Well Posedness of Perfectly Matched or Dissipative Boundary Conditions with Trihedral Corners

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Dedication. *It is a pleasure to contribute this paper to honor of the* 90^{th} *birthday of Peter Lax. Forty eight years ago Peter suggested the study of mixed initial boundary value problems for hyperbolic equations as a thesis topic for JBR. This article returns to this rich area. We thank Peter for his inspiration to the entire field of mathematics. And for his friendship. We offer our best wishes on this landmark birthday.*

Abstract. Existence and uniqueness theorems are proved for boundary value problems with trihedral corners and distinct boundary conditions on the faces. Part I treats strictly dissipative boundary conditions for symmetric hyperbolic systems with elliptic or hidden elliptic generators. Part II treats the Bérenger split Maxwell equations in three dimensions with possibly discontinuous absorptions. The discontinuity set of the absorptions or their derivatives has trihedral corners. Surprisingly, there is almost no loss of derivatives for the Bérenger split problem. Both problems have their origins in numerical methods with artificial boundaries.

Keywords trihedral angle, Bérenger's layers, strictly dissipative boundaries, symmetric hyperbolic systems, Maxwell's equations

1 Introduction

1.1 Overview

This paper analyses mixed initial boundary value problems in domains with corners that arise when one computes approximate solutions of hyperbolic equations on unbounded or large domains by simulations on a smaller computational domain. The computational domain is very often a ball or a

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rectangle. The latter is the most common and has corners as in the figure 1 below.

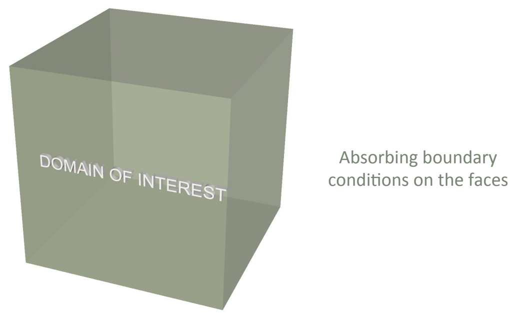

Figure 1: Artificial boundary

At the external boundaries absorbing boundary conditions are imposed. The boundary conditions on faces that are orthogonal are usually different so the initial boundary boundary value problem is of mixed type because of the change in boundary condition.

In spatial dimension $d = 3$ the external corner is a meeting point of three orthogonal faces making a trihedral angle. The study of hyperbolic problems in such regions is very little developed. For nontrivial absorbing conditions we know of no previous work asserting existence and uniqueness with trihedral angles. It is often easy to prove existence of fairly weak solutions and uniqueness of fairly regular ones. Closing the gap for these **external corners** is the subject of Part I.

A second set of problems leading to domains with trihedral angles is the use of the perfectly matched layers of Bérenger. The geometry of such a method in dimension $d = 2$ is a rectangular domain including in its interior the domain of interest, surrounded by absorbing layers where Bérenger split equations are satisfied with transmission conditions on all the solid horizontal and vertical lines in the figure 2.

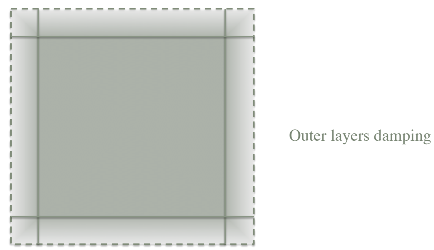

Figure 2: Internal corner in two dimensions

On the dotted lines absorbing boundary conditions are prescribed. Note in particular the **interior corners**. In dimension three the interior domain is a cube and the interior corners are trihedral.

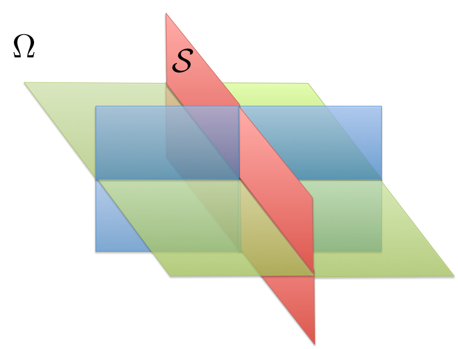

Figure 3: Internal corner : $\{x_1\, x_2\, x_3 \;=\; 0\}$

We study the Bérenger transmission problems for Maxwell's equations in $\mathbb{R}^3$. At the intersection of the 3 planes parallel to the coordinate planes in $\mathbb{R}^3$, transmission conditions are prescribed. We give the first existence proof for the Bérenger split problem with more than one absorption coefficient discontinuous. The original prescription of Bérenger was of this type, though in common practice one uses smoother coefficients. With more than twenty years of computational experience, it is not surprising that the problem is well set. Even in the case of smooth absorptions our theorem is surprising because it has almost no loss of derivatives. Sources in H^1 yield solutions in H^1. Shortly after the introduction of Bérenger's method, Abarbanel and

Gottlieb [1] proved that the split Maxwell equations are only weakly hyperbolic. Sources in H^s yield solutions in H^{s-1} and not better. The resolution of this apparent contradiction between our result and theirs is that the split system loses a derivative for general initial data. It does not lose a derivative for the divergence free solutions of Maxwell's equations (see Section 3.5). Our earlier paper [8] introduced the scheme of the demonstration analysing the first order system version of the $2d$ wave equation. S. Petit in [17], [8] showed that the split equations are lossless for elliptic generators. Here we treat the much subtler case of Maxwell's equation.

