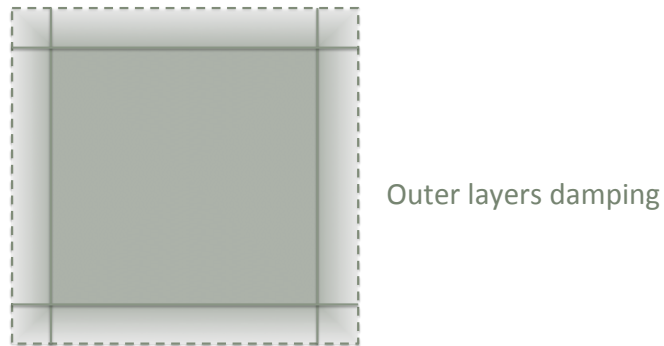
FIGURE 2. Internal corner in two dimensions

in particular the **interior corners**. In dimension three the interior domain is a cube and the interior corners are trihedral. We study the Bérenger transmission problems for Maxwell's equations in $\mathbb{R}^3$. At the intersection of the 3 planes parallel to the coordinate planes in $\mathbb{R}^3$, transmission conditions are prescribed. We give the first proof of existence and uniqueness for the Bérenger split problem with
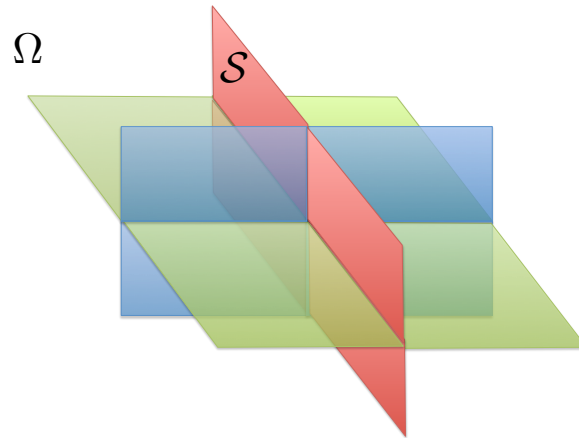
FIGURE 3. Internal corner : $\{x_1\, x_2\, x_3\, =\, 0\}$

more than one absorption coefficient discontinuous. The original prescription of Bérenger was of this type, though in common practice one uses smoother coefficients. With more than twenty years of computational experience, it is not surprising that the problem is well set. Even in the case of smooth absorptions our theorem is surprising because it has almost no loss of derivatives. Sources in H^1 yield solutions in H^1. Shortly after the introduction of Bérenger's method, Abarbanel and Gottlieb [1] proved that the split Maxwell equations are only weakly hyperbolic. Sources in H^s yield solutions in H^{s-1} and not better. The resolution of this apparent contradiction between our result and theirs is that the split system loses a derivative for general initial data. It does not lose a derivative for the divergence free solutions of Maxwell's equations (see Section 3.5). Our earlier paper [10] introduced the scheme of the demonstration analysing the first order system version of the $2d$ wave equation. S. Petit in [21], [10] showed that the split equations are lossless for elliptic generators. We treat the much subtler case of Maxwell's equations.

Our well posedness results apply to the Bérenger split system even when the permittivities are not scalar provided that the non scalar values are constrained to take place on a compact subset of the domain of interest. This is the first such result, with or without loss of derivatives.

The analysis of the Bérenger method for Maxwell's equation answers some important questions but leaves some open. For example at the external boundary one imposes boundary conditions for the Bérenger split system hoped to be absorbing. To our knowledge there are no existence or uniqueness proofs for such exterior corner problems for the split equations.

The analysis of the two problems treated have six common elements. They treat trihedral corners. They proceed by Laplace transform. They rely on elliptic estimates. They use capacity at key points. They both come from numerical methods with artificial boundaries. The estimates of the existence results correspond to stability results for numerical methods.

1.2. **Part I. Dissipative boundaries for elliptic generators.** For symmetric hyperbolic problems, the simplest natural artificial boundary conditions are dissipative. With the aim of absorbing as much as possible, the most natural choices are

strictly dissipative. That is the context of the first part of this paper, strictly dissipative conditions on the faces of rectangular domains. As the faces have different directions, the boundary conditions imposed on adjacent faces are usually different.

Our main result asserts existence, uniqueness, and limited regularity for such problems. Existence is a fairly easy consequence of energy dissipation. It is uniqueness that is difficult. The constructed solutions do not have sufficient regularity to justify an integration by parts. Friedrichs' method of mollifiers does not save the day as there are few tangential directions at corners (see §2.1.2).

1.2.1. *Regularity and incoming corner waves.* A key idea in the analysis is to take advantage of ellipticity or hidden ellipticity in the case of Maxwell's equations. A second idea is to take advantage of estimates on the trace of solutions at the boundary that one gets from strict dissipativity.

Uniqueness asserts that solutions with homogeneous initial and boundary conditions must vanish. How could there be waves in such circumstances? Consider an initial boundary value problem in $\mathbb{R}_t \times \mathcal{O}$ with $\mathcal{O}$ equal to the set of vectors with strictly positive components. The zero initial conditions give the idea that the energy must come from the lateral boundary $\mathbb{R}_t \times \partial\mathcal{O}$. At the flat faces of $\partial\mathcal{O}$ the dissipativity assumption shows that energy is absorbed not emitted. The enemies are the singular parts of $\partial\mathcal{O}$. One must show that energy does not sneak into the domain through those sets, for example the edges of codimension $2, 3, \ldots, d$.

Considering radiation problems on $\mathbb{R}^{1+d}$ with sources $f(t)\delta(x_1)\delta(x_2)\cdots\delta(x_k)$ shows that waves can emerge from sets of dimension $k < d$. The proofs show that energy emerging from sets of codimension ≥ 2, corresponding to the singularities of $\partial\mathcal{O}$ is incompatible with the square integrability and square integrable traces of the objects constructed in the existence theory.

1.2.2. *Corner problems.* Problems with corners have a rich literature some of it very well known. For example, the study of the Dirichlet and Neumann problems in lipshitz domains notably by Jerison and Kenig in the eighties. We appeal to their results at two junctures in the analysis of problems with hidden ellipticity. Their results are used to prove regularity of potentials. They do not treat problems where the boundary conditions change from face to face. Another class of problems concern the diffraction by conical singularities where again the boundary conditions do not change from face to face.

A recent reference that treats polyhedral domains with different boundary conditions on different faces and that includes extensive reference to earlier work is [3]. However, the boundary conditions treated are restricted to elliptic problems with conditions associated to coercive bilinear forms. Our boundary conditions are motivated by absorbing conditions at the edge of computational domains. They usually do not fall under this umbrella.

Higher dimensional corners are discussed by Kupka-Osher in [15] for the constant coefficient scalar wave equation. They employ an explicit solution technique. For uniqueness they merely observe that the conditions are dissipative so uniqueness is a consequence of the energy identity. The oversignt is that the integrations by parts needed to prove the identity require more regularity than the solutions constructed possess. The most famous such example is the Clay Millenium problem concerning the Navier Stokes equations. Existence of not very regular solutions of Navier Stokes was proved by Leray in the thirties. Uniqueness of more regular solutions is easy.

Closing the gap is the problem. Addressing this difficulty for absorbing conditions at a trihedral corner is the problem attacked in Part I.

Taniguchi in a series of papers starting with [26] considered gluing two dissipative problems together at a dihedral corner when one of the problems is strictly dissipative. With respect to the corner variables Taniguchi's coefficients are constant. The analysis is by a Fourier-Laplace transform in those variables. Advantage is taken of the strong trace estimates from the strictly dissipative problem. For the trihedral problem this strategy hits a serious obstruction.

There are points of contact of Part I with Sarason's article [24] largely devoted to dihedral corners. The simple existence proof we give by non characteristic smooth perturbation of the corner is an example of what he calls a *strongly non characteristic boundary* on page 284. His paper includes some multihedral angles in Sections 14-16. The domains are small perturbations, *in the lipshitz norm*, of smooth non characteristic boundaries. The corner of a cube is not of this form. His main thrust is a detailed case by case analysis of *two dimensional corners*. His hypotheses exclude the nonuniqueness example in [17] and are used for uniqueness in [4]. Our sufficient condition (2.6) for the energy identity is sharper than but closely related to Sarason's Theorem 11.1. His strategy as well as that in [12] is to decompose the corner problem into model problems. The meaty part of their demonstrations is the treatment of the elliptic models in two dimensions using functions of one complex variable. That strategy does not extend to the multihedral context.

One can also compare our work with that of Grisvard [8] who shows that in many circumstances, the failure of the standard gain of m elliptic regularity results for m^{th} order coercive boundary value problems at high dimensional corners is due to a finite number of singular corrector functions at the corner. Our $H^{1/2}$ regularity Corollary 2.9 shows that if a result of Grisvard type held in our situation, then the least regular of the possible corrector functions would have to be at least $H^{1/2}$.

The papers by Osher [20] and Sarason-Smoller [25] show how geometric optics constructions can reveal pathological behavior at corners. They inspire some of the examples in Section 2.4.

1.2.3. *Main result.* Part I treats two classes of problem. The easiest to describe is the case where the generator is elliptic. Analogous results are obtained for Maxwell's equation and the linearized compressible Euler equations. For Maxwell the divergence is independent of time while for Euler linearized at a constant state the curl is independent of time. In both cases this allows one to recover estimates resembling those for problems with elliptic generators. In the introduction only the case of operators satisfying the ellipticity hypothesis that is part **ii** of Assumption 1.1 is presented. Problems with hidden ellipticity are treated in Section 2.3.

Consider the case of a single multihedral corner. Using a partition of unity reduces the general case to this one.

Definition 1.1. Denote
$$\mathcal{O} \;:=\; \left\{ x \in \mathbb{R}^d \;:\; x_j > 0, \quad j = 1, \ldots, d \right\}.$$
The **singular subset** of $\partial \mathcal{O}$ is
$$S \;:=\; \left\{ x \in \overline{\mathcal{O}} \;:\; x_j = 0 \;\text{ for at least two values of } j \right\}.$$

Assumption 1.1. i. *The matrix valued functions $A_j(x)$ and $B(x)$ are smooth with partial derivatives of all orders belonging to $L^\infty(\mathbb{R}^d)$. For each x, $A_j(x)$ is hermitian symmetric. The coefficients A_j are constant outside a compact subset of $\mathbb{R}^d$.*

ii. *The differential operator $\sum_j A_j(x)\partial_j$ is elliptic for all $x \in \partial\mathcal{O}$.*

iii. *The subspace $\mathcal{N}_j(x)$ is defined for x belonging to the hyperplane $\{x \in \mathbb{R}^d : x_j = 0\}$ is a smoothly varying subspace, called the boundary subspace, constant outside a compact subset and maximal strictly dissipative for the boundary matrix $-A_j(x)\big|_{x_j=0}$. This means (see §2.1.1) that the dimension of $\mathcal{N}_j$ is equal to the number of positive eigenvalues of $-A_j$ and there is a $c > 0$ so that for all j and all x with $x_j = 0$ and all $v \in \mathcal{N}_j(x)$*

$$\big(- A_j(x)\,v\,,\,v\big)_{\mathbb{C}^N} \;\geq\; c\,\|v\|_{\mathbb{C}^N}^2\,.$$

Here and in the sequel the standard Euclidean scalar product and norm are used on $\mathbb{C}^N$ unless explicitly stated otherwise.

Definition 1.2. Denote

$$A(x,\xi) \;:=\; \sum_j A_j(x)\xi_j\,, \qquad G(x,\partial) \;:=\; A(x,\partial) + B(x)\,,$$

$$L \;:=\; \partial_t + G(x,\partial), \qquad Z(x) \;:=\; B(x) + B(x)^* \;-\; \sum_j \partial_j A_j(x)\,.$$

Denote by L^* the adjoint differential operator with respect to the $L^2(\mathbb{R}^{1+d})$ scalar product, $L^\star\Phi := -\partial_t\Phi - \sum \partial_j(A_j^*\Phi) + B^*\Phi$. The symmetry, $A_j = A_j^*$, implies that

$$L + L^* \;=\; G + G^* = Z(x)\,.$$

Condition **iii** asserts that the boundary space is dissipative for $A(x,\nu(x))$ where $\nu(x)$ is an outward unit normal on $\partial\mathcal{O} \setminus \mathcal{S}$. The minus sign comes from the fact that the outward normal is $-\mathbf{e}_j$ where $\{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ is the standard basis in $\mathbb{R}^d$.

The change of variable $v = e^{-\lambda t}u$ yields an equation of the same type with Z replaced by $Z + \lambda I$. Thus the next assumption entails no loss of generality.

Assumption 1.2. *There is a $\mu > 0$ so that for all x, $Z(x) \geq \mu\,I$.*

Definition 1.3. With the notations of Assumption 1.1, a function $h \in L^2(\partial\mathcal{O} \setminus \mathcal{S})$ is said to satisfy the boundary condition $h \in \mathcal{N}$, when for $1 \leq j \leq d$,

$$h\big|_{\{x_j=0\}\cap\{\partial\mathcal{O}\setminus\mathcal{S}\}} \;\in\; \mathcal{N}_j(x) \quad \text{a.e.}$$

The boundary traces appearing in the next theorem are discussed in Section 2.1.3.

Theorem 1.4. *With Assumptions 1.1 and 1.2, and Definition 1.3, for each $g \in L^2(\mathcal{O})$ there is one and only one $u \in L^\infty\big(]0,\infty[\,;\,L^2(\mathcal{O})\big)$ with*

$$u\big|_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})} \in L^2(]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})), \quad Lu = 0, \quad u(0) = g, \quad u\big|_{\partial\mathcal{O}\setminus\mathcal{S}} \in \mathcal{N}.$$

In addition, $u \in C\big([0,\infty[\,;\,L^2(\mathcal{O})\big)$ and, for all $0 \leq t < T < \infty$ satisfies the energy identity,

$$\|u(T)\|^2 + \int\limits_{[t,T]\times\partial(\mathcal{O}\setminus\mathcal{S})} \big(A\big(x,\nu(x)\big)u, u\big)\,dtd\Sigma + \int\limits_{[t,T]\times\mathcal{O}} \big(Z(x)u, u\big)\,dtdx = \|u(t)\|^2. \quad (1.1)$$

Remark 1.1. 1. Taking $t = 0$ and applying Gronwall's inequality yields

$$\sup_{0\leq s<\infty} \big\|e^{\mu s}\,u(s)\big\|_{L^2(\mathcal{O})}^2 \;+\; \int\limits_{[0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})} \|u\|^2\,dt\,d\Sigma \;\lesssim\; \|u(0)\|_{L^2(\mathcal{O})}^2\,.$$

2. Additional information on the boundary trace and the energy flux is proved in §2.2.4 and §2.2.6.

1.3. **Part II. Internal trihedral angles for Bérenger's strategy.**

1.3.1. *Bérenger's split Maxwell equations.* In contrast to Part I that treats general symmetric systems, the results of the second part are limited to systems that are close cousins of the wave equation, notably Maxwell's equations. Proofs rely on an analysis of equations that are relatives of the Helmholtz equation.

Definition 1.5. The set $\Omega := \{x_1 x_2 x_3 \neq 0\} \subset \mathbb{R}^3$ is the disjoint union of eight open octants. $\mathcal{O} := \{x_j > 0 \text{ for all } j\}$ plays the role of domain of interest. The other seven octants are denoted $\mathcal{O}_\kappa$ with $1 \leq \kappa \leq 7$.

The dynamic Maxwell's equations in time independent media are

$$\varepsilon(x)\, E_t \;=\; \operatorname{curl} B \;-\; 4\pi j, \qquad \mu(x)\, B_t \;=\; -\operatorname{curl} E\,. \tag{1.2}$$

The charge density ρ and current j satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} \;=\; -\operatorname{div} j\,. \tag{1.3}$$

The physically relevant solutions are those satisfying

$$\operatorname{div} \varepsilon E \;=\; 4\pi\rho, \qquad \operatorname{div} \mu B \;=\; 0\,. \tag{1.4}$$

Equation (1.4) is satisfied for all time as soon as it is satisfied at $t = 0$.

Assumption 1.3. i. *In Part II, we suppose that $\varepsilon(x)$ and $\mu(x)$ are C^2 matrix valued functions so that $\partial^\alpha\{\varepsilon, \mu\} \in L^\infty(\mathbb{R}^3)$ for all $|\alpha| \leq 2$, and there is a $C > 0$ so that for all x, $\varepsilon \geq CI$ and $\mu \geq CI$.*

ii. *There is a compact subset $K \subset \mathcal{O}$ with the property that ε and μ are scalar valued on $\mathcal{B} := \mathbb{R}^3 \setminus K$.*

Write

$$\operatorname{curl} \;=\; \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \;=\; \sum C_j\, \partial_j\,, \tag{1.5}$$

$$C_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

Definition 1.6. The Bérenger splitting involves two vector valued functions E, B on $\mathbb{R}_t \times \mathcal{O}$ and *three pairs* of vector valued functions E^j, B^j for $j = 1, 2, 3$ on each of the octants $\mathbb{R}_t \times \mathcal{O}_\kappa$.

The pair E, B satisfies Maxwell's equations (1.2) and (1.4) on $\mathbb{R} \times \mathcal{O}$. On each $\mathbb{R} \times \mathcal{O}_\kappa$ the split variables E^j, B^j satisfy the split system

$$\varepsilon(\partial_t + \sigma_j(x_j))E^j \;=\; C_j \partial_j \sum_{k=1}^{k=3} B^k\,,$$
$$\hspace{6cm} \text{for } \; j = 1, 2, 3. \tag{1.7}$$
$$\mu(\partial_t + \sigma_j(x_j))B^j \;=\; -C_j \partial_j \sum_{k=1}^{k=3} E^k\,.$$

In these equations the reader is warned that $C_j \partial_j$ is a single term. No summation notation is intended.

Abusing notation define the total fields $U := (E, B)$ on all of Ω by

$$U \; := \; (E, B) \; := \; \begin{cases} (E, B) & \text{on} & \mathbb{R} \times \mathcal{O} , \\ \\ \left(\sum E^j , \sum B^j \right) & \text{on} & \mathbb{R} \times (\Omega \setminus \mathcal{O}) . \end{cases} \tag{1.8}$$

The Bérenger split system is completed by the transmission conditions demanding that the tangential components of the function U on the left of (1.8) are continuous across the two dimensional interfaces in $\partial\Omega$. In Section 3.2.1 it is proved that for solutions of the split Maxwell equations, the continuity of the tangential components of U implies the continuity of all components.

1.3.2. *Main result.* Consider sources and solutions supported in $t \geq 0$. In particular, with initial values equal to zero. It is only in this situation that we prove results with essentially no loss of derivatives.

Theorem 1.7. *Suppose that Assumption 1.3 is satisfied and $\omega \supset K$ is open with compact closure $\overline{\omega} \subset \mathcal{O}$.*

i. *There are constants C, λ_0, depending on $\overline{\omega}$, with the following properties. If $\lambda > \lambda_0$, $\operatorname{supp} \mathbf{j} \subset [0, \infty[\times \overline{\omega}$, and*

$$\forall |\alpha| \leq 1, \quad \partial_{t,x}^\alpha \mathbf{j} \; \in \; e^{\lambda t} L^2 \big(\mathbb{R} \, ; \, L^2(\mathbb{R}^3) \big) \; := \; \{ e^{\lambda t} f \; : \; f \in L^2 \big(\mathbb{R} \, ; \, L^2(\mathbb{R}^3) \big) \},$$

then there are E, B defined on $\mathbb{R} \times \mathcal{O}$ and split functions E^j, B^j defined on $\mathbb{R} \times \cup \mathcal{O}_\kappa$, supported in $t \geq 0$, that satisfy the Bérenger split equations and $U \in e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$. The last implies the transmission conditions.

ii. *Any solution with $U \in e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$ satisfies for $\lambda > \lambda_0$,*

$$\int e^{-2\lambda t} \big\| \lambda U \, , \, \nabla_{t,x} U \, , \, \lambda \nabla_{t,x} U \big|_{\overline{\omega}} \big\|_{L^2(\mathbb{R}^3)}^2 \, dt$$
$$\leq \; C \int e^{-2\lambda t} \sum_{|\alpha| \leq 1} \big\| \partial_{t,x}^\alpha \mathbf{j}(t) \big\|_{L^2(\mathbb{R}^3)}^2 \, dt . \tag{1.9}$$

On each octant $\mathcal{O}_\kappa$, the split fields satisfy for each j $E_j^j = B_j^j = 0$ and

$$\int e^{-2\lambda t} \big\| E^j , B^j \, , \, \partial_t E^j , \partial_t B^j \big\|_{L^2(\mathcal{O}_\kappa)}^2 \, dt$$
$$\leq \; C \int e^{-2\lambda t} \sum_{|\alpha| \leq 1} \big\| \partial_{t,x}^\alpha \mathbf{j}(t) \big\|_{L^2(\mathbb{R}^3)}^2 \, dt . \tag{1.10}$$

In particular, such solutions are unique.