Our well posedness results apply to the Bérenger split system even when the permittivities are not scalar provided that the non scalar values are constrained to take place on a compact subset of the domain of interest. This is the first such result, with or without loss of derivatives.

The analysis of the Bérenger method for Maxwell's equation answers some important questions but leaves some open. For example at the external boundary one imposes boundary conditions for the Bérenger split system hoped to be absorbing. To our knowledge there are no existence or uniqueness proofs for such exterior corner problems for the split equations.

The analysis of the two problems treated have five common elements. They treat trihedral corners. They proceed by Laplace transform. They rely on elliptic estimates. They use capacity at key points. They are both come from numerical methods with artificial boundaries.

1.2 Part I. Dissipative boundaries for elliptic generators

For symmetric hyperbolic problems, the simplest natural artificial boundary conditions are dissipative. With the aim of absorbing as much as possible the most natural choices are strictly dissipative. That is the context of the first part of this paper, strictly dissipative conditions on the faces of rectangular domains. As the faces have different directions, the boundary conditions imposed on adjacent faces are usually different.

Our main result asserts existence, uniqueness, and limited regularity for such problems. Existence is a fairly easy consequence of energy dissipation. It is uniqueness that is difficult. The constructed solutions do not have sufficient regularity to justify an integration by parts. Friedrichs' method of mollifiers does not save the day as there are few tangential directions at corners (see §2.1.2).

1.2.1 Regularity and incoming corner waves

A key idea is to take advantage of estimates on the trace of solutions at the boundary that comes from strict dissipativity.

Uniqueness asserts that solutions with homogeneous initial and boundary conditions must vanish. How could there be waves in such circumstances? Consider an initial boundary value problem in $\mathbb{R}_t \times \mathcal{O}$ with $\mathcal{O}$ equal to the set of vectors with strictly positive components. The zero initial conditions give the idea that the energy must come from the lateral boundary $\mathbb{R}_t \times \partial\mathcal{O}$. At the flat faces of $\partial\mathcal{O}$ a dissipativity assumption shows that energy is absorbed not emitted. The enemies are the singular parts of $\partial\mathcal{O}$. One must show that energy does not sneak into the domain through those sets, for example the edges of codimension $2, 3, \ldots, d$. Considering radiation problems on $\mathbb{R}^{1+d}$ with sources $f(t)\delta(x_1)\delta(x_2)\cdots\delta(x_k)$ shows that waves can emerge from sets of dimension $k < d$. The proofs show that energy emerging from sets of codimension ≥ 2, corresponding to the singularities of $\partial\mathcal{O}$ is incompatible with the square integrability of the objects constructed in the existence theory.

1.2.2 Corner problems

Problems with corners have a rich literature some of it very well known. For example, the study of the Dirichlet and Neumann problems in lipshitz domains notably by Jerison and Kenig in the eighties. We appeal to their results at two junctures in the analysis of problems with hidden ellipticity. Their results are used to prove regularity of potentials. They do not treat problems of mixed type where the boundary conditions change from face to face. Another class of problems concern the diffraction by conical singularities where again the boundary conditions do not change from face to face.

A recent reference that treats polyhedral domains with different boundary conditions on different faces and that includes extensive reference to earlier work is [3]. However, the boundary conditions treated are restricted to elliptic problems with conditions associated to coercive bilinear forms. Our boundary conditions are motivated by absorbing conditions at the edge of computational domains. They usually do not fall under this umbrella.

Higher dimensional corners are discussed by Kupka-Osher in [12] for constant coefficient scalar wave equations, with an explicit solution technique. Their proof of uniqueness seems incomplete. They merely observe that the conditions are dissipative so uniqueness is a consequence of the energy identity. The integrations by parts needed to prove the identity requires extra regularity. This sort of difficulty is classic. For example, existence of not too regular solutions of Navier Stokes was proved by Leray in the thirties and uniqueness or regularity is a Clay Millenium problem. Addressing this difficulty for absorbing conditions at a trihedral corner is the problem attacked in Part I.

Taniguchi in a series of papers starting with [21] considered gluing two dissipative problems together at a dihedral corner when one of the problems is strictly dissipative. As in our case, existence is easy and uniqueness hard. With respect to the corner variables Taniguchi's coefficients are constant. The analysis is by a Fourier- Laplace transform in those variables. Advantage is taken of the strong trace estimates from the strictly dissipative problem. For the trihedral problem this strategy hits a serious obstruction.

The papers by Osher [16] and Sarason-Smoller [20] show how geometric optics constructions can reveal pathological behavior at corners. These papers inspire the counterexamples in Section 2.4.

1.2.3 Main result

Part I treats two classes of problem. The easiest to describe is the case where the generator is elliptic. Analogous results are obtained for Maxwell's equation and the linearized compressible Euler equations. For Maxwell the divergence is independent of time while for Euler linearized at a constant state the curl is independent of time. In both cases this allows one to recover estimates resembling those for problems with elliptic generators. In the introduction only the elliptic case is presented. Problems with hidden ellipticity are treated in Section 2.3.

To concentrate on the essential difficulty, consider the case of a single multihedral corner. Using a partition of unity one can reduce the general case to this one. Denote

$$\mathcal{O} \ := \ \big\{x \in \mathbb{R}^d \ : \ x_j > 0, \quad j = 1, \dots, d\big\}\,.$$

The **singular subset** of $\partial\mathcal{O}$ is

$$S \ := \ \{x \in \overline{\mathcal{O}} \,:\, x_j = 0 \ \text{ for at least two values of } j\}\,.$$

Assumption 1.1. i. *The matrix valued functions $A_j(x)$ and $B(x)$ are smooth with partial derivatives of all orders belonging to $L^\infty(\mathbb{R}^d)$. For each x, $A_j(x)$ is hermitian symmetric.*

ii. *The differential operator $\sum_j A_j(x)\partial_j$ is uniformly elliptic for all $x \in \partial\mathcal{O}$.*

iii. *The subspace $\mathcal{N}_j(x)$ is defined for x belonging to the hyperplane $\{x \in \mathbb{R}^d : x_j = 0\}$ and is a smoothly varying subspace that is maximal strictly dissipative (see §2.1.1) for the boundary matrix $-A_j(x)\big|_{x_j=0}$.*

Definition 1.1. *Denote*

$$A(x,\xi) \ := \ \sum_j A_j(x)\xi_j\,, \qquad G(x,\partial) \ := \ A(x,\partial) + B(x)\,,$$

$$L \ := \ \partial_t + G(x,\partial), \qquad Z(x) \ := \ B(x) + B(x)^* \ - \ \sum_j \partial_j A_j(x)\,.$$

Denote by L^* the adjoint differential operator with respect to the $L^2(\mathbb{R}^{1+d})$ scalar product, $L^\star\Phi = -\partial_t\Phi - \sum \partial_j(A_j^*\Phi) + B^*\Phi$. The symmetry of the $A_j(x)$ implies that

$$L + L^* \;=\; G + G^* = Z(x)\,.$$

Condition **iii** asserts that the boundary space is dissipative for $A(x,\nu(x))$ where $\nu(x)$ is an outward conormal to $\partial\mathcal{O}$. The minus sign comes from the fact that the outward normal is $-\mathbf{e}_j$ where $\{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ is the standard basis in $\mathbb{R}^d$.

The change of variable $v = e^{-\lambda t}u$ yields an equation of the same type with Z replaced by $Z+\lambda I$. Thus the next assumption entails no loss of generality.

Assumption 1.2. *There is a $\mu > 0$ so that for all t, x, $Z(t,x) \geq \mu\, I$.*

Definition 1.2. *With the notations of Assumption 1.1, a function $h \in L^2(\partial\mathcal{O})$ is said to satisfy the boundary condition $h \in \mathcal{N}$, when for $1 \leq j \leq d$,*

$$h\big|_{\{x_j=0\}\cap\{\partial\mathcal{O}\setminus\mathcal{S}\}} \;\in\; \mathcal{N}_j(x) \quad a.\ e.$$

The boundary traces appearing in the next theorem are discussed in Section 2.1.3.

Theorem 1.1. *With Assumptions 1.1 and 1.2, and Definition 1.2, for each $g \in L^2(\mathcal{O})$ there is one and only one $u \in L^\infty\big(]0,\infty[\,;\,L^2(\mathcal{O})\big)$ with $u\big|_{\partial(]0,\infty[\times\mathcal{O})} \in L^2(\partial(]0,\infty[\times\mathcal{O}))$, $Lu = 0$, $u(0) = g$, and, $u\big|_{\partial O} \in \mathcal{N}$. In addition for all $0 \leq t < T < \infty$ one has the energy identity,*

$$\|u(T)\|^2 + \int\limits_{[t,T]\times\partial\mathcal{O}} \big(A\big(x,\nu(x)\big)u,u\big)\,dtd\Sigma + \int\limits_{[t,T]\times\mathcal{O}} \big(Z(x)u,u\big)\,dtdx = \|u(t)\|^2. \tag{1.1}$$

Remark 1.1. 1. *Strict dissipativity in the elliptic context is equivalent to the existence of $c > 0$ so that for all $x \in \partial\mathcal{O}$ and u satisfying $u|_{\partial O} \in \mathcal{N}$*

$$\big(A\big(x,\nu(x)\big)u\,,\,u\big) \;\geq\; c\|u\|^2_{\mathbb{C}^N}\,.$$

2. *Taking $t = 0$ and applying Gronwall's inequality yields*

$$\sup_{0\leq s<\infty}\big\|e^{\mu t}\,u(s)\big\|^2_{L^2(\mathcal{O})} \;+\; \int\limits_{[0,\infty[\times\partial\mathcal{O}} \big\|u\big\|^2\,dt\,d\Sigma \;\lesssim\; \big\|u(0)\big\|^2_{L^2(\mathcal{O})}\,.$$

1.3 Part II. Internal trihedral angles for Bérenger split Maxwell

1.3.1 Bérenger split Maxwell

In contrast to Part I that treats general symmetric systems, the results of the second part are limited to systems that are close cousins of the wave equation, notably Maxwell's equations. Proofs pass by an analysis of equations that are relatives of the Helmholtz equation.