Remark 1.2. i. Formula (1.9) has derivatives of order less than or equal to one on both sides. The only possible loss of derivatives is for the split variable E^j, B^j outside the the domain of interest $\mathcal{O}$. The loss is restricted microlocally to $\{\tau = 0\}$.
ii. The estimate for the quantities of interest, namely the restriction of E, B to $\overline{\omega}$ is

$$\lambda^2 \int e^{-2\lambda t} \sum_{|\alpha| \leq 1} \big\| \partial_{t,x}^\alpha E, \partial_{t,x}^\alpha B \big\|_{L^2(\overline{\omega})}^2 \, dt \; \leq \; C \int e^{-2\lambda t} \sum_{|\alpha| \leq 1} \big\| \partial_{t,x}^\alpha \mathbf{j}(t) \big\|_{L^2(\mathbb{R}^3)}^2 \, dt ,$$

identical to the estimates that would hold for the Maxwell equations. The estimates for the Bérenger split equations are somewhat weaker, but only outside the set $\overline{\omega}$. The compact $\overline{\omega}$ can be chosen as large as one likes within the domain of interest $\mathcal{O}$.
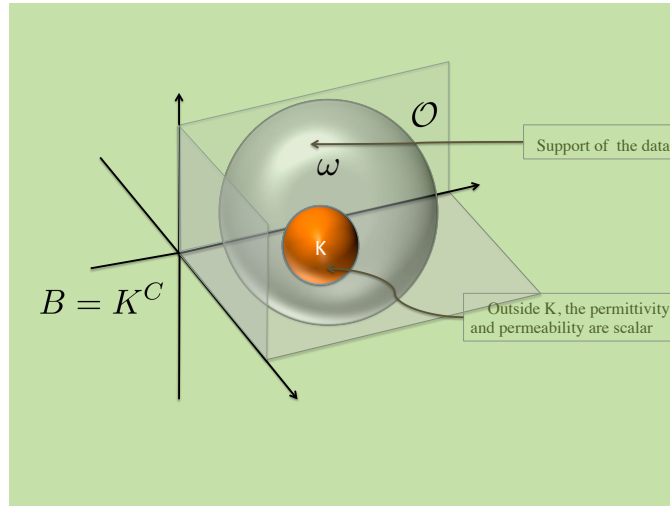
FIGURE 4. Definitions of supports in Theorem 1.7

iii. The solutions constructed above satisfy div $\varepsilon E = 4\pi\rho$, div $\mu B = 0$. Section 3.5 presents a numerical study that contrasts the behavior of the Bérenger splitting for data that satisfy and data that does not satisfy the divergence constraints. **When the divergence constraint is violated**, the loss of derivatives from the Bérenger splitting can occur.

iv. The uniqueness proof uses the Laplace transform. To prove uniqueness of solutions defined only for $t \leq T$ it suffices to continue them using the existence theorem to global solutions and then to apply the global uniqueness result.

v. If one has $\omega \supset \omega_1 \supset K$ then one can construct solutions satisfying divergence free initial conditions $E(0, x), B(0, x) = \mathbf{e}(x), \mathbf{b}(x)$ supported in ω_1 by the following device. Define $t_0 = (1/2)\mathrm{dist}\{\omega_1, \partial\omega\}$. Choose a smooth scalar cutoff function, $\psi(t, x)$, supported in $\mathbb{R}_t \times \omega$, identically equal to one on a neighborhood of $\{t = 0\} \times \omega_1$. and vanishing for $t \geq t_0$. Denote by $\underline{E}, \underline{B}$ the solution of Maxwell's equation on $\mathbb{R}^{1+3}$ with these initial data and $\rho = j = 0$. Then finite speed guarantees that $\underline{E}, \underline{B}$ vanishes outside ω for $0 \leq t \leq t_0$. Subtracting $\psi\underline{E}, \psi\underline{B}$ reduces the inhomogeneous initial value problem to a problem with new source terms on the right. The new source terms, including a divergence free source in the B_t equation, belong to $H_0^1(]0, \infty[\times\omega)$ and cause no trouble.

Remark 1.3. The Bérenger splitting is perfectly matched provided that the permittivities are **constant** outside a compact subset of $\mathcal{O}$. Under this hypothesis, as soon as one proves that the transmission problem is well posed as in Theorem 1.7, it follows that the interfaces are reflectionless and that the restriction of the solution to $\mathcal{O}$ is **exactly** equal to the restriction to $\mathcal{O}$ of the solution of Maxwell's equations. The proof in [9] applies without modification.

2. **Part I. Dissipative boundary conditions for symmetric systems.**

2.1. **Five preliminary results.**

2.1.1. *Nonegative subspaces.* **Notation.** Suppose that $\mathbb{V}$ is a finite dimensional complex scalar product space and $A \in \mathrm{Hom}(\mathbb{V})$ is a hermitian symmetric linear

transformation. Denote by $E_{\geq 0}(A)$ the nonnegative spectral subspace of A and similarly the strictly positive and strictly negative spectral subspaces E_+ and E_-. The transformation is omitted for ease of reading when there is little chance of confusion. Denote by $\Pi_{\geq 0}(A)$, Π_+, and Π_- the associated orthogonal projections.

Definition 2.1. For the transformation $A = A^*$, a linear subspace $\mathcal{N} \subset \mathbb{V}$ is **dissipative** when for all $v \in \mathcal{N}$ one has $(Av, v) \geq 0$. It is **strictly dissipative** when there is a constant $c > 0$ so that for all $v \in \mathcal{N}$

$$(Av, \, v) \ \geq \ c \, \|\Pi_+ v\|^2 \, .$$

It is **maximal dissipative** when in addition $\dim \mathcal{N} = \operatorname{rank} \Pi_{\geq 0}(A)$.

The maximality is equivalent to the fact that there is no strictly larger dissipative subspace.

Lemma 2.2. *For* $\mathbb{V} = E_{\geq 0} \oplus_\perp E_-$, *denote the natural decomposition* $v = v_{\geq 0} + v_-$. *Every maximal dissipative subspace is a graph*

$$v_- = M v_{\geq 0}$$

for a unique linear $M : E_{\geq 0} \to E_-$.

Proof. Suppose that $\mathcal{N}$ is maximal dissipative. The assertion is equivalent to the fact that $\Pi_{\geq 0} : \mathcal{N} \to E_{\geq 0}$ is bijective. Since the dimensions are equal this is equivalent to injectivity.

Suppose that $v \in \mathcal{N}$ and $\Pi_{\geq 0} v = 0$. Then $v \in E_-$. On the other hand, $(Av, v) \geq 0$ by dissipativity. The only $v \in E_-$ for which this is possible is $v = 0$ proving injectivity. $\qquad\square$

Example 2.1. The lemma is used to construct smooth deformations of any maximal dissipative space $\mathcal{N}$ to $E_{\geq 0}$. Precisely choose $\phi \in C^\infty(\mathbb{R})$ with

$$\phi(s) \ = \ \begin{cases} 0 & \text{for } s \leq 1/2 \\ 1 & \text{for } s \geq 1 \end{cases} \ .$$

If $\mathcal{N}$ is the graph of M then the graph of $\phi(s)M$ is maximal dissipative for all s and connects $\mathcal{N}$ for $s \geq 1$ to $E_{\geq 0}$ for $s \leq 1/2$.

2.1.2. Geometry at a corner. In dimension $d > 2$ the study of boundary value problems in a corner is harder and much less developed than the study in regions with a conical singularity with smooth crosssection. The singular set $\mathcal{S}$ includes strata of dimensions $0, 1, 2, 3, \ldots, d - 2$. For example in dimension $d = 2$, the only singularities are corners of dimension 0. In dimension $d = 3$ there are edges of dimension 1 and the corner of dimension 0.

Figure 5 represents a corner of a cube in three dimensions. Figure 6 shows that the corner in $\mathbb{R}^3$ is a cone with triangular cross section. Contrast this with a cone with circular cross section, $\{x_1^2 > x_2^2 + x_3^3, \ x_1 > 0\}$, sketched in Figure 7. At all points other than the corner, the space of tangents is two dimensional.

2.1.3. Traces of solutions of first order systems. The domains with corners, $\mathcal{O} \subset \mathbb{R}^d$ and $I \times \mathcal{O} \subset \mathbb{R}^{1+d}$ with I an open interval bounded or not enter our analysis. They are all lipshitzian domains, but simple ones for which the unit outward normal ν

Codimension three corner.
Zero dimensional space of tangents

Codimension two edge.
One dimensional space of tangents

Codimension one face.
Two dimensional space of tangents
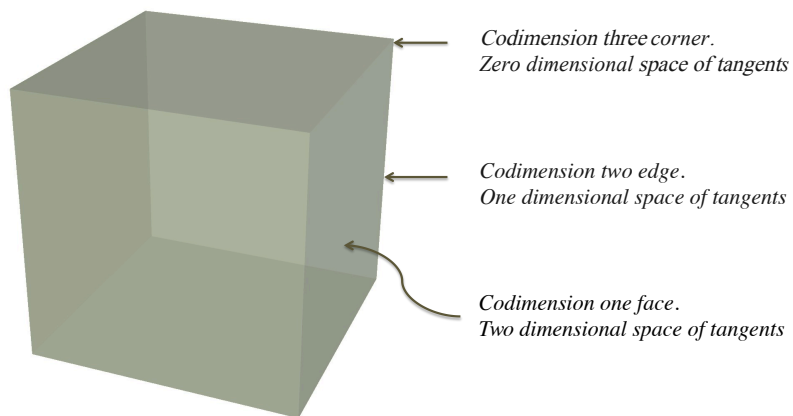
FIGURE 5. Corners and edges in three dimensions.
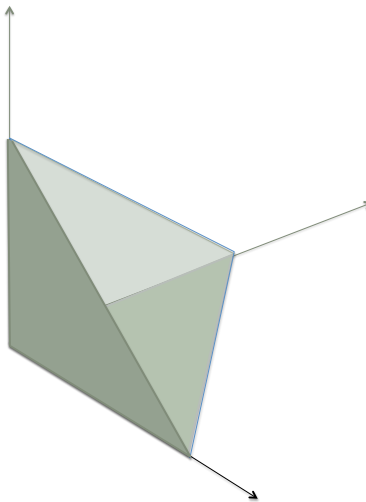
FIGURE 6. Corner in $\mathbb{R}^3$ is a cone on a triangle.

is easily defined and Green's identities are elementary. Denote by $\mathcal{D}$ such a nice domain in $\mathbb{R}^m_y$. Suppose that

$$A(y, \partial) \;=\; \sum_1^m A_\mu(y)\,\frac{\partial}{\partial y_\mu} \;+\; B(y)$$

is an $N \times N$ system with uniformly lipshitzian matrix valued coefficients. The adjoint operator is

$$A(y, \partial)^* w \;:=\; B^* w \;-\; \sum_1^m \frac{\partial}{\partial y_\mu}\Big(A_\mu(y) w\Big).$$

FIGURE 7. Circular cone.

For $u, w \in H^1(\mathcal{D})$ one has

$$\int_{\mathcal{D}} (A(y, \partial)u, w) \, dy \;=\; \int_{\mathcal{D}} (u, A(y, \partial)^* w) \, dy \;+\; \int_{\partial \mathcal{D}} (A(y, \nu(y)) \, u \,,\, w) \, d\Sigma \,.$$

Definition 2.3. Define the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} \;:=\; \Big\{ u \in L^2(\mathcal{D}) \,:\, A(x, \partial)u \in L^2(\mathcal{D}) \Big\} \,.$$

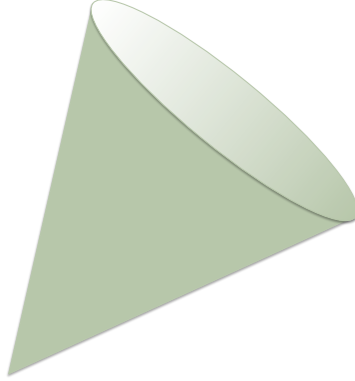Denote by $C^1_{(0)}(\overline{\mathcal{D}})$ the restriction to $\overline{\mathcal{D}}$ of elements in $C^1_0(\mathbb{R}^d)$. Then $C^1_{(0)}(\overline{\mathcal{D}})$ is dense in $\mathcal{H}$. The proof of Friedrich's lemma in [16] works for lipshitzian domains after a bilipshitzian flattening of the boundary. The next result follows [22].

Proposition 2.1. *If $\mathcal{D}$ is a lipshitzian domain, the map*

$$u \quad \mapsto \quad A(y, \nu(y)) \, u \big|_{\partial \mathcal{D}} \;:=\; \gamma$$

has a unique extension from $C^1_{(0)}(\overline{\mathcal{D}})$ to a continuous map from $\mathcal{H}$ to the dual of $H^{1/2}(\partial \mathcal{D})$. If $\phi \in H^{1/2}(\partial \mathcal{D})$ and $\Phi \in H^1(\overline{\mathcal{D}})$ with $\Phi|_{\partial \mathcal{D}} = \phi$ then the trace γ satisfies

$$\langle \gamma \,,\, \phi \rangle \;=\; \int_{\mathcal{D}} \big(A(x, \partial)u \,,\, \Phi \big) - \big(A(x, \partial)^* \Phi \,,\, u \big) \, dy \,.$$

2.1.4. *Layer potentials.* With $\langle \xi \rangle := \big(1 + |\xi|^2 \big)^{1/2}$, denote by $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ the set of symbols satisfying

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \;\leq\; C_{\alpha\beta} \, \langle \xi \rangle^{m - |\beta|} \,,$$

uniformly on $\mathbb{R}^d \times \mathbb{R}^d$. With $G(x, \partial)$ from Definition 1.2 and relying on part **i** of Assumption 1.1, choose $r > 0$ and $p(x, D) \in Op(S^{-1}(\mathbb{R}^d \times \mathbb{R}^d))$ a pseudodifferential parametrix,

$$p(x, D)G(x, \partial) \;-\; I \;\in\; Op(S^{-\infty}(\{x \,:\, \text{dist}(x, \partial\mathcal{O}) < r\} \times \mathbb{R}^d)) \,.$$

The next result on layer potentials can be found on pages 37-38 of [27].

Proposition 2.2. *Denote by H the open half space $\{x_1 > 0\}$ and $d\Sigma$ the element of surface on ∂H. Suppose that $p(x,\xi) \in S^{-1}(\mathbb{R}^d \times \mathbb{R}^d)$ has an asymptotic expansion as a sum of j-homogeneous symbols*

$$p \sim \sum_{j=-1}^{-\infty} p_j(x,\xi)$$

satisfying the transmission condition

$$p_{-1}(x, \xi_1, 0, \dots, 0) = -p_{-1}(x, -\xi_1, 0, \dots, 0).$$

Then there is a $q \in S^0(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ so that for $g \in L^2(\partial H)$ the distribution $p(x, D)(gd\Sigma)$ has trace on the boundary of H given by

$$\Big(p(x, D)(g\, d\Sigma) \Big)(0+, x') = q(x', D')g.$$

2.1.5. *Negligible sets for $\boldsymbol{H^{1/2}(\mathbb{R}^2)}$ and $\boldsymbol{H^1(\mathbb{R}^3)}$.* We prove that sets of codimension 1 are negligible for $H^{1/2}(\mathbb{R}^2)$ and those of codimension 2 are negligible for $H^1(\mathbb{R}^3)$. The sets are not negligible for $H^{1/2+\varepsilon}(\mathbb{R}^2)$ and $H^{1+\varepsilon}(\mathbb{R}^3)$ respectively. Lemma 2.6 is used in Part I and Lemma 2.7 in Parts I and II.

Lemma 2.4. *There is $C > 0$ independent of ε so that the following hold.*
   i. *If $|D|^{1/2}w \in L^2(\mathbb{R}^2)$ then $w \in L^4(\mathbb{R}^2)$ and if $w \neq 0$,*

$$\int_{|x|<\varepsilon} |w|^2\, dx \leq C\varepsilon \frac{\left(\int_{|x|<\varepsilon} |w|^4\, dx \right)^{1/2}}{\left(\int_{\mathbb{R}^2} |w|^4\, dx \right)^{1/2}} \int_{\mathbb{R}^2} |\xi|\, |\widehat{w}(\xi)|^2\, d\xi. \qquad (2.1)$$

   ii. *If $|D|w \in L^2(\mathbb{R}^3)$ then $w \in L^6(\mathbb{R}^3)$ and if $w \neq 0$,*

$$\int_{|x|<\varepsilon} |w|^2\, dx \leq C\varepsilon^2 \frac{\left(\int_{|x|<\varepsilon} |w|^6\, dx \right)^{1/3}}{\left(\int_{\mathbb{R}^3} |w|^6\, dx \right)^{1/3}} \int_{\mathbb{R}^3} |\xi|^2\, |\widehat{w}(\xi)|^2\, d\xi. \qquad (2.2)$$

Proof. Following [6], the space $\dot{H}^s(\mathbb{R}^d)$ is the set of tempered distributions with Fourier transforms in L^1_{loc} and

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2\, d\xi < \infty.$$

If $0 < s < d/2$, then the space $\dot{H}^s(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$.
   i. Using the previous result with $d = 2$ and $s = 1/2$ yields

$$\left(\int_{\mathbb{R}^2} |w(x)|^4\, dx \right)^{1/4} dx \lesssim \left(\int |\xi|\, |\widehat{w}(\xi)|^2\, d\xi \right)^{1/2}. \qquad (2.3)$$

The Cauchy-Schwartz inequality yields

$$\int_{|x|<\varepsilon} |w|^2\, dx \leq \left(\int_{|x|<\varepsilon} \left(|w|^2 \right)^2 dx \right)^{1/2} \left(\int_{|x|<\varepsilon} 1^2\, dx \right)^{1/2}$$

$$= \sqrt{2\pi\varepsilon^2} \left(\int_{|x|<\varepsilon} |w|^4\, dx \right)^{1/2} = C\varepsilon \frac{\left(\int_{|x|<\varepsilon} |w|^4\, dx \right)^{1/2}}{\left(\int_{\mathbb{R}^2} |w|^4\, dx \right)^{1/2}} \left(\int_{\mathbb{R}^2} |w|^4\, dx \right)^{1/2}.$$

The proof of **i** is completed by (2.3).

ii. The case $d = 3, s = 1$ yields

$$\left(\int_{\mathbb{R}^3} |w(x)|^6 \, dx \right)^{1/6} dx \ \lesssim \ \left(\int_{\mathbb{R}^3} |\xi|^2 \, |\widehat{w}(\xi)|^2 \, d\xi \right)^{1/2}. \qquad (2.4)$$

Estimate using Hölder's inequality with exponents 3 and $3/2$,

$$\int_{|x|<\varepsilon} |w|^2 \, dx \ \leq \ \left(\int_{|x|<\varepsilon} \left(|w|^2 \right)^3 dx \right)^{1/3} \left(\int_{|x|<\varepsilon} 1^{3/2} \, dx \right)^{2/3} \ =$$

$$\left(\frac{4\pi}{3} \right)^{\frac{2}{3}} \varepsilon^2 \left(\int_{|x|<\varepsilon} |w|^6 \, dx \right)^{\frac{1}{3}} \ = \ C \, \varepsilon^2 \, \frac{\left(\int_{|x|<\varepsilon} |w|^6 \, dx \right)^{\frac{1}{3}}}{\left(\int_{\mathbb{R}^3} |w|^6 \, dx \right)^{\frac{1}{3}}} \left(\int_{\mathbb{R}^3} |w|^6 \, dx \right)^{\frac{1}{3}}.$$

The proof of **ii** is completed by (2.4). $\hspace{3cm}\square$

Lemma 2.5. i. *If $d \geq 2$ and $|D|^{1/2}w \in L^2(\mathbb{R}^d)$ then*

$$\frac{1}{\varepsilon} \int_{|x_1,x_2|<\varepsilon} |w|^2 \, dx \lesssim \||D|^{1/2}w\|^2_{L^2(\mathbb{R}^d)}, \quad \text{and as } \varepsilon \to 0, \quad \frac{1}{\varepsilon} \int_{|x_1,x_2|<\varepsilon} |w|^2 \, dx \ \to \ 0.$$

ii. *If $d \geq 3$ and $\nabla w \in L^2(\mathbb{R}^d)$ then*

$$\frac{1}{\varepsilon^2} \int_{|x_1,x_2,x_3|<\varepsilon} |w|^2 \, dx \lesssim \|\nabla w\|^2_{L^2(\mathbb{R}^d)}, \quad \text{and as } \varepsilon \to 0, \quad \frac{1}{\varepsilon^2} \int_{|x_1,x_2,x_3|<\varepsilon} |w|^2 \, dx \ \to \ 0.$$

Proof. **i.** Denote by $\widehat{w}(\xi_1,\xi_2,x')$ the partial Fourier transform with $x' := (x_3, x_4, \ldots, x_d)$. Then

$$\int |\xi_1,\xi_2| \, |\widehat{w}(\xi_1,\xi_2,x')|^2 \, d\xi_1 d\xi_2 dx' \ \leq \ \left\| \, |D|^{1/2}w \, \right\|^2_{L^2(\mathbb{R}^d)}.$$

Define

$$0 \ \leq \ f(\varepsilon,x') \ := \ \frac{\left(\int_{|x_1,x_2|<\varepsilon} |w(x_1,x_2,x')|^4 \, dx_1 dx_2 \right)^{1/2}}{\left(\int_{\mathbb{R}^2} |w(x_1,x_2,x')|^4 \, dx_1 dx_2 \right)^{1/2}} \ \leq \ 1.$$

The function f is decreasing in ε. Lebesgue's monotone convergence theorem implies that as $\varepsilon \to 0$,

$$\int f^2(\varepsilon,x') \, dx' \ = \ C(d) \int_{|x_1,x_2|<\varepsilon} |w(x_1,x_2,x')|^4 \, dx_1 dx_2 \ \to \ 0 \, .$$

It follows that f tends to zero for almost all x'.

The estimate of part **i** of Lemma 2.4 above implies that

$$\left(\frac{1}{\varepsilon} \int_{|x_1,x_2|<\varepsilon} |w|^2 \, dx \right)^2 \lesssim \int_{\mathbb{R}^d} f^2(\varepsilon,x') \, |\xi_1,\xi_2| \, |\widehat{w}(\xi_1,\xi_2,x')|^2 \, d\xi_1 d\xi_2 dx' \, .$$

Part **i** follows from Lebesgue's dominated convergence theorem.

ii. Exactly analogous using part **ii** of Lemma 2.4. $\hspace{3cm}\square$

Lemma 2.6. *With $\mathcal{O}$ as in Definition 1.1, if $v \in H^{1/2}(\mathcal{O})$ then as $\varepsilon \to 0$,*

$$\frac{1}{\varepsilon} \int_{\text{dist}(x,S)<\varepsilon} |v|^2 \, dx \ \to \ 0 \, .$$

Proof. Extend v to an element of $H^{1/2}(\mathbb{R}^d)$. The set of points at distance less than ε from S is contained in a finite union of the cylinders $|x_i, x_j| < \varepsilon$ with $1 \leq i < j \leq d$. Apply the preceding lemma to each cylinder. $\hspace{2cm}\square$

Lemma 2.7. i. *If $w \in H^{1/2}(\mathbb{R}^d)$ is supported on a finite union of codimension one affine subspaces, then $w = 0$.*

ii. *If $w \in H^1(\mathbb{R}^d)$ is supported on a finite union of codimension two affine subspaces, then $w = 0$.*

Proof of ii. This case is somewhat harder. Part **i** is left to the reader. Denote by Γ the finite union. Construct a sequence of cutoff functions $\psi_\varepsilon(x) \in C^\infty(\mathbb{R}^d)$ so that $\psi_\varepsilon = 1$ at points x with $\mathrm{dist}(x, \Gamma) \geq \varepsilon$, $\psi_\varepsilon = 0$ at points x with $\mathrm{dist}(x, \Gamma) \leq \varepsilon/2$, and $\|\nabla \psi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq C/\varepsilon$.

By hypothesis, $\psi_\varepsilon w = 0$. The proof is completed by showing that for all $u \in H^1(\mathbb{R}^d)$, $\lim \|\psi_\varepsilon u - u\|_{H^1} = 0$. For the dense set $u \in C_0^\infty(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ this is elementary.