Definition 1.3. *The set* $\Omega := \{x_1 x_2 x_3 \neq 0\} \subset \mathbb{R}^3$ *is the disjoint union of eight open octants.* $\mathcal{O} := \{x_j > 0 \text{ for all } j\}$ *plays the role of domain of interest. The other seven octants are denoted* $\mathcal{O}_\kappa$ *with* $1 \leq \kappa \leq 7$*.*

The dynamic Maxwell's equations in time independent media are

$$\epsilon(x)\, E_t \;=\; \operatorname{curl} B \;-\; 4\pi j, \qquad \mu(x)\, B_t \;= -\operatorname{curl} E\,. \tag{1.2}$$

The charge density ρ and current j satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} \;=\; -\operatorname{div} j\,. \tag{1.3}$$

The physically relevant solutions are those satisfying

$$\operatorname{div} \epsilon E \;=\; 4\pi\rho\,, \qquad \operatorname{div} \mu B \;=\; 0\,. \tag{1.4}$$

Equation (1.4) is satisfied for all time as soon as it is satisfied at $t = 0$.

Assumption 1.3. *In Part II, we suppose that* $\epsilon(x)$ *and* $\mu(x)$ *are* C^2 *matrix valued functions so that* $\partial^\alpha\{\epsilon, \mu\} \in L^\infty(\mathbb{R}^3)$ *for all* $|\alpha| \leq 2$*, and there is a* $C > 0$ *so that for all* x*,* $\epsilon \geq CI$ *and* $\mu \geq CI$*.*

There is a compact subset $K \subset \mathcal{O}$ *with the property that* ϵ *and* μ *are scalar valued on* $\mathcal{B} := \mathbb{R}^3 \setminus K$*.*

Write

$$\operatorname{curl} \;=\; \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \;=\; \sum C_j\, \partial_j\,, \tag{1.5}$$

$$C_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

Definition 1.4. *The Bérenger splitting involves two vector valued functions* E, B *on* $\mathbb{R}_t \times \mathcal{O}$ *and three pairs of vector valued functions* E^j, B^j *for* $j = 1, 2, 3$ *on each of the octants* $\mathbb{R}_t \times \mathcal{O}_\kappa$*.*

The pair E, B *satisfies Maxwell's equations* (1.2) *and* (1.4) *on* $\mathcal{O}$*. For each* κ *the split variables* E^j, B^j *satisfy the split system with the warning that* $C_j \partial_j$ *is a single term not summation notation*

$$\left.\begin{aligned} \epsilon(\partial_t + \sigma_j(x_j))E^j \;&=\; C_j \partial_j \sum_{k=1}^{k=3} B^k\,, \\ \mu(\partial_t + \sigma_j(x_j))B^j \;&=\; -\,C_j \partial_j \sum_{k=1}^{k=3} E^k\,. \end{aligned}\right\} \quad \text{for} \quad j = 1, 2, 3. \tag{1.7}$$

Abusing notation define the total fields $U := (E, B)$ on all of Ω by

$$U \;:=\; (E, B) \;:=\; \begin{cases} \qquad (E, B) & \text{on} \qquad \mathcal{O}\,, \\ \big(\sum E^j\,,\,\sum B^j\big) & \text{on} \qquad \Omega\setminus\mathcal{O}\,. \end{cases} \tag{1.8}$$

The Bérenger split system is completed by the transmission conditions demanding that the tangential components of the function E, B on the left of (1.8) are continuous across the <u>two dimensional</u> interfaces in $\partial\Omega$.

1.3.2 Main result

Consider sources and solutions supported in $t \geq 0$. In particular, with initial values equal to zero. It is only in this situation that we can prove results with essentially no loss of derivatives.

Theorem 1.2. *Suppose that Assumption 1.3 is satisfied and $\omega \supset K$ is open with compact closure $\overline{\omega} \subset \mathcal{O}$. There are constants C, λ_0, depending on $\overline{\omega}$, with the following properties. If $\lambda > \lambda_0$, $\operatorname{supp}\mathbf{j} \subset [0, \infty[\times\overline{\omega}$, and*

$$\forall |\alpha| \leq 1, \qquad \partial^\alpha_{t,x}\mathbf{j} \;\in\; e^{\lambda t}\, L^2\big(\mathbb{R}\,;\, L^2(\mathbb{R}^3)\big)\,,$$

then there are E, B defined on $\mathbb{R}_t \times \mathcal{O}$ and split functions E^j, B^j defined on $\mathbb{R}_t \times \cup\mathcal{O}_\kappa$, supported in $t \geq 0$, so that the total field U defined on the left hand side of (1.8) belongs to $e^{\lambda t}H^1(\mathbb{R}\times\mathbb{R}^3)$ and satisfies the Bérenger differential equations. The transmission condition is guaranteed by $U \in e^{\lambda t}H^1(\mathbb{R}\times\mathbb{R}^3)$. Any solution with $U \in e^{\lambda t}H^1(\mathbb{R}\times\mathbb{R}^3)$ satisfies for $\lambda > \lambda_0$

$$\begin{aligned} \int e^{-2\lambda t}\big\|\lambda U\,,\,\nabla_{t,x}U\,,\lambda\nabla_{t,x}U\big|_{\overline{\omega}}\,\big\|^2_{L^2(\mathbb{R}^3)}\,dt \\ \leq\; C\int e^{-2\lambda t}\sum_{|\alpha|\leq 1}\big\|\partial^\alpha_{t,x}\mathbf{j}(t)\big\|^2_{L^2(\mathbb{R}^3)}\;dt\,. \end{aligned} \tag{1.9}$$

On each octant $\mathcal{O}_\kappa$, the split fields satisfy for each j $E^j_j = B^j_j = 0$ and

$$\begin{aligned} \int e^{-2\lambda t}\big\|E^j, B^j\,, \partial_t E^j, \partial_t B^j\big\|^2_{L^2(\mathcal{O}_\kappa)}\,dt \\ \leq\; C\int e^{-2\lambda t}\sum_{|\alpha|\leq 1}\big\|\partial^\alpha_{t,x}\mathbf{j}(t)\big\|^2_{L^2(\mathbb{R}^3)}\;dt\,. \end{aligned} \tag{1.10}$$

In particular there is uniqueness for such solutions.

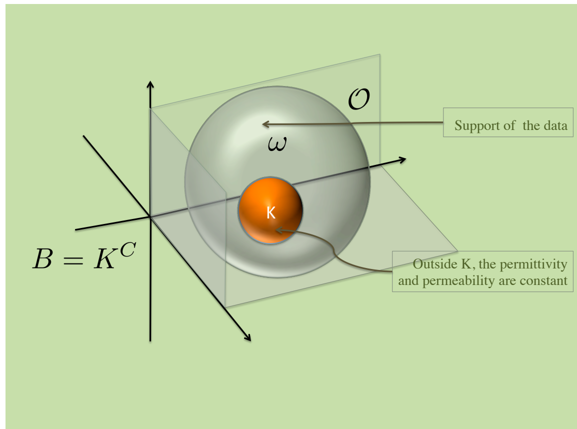

Figure 4: Definitions of supports in Theorem 1.2

Remark 1.2. i. *Formula (1.9) has derivatives of order less than or equal to one on both sides. The only possible loss of derivatives is for the split variable E^j, B^j outside the the domain of interest $\mathcal{O}$. There the loss is restricted microlocally to $\{\tau = 0\}$.*

ii. *The estimate for the quantities of interest, namely the restriction of E, B to $\overline{\omega}$ is*

$$\lambda^2 \int e^{-2\lambda t} \sum_{|\alpha|\leq 1} \|\partial^\alpha_{t,x} E, \partial^\alpha_{t,x} B\|^2_{L^2(\overline{\omega})}\ dt$$
$$\leq\ C \int e^{-2\lambda t} \sum_{|\alpha|\leq 1} \left\|\partial^\alpha_{t,x}\mathbf{j}(t)\, ,\, \partial^\alpha_{t,x}\rho(t)\right\|^2_{L^2(\mathbb{R}^3)}\ dt\,,$$

like the estimates that would hold for the Maxwell equations. The estimates for the Bérenger split equations are somewhat weaker, but only outside the set $\overline{\omega}$. The compact $\overline{\omega}$ can be chosen as large as one likes within the domain of interest $\mathcal{O}$.

iii. *The solutions constructed above satisfy* $\mathrm{div}\, \epsilon E = \rho$, $\mathrm{div}\, \mu B = 0$. *Section 3.5 presents a numerical study that contrasts the behavior of the Bérenger splitting for data that satisfy and data that does not satisfy the divergence constraints.* **When the divergence constraint is violated**, *the loss of derivatives from the Bérenger splitting occurs.*

iv. *The uniqueness proof passes through the Laplace transform. To prove uniqueness of solutions defined only for $t \leq T$ it suffices to continue them using the existence theorem to global solutions and then to apply the global uniqueness result.*

Remark 1.3. *The Bérenger splitting is perfectly matched provided that the permittivities are constant outside a compact subset of $\mathcal{O}$. As soon as one proves that the transmission problem is well posed as in Theorem 1.2, it follows that the interfaces are reflectionless and that the restriction of the solution to $\mathcal{O}$ is* **exactly** *equal to the restriction to $\mathcal{O}$ of the solution of Maxwell's equations (see [7]).*

2 Part I. Dissipative boundary conditions for elliptic symmetric systems

2.1 Four preliminary results

2.1.1 Nonegative subspaces.

Suppose that $\mathbb{V}$ is a finite dimensional complex scalar product space and $A \in \mathrm{Hom}(\mathbb{V})$ is a hermitian symmetric linear transformation. Denote by $E_{\geq 0}(A)$ the nonnegative spectral subspace of A and similarly the strictly positive and strictly negative spectral subspaces E_+ and E_-. The transformation is omitted for ease of reading when there is little chance of confusion. Denote by $\Pi_{\geq 0}(A)$, Π_+, and Π_- the associated orthogonal projections.

Definition 2.1. *For the transformation $A = A^*$, a linear subspace $\mathcal{N} \subset \mathbb{V}$ is* **dissipative** *when for all $v \in \mathcal{N}$ one has $(Av, v) \geq 0$. It is* **strictly dissipative** *when there is a constant $c > 0$ so that for all $v \in \mathcal{N}$*

$$(Av\,,\,v)\ \geq\ c\,\|\Pi_+ v\|^2\,.$$

It is **maximal dissipative** *when in addition* $\dim \mathcal{N} = \operatorname{rank} \Pi_{\geq 0}(A)$.

The maximality is equivalent to the fact that there is no strictly larger dissipative subspace.