The proof is completed by showing that uniformly in ε,

$$\|\psi_\varepsilon u\|_{H^1(\mathbb{R}^d)} \lesssim \|u\|_{H^1(\mathbb{R}^d)}.$$

For this, estimate

$$\|\nabla(\psi_\varepsilon u) - \psi_\varepsilon \nabla u)\|_{H^1(\mathbb{R}^d)}^2 \leq \frac{C}{\varepsilon^2} \int_{\mathrm{dist}(x,\Gamma)<\varepsilon} |u|^2 \, dx \lesssim \int_{\mathbb{R}^d} |\nabla u|^2 \, dx$$

by part **ii** of Lemma 2.5. □

2.2. **Proof of Theorem 1.4.**

2.2.1. *Step 1. Construction of solutions.* Existence is proved by constructing u as the limit of solutions u^ε to problems in domains $\mathcal{O}^\varepsilon$ obtained by smoothing $\mathcal{O}$. Take $\phi \in C^\infty(\mathbb{R})$ from Example 2.1. The ellipticity hypothesis Assumption 1.1.ii implies that $\mathbb{R} \times (\partial \mathcal{O} \setminus \mathcal{S})$ is noncharacteristic so the maximal dissipative boundary condition $\mathcal{N}_j$ is given by an equation

$$\mathbf{v}_- = M_j(x)\mathbf{v}_+ .$$

Define $\mathcal{N}_j^\varepsilon$ to be the maximal dissipative space defined by

$$\mathbf{v}_- = \phi(|x_j|/\varepsilon) \, M_j(x) \, \mathbf{v}_+ .$$

Define $\mathcal{O}^\varepsilon$ by smoothing the edges of Ω leaving the boundary unchanged where all of the x_j are greater than $\varepsilon/2$, see figure 8. For $\varepsilon > 0$ sufficiently small, $\partial \mathcal{O}^\varepsilon$ lies in the domain of ellipticity of $A(x, D)$ so is noncharacteristic. Define a boundary space $\mathcal{N}^\varepsilon$ on the boundary of $\mathcal{O}^\varepsilon$ to be equal to $\mathcal{N}_j^\varepsilon$ on the unchanged part and given by the equation $E_+(A(x, \nu(x)))$ on the parts that have been smoothed. Thanks to the ellipticity, this is a smooth maximally dissipative boundary condition. The standard theory constructs $u^\varepsilon \in C([0, \infty[; L^2(\mathcal{O}^\varepsilon))$ a solution of the mixed problem with initial value equal to the restriction of g to $\mathcal{O}^\varepsilon$.

The function u^ε satisfies the energy identity for all $T > 0$,

$$\|u^\varepsilon(T)\|^2 + \int_{[0,T]\times\partial\mathcal{O}^\varepsilon} \big(A(x,\nu(x))u^\varepsilon, u^\varepsilon\big)dtd\Sigma + \int_{[0,T]\times\mathcal{O}^\varepsilon} \big(Z(x)u^\varepsilon, u^\varepsilon\big)dtdx = \|u^\varepsilon(0)\|^2.$$

Therefore

$$\sup_{t\geq 0} e^{2\mu t} \|u^\varepsilon(t)\|^2 + \int_0^\infty \|u^\varepsilon(t)|_{\partial\mathcal{O}^\varepsilon}\|^2 dt \lesssim \|g\|^2 .$$

By the Cantor diagonal process, choose a subsequence $\varepsilon(k) \to 0$, a $u \in e^{-\mu t}L^\infty([0, \infty[; L^2(\mathcal{O}))$ and a $\gamma \in L^2\big([0, \infty[\times\partial\mathcal{O})$ so that for all $\delta > 0$ and $0 < T$,

$$u^{\varepsilon(k)} \rightharpoonup u \qquad \text{weak star in } L^\infty(]0, \infty[; L^2(\mathcal{O}^\delta)), \quad \text{and}$$
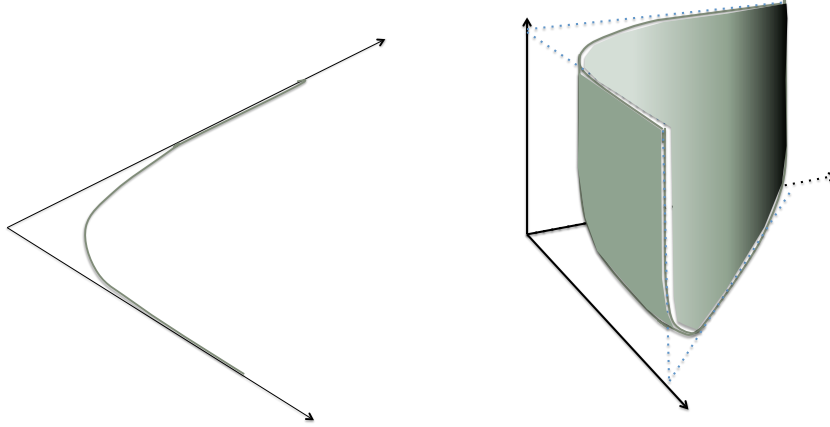
FIGURE 8. Construction of $\mathcal{O}_\epsilon$ when $d = 2$ (left) and $d = 3$ (right) .

$$A(x, \nu(x))u^{\varepsilon(k)}\big|_{]0,T[\times(\partial\mathcal{O}\cap\{\mathrm{dist}(x,\mathcal{S})>\delta\})} \quad \rightharpoonup \quad \gamma$$

weak star in $L^2\big([0,T] \times (\partial\mathcal{O} \cap \{\mathrm{dist}(x, \mathcal{S}) > \delta\})\big)$. It follows that $Lu = 0$, $u(0) = g$ and $u|_{\{x_j=0\}\cap\partial\mathcal{O}\setminus\mathcal{S}}$ belongs to $\mathcal{N}_j$.

Since this is true for all T and all $\delta > 0$ it follows that

$$A(x, \nu(x))u\big|_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})} \;=\; \gamma$$

This completes the construction of a solution with $u \in e^{-\mu t}L^\infty([0, \infty[\,;\, L^2(\mathcal{O}))$ and $u|_{\partial\mathcal{O}\setminus\mathcal{S}} \in L^2(]0, \infty[\times(\partial\mathcal{O} \setminus \mathcal{S}))$. That the solution satisfies the additional condition of belonging to $C([0, \infty[\,:\, L^2(\mathcal{O}))$ and satisfies the energy identity will be proved after uniqueness is proved.

2.2.2. *Step 2. Uniqueness of solutions.* That two solutions with the same data must coincide is proved by showing that their Laplace transforms are equal. This reduces to uniqueness for a problem that is elliptic at the boundary.

Laplace transformation. Consider solutions satisfying

$$u \in L^\infty([0, \infty[\,;\, L^2(\mathcal{O})), \qquad u\big|_{\partial\mathcal{O}\setminus\mathcal{S}} \in L^2(]0, \infty[\,;\, L^2(\partial\mathcal{O} \setminus \mathcal{S}))\,.$$

The difference of two such solutions that have the same initial value has Laplace transform $\widetilde{u}(\tau)$ analytic in $\mathrm{Re}\,\tau > 0$ with values in $L^2(\mathcal{O})$ and with $\widetilde{u}|_{\partial\mathcal{O}\setminus\mathcal{S}}$ analytic with values in $L^2(\partial\mathcal{O} \setminus \mathcal{S})$ and satisfying

$$\tau\widetilde{u} + G(x, \partial_x)\widetilde{u} \;=\; 0\,, \qquad \widetilde{u}|_{\partial\mathcal{O}\setminus\mathcal{S}} \in \mathcal{N}\,. \tag{2.5}$$

Uniqueness is therefore a consequence of the following result for the transformed problem.

Theorem 2.8. *If* $\mathrm{Re}\,\tau > 0$ *and* $v \in L^2(\mathcal{O})$ *satisfies*

$$\tau v + G(x, \partial_x)v = 0, \quad v\big|_{\partial\mathcal{O}\setminus\mathcal{S}} \in \mathcal{N}, \quad \text{and} \quad v|_{\partial\mathcal{O}} \in L^2(\partial\mathcal{O} \setminus \mathcal{S}), \tag{2.6}$$

then $v = 0$.

Remark 2.1. i. Since $\partial\mathcal{O}$ is lipshitzian the Sobolev spaces $H^s(\partial\mathcal{O})$ are well defined for $|s| \leq 1$. **ii.** The second condition in (2.6) makes sense for $v, Gv \in L^2(\mathcal{O})$. **iii.** The third asserts a regularity that is true for the solutions constructed.

At a formal level the result is immediate. If the integrations by parts were justified, for example if the solution was $H^1(\mathcal{O})$ rather than $L^2(\mathcal{O})$, Theorem 2.8 would follow for $\operatorname{Re}\tau > 0$ from

$$
\begin{aligned}
0 &= 2\operatorname{Re}\int_{\mathcal{O}} \big(\tau v + G(x,\partial)v\,,\,v\big)\,dx \\
&= 2\operatorname{Re}\tau\|v\|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}}(Zv,v)dx + \int_{\partial\mathcal{O}}\big(A(x,\nu(x))v,v\big)\,dx \qquad (2.7) \\
&\geq 2\operatorname{Re}\tau\,\|v\|^2_{L^2(\mathcal{O})}\,.
\end{aligned}
$$

In the last step the positivity of Z and the dissipativity of the boundary condition are used. The method is to justify the integration by parts. That is done in several steps.

Lopatinski's condition.

Proposition 2.3. *Suppose that*

$$
A(\partial) \ := \ \sum_{j=1}^{d} A_j\partial_j
$$

is an elliptic operator with hermitian constant coefficients. Suppose that $H := \{x : x \cdot \xi > 0\}$ is an open half space so $A(\nu) = -A(\xi/|\xi|)$. Suppose that $\mathcal{N}$ is a maximal strictly dissipative subspace for $A(\nu)$. Then $\mathcal{N}$ satisfies the coercivity condition of Lopatinski for the half space H.

Proof. A linear change of coordinates reduces to the case $H = \{x_1 > 0\}$. In that case, Lopatinski's condition is that for all $0 \neq \xi' \in \mathbb{R}^{d-1}_{x'}$, if $w(x_1)$ satisfies the ordinary differential boundary value problem

$$
A(\partial_1, i\xi')w(x_1) \ = \ 0, \quad w(0) \in \mathcal{N}, \quad \text{and} \quad \lim_{x_1 \to \infty} w(x_1) \ = \ 0,
$$

then $w = 0$.

Under the above hypotheses, the function $u = e^{ix'\xi'}w(x_1)$ is a stationary solution of the hyperbolic equation $\big(\partial_t + A(\partial)\big)u = 0$.

Denote by $\mathbf{e}_j$ the standard basis for $\mathbb{C}^d$. For $j = 2,\ldots,d$ choose nonzero real numbers α_j so that $\alpha_j\xi'_j = 2\pi$. Define for $j \geq 2$, $\mathbf{v}_j := \alpha_j\mathbf{e}_j$. Introduce the lattice $\mathcal{L} \in \mathbb{R}^{d-1}_{x'}$ consisting of vectors $\sum_j n_j\mathbf{v}_j$ with $n_j \in \mathbb{Z}$. The stationary solution $u(x) = e^{i\xi'x'}w(x_1)$ is then $\mathcal{L}$-periodic in x'.

The energy identity for $\mathcal{L}$-periodic solutions of $Lu = 0$ then asserts that

$$
\frac{d}{dt}\big\|u(t)\big\|^2_{L^2(\mathbb{R}^3_+/\mathcal{L})} \ + \ \int_{\{x_1=0\}/\mathcal{L}}\big(-A_1u(t,0,x')\,,\,u(t,0,x')\big)\,dx' \ = \ 0\,.
$$

For the stationary solution one finds

$$
\int_{\{x_1=0\}/\mathcal{L}}\big(-A_1u(0,x')\,,\,u(0,x')\big)\,dx' \ = \ 0\,.
$$

The strict dissipativity of $\mathcal{N}$ implies that $u|_{x_1=0} = 0$. Therefore $w(0) = 0$. Uniqueness for the ordinary differential equation initial value problem implies that w is identically equal to zero. Therefore u is identically equal to zero. $\qquad\square$

Corollary 2.9. *If H is one of the half spaces $\{x_j > 0\}$, r as in Section 2.1.4,*

$$v \in L^2(H), \qquad Gv \in H^{-1/2}(H), \qquad \text{and} \qquad \Pi_+ v\big|_{\partial H} \in L^2(\partial H),$$

then $v \in H^{1/2}(\{0 \le x_j \le r/2\})$.

Proof. Choose a locally finite cover of $\{0 \le x_j \le r/2\}$ by balls B^k of radius $3r/4$ with centers on ∂H. Denote by $\widetilde{B}^k$ the ball with the same center and radius $4r/5$. The classical boundary regularity estimate that follows from the Lopatinski condition ([11], §20.1) asserts that for any $s \ge 0$,

$$\|v\|^2_{H^s(B^k \cap H)} \lesssim \|Gv\|^2_{H^{s-1}(\widetilde{B}^k \cap H)} + \|\Pi_+ v\|^2_{H^{s-1/2}(\widetilde{B}^k \cap \partial H)} + \|v\|^2_{L^2(\widetilde{B} \cap H)}. \qquad (2.8)$$

The uniformity of these estimates relies on Assumption 1.1 guaranteeing that A_j and $\mathcal{N}_j$ are constant outside a compact set.

For v satisfying the hypothesis, the right hand side of (2.8) is finite for $s \le 1/2$. The limit is imposed by the boundary term that for $s > 1/2$ requires more smoothness than $\Pi_+ v|_{\partial H} \in L^2$. Taking the limiting case, $s = 1/2$ and summing on k proves the Corollary. $\qquad\square$

Sufficient regularity at the singular set S. Choose $\chi \in C^\infty(\mathbb{R}^d)$ with

$$0 \le \chi \le 1, \qquad \chi = 1 \text{ if } \operatorname{dist}(x, \mathcal{S}) \ge 1, \qquad \chi = 0 \text{ if } \operatorname{dist}(x, \mathcal{S}) \le 1/2,$$

and for all α, $\partial^\alpha \chi \in L^\infty(\mathbb{R}^d)$. Define $\chi_\varepsilon(x) := \chi(x/\varepsilon)$. Multiplying by χ_ε excises an ε-neighborhood of $\mathcal{S}$.

With v from Theorem 2.8, define $v^\varepsilon := \chi_\varepsilon(x)v \in \cap_s H^s(\mathcal{O})$. For this function integration by parts is justified and yields

$$\begin{aligned} 0 &= 2\operatorname{Re} \int_{\mathcal{O}} \langle \tau v^\varepsilon + G(x, \partial)v^\varepsilon, v^\varepsilon \rangle \, dx \\ &= 2\operatorname{Re}\tau \|v^\varepsilon\|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}} (Zv^\varepsilon, v^\varepsilon)dx + \int_{\partial\mathcal{O}} \big(A(x, \nu(x))v^\varepsilon, v^\varepsilon\big) \, dx. \end{aligned} \qquad (2.9)$$

The boundary term is nonnegative.

Write

$$G v^\varepsilon = G(\chi_\varepsilon v) = \chi_\varepsilon Gv - [G, \chi_\varepsilon]v.$$

Want to pass to the limit in (2.9). The Z term, boundary term, and $\chi_\varepsilon Gv$ term pose no problem. The commutator is a multiplication operator by matrices with coefficients bounded in magnitude by C/ε so

$$\Big| \int_{\mathcal{O}} \langle [G, \chi_\varepsilon]v, v \rangle \, dx \Big| \le \frac{C}{\varepsilon} \int_{\operatorname{dist}(x,S)<\varepsilon} |v|^2 \, dx.$$

To complete the proof of uniqueness it suffices to show that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\operatorname{dist}(x,S)<\varepsilon} |v|^2 \, dx = 0. \qquad (2.10)$$

Once established this implies that $\int \langle v^\varepsilon, [G, \chi_\varepsilon]v^\varepsilon \rangle \, dx \to 0$.

The conclusion of Lemma 2.6 is exactly (2.10). *Therefore Lemma 2.6 implies that to complete the proof of uniqueness it suffices to show that $v \in H^{1/2}(\mathcal{O})$.*

Proof that $v \in H^{1/2}(\mathcal{O})$.

Lemma 2.10. *If*

$$w \in L^2(\mathcal{O}), \qquad Gw := f \in L^2(\mathcal{O}), \qquad \text{and,} \qquad w|_{\partial\mathcal{O}\setminus\mathcal{S}} \in L^2(\partial\mathcal{O}\setminus\mathcal{S}),$$

denote by $\underline{f} \in L^2(\mathbb{R}^d)$ and $\underline{w}$ the extension by zero of $f \in L^2(\mathcal{O})$ and $w \in L^2(\mathcal{O})$. Then in the sense of distributions on $\mathbb{R}^d$,

$$G\underline{w} \;=\; \underline{f} \;+\; A(x, \nu(x))\, w|_{\partial\mathcal{O}\setminus\mathcal{S}}\, d\Sigma. \tag{2.11}$$

Proof. For $x \in \mathcal{O}$, identity (2.11) holds in a neighborhood of x thanks to the equation $Gw = f$ in $\mathcal{O}$. If $x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}$ then both sides of (2.11) vanish on a neighborhood of x.

If x is a point of $\partial\mathcal{O} \setminus \mathcal{S}$ and B is a small ball centered at x then $w|_{B\cap\overline{\mathcal{O}}}$ is smooth on $B \cap \overline{\mathcal{O}}$ thanks to the Lopatinski condition. Therefore (2.11) holds on a neighborhood of x.

Therefore, the difference between the left and right hand sides of (2.11) is an element of $H^{-1}(\mathbb{R}^d)$ supported on the finite union of a codimension two affine subspaces of $\mathbb{R}^d$. The difference vanishes by part **ii** of Lemma 2.7. $\qquad\square$

Theorem 2.11. *If*

$$v \in L^2(\mathcal{O}), \qquad Gv \in L^2(\mathcal{O}), \qquad \text{and,} \qquad v|_{\partial\mathcal{O}\setminus\mathcal{S}} \in L^2(\partial\mathcal{O}\setminus\mathcal{S})$$

then $v \in H^{1/2}(\mathcal{O})$.

Proof of Theorem 2.11. At points $x \in \partial\mathcal{O} \setminus \mathcal{S}$, Lopatinski's condition implies that v belongs to H^1 on a neighborhood of x in $\mathcal{O}$. What is needed is to show that the $H^{1/2}$ regularity holds on a neighborhood of the singular points $\mathcal{S} \subset \partial\mathcal{O}$.

Choose r and $p(x,\xi)$ as in Section 2.1.4. By elliptic regularity one has

$$v \;\in\; H^1\left(\left\{x \in \mathcal{O} : \frac{r}{4} < \operatorname{dist}(x, \partial\mathcal{O}) < \frac{r}{2}\right\}\right).$$

It suffices to show that

$$v \;\in\; H^{1/2}\left(\left\{x \in \mathcal{O} : \operatorname{dist}(x, \partial\mathcal{O}) \le \frac{r}{4}\right\}\right).$$

Denote $f = Gv \in L^2(\mathcal{O})$. Denote by $\underline{v}$ and $\underline{f}$ in $L^2(\mathbb{R}^3)$ the extensions by zero. Denote $g = A(x, \nu(x))w|_{\partial\mathcal{O}\setminus\mathcal{S}} \in L^2(\partial\mathcal{O}\setminus\mathcal{S})$. Lemma 2.10 together with the fact that $p(x, D)$ is a parametrix imply that

$$G\Big(p(x, D)(\underline{f} + g\, d\Sigma)\Big) - G\underline{v} \;\in\; \cap_s H^s\left(\left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\mathcal{O}) \le \frac{3r}{4}\right\}\right).$$

Elliptic regularity on $\mathbb{R}^d$ implies that

$$p(x, D)(\underline{f} + g\, d\Sigma) - \underline{v} \;\in\; \cap_s H^s\left(\left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\mathcal{O}) \le \frac{r}{2}\right\}\right).$$

Since p is of order -1, $p(x, D)\underline{f} \in H^1(\mathbb{R}^d)$ so

$$p(x, D)(g\, d\Sigma) - \underline{v} \;\in\; H^1\left(\left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\mathcal{O}) \le \frac{r}{2}\right\}\right).$$

The proof is completed by showing that $p(x, D)(g\, d\Sigma) \in H^{1/2}(\{x \in \mathcal{O} : \operatorname{dist}(x, \partial\mathcal{O}) < r/2\})$.

Denote $H_j := \{x_j > 0\}$. Decompose

$$g \;=\; \sum g_j, \qquad g_j \;\in\; L^2(\partial H_j \cap \partial\mathcal{O}).$$

It is sufficient to show that $p(x, D)(g_j\, d\Sigma) \in H^{1/2}(\{x \in \mathcal{O} \,:\, \mathrm{dist}(x, \partial\mathcal{O}) \le r/2\})$. We prove the stronger assertion that $p(x, D)(g_j\, d\Sigma) \in H^{1/2}(\{x \in \mathbb{R}^3 \,:\, 0 \le x_j \le r/2\})$. The last assertion does not involve corners.

The trace inequality in $\mathbb{R}^d$,

$$\left(\int_{\partial H_j} |\psi(0, x')|^2\, dx' \right)^{1/2} \lesssim \left(\int_{\mathbb{R}^d} |\nabla\psi(x)|^k\, dx \right)^{1/k}, \qquad k = \frac{2d}{d+1} < 2,$$

together with $g_j \in L^2(\partial H_j)$ imply that with $d = 3$ and $k = 3/2$

$$g_j\, d\Sigma \in \left(W^{1,k}(\mathbb{R}^3) \right)' = W^{-1,q}(\mathbb{R}^3), \qquad \frac{1}{q} + \frac{1}{k} = 1, \qquad q = 3\,.$$

Since p is order -1,

$$w_j := p(x, D)\left(g_j\, d\Sigma\right) \in W^{0,q}(\mathbb{R}^3) = L^3(\mathbb{R}^3)\,.$$

The local regularity of w_j is better than L^2.

Since the distribution kernel of $p(x, D)$ is smooth off the diagonal and rapidly decaying with all derivatives as the distance to the diagonal grows this is sufficient to conclude that

$$w_j \in L^2(\mathbb{R}^3)\,. \tag{2.12}$$

In addition

$$Gw_j - g_j\, d\Sigma \in \cap_s H^s(\mathbb{R}^3)\,.$$

Since $g_j\, d\Sigma$ vanishes on $\mathcal{O}$ this implies that

$$(Gw_j)|_{H_j} \in \cap_s H^s(H_j)\,. \tag{2.13}$$

Proposition 2.2 implies that the trace of $w_j|_{H_j}$ at ∂H_j is square integrable,

$$w_j|_{\partial H_j} \in L^2(\partial H_j)\,. \tag{2.14}$$

The three last numbered equations and Corollary 2.9 imply that $w_j|_{H_j} \in H^{1/2}(\{x \in \mathbb{R}^3 \,:\, 0 \le x_j \le r/2\})$. The proof is complete. $\qquad\square$

This completes the verification of (2.10) and thereby completes the proof of uniqueness.