Lemma 2.1. *For $\mathbb{V} = E_{\geq 0} \oplus_\perp E_-$, denote the natural decomposition $v = v_{\geq 0} + v_-$. Every maximal dissipative subspace is a graph*

$$v_- = M v_{\geq 0}$$

for a unique linear $M : E_{\geq 0} \to E_-$.

Proof. Suppose that $\mathcal{N}$ is maximal dissipative. The assertion is equivalent to the fact that $\Pi_{\geq 0} : \mathcal{N} \to E_{\geq 0}$ is bijective. Since the dimensions are equal this is equivalent to injectivity.

Suppose that $v \in \mathcal{N}$ and $\Pi_{\geq 0} v = 0$. Then $v \in E_-$. On the other hand, $(Av, v) \geq 0$ by dissipativity. The only $v \in E_-$ for which this is possible is $v = 0$ proving injectivity. $\square$

Example 2.1. *The lemma is used to construct smooth deformations of any maximal dissipative space* $\mathcal{N}$ *to* $E_{\geq 0}$*. Precisely choose* $\phi \in C^\infty(\mathbb{R})$ *with*

$$\phi(s) \;=\; \begin{cases} 0 & \text{for } s \leq 1/2 \\ 1 & \text{for } s \geq 1 \end{cases} .$$

Then if $\mathcal{N}$ *is the graph of* M *then the graph of* $\phi(s)M$ *is maximal dissipative for all* s *and connects* $\mathcal{N}$ *for* $s \geq 1$ *to* $E_{\geq 0}$ *for* $s \leq 1/2$.

2.1.2 Geometry at a corner.

In dimension $d > 2$ the study of boundary value problems in a corner is harder and much less developed than the study in regions with a conical singularity with smooth crosssection. The singular set $\mathcal{S}$ includes strata of dimensions $0, 1, 2, 3, \ldots, d-2$. For example in dimension $d = 2$, the only singularities are corners of dimension 0. In dimension $d = 3$ there are edges of dimension 1 and the corner of dimension 0.

Figure 5 represents a corner of a cube in three dimensions.

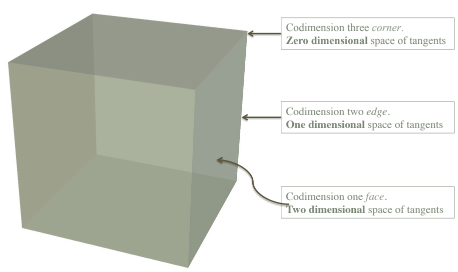

Figure 5: Corners and edges in three dimensions.

Figure 6 shows that the corner in $\mathbb{R}^3$ is a cone with triangular cross section.

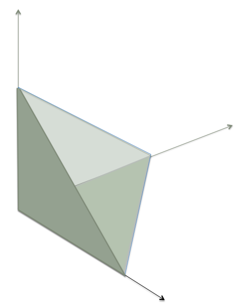

Figure 6: Corner in $\mathbb{R}^3$ is a cone on a triangle.

Contrast this with a cone with circular cross section, $\{x_1^2 > x_2^2 + x_3^3,\ x_1 > 0\}$, sketched in Figure 7. At all points other than the corner, the space of tangents is two dimensional.

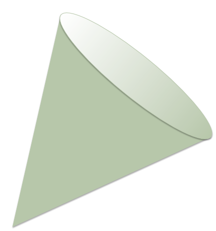

Figure 7: Circular cone.

2.1.3 Traces of solutions of first order systems.

Definition 2.2. *Define the Hilbert space $\mathcal{H}$ by*

$$\mathcal{H} \ := \ \Big\{ u \in L^2(\mathcal{O}) \ : \ A(x, \partial) u \in L^2(\mathcal{O}) \Big\} .$$

Denote by $C^1_{(0)}(\overline{\mathcal{O}})$ the restriction to $\overline{\mathcal{O}}$ of elements in $C^1_0(\mathbb{R}^d)$. Friedrichs' Lemma implies that if $\mathcal{O} \in \mathbb{R}^n$ is a uniformly lipschitzian domain, then $C^1_{(0)}(\overline{\mathcal{O}})$ is dense in $\mathcal{H}$. The proof of Friedrich's lemma in [13] works for lipshitzian domains after a bilipschitzian flattening of the boundary.

The next result of this section is from [18] whose proof works after flattening the boundary. The operator $A(x, \partial)^\dagger$ denotes the transpose with respect to the bilinear form $\phi, \psi \mapsto \int \phi\, \psi\, dx$. Thus, $A^\dagger \Phi = -\sum_j \partial_j (A_j(x)\Phi)$. Similarly $L^\dagger$ on $\mathbb{R}^{1+d}$ and the pairing $\int \phi\, \psi\, dxdt$.

Proposition 2.2. *If $\mathcal{O} \in \mathbb{R}^n$ is a uniformly lipschitzian domain and $A(x, \partial)$ is a first order system with uniformly lipschitzian coefficients then the map*

$$u \quad \mapsto \quad A(x, \nu(x))u\big|_{\partial\mathcal{O}} \;:=\; \gamma$$

has a unique continuous extension from $C^1_{(0)}(\overline{\mathcal{O}})$ to a map from $\mathcal{H}$ to the dual of $Lip(\partial\mathcal{O})$. If $\phi \in Lip(\partial\mathcal{O})$ and $\Phi \in Lip(\overline{\mathcal{O}})$ with $\Phi|_{\partial\mathcal{O}} = \phi$ then the trace γ satisfies

$$\big\langle \gamma\,,\, \phi \big\rangle := \int_{\mathcal{O}} \big\langle A(x,\partial)u\,,\, \Phi \big\rangle - \big\langle A(x,\partial)^\dagger \Phi\,,\, u \big\rangle\; dx.$$

2.1.4 Layer potentials.

With $\langle \xi \rangle := \big(1+|\xi|^2\big)^{1/2}$, denote by $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ the set of symbols satisfying

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \;\leq\; C_{\alpha\beta}\, \langle \xi \rangle^{m-|\beta|},$$

uniformly on $\mathbb{R}^d \times \mathbb{R}^d$. With $G(x, \partial)$ from Definition 1.1 choose $r > 0$ and $p(x, D) \in Op(S^{-1}(\mathbb{R}^d \times \mathbb{R}^d))$ a pseudodifferential parametrix,

$$p(x,D)G(x,\partial) \;-\; I \;\in\; Op(S^{-\infty}(\{x\,:\, \mathrm{dist}(x,\partial\mathcal{O}) < r\} \times \mathbb{R}^d)).$$

The next result on layer potentials can be found on pages 37-38 of [22].

Proposition 2.3. *Denote by H the open half space $\{x_1 > 0\}$ and $d\Sigma$ the element of surface on ∂H. Suppose that $p(x, \xi) \in S^{-1}(\mathbb{R}^d \times \mathbb{R}^d)$ has an asymptotic expansion as a sum of j-homogeneous symbols*

$$p \;\sim\; \sum_{j=-1}^{-\infty} p_j(x,\xi)$$

satisfying the transmission conditions

$$p_{-1}(x, \xi_1, 0, \dots, 0) \;=\; -\, p_{-1}(x, -\xi_1, 0, \dots, 0)\,.$$

Then there is a $q \in S^0(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ so that for $g \in L^2(\partial H)$ the distribution $p(x, D)(gd\Sigma)$ has trace on the boundary of H given by

$$\Big(p(x,D)(g\, d\Sigma)\Big)(0+\,, x') \;=\; q(x', D')g\,.$$

2.2 Dissipative elliptic corners, Theorem 1.1

2.2.1 Step 1. Proof of existence of solutions

Existence is proved by constructing u as the limit of solutions u^ϵ to problems in domains $\mathcal{O}^\epsilon$ obtained by smoothing $\mathcal{O}$. Take $\phi \in C^\infty(\mathbb{R})$ from Example 2.1. The ellipticity hypothesis implies that the boundary is noncharacteristic so the maximal dissipative boundary condition $\mathcal{N}_j$ is given by an equation

$$\mathbf{v}_- \;=\; M_j(x)\mathbf{v}_+\,.$$

Define $\mathcal{N}_j^\epsilon$ to be the maximal dissipative space defined by

$$\mathbf{v}_- \;=\; \phi(|x_j|/\epsilon)\, M_j(x)\, \mathbf{v}_+\,.$$

Define $\mathcal{O}^\epsilon$ by smoothing the edges of Ω leaving the boundary unchanged where all of the x_j are greater than $\epsilon/2$, see figure 8. Define a boundary

Figure 8: Construction of $\mathcal{O}_\epsilon$ when $d = 2$ (left) and $d = 3$ (right) .

space $\mathcal{N}^\epsilon$ on the boundary of $\mathcal{O}^\epsilon$ to be equal to $\mathcal{N}_j^\epsilon$ on the unchanged part and given by the equation $E_+(A(x,\nu(x)))$ on the parts that have been smoothed. The domain $\mathcal{O}^\epsilon$ has smooth noncharacteristic boundary so the standard theory constructs u^ϵ a solution of the mixed problem with initial value equal to the restriction of g to $\mathcal{O}^\epsilon$.

The function u^ϵ satisfies the energy identity for all $T > 0$,

$$\|u^\epsilon(T)\|^2 + \int_{[0,T]\times\partial\mathcal{O}^\epsilon} \big(A(x,\nu(x))u^\epsilon, u^\epsilon\big)dtd\Sigma + \int_{[0,T]\times\mathcal{O}^\epsilon} \big(Z(x)u^\epsilon, u^\epsilon\big)dtdx = \|u^\epsilon(0)\|^2.$$

Therefore

$$\sup_{t\ge 0}\; e^{2\mu t}\,\|u^\epsilon(t)\|^2 \;+\; \int_0^\infty \|u^\epsilon(t)|_{\partial\mathcal{O}}\|^2 dt \;\lesssim\; \|g\|^2\,.$$

By the Cantor diagonal process choose a subsequence $\epsilon(k) \to 0$, a $u \in e^{-\mu t} L^\infty([0,\infty[: L^2(\mathcal{O}))$ and a $\gamma \in L^2\big([0,\infty[\times \partial\mathcal{O}\big)$ so that for all $\delta > 0$

$$u^{\epsilon(k)} \quad \rightharpoonup \quad u \qquad \text{weak star in } L^\infty([0,\infty]; L^2(\mathcal{O}^\delta))\,, \quad \text{and}$$

$$u^{\epsilon(k)}\big|_{\partial\mathcal{O}^\epsilon \cap \{|x|>\delta\}} \quad \rightharpoonup \quad \gamma \qquad \text{weak star in } L^2\big([0,\infty] \times (\partial\mathcal{O} \cap \{\mathrm{dist}(x,\mathcal{S}) > \delta\})\big).$$