2.2.3. *Step 3. Proof of continuity in time and the energy equality.* With the $H^{1/2}(\mathcal{O})$ regularity in hand, one can justify the integration by parts in the basic *a priori* estimate (2.7) for the operator $\Lambda + G$ with real $\Lambda \ge 0$,

$$\Lambda\, \|u\| \le \|(\Lambda + G)u\| \quad \text{provided} \quad u \in \mathcal{N} \text{ on } \partial\mathcal{O}\,.$$

In particular the family of maps $\Lambda(\Lambda + G)^{-1}$ is uniformly bounded from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ as $\Lambda \to \infty$. Therefore

$$(I + \varepsilon G)^{-1} = \varepsilon^{-1}(\varepsilon^{-1}I + G)^{-1}$$

is uniformly bounded in $\mathrm{Hom}(L^2(\mathcal{O}))$. Define for $0 < \varepsilon << 1$

$$g^\varepsilon := (I + \varepsilon G)^{-1}g\,.$$

Lemma 2.12. *As $\varepsilon \to 0$ one has for all $g \in L^2(\mathcal{O})$*

$$\left\| g^\varepsilon - g \right\|_{L^2(\mathcal{O})} \longrightarrow 0\,.$$

Proof. From the uniform boundedness it is sufficient to prove the result for g in a dense subset of $L^2(\mathcal{O})$. Compute

$$I \;-\; (I + \varepsilon G)^{-1} \;=\; \varepsilon G (I + \varepsilon G)^{-1}\,.$$

For g belonging to the dense subset $C_0^\infty(\mathcal{O})$,

$$\left\| g^\varepsilon - g \right\| \;\leq\; \varepsilon\, \left\| (I + \varepsilon G)^{-1} \right\|\, \left\| G g \right\| \;\to\; 0$$

since the last two factors are bounded.   □

Thanks to uniqueness we can define u^ε to be the only solution with initial value g^ε. The function $\partial_t u^\varepsilon$ is the solution with initial value $G g^\varepsilon \in L^2(\mathcal{O})$. In particular $\partial_t u^\varepsilon \in L^\infty([0,T]; L^2(\mathcal{O}))$. Therefore, $G u^\varepsilon \in L^\infty([0,\infty[;\, L^2(\mathcal{O}))$. Theorem 2.11 implies that $u^\varepsilon \in L^\infty([0,\infty[;\, H^{1/2}(\mathcal{O}))$.

This suffices to justify the integration by parts in the energy identity for u^ε and also for $u^\varepsilon - u^\delta$. The latter implies that

$$\sup_{0 < t < T} \| u^\varepsilon(t) - u^\delta(t) \| \;+\; \left\| (u^\varepsilon - u^\delta)\big|_{\partial\mathcal{O}} \right\|_{L^2([0,T] \times \partial\mathcal{O})} \;\leq\; C\, \| g^\varepsilon - g^\delta \|\,.$$

Therefore u^ε is a Cauchy sequence in $C([0,T] : L^2(\mathcal{O}))$ and $u^\varepsilon\big|_{\partial\mathcal{O}}$ is a Cauchy sequence in $L^2([0,T] \times \partial\mathcal{O})$. It follows that the limit $u \in C([0,\infty[;\, L^2(\mathcal{O}))$. Passing to the limit in the energy identity for u^ε on $[t,T] \times \mathcal{O}$ yields the energy identity for u. This completes the proof of the last remaining part of Theorem 1.4   □

2.2.4. *Additional information about the trace* $A(x, \nu(x)) u|_{\partial\mathcal{O}}$. The trace $A(x, \nu(x)) u$ at the boundary belongs to $H^{-1/2}(]0, T[\times \partial\mathcal{O})$ for any $T > 0$. We have shown that its restriction to $]0, T[\times (\partial\mathcal{O} \setminus \mathcal{S})$ is equal to γ. *In this section this is strengthened to equality on* $]0, T[\times \partial\mathcal{O}$.

The preceding result asserts that

$$\operatorname{supp}\Big(A(x, \nu(x)) u\big|_{]0,T[\times\partial\mathcal{O}} \;-\; \gamma \Big) \quad \subset \quad]0, T[\times \mathcal{S}\,.$$

The quantity in parentheses on the left hand side is an element of $H^{-1/2}(]0, T[\times \partial\mathcal{O})$ with support contained in a finite union of codimension 1 affine subspaces of $\partial\mathcal{O}$. Lemma 2.7 proves that this can only happen if the quantity in parentheses vanishes identically.

Since this holds for all $T \in \mathbb{N}_+$ it follows that

$$A(x, \nu(x)) u\big|_{]0,\infty[\times\partial\mathcal{O}} \;-\; \gamma \;=\; 0\,.$$

It is interesting that the proof of uniqueness did not require this additional information.

2.2.5. *Semigroup generator.* Define a linear operator G on $L^2(\mathcal{O})$ by

$$\mathcal{D}(G) := \big\{ v \in L^2(\mathcal{O}) \,:\, Gv \in L^2(\mathcal{O}),\ \ v\big|_{\partial\mathcal{O}\setminus\mathcal{S}} \in L^2(\partial\mathcal{O} \setminus \mathcal{S}),\ \ v \in \mathcal{N} \text{ on } \partial\mathcal{O} \setminus \mathcal{S} \big\}\,.$$

The proof of uniqueness verifies that G satisfies the conditions of the Hille-Yosida Theorem characterizing generators of contraction semigroups. Thus applying the Hille-Yosida Theorem, the uniqueness proof suffices for existence. This is the strategy for planar corners in [12]. Our bare hands existence proof applies without modification to problems with time dependent coefficients. We do not know how to prove uniqueness for the problems with time dependent coefficients.

The proof of §2.2.4 shows that for elements of $\mathcal{D}(G)$ one has $A(x, \nu(x))v|_{\partial\mathcal{O}} \in L^2(\partial\mathcal{O})$. Therefore putting this seemingly more restrictive condition in the definition of the domain would not change the domain.

2.2.6. *Additional information about the energy flux.* Following [22], if u and Lu belong to L^2 then extending by continuity as in Proposition 2.1 from the dense set $C_{(0)}^\infty$ defines

$$\left(A(x, \nu(x))\, u\, ,\, u\right)\big|_{]0,\infty[\times\partial\mathcal{O}} \ \in\ \mathrm{Lip}_0(]0,\infty[\times\partial\mathcal{O})', \tag{2.15}$$

the right hand side denoting the dual of the compactly supported lipshitzian functions.

In this section we show that for our solutions one has

$$\left(A(x, \nu(x))\, u\, ,\, u\right)\big|_{]0,\infty[\times\partial\mathcal{O}} \ =$$
$$\left(A(x, \nu(x))\, u\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})}\, ,\, u\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})}\right) \ \in\ L^1(]0,T[\times\partial\mathcal{O}). \tag{2.16}$$

The last inclusion since $u\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})} \in L^2(]0,\infty[\times\partial\mathcal{O})$.

Define $u^{\delta,\varepsilon} := \chi_\delta(x)u^\varepsilon$ with the cutoff function that appears in (2.9). Then as $\delta \to 0$, $u^{\delta,\varepsilon} \to u^\varepsilon$ in $L^2(]0,T] \times \partial\mathcal{O})$. Since $u^\varepsilon \in L^\infty(]0,T[: H^{1/2}(\mathcal{O}))$, Lemma 2.6 implies, as in the proof of uniqueness, that

$$(Lu^{\delta,\varepsilon}\, ,\, u^{\delta,\varepsilon}) \ \to\ (Lu^\varepsilon\, ,\, u^\varepsilon) \qquad \text{in}\ \ L^1(]0,T] \times \partial\mathcal{O}) \quad \text{as}\quad \delta \to 0.$$

It follows that the energy fluxes $\left(A(x, \nu(x))\, u^{\delta,\varepsilon}\, ,\, u^{\delta,\varepsilon}\right)\big|_{]0,\infty[\times\partial\mathcal{O}}$ converge weak star in Lip_0' to the flux $\left(A(x, \nu(x))\, u^\varepsilon\, ,\, u^\varepsilon\right)\big|_{]0,\infty[\times\partial\mathcal{O}}$.

Since the indicator function is equal to one on a neighborhood of the support of $u^{\delta,\varepsilon}$,

$$\left(A(x, \nu(x))\, u^{\delta,\varepsilon}\, ,\, u^{\delta,\varepsilon}\right)\big|_{]0,\infty[\times\partial\mathcal{O}}$$
$$=\left(A(x, \nu(x))\, u^{\delta,\varepsilon}\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})}\, ,\, u^{\delta,\varepsilon}\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})}\right). \tag{2.17}$$

Strongly in $L^2(]0,\infty[\times\partial\mathcal{O})$ one has,

$$\lim_{\delta\to 0}\ u^{\delta,\varepsilon}\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})} \ =\ u^\varepsilon\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})},$$

and

$$\lim_{\varepsilon\to 0}\ u^\varepsilon\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})} \ =\ u\, \mathbf{1}_{]0,\infty[\times(\partial\mathcal{O}\setminus\mathcal{S})}.$$

Passing to the limit $\delta \to 0$ then $\varepsilon \to 0$ in (2.17) proves (2.16).

This strengthens the conclusion of Theorem 1.4 that shows that the flux is L^1 off the singular set $]0,\infty[\times\mathcal{S}$. In contrast to the proof in the §2.2.3, there are elements of Lip' supported on $]0,T[\times\mathcal{S}$.

Remark 2.2. In the same way, one proves that the energy fluxes of the solutions in Theorems 2.13 and 2.15 belong to $L^1(]0,\infty[\times\partial\mathcal{O})$. Analogues of the trace results of the preceding section are also valid.

2.3. **Maxwell and Euler, hidden ellipticity.** This section proves existence and uniqueness for *some* problems with strictly dissipative corners for which G is not elliptic. A common feature is that the kernel of $G(\xi)$ has dimension independent of $\xi \neq 0$. In this situation it is always true that there is a partial differential operator $Q(D)$ with $QG = GQ = 0$ and so that G, Q is an overdetermined elliptic system, see [7]. Solutions of $\partial_t u + Gu = 0$ satisfy $\partial_t Qu = 0$ so one trivially knows the exact regularity of Qu for all time. The idea is to take advantage of the ellipticity of G, Q. This is what we call *hidden ellipticity*. We consider only the case where there

is a Q of first order, a class of problems introduced by Majda in [2]. We do not propose a general strategy but treat three important examples; Maxwell's equations, the compressible Euler equations linearized about the stationary solution, and the wave equation written as a first order system.

2.3.1. *Maxwell's equations.* Taking the divergence of the dynamic Maxwell equations (1.2) yields

$$\partial_t\big(\operatorname{div}\varepsilon(x)E \ - \ 4\pi\operatorname{div}j\big) \ = \ 0\,. \qquad \partial_t\big(\operatorname{div}\mu(x)B\big) = 0\,. \qquad (2.18)$$

The continuity equation (1.3) implies

$$\partial_t\big(\operatorname{div}\varepsilon E - 4\pi\rho\big) \ = \ 0\,. \qquad (2.19)$$

The physical solutions satisfy (1.4) asserting that the time independent quantities $\operatorname{div}\varepsilon E - 4\pi\rho$ and $\operatorname{div}\mu B$ vanish identically.

In regions without currents and charges, equations (2.18) and (2.19) express hidden ellipticity with

$$Q \ := \ \begin{pmatrix} \varepsilon(x)\operatorname{div} & 0 \\ 0 & \mu(x)\operatorname{div} \end{pmatrix}.$$

Equations (1.2) have form $Lu = 0$ with

$$L \ = \ A_0(x)\,\partial_t \ + \ \sum_{j=1}^{3} A_j\partial_j\,, \qquad A_0(x) \ := \ \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix},$$

and 6×6 constant symmetric real matrices A_j for $j = 1, 2, 3$.

The nullspace of $\sum A_j\nu_j$ consists of the vectors E, B with both E and B parallel to ν. The range is the set of vectors E, B with both E and B tangent to the boundary. Introduce the tangential components

$$E_{tan} \ = \ E - (E\cdot\nu)\nu\,, \qquad B_{tan} \ = \ B - (B\cdot\nu)\nu\,.$$

For the halfspace $x_1 < 0$, the strictly dissipative subspaces $\mathcal{N}$ are those for which

$$(A_1u\,,\,u) \geq c\left(E_2^2 + E_3^2 + B_2^2 + B_3^2\right), \qquad E_2^2 + E_3^2 + B_2^2 + B_3^2 = \big\|E_{tan}, B_{tan}\big\|^2.$$

Assumption 2.1. i. *In Part I, the strictly positive matrix valued permittivities are assumed to be infinitely differentiable with partial derivatives of all orders belonging to $L^\infty(\mathbb{R}^3)$*

ii. *For each $1 \leq j \leq 3$, suppose that $\mathcal{N}_j$ is a strictly dissipative subspace for $-A_j$. That is for each j, $\mathcal{N}_j$ has dimension 4 and there is a constant $c_j > 0$ so that for all $u \in \mathcal{N}_j$*

$$\big(-A_ju\,,\,u\big) \ \geq \ c_j\,\big\|E_{tan}, B_{tan}\big\|^2\,.$$

Theorem 2.13. *With Assumption 2.1, for each $f, g \in L^2(\mathcal{O})$ with $\operatorname{div}\varepsilon(x)f = \operatorname{div}\mu(x)g = 0$, there is one and only one $u = (E, B) \in L^\infty\big(]0, \infty[\,;\, L^2(\mathcal{O})\big)$ with*

$$\big\{E_{tan}, B_{tan}\big\}\big|_{\partial\mathcal{O}\setminus\mathcal{S}} \ \in \ L^2(]0, \infty[\times(\partial\mathcal{O}\setminus\mathcal{S})),$$

$$Lu = 0, \quad u(0) = (f, g), \quad \text{and,} \quad u|_{\{x_j=0\}\cap\{\partial\mathcal{O}\setminus S\}} \in \mathcal{N}_j \ \text{ for } \ 1 \leq j \leq 3\,.$$

In addition, $u \in C\big([0, \infty[\,;\, L^2(\mathcal{O})\big)$, and for all $0 \leq t < T < \infty$ satisfies the energy identity

$$
\int_{\mathcal{O}} \big(A_0(x)\, , u(T,x)\big),\, u(T,x)\big)\big)\ dx \ +
$$
$$
\int_{[t,T] \times \partial(\mathcal{O} \setminus \mathcal{S})} \big(A(\nu(x))\, u, u\big)\, dt d\Sigma \ = \ \int_{\mathcal{O}} \big(A_0(x)\, u(0,x)\big),\, u(T,x)\big)\big)\ dx\,. \tag{2.20}
$$

Furthermore for $0 \leq t < \infty$, $\operatorname{div} \varepsilon(x) E = \operatorname{div} \mu(x) B = 0$.

Proof. The existence of a solution is proved as before. That is, the domain is smoothed and the boundary condition on the smoothed part is of the form

$$
\Pi_- u \ = \ M(x)\, \Pi_{\geq 0} u\,,
$$

with $M = 0$ on the rounded portions of $\partial \mathcal{O}^\varepsilon$. A passage to the limit removes the smoothing.

The difficult part is uniqueness. The strategy is to show that for a solution with data equal to zero, the Laplace transform vanishes. The essential step is to show that the Laplace transform $\widehat{u} = \{\widehat{E}\,,\, \widehat{B}\}$ belongs to $H^{1/2}(\mathcal{O})$.

Proof that $\widehat{E}, \widehat{B} \in H^{1/2}(\mathcal{O})$.

Drop the hats, but remember that the analysis is of the Laplace transform of a solution with zero data, and the goal is to prove that the transform vanishes for $\operatorname{Re} \tau \gg 1$.

The tangential components $\{E_{tan}, B_{tan}\}|_{\partial \mathcal{O} \setminus \mathcal{S}}$ are square integrable from strict dissipativity. From $E, \operatorname{div} E, \operatorname{curl} E \in L^2(\mathcal{O})$ it follows that $E|_{\partial \mathcal{O}} \in H^{-1/2}(\partial \mathcal{O})$. Denote by γ the unique element of $L^2(\partial \mathcal{O})$ so that $\{E_{tan}, B_{tan}\}|_{\partial \mathcal{O} \setminus \mathcal{S}} = \gamma|_{\mathcal{O} \setminus \mathcal{S}}$. Then $\{E_{tan}, B_{tan}\}|_{\partial \mathcal{O} \setminus \mathcal{S}} - \gamma$ is an element of $H^{-1/2}(\partial \mathcal{O})$ supported on the codimension one subset $\mathcal{S}$ so is identically equal to zero by Lemma 2.7. Thus, $\{E_{tan}, B_{tan}\}|_{\partial \mathcal{O}} = \gamma \in L^2(\partial \mathcal{O})$.

To take advantage of this, introduce potentials.

Lemma 2.14. *If $E \in L^2(\mathcal{O})$ satisfies*

$$
E_{tan} \in L^2(\partial \mathcal{O} \setminus \mathcal{S}), \quad \operatorname{div} \varepsilon(x) E = f \in L^2(\mathcal{O}), \quad \operatorname{curl} E = h \in L^2(\mathcal{O}),
$$

then $E \in H^{1/2}(\mathcal{O})$ and

$$
\|E\|_{H^{1/2}(\mathcal{O})} \ \lesssim \ \|E_{tan}\|_{L^2(\partial \mathcal{O} \setminus \mathcal{S})} \ + \ \|E\|_{L^2(\mathcal{O})} \ + \ \|\operatorname{div} \varepsilon E\,,\, \operatorname{curl} E\|_{L^2(\mathcal{O})}\,.
$$

Proof. • Reason locally. Introduce concentric balls B_1 of radius 1 and $B_{1/2}$ of radius $1/2$. The balls have centers either at distance > 1 to $\partial \mathcal{O}$ or are on a face of the boundary and distance > 1 to the singular points of the boundary, or are at a singular point and at distance > 1 to the corner $x = 0$ or are at the corner.

Lemma 2.14 folllows on taking a locally finite cover of $\overline{\mathcal{O}}$ by balls B_1^k, a partition of unity ψ_k subordinate to the cover, and summing over k the local estimates

$$
\|\psi_k E\|^2_{H^{1/2}(\mathcal{O})} \lesssim \|\operatorname{div} \varepsilon \psi_k E\,,\, \operatorname{curl} \psi_k E\|_{L^2(\mathcal{O})}
$$
$$
+ \|\psi_k E\|_{L^2(\mathcal{O})} + \|\psi_k E_{tan}\|^2_{L^2(\partial \mathcal{O} \setminus \mathcal{S})}\,.
$$

In this way the proof of the estimate is reduced to the case of functions supported in the balls B_1.

• Consider E supported in one of the balls of radius one. The interesting case is when the ball is one whose center lies on $\partial\mathcal{O}$. Choose an extension $\underline{h} \in L^2(\mathbb{R}^3)$ of $\operatorname{curl} E$ to $\mathbb{R}^3$ satisfying

$$\operatorname{supp} \underline{h} \subset B_1 \,, \qquad \|\underline{h}\|_{L^2(\mathbb{R}^3)} \lesssim \|h\|_{L^2(\mathcal{O} \cap B_1)} \,, \qquad \int_{\mathbb{R}^3} \underline{h} \, dx \;=\; 0 \,.$$

Since $\int \underline{h} \, dx = 0$, explicit solution in Fourier constructs a unique solution $\underline{E} \in H^1(\mathbb{R}^3)$ to

$$\operatorname{curl} \underline{E} \;=\; \underline{h}, \qquad \operatorname{div} \underline{E} \;=\; 0 \,,$$

and $\|\underline{E}\|_{H^1(\mathbb{R}^3)} \lesssim \|h\|_{L^2(\mathcal{O})}$. From $\underline{E} \in H^1(\mathbb{R}^3)$ it follows that $\underline{E}|_{\partial\mathcal{O}} \in H^{1/2}(\mathcal{O}) \subset L^2(\partial\mathcal{O})$.

• To complete the proof it suffices to show that $E - \underline{E} \in H^{1/2}(\mathcal{O} \cap B_1)$. Since $\operatorname{curl}(E - \underline{E}) = 0$ on $B_1 \cap \mathcal{O}$ there is a potential $\phi \in H^1(\mathcal{O} \cap B_1)$ so that

$$E - \underline{E} \;=\; -\operatorname{grad} \phi \quad \text{in} \ \ \overline{\mathcal{O}} \cap B_1 \,. \tag{2.21}$$

Denote by $\zeta := \phi|_{\partial\mathcal{O}}$. As the trace of an element of H^1 it is automatically in $H^{1/2}(B_1 \cap \partial\mathcal{O})$. Next show that $\zeta \in H^1(B_1 \cap \partial\mathcal{O})$. This space is defined by performing a bilipschitzian flattening of the boundary. In this way, the assertion $D\zeta \in L^p$ makes sense and it suffices to show that $D\zeta \in L^2$. Equation (2.21) implies that

$$D\zeta|_{B_1 \cap (\partial\mathcal{O} \setminus \mathcal{S})} \;=\; (E - \underline{E})_{tan} \;\in\; L^2(\mathcal{O} \setminus \mathcal{S}) \,.$$

Denote by $\gamma \in L^2(B_1 \cap \partial\mathcal{O})$ the unique element of $L^2(B_1 \cap \partial\mathcal{O})$ whose restriction to $\partial\mathcal{O} \setminus \mathcal{S}$ is equal to $(E - \underline{E})_{tan}$. Then $D\zeta - \gamma$ is an element of $H^{-1/2}(B_1 \cap \partial\mathcal{O})$ supported on the codimension one subset $\mathcal{S} \cap (B_1 \cap \partial\mathcal{O})$. Lemma 2.7 implies that $D\zeta - \gamma$ vanishes identically completing the proof that $\phi|_{\partial\mathcal{O}} \in H^1(B_1 \cap \partial\mathcal{O})$.

The potential is unique up to an additive constant. The potential is uniquely determined by imposing

$$\int_{B_{1/2} \cap \partial\mathcal{O}} \phi \, dx \;=\; 0 \,.$$

This is used with the inequality of Poincaré type on $\partial\mathcal{O}$,

$$\|\zeta\|_{H^1(B_1 \cap \partial\mathcal{O})} \;\lesssim\; \|D\zeta\|_{L^2(B_1 \cap \partial\mathcal{O})} \;+\; \Big| \int_{B_{1/2} \cap \partial\mathcal{O}} \zeta \, dx \Big|$$

to show that

$$\|\phi\|_{H^1(B_1 \cap \partial\mathcal{O})} \;\lesssim\; \|(E - \underline{E})_{tan}\|_{L^2(B_1 \cap (\partial\mathcal{O} \setminus \mathcal{S}))} \,. \tag{2.22}$$

On B_1 one has the scalar elliptic equation

$$-\operatorname{div} \varepsilon \operatorname{grad} \phi \;=\; f - \operatorname{div} \varepsilon \underline{E} \;\in\; L^2(B_1) \,.$$

Equation (2.22) asserts that the values of ϕ on $\partial\mathcal{O}$ belong to $H^1(B_1 \cap \partial\mathcal{O})$. The limit case elliptic regularity for the Dirichlet problem [14] yields $\phi \in H^{3/2}(\mathcal{O} \cap B_{1/2})$. In addition, with constants independent of the balls,

$$\|\phi\|_{H^{3/2}(\mathcal{O} \cap B_{1/2})} \;\lesssim\; \|\phi\|_{H^1(B_1 \cap \partial\mathcal{O})} \;+\; \|f, h\|_{L^2(B_1 \cap \partial\mathcal{O})} \,.$$

This completes the proof of the Lemma 2.14. $\qquad\qquad\qquad\qquad\qquad\square$

Given Lemma 2.14, the proof of Theorem 2.13 is completed in the same way as the proof of Theorem 1.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.3.2. *Linearized Euler at velocity zero.* Consider the inviscid compressible Euler equations linearized about a state of constant density and velocity. Computing such a solution it is intelligent to make a galilean transformation moving at the background speed, thus reducing to the case of background speed equal to zero. That linearization in non dimensionalized form is,

$$\partial_t u \; + \; \operatorname{grad} p \; = \; h \,, \qquad \partial_t p \; + \; \operatorname{div} u \; = \; 0 \,. \qquad (2.23)$$

We study the case $h = 0$. By Duhamel's principal that is sufficient. The unknown is a $d+1$ vctor (u,p). The system

$$\partial_t \begin{pmatrix} u_1 \\ \vdots \\ u_d \\ p \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & \partial_1 \\ 0 & 0 & \cdots & \partial_2 \\ \vdots & \vdots & & \vdots \\ \partial_1 & \partial_2 & \cdots & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_d \\ p \end{pmatrix} = 0$$

is clearly symmetric. This characteristic equation is $(\tau^2 \, - \, |\xi|^2) \, \tau^d \; = \; 0$.

For a half plane with unit conormal ν the kernel of the normal matrix $A(\nu)$ consists of the states with $u \cdot \nu = 0 = p$. The nonzero spectral components satisfy

$$\left\| \Pi_+(u,p) \right\|^2 \; + \; \left\| \Pi_-(u,p) \right\|^2 \; = \; \left| u \cdot \nu \right|^2 \; + \; \left| p \right|^2 .$$

The strictly dissipative subspaces are those on which

$$\Big(A(\nu)(u,p) \,, \, (u,p) \Big) \; \geq \; c \Big(\left| u \cdot \nu \right|^2 \; + \; \left| p \right|^2 \Big), \qquad c > 0 \,.$$

Taking the curl of the first equation in (2.23) yields

$$\partial_t \big(\operatorname{curl} u \big) \; = \; \operatorname{curl} h \,. \qquad (2.24)$$

This is an example of the hidden ellipticity with

$$Q \; := \; \begin{pmatrix} \operatorname{curl} & 0 \\ 0 & 0 \end{pmatrix} .$$

Theorem 2.15. *Suppose that for $1 \leq j \leq d$, $\mathcal{N}_j$ is a maximal strictly dissipative subspace for A_j in the Euler system, and $f \in L^2(\mathcal{O})$. Then there is a unique $u \in L^\infty(]0, \infty[\,; L^2(\mathcal{O}))$ so that for all $T > 0$*

$$\big(u \cdot \nu \,, \, p \big) \big|_{]0,T[\times (\partial\mathcal{O} \setminus \mathcal{S})} \; \in \; L^2(]0, T[\times (\partial\mathcal{O} \setminus \mathcal{S}))$$

satisfying (2.23), the boundary condition $u \in \mathcal{N}_j$ on $\{(\mathcal{O} \setminus \mathcal{S}) \cap \{x_j = 0\}\}$, and, the initial condition $(u(0), p(0)) = f$. In addition, $u \in C([0, \infty[\,; L^2(\mathcal{O}))$ and for any $0 \leq t_1 < t_2 \leq T$ the solution satisfies the energy identity

$$\big\| (u(t), p(t)) \big\|^2 \Big|_{t_1}^{t_2} \; + \; \int_{]t_1.t_2[\times (\partial\mathcal{O} \setminus \mathcal{S})} \big(A(\nu)(u,p) \,, \, (u,p) \big) \; dt \, d\Sigma \; = \; 0 \,.$$

Proof. The proof of existence is by rounding the corner and then passing to the limit. The hard part is uniqueness. Need to show that the only solution with $g = h = 0$ is the zero solution. This is proved by showing that the Laplace transform vanishes for all τ with $\operatorname{Re} \tau > 0$. Risking confusion the Laplace transform is written (u, p) and satisfies

$$\tau u \; + \; \operatorname{grad} p \; = \; 0, \qquad \tau p \; + \; \operatorname{div} u \; = \; 0, \qquad (2.25)$$

together with

$$(u,p) \in L^2(\mathcal{O}), \quad (u \cdot \nu \,, \, p) \in L^2(\partial\mathcal{O} \setminus \mathcal{S}), \quad (u,p) \in \mathcal{N}_j \; \text{ on } \; \partial\mathcal{O} \setminus \mathcal{S}. \qquad (2.26)$$

Multiplying the equation by (u, p) and integrating by parts would show that $u = p = 0$ because the boundary conditions are dissipative. To prove uniqueness it suffices to justify the integration by parts.

The first step is to take the curl of the first equation in (2.25) to find $\operatorname{curl} u = 0$. The integration by parts is justified by showing that u, p belongs to $H^{1/2}(\mathcal{O})$.

Lemma 2.16. *If* $\operatorname{Re} \tau > 0$ *and* (u, p) *satisfies* (2.25) *and* (2.26) *then* $(u, p) \in H^{1/2}(\mathcal{O})$.

Proof. Since $\operatorname{grad} p = -\tau u \in L^2$, one has $p \in H^1(\mathcal{O})$. It suffices to show that $u \in H^{1/2}(\mathcal{O})$.

Since $\operatorname{curl} u = 0$ it follows that there is a $\phi \in \mathcal{D}'(\mathcal{O})$ with $u = \operatorname{grad} \phi$ on $\mathcal{O}$. The potential ϕ is unique up to an additive constant. Taking divergence yields

$$\Delta \phi \;=\; \operatorname{div} \operatorname{grad} \phi \;=\; \operatorname{div} u \;=\; -\tau p \;\in\; H^1(\mathcal{O}).$$

On $\partial\mathcal{O}$ one has

$$\frac{\partial \phi}{\partial \nu} \;=\; \nu \cdot \operatorname{grad} \phi \;=\; \nu \cdot u \;\in\; L^2(\partial\mathcal{O} \setminus \mathcal{S}).$$

Define $\gamma \in L^2(\partial\mathcal{O})$ to be the unique element equal to $\nu \cdot u$ on the set of full measure $\partial\mathcal{O} \setminus \mathcal{S}$. Since $\Delta \phi \in H^1(\mathcal{O})$ it follows that $\partial\phi/\partial\nu|_{\partial\mathcal{O}}$ is a well defined element of $H^{-1/2}(\partial\mathcal{O})$. Then

$$\partial\phi/\partial\nu|_{\partial\mathcal{O}} \;-\; \gamma$$

is an element of $H^{-1/2}(\partial\mathcal{O})$ supported on the codimension one subset $\mathcal{S}$. Lemma 2.7 implies that the difference vanishes so $\partial\phi/\partial\nu|_{\partial\mathcal{O}} \in L^2(\partial\mathcal{O})$.

Regularity for the Neumann problem in the limiting case (see [13]) implies

$$u \;=\; \operatorname{grad} \phi \;\in\; H^{1/2}(\mathcal{O}),$$

completing the proof of the Lemma. $\qquad \square$

The lemma provides the regularity necessary to justify the integration by parts in the energy identity that yields uniqueness. The proof of the final parts of Theorem 2.15 are as in the proof of Theorems 1.4. $\qquad \square$

2.3.3. *Wave equation written as a system.* Reducing the wave equation to a first order differential system in dimension $d > 1$ introduces non physical stationary modes. As for Maxwell, the solutions of interest do not excite these modes.

Treat the case of the wave equation in dimension d. It is converted to a first order system for the $d + 1$ derivatives $v_\mu := \partial_\mu u$, $\mu = 0, 1, \ldots, d$ where $\partial_0 = \partial_t$. Write the variables as

$$v \;=\; (v_0, \mathbf{v}).$$

The equations are

$$\partial_t v_j \;=\; \partial_j v_0, \quad j = 1, \ldots, d, \qquad \partial_t v_0 \;=\; \operatorname{div} \mathbf{v}.$$

Up to a change of sign these are exactly the linearized Euler equations treated in the preceeding section.

2.4. **Counterexamples.** The first examples are related to those of Osher [20] and Sarason-Smoller [25] while the last elaborates §1.2.1. In addition to these examples we mention the strictly dissipative example of nonuniqueness in [17] that violates both ellipticity and square integrable traces.

2.4.1. *Counterexamples from two transport equations.* The examples are in dimension $d = 2$ where

$$\mathcal{O} := \left\{ x \in \mathbb{R}^2 : x_1 > 0, \text{ and } x_2 > 0 \right\}.$$

Introduce the shorthand $\mathbf{U} = (U_1, U_2)$ and $\mathbf{U}\partial_x := U_1\partial_{x_1} + U_2\partial_{x_2}$. Consider the pair of transport equations

$$(\partial_t + \mathbf{U}\partial_x)u = 0, \qquad (\partial_t + \mathbf{V}\partial_x)v = 0$$

for the unknown $(u(t,x), v(t,x))$. The vectors $\mathbf{U}$ and $\mathbf{V}$ satisfy

$$U_1 < 0 < U_2, \qquad \text{and} \qquad V_2 < 0 < V_1.$$

The boundary segments of $\mathcal{O}$ are non characteristic. The vector field $\mathbf{U}$ (resp. $\mathbf{V}$) points into the northwest (resp. southeast) quadrant. Every ray starting in $\mathcal{O}$ reaches the boundary at a point other than the origin after a finite amount of time. Broken ray paths suitably reflected at the boundary meet the boundary infinitely often and never at the origin.

On the x_1 axis, a homogeneous boundary condition $u = \alpha v$ with $\alpha \in \mathbb{C}$ is imposed. On the x_2 axis one imposes $v = \beta u$ with $\beta \in \mathbb{C}$.

Osher [19] proves by that the norm

$$\int_{\mathcal{O}} M^2 |u|^2 + |v|^2 \, dx, \qquad M \in \,]0, \infty[\tag{2.27}$$

is dissipative at both boundary segments if and only if

$$M^2|\alpha|^2 \leq -\frac{V_2}{U_2}, \qquad \text{and} \qquad |\beta|^2 \leq -\frac{U_1}{V_1}M^2.$$

Equivalently $|\alpha\beta|^2 \leq U_1 V_2 / V_1 U_2$. In this case not hard to construct square integrable solutions with square integrable traces and prove their uniqueness [24].

The case $\mathbf{U} = -\mathbf{V}$. In this case, the simultaneously dissipated case is $|\alpha\beta| \leq 1$. In the opposite case $|\alpha\beta| > 1$ the system misbehaves. Our ray tracing analysis is as in [25].

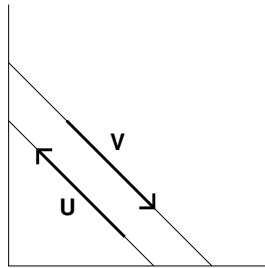When $\mathbf{U} = -\mathbf{V}$, the values of u are rigidly transported with speed $\mathbf{U}$ till they

FIGURE 9. The case $\mathbf{U} = -\mathbf{V}$.

reach the x_2 axis where their value is multiplied by β and fed to the v equation where they are transported to the x_1 axis *along the same ray traversed in the opposite direction.* Then they are multiplied by α and fed to the u equation. And so on.

When $|\alpha\beta| > 1$ one circuit with two reflections leads to an amplification. In the absence of corners, this would cause no problem at all.

For initial data supported in $x_1 + x_2 > \delta$ the shortest circuit has length $2\delta\sqrt{2}$ and solutions are amplified by no more than $(\alpha\beta)^{T/(2\delta\sqrt{2})}$. One easily shows existence and uniqueness of solutions supported in $\{x_1 + x_2 > \delta\}$.

As one approaches the corner the amplification becomes more and more extreme. Data supported in $2^{-(n+1)} < x_1 + x_2 < 2^{-n}$ is amplified at time $t \sim 1$ by $|\alpha\beta|^{2^n}$. In order for a solution to have finite L^2 norm, the initial data must satisfy

$$\sum_{n \geq 0} |\alpha\beta|^{2^n} \int_{2^{-(n+1)} < x_1 + x_2 < 2^{-n}} |f(x)|^2 \, dx \; < \; \infty.$$

This is a dense set of data. For f vanishing outside a bounded subset of $\overline{\mathcal{O}}$, the corrsponding sums are finite for all $t > 0$ if and only if

$$\forall \, K > 0, \qquad \int_{\mathcal{O}} e^{K/|x|} \, |f(x)|^2 \, dx \; < \; \infty.$$

Summary. *For the case $\mathbf{U} = -\mathbf{V}$ the direct and adjoint problems glue together problems satisfying the uniform Kreiss-Sakamoto condition. When $|\alpha\beta| > 1$ there is existence only for a dense set of data that are small near the corner. Uniqueness is proved locally in each set $\{2^{-n} < x_1 + x_2 < 2^n\}$, $n > 0$.*

Nonuniqueness for $\mathbf{U} = (-1, 1)$ and $\mathbf{V} = (1, -a)$, $1 < a$. For these $\mathbf{U}, \mathbf{V}$, an initial ray and its successive reflections are sketched Figure 10. Each cycle of two
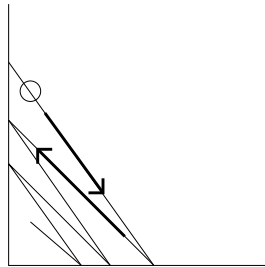

FIGURE 10. Multiply reflected rays.

reflections brings one closer to the origin by a fixed factor < 1. A ray starting in the little disk in the figure approaches the origin in finite time with infinitely many reflections.

Suppose that $\alpha > 0$, $\beta > 0$, and $\alpha\beta < 1$. In each cycle there is decay. Suppose that at $t = 0$, u and v are nonnegative, not identically zero, and supported in the little disk. Consider the value of the solution transported along the broken ray starting at $t = 0$ at the center of the disk in the $\partial_t + \mathbf{V}\partial_x$ direction.

On the initial segment, v is constant. On the reflected ray, the value of u is α times the value of v on the incoming ray. In each cycle of two reflections the value of v is multiplied by $\alpha\beta$. Along the ray the value of v converges to 0 as t increases to T. The value of $|x|$ also converges to zero and the solution is $\leq C|x|$.

An entirely analogous argument shows that the initial values of u when traced forward in time yields wave approaching the origin where they are $O(|x|)$.

The reflecting waves constructed above have nonnegative components and move steadily toward the origin as t increases. Ray by ray they are absorbed at the origin. After the last ray starting in the disk arrives at the origin, say at time $\underline{T}$, there is

nothing left. Define a candidate solution to be these values extended by zero at all points that are not reached by the broken rays starting at $t = 0$ in the disk.

Now play the candidate solution backward in time. This backward problem is amplifying in each cycle. The solution just constructed vanishes for large time. And as time decreases, a positive wave emerges from the corner.

It is not hard to show that the resulting candidate is indeed a weak solution of the boundary value problem. It is an example of non uniqueness.

The boundary conditions, though amplifying, both satisfy the uniform Kreiss-Sakamoto condition. Each is strictly dissipative with respect to a scalar product equivalent to that in L^2. The scalar products associated to the different boundary edges are different.

Summary. *Gluing two boundary conditions together at a corner can yield nonuniqueness. A non zero wave can emerge from the corner. Even when each individual condition is strictly dissipative with respect to an L^2 scalar product of the form $\int M|u|^2 + |v|^2 \, dx$ with different M for the two conditions.*

The existence and uniqueness theorem that we proved differs from these examples in two respects. First the generator $A(x, \partial)$ is elliptic, at least at the boundary. Second, the boundary conditions are strictly dissipative *with respect to the same scalar product*. The counterexample satisfies the hypothesis of square integrable traces on the boundaries.

2.4.2. *Elliptic counterexamples.* The transport counterexamples violate two hypotheses, ellipticity and strict dissipativity *with respect to a fixed scalar product*. We sketch elliptic versions that violate only the second.

Let

$$G := A_1\partial_1 + A_2\partial_2 \,, \qquad A_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad A_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Modify the equations to be

$$(\partial_t + \mathbf{U}\partial_x + \mu G)u = 0 \,, \qquad (\partial_t - \mathbf{U}\partial_x + \mu G)v = 0 \,, \qquad 0 < \mu \ll 1,$$

where now both u and v are $\mathbb{C}^2$ valued. The characteristic variety is given by

$$\left((\tau + \mathbf{U} \cdot \xi)^2 - \mu|\xi|^2 \right) \left((\tau - \mathbf{U} \cdot \xi)^2 - \mu|\xi|^2 \right) = 0 \,.$$

The boundary conditions are as before.

When $|\alpha\beta| > 1$, the problem is strongly explosive as one shows by geometric optics. The convenient choice is to take ξ parallel to $\mathbf{U}$ in which case the transport directions of geometric optics are parallel to $\mathbf{U}$. The leftward going wave will have amplitude τ_{left} and the rightward waves a τ_{right}. The transport equations of geometric optics are exactly the equations of the Osher example. They are explosive.

Prove ill posedness as follows. Given a challenge number M, choose a ray and $T > 0$ so that the transport equation of geometric optics yields amplification $> M$ at time T. Choose a very small disk so that along this ray the translated disks and their successive reflections do not meet the corner. Then choose an initial amplitude function supported in the disk and consider the approximate solutions in the limit of wavelength tending to zero to find solutions whose L^2 norm is amplified by more than M.

Summary. *Two strictly dissipative Kreiss well posed conditions glued together at a corner yield an ill posed problem. The generators, $A(x, \partial)$ are elliptic. The problems are strictly dissipative for **different** but equivalent L^2 norms.*

2.4.3. *A radiation problem example.* For $d \geq 2$, consider the solution of the radiation problem in $\mathbb{R}^d$

$$\Box_{1+d} w \;=\; f(t)\, \delta(x)\,, \qquad w = f = 0 \quad \text{for} \quad t \leq 0\,, \qquad f \in C_0^\infty(\mathbb{R})\,.$$

Since δ is even in each x_j, the same is true of w. Therefore, on each face of $\mathbb{R}_t \times (\partial \mathcal{O} \setminus \mathcal{S})$, w satisfies the homogeneous Neumann condition $\nu \cdot \partial_x w = 0$.

It satisfies $\Box w = 0$ in $\mathbb{R}_t \times \mathcal{O}$ and has vanishing Cauchy data for $t < 0$ and homogeneous boundary condition, yet there are waves in $\mathcal{O}$. The waves come out of the corner.

The Neumann multihedral corner problem is easily proved to be well set for solutions in H^1. A short proof uses the self adjoint operator $-\Delta$ with form domain $\mathcal{D}((-\Delta)^{1/2}) = H^1(\mathcal{O})$. Finite energy solutions are those with $u_t(t) \in L^2(\mathcal{O})$ and $u(t) \in \mathcal{D}((-\Delta)^{1/2})$. The Neumann condition is the associated natural boundary condition. The counterexample to uniqueness does not belong to $H^1(\mathcal{O})$, has infinite energy, and therefore does not satisfy the hypotheses of the uniqueness result. This example has elliptic spatial part. The solution violates three hypotheses of our uniqueness theorem. The boundary conditions are conservative rather than strictly dissipative. The analogue of L^2 regularity, in this case H^1 regularity (and H^1 boundary trace), is violated. The boundary traces are not H^1.

3. **Part II. Trihedral internal corners for Bérenger's method.** The set Ω, comprised of eight octants $\mathcal{O}$ and $\mathcal{O}_\kappa$, is defined in Definition 1.5 page 4409 and sketched in Figure 3 page 4405.

The nonnegative absorptions $0 \leq \sigma_j(x_j)$ are uniformly bounded, $\sigma_j \in L^\infty(\mathbb{R})$. The original absorption coefficients of Bérenger were chosen as Heaviside functions. In dimension $d = 3$ the present paper is the first demonstrating that the Bérenger split Maxwell equations are well posed for three absorptions less regular than C^2.

3.1. **Splitting Maxwell.**

3.1.1. *Vector calculus.* Denote by $\mathbf{e}_1 := (1, 0, 0)$, $\mathbf{e}_2 := (0, 1, 0)$, and $\mathbf{e}_3 := (0, 0, 1)$ the standard basis elements of $\mathbb{C}^3$. Denote by $p_j : \mathbb{C}^3 \to \mathbb{C}^1$ the linear transformation $p_j(v_1, v_2, v_3) := v_j$. If π_j denotes the orthogonal projection on the jth coordinate axis, then $\pi_j v = (p_j v) \mathbf{e}_j$. The 1×3 matrices of p_j are

$$p_1 := (1, 0, 0), \qquad p_2 := (0, 1, 0), \qquad p_3 := (0, 0, 1).$$

Then,

$$\text{div} \;=\; (\partial_1, \partial_2, \partial_3) \;=\; \sum p_j\, \partial_j\,.$$

The C_j from (1.5), (1.6) satisfy

$$C_1^2 \;:=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_2^2 \;:=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_3^2 \;:=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_j^* = -C_j\,, \qquad \left(C_i C_j\right)^* \;=\; C_j C_i\,, \qquad C_1 C_2 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Denote by $m_{i,j} \in \mathrm{Hom}(\mathbb{C}^3)$ the linear transformation whose matrix has a 1 in the i,j position and zeroes elsewhere. Then

$$C_1 C_2 = m_{2,1}, \qquad \text{and,} \qquad \text{for } i \neq j, \qquad C_i C_j = m_{j,i}.$$

Expand

$$\mathrm{div}\,\mathrm{curl} = \sum_j p_j \partial_j \sum_k C_k \partial_k = \sum_{j,k} p_j\, C_k\, \partial_j \partial_k.$$

The identity $0 = \mathrm{div}\,\mathrm{curl}$ is equivalent to the matrix identity

$$p_j C_k + p_k C_j = 0 \quad \text{for all } j,k. \tag{3.1}$$

The Laplace transform of the split Bérenger Maxwell equations (1.7) is

$$\begin{aligned}
\varepsilon(\tau + \sigma_j(x_j))\widehat{E}^j &= C_j \partial_j \sum \widehat{B}^k, \\
\mu(\tau + \sigma_j(x_j))\widehat{B}^j &= -C_j \partial_j \sum \widehat{E}^k.
\end{aligned} \qquad \text{for } j = 1,2,3. \tag{3.2}$$

Mulitply by $\tau/(\tau + \sigma_j)$ to find for $j = 1,2,3$

$$\varepsilon\tau\,\widehat{E}^j = \frac{\tau}{\tau + \sigma_j}\, C_j \partial_j \sum \widehat{B}^k, \qquad \mu\tau\,\widehat{B}^j = -\frac{\tau}{\tau + \sigma_j}\, C_j \partial_j \sum \widehat{E}^k. \tag{3.3}$$

Care must be taken to keep the factors in this order since the absorptions σ_j depend on x_j so do not commute with ∂_j.

Introduce the total fields $E = \sum_k E^k$, $B = \sum_k B^k$ on $\Omega \setminus \mathcal{O}$ to find

$$\varepsilon\tau\,\widehat{E}^j = \frac{\tau}{\tau + \sigma_j}\, C_j \partial_j \widehat{B}, \qquad \mu\tau\widehat{B}^j = -\frac{\tau}{\tau + \sigma_j}\, C_j \partial_j \widehat{E}. \tag{3.4}$$

Summing on j yields on $\Omega \setminus \mathcal{O}$

$$\varepsilon\tau\,\widehat{E} = \sum_j \frac{\tau}{\tau + \sigma_j}\, C_j \partial_j \widehat{B}, \qquad \mu\tau\,\widehat{B} = -\sum_j \frac{\tau}{\tau + \sigma_j}\, C_j \partial_j \widehat{E}.$$

Including the Maxwell equation on $\mathcal{O}$ yields everywhere in Ω

$$\tau\varepsilon\,\widehat{E} = \sum_j \frac{\tau}{\tau + \sigma_j} C_j \partial_j\,\widehat{B} + 4\pi\widehat{\mathbf{j}}, \quad \tau\mu\,\widehat{B} = -\sum_j \frac{\tau}{\tau + \sigma_j}\, C_j \partial_j\,\widehat{E}. \tag{3.5}$$

One can recover $\widehat{E}^j, \widehat{B}^j$ in $\mathcal{O}_\kappa$ from the values of $\widehat{E}, \widehat{B}$ in $\mathcal{O}_\kappa$ using (3.4).

The next sections analyse the sytem of equations (3.5) satisfied by $\widehat{E}, \widehat{B}$.

3.1.2. *Tilde operators.* Introduce operators so that equations (3.5) in the domains $\mathcal{O}_\kappa$ resemble Maxwell's equations. They replace ∂_j by $\tau/(\tau + \sigma_j)\,\partial_j$ paying attention to the order of factors.

Definition 3.1. In each of the eight octants define differential operators

$$\widetilde{\mathrm{div}} := \sum_j \frac{\tau}{\tau + \sigma_j}\, p_j\, \partial_j, \qquad \widetilde{\mathrm{curl}} := \sum_j \frac{\tau}{\tau + \sigma_j}\, C_j \partial_j,$$

$$\widetilde{\mathrm{grad}}\,\phi := \left(\frac{\tau}{\tau + \sigma_1}\partial_1\phi, \frac{\tau}{\tau + \sigma_2}\partial_2\phi, \frac{\tau}{\tau + \sigma_3}\partial_3\phi \right),$$

$$\widetilde{\Delta}\,\phi := \widetilde{\mathrm{div}}\,\widetilde{\mathrm{grad}}\,\phi = \sum_j \frac{\tau}{\tau + \sigma_j}\, \partial_j \left(\frac{\tau}{\tau + \sigma_j}\, \partial_j\,\phi \right).$$

Lemma 3.2. *The tilde operators satisfy in each octant*

$$\widetilde{\operatorname{div}}\,\widetilde{\operatorname{curl}} \;=\; 0, \qquad \widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}} \;-\; \widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}} \;=\; \begin{pmatrix} \widetilde{\Delta} & 0 & 0 \\ 0 & \widetilde{\Delta} & 0 \\ 0 & 0 & \widetilde{\Delta} \end{pmatrix}.$$

Proof. For the first identity compute

$$\begin{aligned}
\widetilde{\operatorname{div}}\,\widetilde{\operatorname{curl}} \;&=\; \sum_k \frac{\tau}{\tau+\sigma_k}\, p_k \partial_k \Big(\sum_j \frac{\tau}{\tau+\sigma_j}\, C_j \partial_j \Big) \\
&=\; \sum_{j,k} \frac{\tau}{\tau+\sigma_k}\, p_k \partial_k \Big(\frac{\tau}{\tau+\sigma_j}\, C_j \partial_j \Big).
\end{aligned}$$

Expand

$$\partial_k \Big(\frac{\tau}{\tau+\sigma_j}\, C_j \partial_j \Big) \;=\; \partial_k \Big(\frac{\tau}{\tau+\sigma_j} \Big) C_j \partial_j \;+\; \frac{\tau}{\tau+\sigma_j}\, C_j \,\partial_k \partial_j.$$

The first term vanishes except when $k = j$. The sum of the resulting contributions to the divergence vanishes since it is equal to

$$\sum_j \frac{\tau}{\tau+\sigma_j}\, \partial_j \Big(\frac{\tau}{\tau+\sigma_j} \Big)\, p_j C_j \partial_j$$

and $p_j C_j = 0$ by (3.1). The sum of the remaining contributions to the divergence is equal to

$$\begin{aligned}
\sum_{j,k} \frac{\tau}{\tau+\sigma_k}\, \frac{\tau}{\tau+\sigma_j}\, p_k\, C_j \partial_k \partial_j \;=\;& \sum_j \Big(\frac{\tau}{\tau+\sigma_j} \Big)^2 p_j\, C_j \partial_j^2 \\
&+ \sum_{j \neq k} \frac{\tau}{\tau+\sigma_k}\, \frac{\tau}{\tau+\sigma_j}\, (p_k\, C_j + p_j\, C_k)\partial_k \partial_j.
\end{aligned}$$

This vanishes since $p_j C_j = 0$ and for $k \neq j$, $p_k C_j + p_j C_k = 0$ completing the proof of the first identity.

For the second identity compute the first component of the left hand side applied to B to find

$$\begin{aligned}
&\frac{\tau}{\tau+\sigma_1}\partial_1 \Big(\frac{\tau}{\tau+\sigma_1}\partial_1 B_1 + \frac{\tau}{\tau+\sigma_2}\partial_2 B_2 + \frac{\tau}{\tau+\sigma_3}\partial_3 B_3 \Big) \\
&\qquad - \frac{\tau}{\tau+\sigma_2}\partial_2 \Big(\frac{\tau}{\tau+\sigma_1}\partial_1 B_2 - \frac{\tau}{\tau+\sigma_2}\partial_2 B_1 \Big) \\
&\qquad\qquad + \frac{\tau}{\tau+\sigma_3}\partial_3 \Big(\frac{\tau}{\tau+\sigma_3}\partial_3 B_1 - \frac{\tau}{\tau+\sigma_1}\partial_1 B_3 \Big).
\end{aligned}$$

Rearrange to find

$$\begin{aligned}
&\frac{\tau}{\tau+\sigma_1}\partial_1 \Big(\frac{\tau}{\tau+\sigma_1}\partial_1 B_1 \Big) + \frac{\tau}{\tau+\sigma_2}\partial_2 \Big(\frac{\tau}{\tau+\sigma_2}\partial_2 B_1 \Big) + \frac{\tau}{\tau+\sigma_3}\partial_3 \Big(\frac{\tau}{\tau+\sigma_3}\partial_3 B_1 \Big) \\
&\qquad + \frac{\tau}{\tau+\sigma_1}\partial_1 \Big(\frac{\tau}{\tau+\sigma_2}\partial_2 B_2 \Big) - \frac{\tau}{\tau+\sigma_2}\partial_2 \Big(\frac{\tau}{\tau+\sigma_1}\partial_1 B_2 \Big) \\
&\qquad\qquad + \frac{\tau}{\tau+\sigma_1}\partial_1 \Big(\frac{\tau}{\tau+\sigma_3}\partial_3 B_3 \Big) - \frac{\tau}{\tau+\sigma_3}\partial_3 \Big(\frac{\tau}{\tau+\sigma_1}\partial_1 B_3 \Big).
\end{aligned}$$

Since σ_j depends on x_j only, the second and third lines vanish. The first line is the tilde laplacian of B_1. The other components are similar. □

Definition 3.3. For

$$U := (E, B) \qquad \text{define} \quad \widetilde{\operatorname{div}} U := (\widetilde{\operatorname{div}} E, \widetilde{\operatorname{div}} B),$$

and

$$LU := \left(\varepsilon \tau E - \widetilde{\operatorname{curl}} B , \, \mu \tau B + \widetilde{\operatorname{curl}} E \right).$$

Remark 3.1. In $\mathcal{O}$ the absorptions vanish and this is Maxwell's operator.

3.2. **Estimates in $\overline{\mathcal{O}}$ and in $\mathcal{B}$.** The permittivities are scalar outside K as in Figure 4. In particular wherever there are boundaries. A partition of unity serves to separate the regions where the permitivities are not scalar from the rest. The function U is split as $U_1 + U_2$ with U_1 supported in $\overline{\mathcal{O}}$ and U_2 supported where the permittivites ε, μ are scalar that is in $\mathcal{B} := \mathbb{R}^3 \setminus K$.

The estimates for U_1 are elementary estimates for symmetric hyperbolic systems that are true in much greater generality [23]. The key point is that they take place where the absorptions vanish identically. The statement and sketch of proof follow.

Proposition 3.1. *There are constants $C, M > 0$ so that if $\operatorname{Re} \tau > M$, $U = (E, B) \in H^1(\mathbb{R}^3)$, $\operatorname{supp} U \subset \overline{\mathcal{O}}$, and $LU \in H^1(\mathbb{R}^3)$ then*

$$\operatorname{Re} \tau \, \|U\|_{H^1(\mathbb{R}^3)} \leq C \, \|LU\|_{H^1(\mathbb{R}^3)} . \qquad (3.6)$$

Remark 3.2. i. Estimates (3.6) together with $\tau \, (\operatorname{div} \varepsilon E , \operatorname{div} \mu B) = \operatorname{div} LU$ valid in $\mathcal{O}$ yield

$$\|\tau \, (\operatorname{div} \varepsilon E , \operatorname{div} \mu B)\|_{L^2(\mathbb{R}^3)} = \left\| \operatorname{div} LU \right\|_{L^2(\mathbb{R}^3)}. \qquad (3.7)$$

ii. The differential equation yields the additional estimate

$$\|\tau \, U\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla_x U\|_{L^2(\mathbb{R}^3)} + \|LU\|_{L^2(\mathbb{R}^3)} . \qquad (3.8)$$

The factor τ is the Laplace transform of ∂_t and is on an even footing with ∇_x.

Sketch of proof of Proposition 3.1. Taking the scalar product of the first equation with E and the second with B, and integrating over $\mathbb{R}^3$ yields after an integration by parts

$$\operatorname{Re} \tau \int (\varepsilon E, E) + (\mu B, B) \, dx = \operatorname{Re} \int (U , LU) \, dx .$$

Using the Cauchy-Schwartz inequality on the right yields

$$\operatorname{Re} \tau \, \|U\|_{L^2(\mathbb{R}^3)} \lesssim \|LU\|_{L^2(\mathbb{R}^3)}$$

Assuming that $U \in H^2(\mathbb{R}^3)$, differentiating the equations with respect to x_j repeating the argument and summing on j yields the derivative estimate

$$\operatorname{Re} \tau \, \|U\|_{H^1(\mathbb{R}^3)} \lesssim \|LU\|_{H^1(\mathbb{R}^3)} .$$

For $U \in H^1$ with $LU \in H^1$ introduce $U^\varepsilon := j_\varepsilon * U \in H^2(\mathbb{R}^3)$ a Friedrichs mollification of U with j_ε supported in $\mathcal{O}$. Apply the estimate to find

$$\operatorname{Re} \tau \, \|U^\varepsilon\|_{H^1(\mathbb{R}^3)} \lesssim \|LU^\varepsilon\|_{H^1(\mathbb{R}^3)} .$$

The norm on the left hand side converges to $\|U\|_{H^1(\mathbb{R}^3)}$. Friedrichs' lemma ([11] Lemma 17.1.5) implies that the norm on the right converges to $\|LU\|_{H^1(\mathbb{R}^3)}$. $\square$

The second main estimate concerns functions supported in the region where the permittivities are scalar and includes the regions where the absorptions are non zero. The estimate is weaker and much harder to prove. The proof uses a well adapted complex Helmholtz equation. When $LU \in H^1(\mathbb{R}^3)$, $\widetilde{\operatorname{div}} LU$ is a well defined element

of $L^2(\mathbb{R}^3)$. The hypothesis $\widetilde{\mathrm{div}}\, LU \in H^1(\mathbb{R}^3)$ in the next proposition means that this element of $L^2(\mathbb{R}^3)$ belongs to $H^1(\mathbb{R}^3)$.

Proposition 3.2. *There are constants $C, M > 0$ so that for $\mathrm{Re}\,\tau > M$ and $U \in H^1(\mathbb{R}^3)$ with $\mathrm{supp}\,U \subset \overline{\mathcal{B}}$, $LU \in H^1(\mathbb{R}^3)$, and $\widetilde{\mathrm{div}}\, LU \in H^1(\mathbb{R}^3)$, one has*

$$\mathrm{Re}\,\tau\,\|U\|_{L^2(\mathbb{R}^3)} + \|\nabla U\|_{L^2(\mathbb{R}^3)}$$
$$\leq C\Big(\|LU\|_{H^1(\mathbb{R}^3)} + \|\tau LU\|_{L^2(\mathbb{R}^3)} + \|\tfrac{1}{\tau}\,\widetilde{\mathrm{div}}\, LU\|_{H^1(\mathbb{R}^3)}\Big). \tag{3.9}$$

In addition,

$$\|\tau\,\widetilde{\mathrm{div}}\, U\|_{H^1(\mathbb{R}^3)} \;\leq\; \|\widetilde{\mathrm{div}}\, LU\|_{H^1(\mathbb{R}^3)}, \tag{3.10}$$

and

$$\|\tau\,U\|_{L^2(\mathbb{R}^3)} \;\lesssim\; \|\nabla_x U\|_{L^2(\mathbb{R}^3)} \;+\; \|LU\|_{L^2(\mathbb{R}^3)}. \tag{3.11}$$

Remark 3.3. i. Estimate (3.9) is unbalanced because the $\|\widetilde{\mathrm{div}}\, LU\|_{H^1}$ is a second derivative estimate on LU and there is no second derivative on the left. However, (3.10) shows that the second derivatives $\|\widetilde{\mathrm{div}}\, U\|_{H^1}$ are also bounded by the righthand side. This balances the $\|\widetilde{\mathrm{div}}\, LU\|_{H^1}$ term.

ii. Similarly, (3.11) shows that one can add $\|\tau U\|_{L^2}$ to the left hand side of (3.9).

iii. The estimates depend only on the L^∞ norms of the absorptions. In particular they are uniform on families of absorptions that are bounded in L^∞.

The proof of Proposition 3.2 occupies the next sections.

3.2.1. *A well adapted Helmholtz operator.* This section makes the delicate selection of a Helmholtz operator for L acting on functions U with $\mathrm{supp}\,U \subset \mathcal{B}$ where the permittivities are scalar valued. This domain includes all the points where the absorptions are non zero. The result is an equation that not only holds in Ω but on the whole of $\mathcal{B}$. A typical selection would yield a second order operator that when applied to $U \in H^1$ would produce delta functions on the boundary surfaces of Ω where $\nabla_x U$ may be discontinuous. The good choice does not. The first step is to identify quantities that are continuous across the smooth parts of $\partial\Omega$.

Quantities that do not jump. The Laplace transformed split Maxwell equations yield

$$LU \;:=\; \big(\varepsilon\tau E - \widetilde{\mathrm{curl}}\, B\,,\; \mu\tau B + \widetilde{\mathrm{curl}}\, E\big) \;=\; \big(\Phi, \Psi\big). \tag{3.12}$$

The Bérenger transmission condition requires that the tangential components of $E = \sum E^k$ and $B = \sum B^k$ are continuous across the smooth parts of the interfaces, namely $\{x_j = 0\,,\; x_k \neq 0 \text{ for } k \neq j\}$

The face $x_1 = 0$ of $\partial\Omega$ consists of the two dimensional smooth stratum where $x_2 \neq 0 \neq x_3$, one dimensional edges where exactly two coordinates vanish, and the origin. Using square brackets, $[\,\cdot\,]$, to denote the jump one has

$$\Big[E_2, E_3, B_2, B_3\Big] \;=\; 0 \quad \text{across} \quad \big\{x_1 = 0,\; x_2 \neq 0 \neq x_3\big\}.$$

Taking tangential derivatives implies that

$$j \geq 2 \;\Rightarrow\; \Big[\partial_j E_2, \partial_j E_3, \partial_j B_2, \partial_j B_3\Big] \;=\; 0 \quad \text{across} \quad \big\{x_1 = 0,\; x_2 \neq 0 \neq x_3\big\}.$$

The coefficients ε and μ are C^2 and scalar valued in $\mathcal{B}$. For $k \geq 2$ the function σ_k is continuous across $\{x_1 = 0,\, x_2 \neq 0 \neq x_3\}$ so

$$j, k \geq 2 \;\Rightarrow\; \Big[\frac{\tau}{\tau + \sigma_k}\partial_j\{\varepsilon E, \mu B\}\Big] = 0 \;\; \text{across} \; \{x_1 = 0,\; x_2 \neq 0 \neq x_3\}. \tag{3.13}$$

Next prove that $[E, B] = 0$ across $\partial\Omega \setminus \mathcal{S}$, that is

$$\big[E\,,\,B\big] \;=\; 0 \quad \text{across} \quad \big\{x_j = 0\,,\; x_k \neq 0 \;\text{ for }\; k \neq j\big\}\,.$$

Treat E and the case $j = 1$. The others are similar. Already know the continuity of E_2, E_3 so need $[E_1] = 0$. Use the equation $\tau\varepsilon E = \widetilde{\operatorname{curl}}\,B$ on a neighborhood of $\partial\Omega$ to find

$$\tau\varepsilon E_1 \;=\; \frac{\tau}{\tau + \sigma_2}\,\partial_2 B_3 \;-\; \frac{\tau}{\tau + \sigma_3}\,\partial_3 B_2\,.$$

Equation (3.13) implies that the right hand side does not jump. Since ε is continuous this implies the desired conclusion that $[E_1] = 0$ across $\{x_1 = 0, x_2 \neq 0 \neq x_3\}$.

For the boundary $x_1 = 0$ analyse derivatives of the fields with respect to x_1. The source terms (Φ, Ψ) are compactly supported in $\overline{\omega}$. Thus on a neighborhood of $\partial\Omega$ the equations $\widetilde{\operatorname{div}}\,(\varepsilon E) = \widetilde{\operatorname{div}}\,(\mu B) = 0$ yield

$$
\begin{aligned}
\frac{\tau}{\tau + \sigma_1}\,\partial_1(\varepsilon E_1) \;+\; \frac{\tau}{\tau + \sigma_2}\,\partial_2(\varepsilon E_2) \;+\; \frac{\tau}{\tau + \sigma_3}\,\partial_3(\varepsilon E_3) \;&=\; 0\,,\\
\frac{\tau}{\tau + \sigma_1}\,\partial_1(\mu B_1) \;+\; \frac{\tau}{\tau + \sigma_2}\,\partial_2(\mu B_2) \;+\; \frac{\tau}{\tau + \sigma_3}\,\partial_3(\mu B_3) \;&=\; 0\,.
\end{aligned}
\tag{3.14}
$$

In particular the first terms on the left have well defined traces in $H^{-1/2}(\{x_1 = 0\} \setminus S)$. Equation (3.13) and (3.14) yield

$$\Big[\frac{\tau}{\tau + \sigma_1}\partial_1\{\varepsilon E_1\,,\,\mu B_1\}\Big] \;=\; 0 \quad \text{across} \quad \{x_1 = 0,\; x_2 \neq 0 \neq x_3\}\,. \tag{3.15}$$

For the second and third components of $\partial_1(\varepsilon E), \partial_1(\mu B)$ use (3.12). The second and third components of those equations express

$$\frac{\tau}{\tau + \sigma_1}\partial_1\{\mu B_2\,,\,\mu B_3\} \qquad \text{and} \qquad \frac{\tau}{\tau + \sigma_1}\partial_1\{\varepsilon E_2\,,\,\varepsilon E_3\} \tag{3.16}$$

as sums of functions continuous across the smooth parts of the boundary face $x_1 = 0$ and their tangential derivatives. Therefore

$$\Big[\frac{\tau}{\tau + \sigma_1}\,\partial_1\{\varepsilon E_2, \varepsilon E_3, \mu B_2, \mu B_3\}\Big] = 0 \quad \text{across} \quad \{x_1 = 0, x_2 \neq 0 \neq x_3\}\,. \tag{3.17}$$

Together with (3.15), this yields

$$\Big[\frac{\tau}{\tau + \sigma_1}\,\partial_1\{\varepsilon E, \mu B\}\Big] \;=\; 0 \quad \text{across} \quad \{x_1 = 0\,,\; x_2 \neq 0 \neq x_3\}\,.$$

By symmetry,

$$\Big[\frac{\tau}{\tau + \sigma_j}\,\partial_j\{\varepsilon E, \mu B\}\Big] \;=\; 0 \quad \text{across} \quad \{x_j = 0\,,\; x_k \neq 0 \;\text{ for }\; k \neq j\}\,. \tag{3.18}$$

The Helmholtz operator. Multiplying (3.12) by $\tau\mu$ yields

$$
\begin{aligned}
\varepsilon\mu\,\tau^2 E \;=\;\;\; & \mu\tau(\Phi + \widetilde{\operatorname{curl}}\,B) \;=\; \mu\tau\Phi + \widetilde{\operatorname{curl}}\,(\mu\tau B) - \widetilde{\operatorname{grad}}\,\mu \wedge (\tau B)\\[4pt]
=\;\;\; & \mu\tau\Phi + \widetilde{\operatorname{curl}}\,(\Psi - \widetilde{\operatorname{curl}}\,E) - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu} \wedge (\Psi - \widetilde{\operatorname{curl}}\,E)\\[4pt]
=\;\;\; & -\widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}}\,E + \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu} \wedge \widetilde{\operatorname{curl}}\,E + \mu\tau\Phi + \widetilde{\operatorname{curl}}\,\Psi - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu} \wedge \Psi\,.
\end{aligned}
$$

Expand $\widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}}\,E = -\widetilde{\operatorname{div}}\,\widetilde{\operatorname{grad}}\,E + \widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\,E$. Introduce εE in the previous quantities to find

$$\widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\,E \;=\; \widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\,\frac{\varepsilon E}{\varepsilon} \;=\; \widetilde{\operatorname{grad}}\,(\widetilde{\operatorname{grad}}\,(\frac{1}{\varepsilon})\cdot\varepsilon E + \frac{1}{\varepsilon}\widetilde{\operatorname{div}}\,(\varepsilon E))$$

$$=\; -\widetilde{\operatorname{grad}}\,(\frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\cdot E) + \widetilde{\operatorname{grad}}\,(\frac{1}{\varepsilon\tau}\widetilde{\operatorname{div}}\,\Phi)\,,$$

$$\widetilde{\operatorname{div}}\,\widetilde{\operatorname{grad}}\,E_j \;=\; \widetilde{\operatorname{div}}\,\widetilde{\operatorname{grad}}\,\frac{\varepsilon E_j}{\varepsilon} \;=\; \widetilde{\operatorname{div}}\,(\frac{1}{\varepsilon}\widetilde{\operatorname{grad}}\,(\varepsilon E_j)) - \widetilde{\operatorname{div}}\,(\frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\,E_j)\,,$$

$$\varepsilon\mu\tau^2 E \;=\; -\widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}}\,E + \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\widetilde{\operatorname{curl}}\,E + \mu\tau\Phi + \widetilde{\operatorname{curl}}\,\Psi - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\Psi$$

$$=\; \widetilde{\operatorname{div}}\,(\frac{1}{\varepsilon}\widetilde{\operatorname{grad}}\,(\varepsilon\{E_j\})) - \widetilde{\operatorname{div}}\,(\frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\,\{E_j\})$$

$$+\,\widetilde{\operatorname{grad}}\,(\frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\cdot E) - \widetilde{\operatorname{grad}}\,(\frac{1}{\varepsilon\tau}\widetilde{\operatorname{div}}\,\Phi)$$

$$+\,\frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\widetilde{\operatorname{curl}}\,E + \mu\tau\Phi + \widetilde{\operatorname{curl}}\,\Psi - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\Psi\,.$$

This yields the system of wave equations, scalar in its principal part,

$$q_E E \;:=\; \varepsilon\mu\tau^2 E \;-\; \widetilde{\operatorname{div}}\,(\frac{1}{\varepsilon}\widetilde{\operatorname{grad}}\,(\varepsilon E)) \;+\; \mathcal{L}_E E \;=\; \Phi_E, \tag{3.19}$$

with lower order terms and right hand side given by

$$\mathcal{L}_E E \;:=\; \{\widetilde{\operatorname{div}}\,(\frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\,E_j)\}_j - \widetilde{\operatorname{grad}}\,(\frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\cdot E) - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\widetilde{\operatorname{curl}}\,E, \tag{3.20}$$

$$\Phi_E \;:=\; -\widetilde{\operatorname{grad}}\,(\frac{1}{\varepsilon\tau}\widetilde{\operatorname{div}}\,\Phi) + \mu\tau\Phi + \widetilde{\operatorname{curl}}\,\Psi - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\Psi. \tag{3.21}$$

The magnetic field satisfies the similar equation

$$q_B B \;:=\; \varepsilon\mu\tau^2 B \;-\; \widetilde{\operatorname{div}}\,(\frac{1}{\mu}\widetilde{\operatorname{grad}}\,(\mu B)) \;+\; \mathcal{L}_B B \;=\; \Phi_B, \tag{3.22}$$

with lower order terms and right hand side given by

$$\mathcal{L}_B B \;:=\; \{\widetilde{\operatorname{div}}\,(\frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\,B_j)\} - \widetilde{\operatorname{grad}}\,(\frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\cdot B) - \frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\wedge\widetilde{\operatorname{curl}}\,B, \tag{3.23}$$

$$\Phi_B \;:=\; -\widetilde{\operatorname{grad}}\,(\frac{1}{\mu\tau}\widetilde{\operatorname{div}}\,\Psi) + \varepsilon\tau\Psi - \widetilde{\operatorname{curl}}\,\Phi + \frac{\widetilde{\operatorname{grad}}\,\varepsilon}{\varepsilon}\wedge\Phi. \tag{3.24}$$

Proposition 3.3. i. *The map $E, B \mapsto q_E E, q_B B$ maps $H^1(\mathbb{R}^3;\mathbb{C}^6)$ to $H^{-1}(\mathbb{R}^3;\mathbb{C}^6)$.*
   ii. *If $u = (E,B) \in \mathcal{D}'(\Omega)$ satisfies (3.5,3.12) with data as in Proposition 3.2 supported in $\overline{\omega}$, then $(q_E E\,,\, q_B B) = (\Phi_E, \Phi_B)$ on Ω.*
   iii. *If $u = (E,B)$ is as in **ii**, then $(q_E E\,,\, q_B B) = (\Phi_E, \Phi_B)$ on $\mathbb{R}^3$. In this case, (E,B) satisfies the Laplace transformed Bérenger system.*

Remark 3.4. The first part of property **iii** distinguishes the well adapted Helmholtz operator. If the operator were not well adapted there would be delta function terms on $\partial\Omega$ from the jumps in $\nabla_x E, \nabla_x B$.

Proof. Part **i** is immediate and part **ii** is proved before the proposition.

iii. We prove that $q_E E = \Phi_E$ on $\mathbb{R}^3$. The proof that $q_B B = \Phi_B$ on $\mathbb{R}^3$ is virtually identical. That these two equations on $\mathbb{R}^3$ imply the transformed Bérenger system is straightforward.

The support of $q_E E - \Phi_E$ is contained in $\mathbb{R}^3 \setminus \Omega = \partial\Omega$. Since $E \in H^1(\mathbb{R}^3)$, it follows that for all j,

$$F_j := \frac{\tau}{\tau + \sigma_j} \partial_j(\varepsilon E) \in L^2(\mathbb{R}^3).$$

Denote by $\chi_\pm$ the characteristic function of $\{\pm x_j > 0\}$. Then $F_j = \chi_+ F_j + \chi_- F_j$.

Denote the face $\{x_j = 0\}$ by $\mathcal{F}$. The singular points in $\mathcal{F}$ are the edges, $\mathcal{G} = \{x : x_j = 0, \ x_{j+1} x_{j+2} = 0\}$. Lemma 2.7 shows that $\mathcal{G}$ is a negligible set for $H^{1/2}(\mathcal{F}) \sim H^{1/2}(\mathbb{R}^2)$. This implies that the open set $C_0^\infty(\mathbb{R}^2 \setminus \mathcal{G})$ is dense in $H^{1/2}(\mathbb{R}^2)$. Therefore the most restrictive definition of $H^{1/2}(\mathbb{R}^2 \setminus \mathcal{G})$, namely this closure, is identical to the least restrictive definition, namely that the restriction to each component of $\mathbb{R}^2 \setminus \mathcal{G}$ has an extension to an element of $H^{1/2}(\mathbb{R}^2)$. Both are equal to the intermediate space defined as restrictions to $\mathbb{R}^2 \setminus \mathcal{G}$ of elements of $H^{1/2}(\mathbb{R}^2)$. Since each element of $H^{1/2}(\mathbb{R}^2 \setminus \mathcal{G})$ is the restriction of exactly one element of $H^{1/2}(\mathbb{R}^2)$, the space $H^{1/2}(\mathbb{R}^2 \setminus \mathcal{G})$ is naturally isomorphic to $H^{1/2}(\mathbb{R}^2)$. An analogous argument shows that $H^{-1/2}(\mathbb{R}^2 \setminus \mathcal{G}) = H^{-1/2}(\mathbb{R}^2)$.

Equation (3.12) implies (3.14) that in turn implies that

$$F_j \in C\big(]-\infty, 0[\,;\, H^{-1/2}(\mathbb{R}^2 \setminus \mathcal{G})\big) \cap C\big(]0, \infty[\,;\, H^{-1/2}(\mathbb{R}^2 \setminus \mathcal{G})\big),$$

and each continuous function has a continuous extension to the closed half line defining the traces from each side. In particular, $\chi_\pm \partial_j F$ makes sense as a piecewise continuous function with values in $H^{-1/2}(\mathcal{F} \setminus \mathcal{G})$. Equation (3.14) implies that

$$\partial_j F_j \in C\big(]-\infty, 0[\,;\, H^{-3/2}(\mathbb{R}^2 \setminus \mathcal{G})\big) \cap C\big(]0, \infty[\,;\, H^{-3/2}(\mathbb{R}^2 \setminus \mathcal{G})\big),$$

and each continuous function has a continuous extension to the closed half line. The formula for the distribution derivative of piecewise C^1 function of x_j implies that on $\mathbb{R}^3 \setminus \mathcal{S}$,

$$\partial_j \frac{\chi_+ F_j}{\varepsilon} = \chi_+ \partial_j \frac{F_j}{\varepsilon} + \frac{F_j}{\varepsilon}\bigg|_{x_j = 0+} \delta(x_j), \quad \partial_j \frac{\chi_- F_j}{\varepsilon} = \chi_- \partial_j \frac{F_j}{\varepsilon} - \frac{F_j}{\varepsilon}\bigg|_{x_j = 0-} \delta(x_j).$$

Define

$$Q := \frac{1}{\varepsilon} F_j\bigg|_{x_j = 0+} - \frac{1}{\varepsilon} F_j\bigg|_{x_j = 0-} \in H^{-1/2}(\mathbb{R}^2 \setminus \mathcal{G}) = H^{-1/2}(\mathbb{R}^2).$$

Equation (3.15) asserting continuity across $\{x_j = 0\} \setminus S$ yields

$$\text{supp } Q \subset \mathcal{S} \cap \{x_j = 0\} = \mathcal{G}.$$

Lemma 2.7 shows that $\mathcal{G}$ is negligible for $H^{-1/2}(\mathbb{R}^2)$. Therefore $Q = 0$.

Since $Q = 0$, adding the $\chi_\pm$ identities above yields

$$\partial_j \frac{F_j}{\varepsilon} = \chi_+ \partial_j \frac{F_j}{\varepsilon} + \chi_- \partial_j \frac{F_j}{\varepsilon} \quad \text{on} \quad \mathbb{R}^3 \setminus \mathcal{S}.$$

Multiplying by $\tau/(\tau + \sigma_j)$ then summing on j shows that the differential equation $q_E E = \Phi_E$, originally known to hold on Ω, in fact holds on the larger set $\mathbb{R}^3 \setminus \mathcal{S}$. Therefore $q_E E - \Phi_E$ is an element of $H^{-1}(\mathbb{R}^3)$ with support in $\mathcal{S}$. Lemma 2.7 proves that it vanishes identically, proving the desired identity. $\qquad\square$

3.2.2. *Estimates for the Helmholtz operators q_E, q_B.*

Proposition 3.4. *If $\sigma_j \in L^\infty(\mathbb{R})$ for $1 \le j \le 3$, then there are constants C, M so that for all $u = (E, B) \in H^1(\mathbb{R}^3)$ with $(q_E E, q_B B) \in L^2(\mathbb{R}^3)$, and $\tau \in \mathbb{C}$ with $\operatorname{Re}\tau > M$,*

$$\operatorname{Re}\tau \, \|u\|_{L^2(\mathbb{R}^3)} \; + \; \|\nabla u\|_{L^2(\mathbb{R}^3)} \; \le \; C \, \|q_E E, \, q_B B\|_{L^2(\mathbb{R}^3)} \,. \tag{3.25}$$

Proof. We prove the estimate for E. The estimate for B is almost identical.
Step I. Derive a modified Helmholtz equation that is prepared for the energy method. Define

$$p(E) := \sum_j \, \partial_j \, \frac{1}{\varepsilon} \frac{(\tau + \sigma_{j+1})(\tau + \sigma_{j+2})}{\tau(\tau + \sigma_j)} \, \partial_j(\varepsilon E) \,. \tag{3.26}$$

The indices are taken modulo 3. In the coefficient between the ∂_j the terms $\tau + \sigma_{j+1}$ and $\tau + \sigma_{j+2}$ are independent of x_j so commute with ∂_j.

Since $q_E E = \Phi(E) \in L^2(\mathbb{R}^3)$, one can multiply by $(\tau + \sigma_1)(\tau + \sigma_2)(\tau + \sigma_3)/\tau^3 \in L^\infty(\mathbb{R}^3)$. This function as well as its inverse is uniformly bounded in $\operatorname{Re}\tau$ large and one has

$$P(\tau, x, \partial_1, \partial_2\,, \partial_3)\, E \;=\; \frac{(\tau + \sigma_1)(\tau + \sigma_2)(\tau + \sigma_3)}{\tau^3} \, \Phi_E \quad \text{on} \;\; \mathbb{R}^3\,, \tag{3.27}$$

with

$$P \; := \; \varepsilon\mu \, \frac{\prod_j (\tau + \sigma_j(x_j))}{\tau} \; - \; p \; + \; \ell, \qquad \ell = \text{lower order}\,.$$

The jth component of ℓ is given by

$$\begin{aligned}
(\ell E)_j =& \partial_k \left(\frac{(\tau + \sigma_j)(\tau + \sigma_l)}{\tau(\tau + \sigma_k)} \frac{\partial_k \varepsilon}{\varepsilon} E_j \right) + \partial_l \left(\frac{(\tau + \sigma_j)(\tau + \sigma_k)}{\tau(\tau + \sigma_l)} \frac{\partial_l \varepsilon}{\varepsilon} E_j \right) \\
& - \partial_j \left(\frac{(\tau + \sigma_l)}{\tau} \frac{\partial_k \varepsilon}{\varepsilon} E_k \right) - \partial_j \left(\frac{(\tau + \sigma_k)}{\tau} \frac{\partial_l \varepsilon}{\varepsilon} E_l \right) \\
& - \frac{(\tau + \sigma_l)(\tau + \sigma_j)}{\tau^2} \frac{\partial_k \mu}{\mu} E_l + \frac{(\tau + \sigma_k)(\tau + \sigma_j)}{\tau^2} \frac{\partial_l \mu}{\mu} E_k.
\end{aligned} \tag{3.28}$$

It suffices to prove the estimate (3.25) with the term $q_E E$ on the right replaced by PE.
Step II. Consider first the case $\varepsilon = 1$ that has all the essential difficulties and is easier to read. In this case, the first order term ℓ vanishes yielding

$$\frac{\prod(\tau + \sigma_j)}{\tau} u \; - \; pu \; = \; f \,.$$

The estimate is proved by considering the real and imaginary parts of the $L^2(\mathbb{R}^3)$ scalar products,

$$(u, f) \; = \; \Big(\frac{\prod(\tau + \sigma_j)}{\tau} u \,, \, u \Big) - (pu, u), \tag{3.29}$$

$$-(pu, u) \; = \; \int \Big(\sum_{j=1}^{3} \frac{\prod_{k \ne j}(\tau + \sigma_k)}{\tau(\tau + \sigma_j)} \, |\partial_j u|^2 \Big) \, dx \,. \tag{3.30}$$

• The right hand side of (3.30) implies that the term (pu, u) is not far from its unperturbed value, that is with vanishing absorptions. Indeed,

$$\frac{(\tau + \sigma_1)(\tau + \sigma_2)}{\tau(\tau + \sigma_3)} \; - \; 1 \; = \; \frac{(\sigma_1 + \sigma_2 - \sigma_3)\tau + \sigma_1 \sigma_2}{\tau(\tau + \sigma_3)} \,.$$

Since the σ_j are uniformly bounded, there is a constant C_1 so that for $\operatorname{Re}\tau$ sufficiently large,

$$\left|\frac{(\tau+\sigma_1)(\tau+\sigma_2)}{\tau(\tau+\sigma_3)} - 1\right| \leq \frac{C_1}{|\tau|},$$

and therefore,

$$\left|(pu,u) + \|\nabla u\|^2\right| \leq C_1\frac{\|\nabla u\|^2}{|\tau|}. \tag{3.31}$$

Therefore, for $\operatorname{Re}\tau$ sufficiently large,

$$|\operatorname{Im}(pu,u)| \leq C_1\frac{\|\nabla u\|^2}{|\tau|}, \qquad \operatorname{Re}(pu,u) + \|\nabla u\|^2 \leq C_1\frac{\|\nabla u\|^2}{|\tau|}. \tag{3.32}$$

- Expand the zero order term

$$\frac{\prod(\tau+\sigma_j)}{\tau} = \frac{\overline{\tau}\prod(\tau+\sigma_j)}{|\tau|^2} = \frac{\overline{\tau}(\tau^3 + \sum\sigma_j\tau^2 + \sum\sigma_i\sigma_j\tau + \sigma_1\sigma_2\sigma_3)}{|\tau|^2}$$

$$= \frac{|\tau|^2\tau^2 + \sum\sigma_j|\tau|^2\tau + \sum\sigma_i\sigma_j|\tau|^2 + \sigma_1\sigma_2\sigma_3\overline{\tau}}{|\tau|^2}.$$

Extract the imaginary part

$$\operatorname{Im}\frac{\prod(\tau+\sigma_j)}{\tau} = \frac{|\tau|^2\operatorname{Im}\tau^2 + \sum\sigma_j|\tau|^2\operatorname{Im}\tau - \sigma_1\sigma_2\sigma_3\operatorname{Im}\tau}{|\tau|^2}$$

$$= \left(2|\tau|^2\operatorname{Re}\tau + \sum\sigma_j|\tau|^2 - \sigma_1\sigma_2\sigma_3\right)\frac{\operatorname{Im}\tau}{|\tau|^2}.$$

Therefore for $\operatorname{Re}\tau$ sufficiently large,

$$\left|\operatorname{Im}\frac{\prod(\tau+\sigma_j)}{\tau}\right| \geq \frac{3}{2}\operatorname{Re}\tau\,|\operatorname{Im}\tau|. \tag{3.33}$$

The real part satisfies

$$\operatorname{Re}\frac{\prod(\tau+\sigma_j)}{\tau} = \frac{|\tau|^2\operatorname{Re}\tau^2 + \sum\sigma_j|\tau|^2\operatorname{Re}\tau + \sum\sigma_i\sigma_j|\tau|^2 + \sigma_1\sigma_2\sigma_3\operatorname{Re}\tau}{|\tau|^2}.$$

Therefore for $\operatorname{Re}\tau$ sufficiently large,

$$\operatorname{Re}\frac{\prod(\tau+\sigma_j)}{\tau} \geq \operatorname{Re}\tau^2 = (\operatorname{Re}\tau)^2 - (\operatorname{Im}\tau)^2. \tag{3.34}$$

- Next use equation (3.29). The imaginary part of (3.29) yields

$$\operatorname{Im}\left(\frac{\prod(\tau+\sigma_j)}{\tau}u, u\right) = \operatorname{Im}(pu,u) + \operatorname{Im}(f,u).$$

Insert (3.33) and (3.32) to find

$$\frac{3}{2}\operatorname{Re}\tau|\operatorname{Im}\tau|\,\|u\|^2 \leq \|f\|\,\|u\| + \frac{C_1}{|\tau|}\|\nabla u\|^2.$$

This is used to estimate $|\operatorname{Im}\tau|\|u\|$. Multiply by $\frac{2}{3}|\operatorname{Im}\tau|/\operatorname{Re}\tau$ to find

$$|\operatorname{Im}\tau|^2\,\|u\|^2 \leq \frac{2}{3}\frac{|\operatorname{Im}\tau|}{\operatorname{Re}\tau}\|u\|\|f\| + \frac{2C_1}{3}\frac{|\operatorname{Im}\tau|}{|\tau|\operatorname{Re}\tau}\|\nabla u\|^2. \tag{3.35}$$

The information in the real part of (3.29) yields,

$$\operatorname{Re}\left(\frac{\prod(\tau+\sigma_j)}{c^2\tau}u, u\right) = \operatorname{Re}(pu,u) + \operatorname{Re}(f,u).$$

Insert (3.34) and (3.32) to find

$$(\operatorname{Re}\tau)^2\|u\|^2 \ + \ \|\nabla u\|^2 \ \le \ |(f,u)| \ + \ (\operatorname{Im}\tau)^2\|u\|^2 \ + \ \frac{C_1\,\|\nabla u\|^2}{|\tau|}\,.$$

On the right use the Cauchy-Schwartz inequality for the first term, and insert (3.35) in the second term to find

$$(\operatorname{Re}\tau)^2\|u\|^2 \ + \ \|\nabla u\|^2 \ \le \ \|f\|\,\|u\|\left(1+\frac{|2\operatorname{Im}\tau|}{3\operatorname{Re}\tau}\right) \ + \ \left(1+\frac{|2\operatorname{Im}\tau|}{3\operatorname{Re}\tau}\right)\frac{C_1\,\|\nabla u\|^2}{|\tau|}\,.$$

Pick $\alpha \in (0,1)$. For $\operatorname{Re}\tau$ sufficiently large one has

$$\left(1+\frac{2|\operatorname{Im}\tau|}{3\operatorname{Re}\tau}\right)\frac{\max(C_1,1)}{|\tau|} \ \le \ \alpha\,.$$

Then

$$(\operatorname{Re}\tau)^2\,\|u\|^2 \ + \ \frac{\|\nabla u\|^2}{2} \ \le \ \alpha\left(\|\nabla u\|^2 + \|\tau u\|\,\|f\|\right). \tag{3.36}$$

- Next estimate τu using the equation $\frac{\Pi(\tau+\sigma_j)}{\tau}u - pu = f$, isolating

$$\tau^2 u = pu + f - \left((\sigma_1+\sigma_2+\sigma_3)\tau + (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \frac{\sigma_1\sigma_2\sigma_3}{\tau}\right)u. \tag{3.37}$$

Choose a constant C_3 so that for $\operatorname{Re}\tau$ sufficiently large,

$$\left|(\sigma_1+\sigma_2+\sigma_3)\tau + (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \sigma_1\sigma_2\sigma_3/\tau\right| \ \le \ C_3\,|\tau|\,. \tag{3.38}$$

Multiply (3.37) by $\bar{u}$ and use (3.38) and (3.31) to find

$$\|\tau u\|^2 \ \le \ \frac{C_1}{|\tau|}\,\|\nabla u\|^2 \ + \ C_3\,\|\tau u\|\,\|u\| \ + \ \|u\|\,\|f\|\,.$$

The Cauchy-Schwarz and elementary Young's inequalities imply that there is a C_4 so that $\operatorname{Re}\tau$ sufficiently large

$$\|\tau\,u\|^2 \ \le \ C_4\left(\|\nabla u\|^2 \ + \ \|u\|^2 \ + \ \|f\|^2\right). \tag{3.39}$$

Using again Young's inequality with coefficient $\beta \in (0,1)$ in (3.36) yields

$$(\operatorname{Re}\tau)^2\,\|u\|^2 \ + \ \frac{\|\nabla u\|^2}{2} \ \le \ \alpha\left(\|\nabla u\|^2 + \frac{\beta}{2}\|\tau u\|^2 + \frac{1}{2\beta}\|f\|^2\right)$$

Inserting estimate (3.39) in the righthand side yields

$$(\operatorname{Re}\tau)^2\,\|u\|^2 \ + \ \frac{\|\nabla u\|^2}{2} \ \le \ \alpha\left((1+\frac{\beta C_4}{2})\,\|\nabla u\|^2 + \frac{\beta C_4}{2}\|u\|^2 + (\frac{\beta C_4}{2}+\frac{1}{2\beta})\|f\|^2\right).$$

Choose α and β so that $\alpha(1+\frac{\beta C_4}{2}) = \frac{1}{4}$ to obtain with a constant C,

$$(\operatorname{Re}\tau)^2\,\|u\|^2 \ + \ \frac{1}{4}.\|\nabla u\|^2 \ \le \ \frac{1}{4}\,\|u\|^2 + C\|f\|^2.$$

This proves (3.25) for $\operatorname{Re}\tau$ sufficiently large in the case $\varepsilon = 1$.

Step III. Endgame. The argument above works without essential modification in case $\ell = 0$. One need only use the fact that ε is uniformly bounded above and below.

The case $\ell \neq 0$ introduces terms $(E, \ell E)$ in the estimates. Since $\|\ell E\| \lesssim \|E\| + \|\nabla_x E\|$ this yields for $\gamma > 0$,

$$
\begin{aligned}
(\operatorname{Re}\tau)^2 \|E\|^2 \;+\; \|\nabla_x E\|^2 \;&\leq\; C\,\|PE\|^2 \;+ C\,|(E, \ell E)| \\
&\leq\; C\,\|PE\|^2 \;+\; \frac{C}{4\gamma}\|E\|^2 \;+\; C\gamma\|\nabla_x E\|^2 \,.
\end{aligned}
$$

Choosing γ so that $C\gamma < 1$, the last term is absorbed by the $\|\nabla E\|^2$ on the left. Then the $C\|E\|^2/4\gamma$ term is absorbed in the $(\operatorname{Re}\tau)^2\|E\|^2$ on the left for $\operatorname{Re}\tau$ large. $\qquad\square$

Proof of Proposition 3.2. It suffices to prove the estimate for E. The function E satisfies

$$
PE \;=\; \Phi_E \;:=\; -\widetilde{\operatorname{grad}}\,\big(\frac{1}{\varepsilon\tau}\widetilde{\operatorname{div}}\,\Phi\big) + \mu\tau\Phi + \widetilde{\operatorname{curl}}\,\Psi - \frac{\widetilde{\operatorname{grad}\mu}}{\mu}\wedge\Psi,
$$

with $(\Phi, \Psi) = LU$. The estimate

$$
\|\Phi_E\| \;\lesssim\; \|\frac{1}{\tau}\widetilde{\operatorname{div}}\,\Phi\|_{H^1} \;+\; \|\tau\Phi\|_{L^2} \;+\; \|\Psi\|_{H^1}\,,
$$

with a similar estimate for the B equation, gives

$$
\|f\| \;\lesssim\; \|LU\|_{H^1} \;+\; \|\tau LU\|_{L^2} \;+\; \|\frac{1}{\tau}\widetilde{\operatorname{div}}\,LU\|_{H^1}\,. \qquad\square
$$

3.3. **Estimates for the Bérenger equations.** The proof of the estimates in Theorem 1.7 proceeds by estimating the Laplace transform. The Laplace transform is split into two pieces U_1 and U_2 that are estimated using the Propositions of Section 3.2. If $U(\tau, x)$ is the Laplace transform then

$$
LU \;=\; (4\pi\widehat{\mathbf{j}},\, 0)\,.
$$

Definition 3.4. Choose $\phi_1(x) \in C_0^\infty(\mathbb{R}^3)$ supported in $\mathcal{O}$ with $\phi_1 = 1$ on a neighborhood of $\overline{\omega}$. Define $\phi_2 := 1 - \phi_1$. Define $U_j := \phi_j U$.

Proposition 3.5. *There are positive constants C, M so that if $\operatorname{Re}\tau > M$, $U, LU \in H^1(\mathbb{R}^3)$, $\operatorname{supp} LU \subset \overline{\omega}$ then*

$$
\operatorname{Re}\tau\,\|U_1\|_{H^1} \;+\; \operatorname{Re}\tau\,\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\leq\; C\,\|LU\|_{H^1}\,. \tag{3.40}
$$

A weaker estimate, as strong as the Helmholtz estimates is

$$
\operatorname{Re}\tau\,\|U\|_{L^2} \;+\; \|\nabla_x U\|_{L^2} \;\leq\; C\,\|LU\|_{H^1}\,. \tag{3.41}
$$

Remark 3.5. The field of interest in the computation is U_1. U_1 satisfies estimates exactly as strong as the estimates for Maxwell's equations. The estimate for U_2 is weaker but still without loss of derivatives.

Proof. Compute a system of equations satisfied by the pair U_1, U_2. Define

$$
M_j(\tau, x) \;:=\; [L, \phi_j] \;=\; \sum_k A_k \partial_k \phi_j \;=\; \begin{pmatrix} 0 & [\widetilde{\operatorname{curl}}, \phi_j] \\ -[\widetilde{\operatorname{curl}}, \phi_j] & 0 \end{pmatrix}.
$$

For each $\tau \in \mathbb{C}$, the M_j are smooth matrix valued functions,

$$
M_j(\tau, \cdot\,) \;\in\; C_0^\infty(\mathcal{O})\,, \qquad \operatorname{supp} M_j \,\cap\, \overline{\omega} \;=\; \emptyset\,. \tag{3.42}
$$

Compute

$$
LU_j \;=\; L(\phi_j U) \;=\; \phi_j LU \;+\; [L, \phi_j]U \;=\; \phi_j LU \;+\; M_j U\,. \tag{3.43}
$$

3.3.1. *Estimate for U_2.* For our problem with $\mathbf{j}$ supported in $\overline{\omega}$, $\phi_2 F = 0$. The equation for U_2 is

$$LU_2 \;=\; M_2 U\,.$$

Since $M_2\, U$ is supported in $\mathcal{O}$ where the absorptions vanish, $\widetilde{\mathrm{div}} = \mathrm{div}\,$. Proposition 3.2 implies that

$$
\begin{aligned}
\operatorname{Re}\tau\,\|U_2\|_{L^2} + \|\nabla_x U_2\|_{L^2} \;&\lesssim\; \|LU_2\|_{H^1} + \|\tau\,LU_2\|_{L^2} + \Big\|\tfrac{1}{\tau}\,\mathrm{div}\,LU_2\Big\|_{H^1} \\
&\lesssim\; \|M_2\,U\|_{H^1} + \|\tau\,M_2\,U\|_{L^2} + \Big\|\tfrac{1}{\tau}\,M_2\,U\Big\|_{H^2}\,.
\end{aligned}
\tag{3.44}
$$

On $\mathcal{O}\setminus\overline{\omega}$, ε and μ are scalar and $LU = 0$. It follows that

$$\big(\varepsilon\mu\,\tau^2 \;-\; \Delta \;+\; q(x,\partial_x)\big)U \;=\; 0 \qquad \text{on}\quad \mathcal{O}\setminus\overline{\omega}, \tag{3.45}$$

with q a system of partial differential operators of degree 1. The terms of degree one have C^1 coefficients with bounded derivatives and the term of order zero has bounded coefficient since ε,μ have derivatives up to order two continuous and bounded.

Equation (3.45) is a homogeneous elliptic equation that holds on a neighborhood of supp M_2. The elliptic regularity theorem for the laplacian is used to estimate the H^2 norm of $M_2 U$.

Define a finite sequence of scalar cutoff functions $\chi_j \in C_0^\infty(\mathcal{B})$. The first is chosen so that χ_1 is identically equal to one on a neighborhood of supp M_2. The succeeding cutoffs are chosen so that χ_{j+1} is identically equal to one on a neighborhood of supp χ_j. Elliptic regularity yields

$$\|\chi_1 U\|_{H^2} \;\lesssim\; \|\chi_2 \Delta U\|_{L^2} \;+\; \|\chi_2 U\|_{L^2}\,.$$

Using (3.45) yields

$$\|\chi_2 \Delta U\|_{L^2} \;\lesssim\; \|\tau^2\,\chi_2 U\|_{L^2} \;+\; \|\chi_3 U\|_{H^1}\,.$$

Using these, estimate

$$\Big\|\tfrac{1}{\tau}\,M_2\,U\Big\|_{H^2} \;\lesssim\; \Big\|\tfrac{1}{\tau}\,\chi_1 U\Big\|_{H^2} \;\lesssim\; \|\tau\,\chi_2 U\|_{L^2} \;+\; \Big\|\tfrac{1}{\tau}\,\chi_3 U\Big\|_{H^1}\,.$$

Inject in (3.44) to find

$$\operatorname{Re}\tau\,\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \|\tau\,\chi_2 U\|_{L^2} \;+\; \|\chi_3 U\|_{H^1}\,. \tag{3.46}$$

On the right express $\tau\,U_2$ in the support of χ_2 hence inside $\mathcal{O}$ using the equation to estimate

$$\|\tau\chi_2 U\|_{L^2} \;\lesssim\; \|\chi_2 \nabla U\|_{L^2} \;+\; \|\chi_1 U\|_{L^2} \;\lesssim\; \|\chi_3 U\|_{H^1}\,.$$

Therefore (3.46) yields

$$\operatorname{Re}\tau\,\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \|\chi_3 U\|_{H^1}\,. \tag{3.47}$$

3.3.2. *Estimate for U_1 and U_2.* The equation for U_1 has source term $\phi_2 F$. The cutoff ϕ_2 was chosen to be identically equal to one in a neighborhood of the support of the source term $\mathbf{j}$ leading to

$$LU_1 \;=\; M_1 U \;+\; 4\pi\widehat{\mathbf{j}}\,.$$

Proposition 3.1 yields

$$\operatorname{Re}\tau\,\|U_1\|_{H^1} \;\lesssim\; \|M_1 U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1} \;\lesssim\; \|\chi_1 U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,.$$

This easy derivation depended only on the fact that the cutoff function ϕ_1 was supported in $\mathcal{O}$. Exactly the same argument yields for all j,

$$\operatorname{Re}\tau \, \|\chi_j U\|_{H^1} \;\lesssim\; \|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,.$$

Combining the last two yields

$$\operatorname{Re}\tau \, \|U_1\|_{H^1} \;\lesssim\; \frac{1}{\operatorname{Re}\tau}\, \|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,. \tag{3.48}$$

This improves (3.47) to

$$\operatorname{Re}\tau \, \|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \frac{1}{\operatorname{Re}\tau}\Big(\|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\Big)\,. \tag{3.49}$$

Summing yields

$$\operatorname{Re}\tau \, \|U_1\|_{H^1} \;+\; \operatorname{Re}\tau \, \|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \frac{1}{\operatorname{Re}\tau}\, \|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,.$$

For $\operatorname{Re}\tau$ large this proves estimate (3.40), completing the proof of Proposition 3.5.

<div align="right">□</div>

3.4. Existence and uniqueness proofs.

3.4.1. *A Paley-Wiener theorem.* The Laplace transform of a distribution F supported in $t \geq 0$ with $e^{-\lambda t}F \in L^1(\mathbb{R})$ for $\lambda > M$, is holomorphic in $\operatorname{Re}\tau > M$, given by

$$\widehat{F}(\tau) \;:=\; \int e^{-\tau t}\, F(t)\, dt\,.$$

The functions F take values in a Hilbert space H. Recall the classical theorem of Paley and Wiener.

Theorem 3.5. *The Laplace transforms of functions $F \in e^{Mt}\,L^2(\mathbb{R}\,;\,H)$ with $\operatorname{supp}F \subset \{t \geq 0\}$ are exactly the functions $G(\tau)$ holomorphic in $\operatorname{Re}\tau > M$ with values in H and so that*

$$\sup_{\lambda > M}\, \int_{\operatorname{Re}\tau = \lambda} \big\|\widehat{F}(\tau)\big\|_H^2 \; |d\tau| \;<\; \infty\,.$$

In this case the functions $\widehat{F}(\tau)$ have non tangential trace at $\operatorname{Re}\tau = M$ that satisfy

$$\int e^{-2Mt}\, \|F(t)\|_H^2\, dt \;=\; \sup_{\lambda > M}\, \int_{\operatorname{Re}\tau = \lambda} \big\|\widehat{F}(\tau)\big\|_H^2 \; |d\tau| \;=\; \int_{\operatorname{Re}\tau = M} \big\|\widehat{F}(\tau)\big\|_H^2 \; |d\tau|\,.$$

3.4.2. *Estimates of Theorem 1.7.* The *a priori* estimates of Theorem 1.7 follow by combining the estimates of Proposition 3.5 with Theorem 3.5.

  • The estimate

$$\int e^{-2\lambda t}\big(\lambda\|U\|_{L^2}^2 \;+\; \|\nabla_x U\|_{L^2}^2\big)\, dt \;\lesssim\; \text{right hand side of (1.9)}$$

follows by combining with the estimate (3.41).

  • The remaining estimate in (1.9) follows from Theorem 3.5 and the estimate for ∇U_1 in (3.40).

  • The estimate for U_t comes from expressing U_t in terms of U_x and LU.

  • The fact that $E_j^j = B_j^j = 0$ follows from (3.3) and the fact that the jth row of the C_j vanishes.

  • The estimate for $\partial_t\{E^j, B^j\}$ follows from the equation (3.4) that shows that $|\tau\widehat{E}^j, \tau\widehat{B}^j| \leq |\nabla_x U|$.

3.4.3. *Proof of Theorem 1.7.* **I. Proof of existence for smooth sources j and smooth absorptions $\sigma_j(x_j)$.**

For $\sigma_j \in W^{2,\infty}(\mathbb{R})$ and $\mathbf{j} \in e^{\lambda t} H^2(\mathbb{R} \times \mathbb{R}^3)$ with support in $t \geq 0$ existence is proved in [9] following [21]. The method is an elaboration of [18]. A Hilbert space norm bounded below by the L^2 norm of E, B, E^j, B^j and above by the H^2 norm of these quantities is constructed so that for solutions the square of the norm, denoted $\mathcal{E}(U)$, satisfies $d\mathcal{E}(U)/dt \lesssim \mathcal{E}(U) + \mathcal{E}(\mathbf{j})$. Similar estimates hold for the semidiscrete scheme that one finds by using the Yee discretization of the x derivatives. Passing to the limit of decreasing mesh size yields a solutions in $e^{\lambda t} L^2(\mathbb{R} \times \mathbb{R}^3)$.

We need solutions with total field U belonging to $e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$.

Since the equation is time translation invariant, if $\mathbf{j}$ and $\partial_t \mathbf{j}$ belong to $e^{\lambda t} H^2(\mathbb{R} \times \mathbb{R}^3)$ one finds a solution with U and U_t in $e^{\lambda t} L^2(\mathbb{R} \times \mathbb{R}^3)$. The Laplace transforms satisfy

$$\tau \varepsilon \widehat{E} \;=\; \widetilde{\mathrm{curl}}\, \widehat{B} \;+\; 4\pi \widehat{\mathbf{j}}, \qquad \tau \mu \widehat{B} \;=\; -\widetilde{\mathrm{curl}}\, \widehat{E}\,.$$

Taking divergence yields

$$\tau \widetilde{\mathrm{div}}\, \varepsilon \widehat{E} \;=\; 4\pi \widetilde{\mathrm{div}}\, \widehat{\mathbf{j}}, \qquad \tau \widetilde{\mathrm{div}}\, \mu \widehat{B} \;=\; 0\,.$$

This bounds

$$\widetilde{\mathrm{div}}\, \varepsilon \widehat{E}\,, \quad \widetilde{\mathrm{curl}}\, \widehat{E}\,, \quad \widetilde{\mathrm{div}}\, \mu \widehat{B} \quad \text{and} \quad \widetilde{\mathrm{curl}}\, \widehat{B}\,.$$

Lemma 3.6. *The overdetermined system* $F \;\mapsto\; (\widetilde{\mathrm{div}}\, \varepsilon F\,, \widetilde{\mathrm{curl}}\, F)$ *is for each* $\mathrm{Re}\,\tau > 0$ *an overdetermined elliptic system. Similarly* $G \;\mapsto\; (\widetilde{\mathrm{div}}\, \mu G\,, \widetilde{\mathrm{curl}}\, G)$. *The ellipticity is uniform in* x.

Proof. Verify the Lopatinski condition for the system with permittivity ε. For $\underline{x}$ fixed the plane wave with real ξ, $e^{i\xi x}\, \mathbf{e}$, is a solution of the frozen $\widetilde{\mathrm{div}}\,, \widetilde{\mathrm{curl}}$ problem if and only if

$$e^{i\xi x}\, \underline{\mathbf{e}}, \qquad \underline{\mathbf{e}}_j := \frac{\tau}{\tau + \sigma_j(\underline{x})}\mathbf{e}_j$$

is a plane wave solution of the overdetermined system $H \mapsto \mathrm{div}\, \varepsilon H, \mathrm{curl}\, H$. For the latter the necessary and sufficient conditions are

$$\xi \cdot \varepsilon \underline{\mathbf{e}} = 0\,, \quad \text{and} \quad \xi \wedge \underline{\mathbf{e}} = 0\,.$$

The second implies that $\xi \parallel \underline{\mathbf{e}}$. Without loss of generality we can take $\underline{\mathbf{e}}$ real. The first condition then yields $\underline{\mathbf{e}} = 0$ because ε is positive definite. $\square$

Elliptic regularity implies that

$$\|\nabla_x \widehat{E}\| \;\lesssim\; \|\widetilde{\mathrm{div}}\, \varepsilon \widehat{E}\| \;+\; \|\widetilde{\mathrm{curl}}\, \widehat{E}\| \;+\; \|\widehat{E}\|.$$

This together with Theorem 3.5 implies that the total field $U \in e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$. This completes the proof of **I.** $\square$

II. Proof of existence. Choose sequences $\sigma_j^n \in W^{2,\infty}(\mathbb{R})$ with support in $]-\infty, 0]$, and $\sigma_j^n \to \sigma_j$ in $L^\infty(\mathbb{R})$ weak star as $n \to \infty$.

Choose $\mathbf{j}^n \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ with support in $\{t \geq 0\}$ and $\mathbf{j}^n \to \mathbf{j}$ in $e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$.

Denote by $E^n, B^n, E^{n,j}, B^{n,j}$ the solution of the Bérenger split problem with absorptions σ_j^n and current $\mathbf{j}^n$ constructed in **I.**.

The estimates of Theorem 1.7, proved in Section 3.4.2 apply to this sequence. The constants C, M associated to the family of absorptions $\{\sigma_j^n\}$ can be chosen uniformly. They depend only on the L^∞ norms of the absorptions, see part **iii** of Remark 3.3.

Extracting a subsequence one may suppose that the total fields

$$U^n \quad \rightharpoonup \quad U \quad \text{weakly in} \quad e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3) \,,$$

and the split fields satisfy for all κ, j,

$$U^{n,j} \quad \rightharpoonup \quad U^j \quad \text{weakly in} \quad e^{\lambda t} L^2(\mathbb{R} \times \mathcal{O}_\kappa) \,.$$

The limits are the desired solutions. This completes the proof of existence.

III. Proof of uniqueness. For a solution with vanishing data, the Laplace transform $\widehat{U}$ of the total field satisfies a homogeneous equation for which uniqueness is proved in Proposition 3.5. This completes the proof of uniqueness and thererfore of Theorem 1.7. □

3.5. **Numerical study of the loss of derivatives.** The simulations in this section do **not** concern corners. We have proved that there is essentially no loss of derivatives for the Bérenger split Maxwell equations even in the presence of corners while it was commonly believed that there was loss *even without corners*. The simulations below show that for the split equations with neither boundaries nor absorptions there is loss for data whose divergence is non vanishing and no loss for divergence free data.

The simulations treat the 2-D transverse electric Maxwell system in $\mathbb{R}^2$, with speed of light equal to one,

$$\partial_t E_1 = \partial_2 B \,, \quad \partial_t E_2 = -\partial_1 B \,, \quad \partial_t B = -\partial_1 E_2 + \partial_2 E_1. \tag{3.50}$$

In the Bérenger split system, the third equation is replaced by two equations

$$\partial_t B_1 = -\partial_1 E_2 - \sigma_1(x_1)B_1, \ \partial_t B_2 = \partial_2 E_1 - \sigma_2(x_2)B_2, \ B = B_1 + B_2. \tag{3.51}$$

Following Bérenger [5] and discussed in [9] §3.3, only the magnetic field is split. The total magnetic field is the sum of the split fields.

The computation concerns the model with absorption $\sigma_j = 0$. It reveals the role of the splitting alone on the loss of derivatives. The field B in the Bérenger model is equal to the B of the Maxwell model. This is also true for the discrete approximations.

The equations are solved in the rectangle $[-1, 1] \times [-1, 1]$, with perfect conductor boundary conditions $n \wedge E = 0$ on the boundary. The equations are discretized by the Yee scheme, see for instance [9]. The mesh sizes are $dx_1 = dx_2 = 10^{-3}$, $dt = 7.0711e - 04$ just inside the CFL limit. The system is run for 40 time steps.

The magnetic field is zero at $t = 0$. The initial electric field has frequency $\omega = 5 \times 2^n$ with $0 \le n \le 5$, are obtained from

$$\Phi = a \, cos(2\pi \, \omega \, \mathbf{v} \cdot x) \,, \qquad \mathbf{v} \ = \ \frac{1}{\sqrt{2}} \left(1, -1 \right) ,$$

where a is a smooth function compactly supported in a disk of center O and radius 0.3.

For the Maxwell system we measure the discrete L^2 norm in space and time of (E, B), for the Bérenger system the norm of (E, B_1, B_2, B). Both are normalized by the L^2 norm of E at the initial time.

In the first set of experiments, the initial electric field $E_0 = (\partial_2 \Phi, -\partial_1 \Phi)$ is divergence free. The divergence of the finite difference approximation remains below 10^{-10}, and the norm of the solution is given in Table 1.

Frequency	5	10	20	40	80	160
Maxwell	0.0852	0.1269	0.1132	0.1162	0.1226	0.1266
Berenger	0.0444	0.0642	0.0568	0.0581	0.0613	0.0633

TABLE 1. L^2 norm as a function of the frequency. Divergence equal 0

In the second set, the initial electric field $E_0 = (\partial_2 \Phi, \partial_1 \Phi)$ differs by a change of sign in the second component and is not divergence free. The norm of the solution is given in Table 2.

Frequency	5	10	20	40	80	160
Maxwell	0.1702	0.1702	0.1703	0.1703	0.1703	0.1703
Berenger	0.1835	0.2121	0.3012	0.5247	1.0036	1.9546

TABLE 2. L^2 norm as a function of the frequency. Divergence not 0

For divergence free initial electric field, the norm of the solution is constant as a function of the frequency. In the second case, the norm of the solution of the Maxwell system is constant, while that of the Bérenger system grows linearly with the frequency. The L^2 norm of the solution grows like the H^1 norm of the data, illustrating a loss of one derivative.

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