It follows that $Lu = 0$, $u(0) = g$ and $u|_{\{x_j=0\}\cap\partial\mathcal{O}\setminus\mathcal{S}}$ belongs to $\mathcal{N}_j$. It remains to study the trace of u at the boundary in a neighborhood of $]0,\infty[\times\mathcal{S}$. The trace belongs to the dual of Lip. A codimension two subset is not negligible for such functionals. The next computation shows that the trace of u at the boundary is the square integrable function equal to γ on $]0,\infty[\times\partial\mathcal{O}$ and equal to g on $\{t = 0\} \times \mathcal{O}$. There is nothing supported on the singular parts of the boundary. This is equivalent to showing that for all compactly supported lipschitzian functions Φ on $\mathbb{R}^{1+d}$,

$$\begin{aligned} &\int_{]0,\infty[\times\mathcal{O}} \big\langle u\,, L^\dagger\Phi\big\rangle\, dxdt \;= \\ &\qquad \int_{]0,\infty[\times\partial\mathcal{O}} \big\langle \gamma\,,\, A(x,\nu(x))\Phi\big\rangle\, d\Sigma dt + \int_{\{t=0\}\times\mathcal{O}} \big\langle g\,,\, \Phi(0,x)\big\rangle\, dx\,. \end{aligned} \tag{2.1}$$

To evaluate the left hand side, choose a family of cutoff function $\psi_\delta \in C^\infty(\mathbb{R}^{1+d})$ so that $\psi_\delta = 1$ (resp. $=0$) if $\mathrm{dist}((t,x),S) > 1$ (resp. $< 1/2$), and $\|\nabla\psi_\delta\|_{L^\infty} \lesssim 1/\delta$. In the left hand side of (2.1) write $\Phi = \psi_\delta\Phi + (1-\psi_\delta)\Phi$ and analyse the resulting two summands.

$(1-\psi_\delta)\Phi$ is supported in a δ neighbhorhood of S intersected with a compact subset $K \subset \mathbb{R}^{1+d}$. The integrand is bounded by $C|u|/\delta$. The Cauchy-Schwartz inequality implies that

$$\begin{aligned} &\Big|\int_{]0,\infty[\times\mathcal{O}} \big\langle u\,, L^\dagger\,(1-\psi_\delta)\Phi\big\rangle\, dxdt\Big| \;\lesssim \\ &\qquad\qquad \Big(\int_{K\cap\mathrm{dist}(x,\mathcal{S})<\delta} \frac{1}{\delta^2}\, dxdt\Big)^{1/2} \Big(\int_{K\cap\mathrm{dist}(x,\mathcal{S})<\delta} |u|^2\, dxdt\Big)^{1/2} \end{aligned}$$

The first term on the left is bounded independent of δ and the second tends to zero as $\delta \to 0$.

The second summand is supported away from the singular set so

$$\int_{]0,\infty[\times\mathcal{O}} \big\langle u\,,\, L^\dagger\,\psi_\delta\Phi\big\rangle\, dxdt \;=\; \lim_{k\to\infty} \int_{]0,\infty[\times\mathcal{O}} \big\langle u^{\epsilon(k)}\,,\, L^\dagger\,\psi_\delta\Phi\big\rangle\, dxdt\,.$$

Since the integrand vanishes near $\mathcal{S}$ the integral on the right can be taken over $]0,\infty[\times\mathcal{O}^\epsilon$. In that set use the equation satisfied by u^ϵ to find

$$\int_{]0,\infty[\times\mathcal{O}^\epsilon} \big\langle u^\epsilon\,,\, L^\dagger\,\psi_\delta\Phi\big\rangle\, dxdt \;=$$

$$\int_{]0,\infty[\times\partial\mathcal{O}^\epsilon} \big\langle u^\epsilon\,,\, A(x,\nu(x))\psi_\delta\Phi\big\rangle\, d\Sigma dt + \int_{\{t=0\}\times\mathcal{O}^\epsilon} \big\langle g\,,\, \psi_\delta\Phi(0,x)\big\rangle\, dx\,.$$

For δ fixed and $k\to\infty$, passing to the limit yields

$$\int_{]0,\infty[\times\mathcal{O}} \big\langle u\,,\, L^\dagger\,\psi_\delta\Phi\big\rangle\, dxdt \;=\; \int_{]0,\infty[\times\partial\mathcal{O}} \big\langle \gamma\,,\, A(x,\nu(x))\psi_\delta\Phi\big\rangle\, d\Sigma dt + \int_{\{t=0\}\times\mathcal{O}} \big\langle g\,,\, \psi_\delta\Phi(0,x)\big\rangle\, dx\,.$$

Therefore

$$\lim_{\delta\to 0}\int_{]0,\infty[\times\mathcal{O}} \big\langle u\,,\, L^\dagger\,\psi_\delta\Phi\big\rangle\, dxdt \;=\; \int_{]0,\infty[\times\partial\mathcal{O}} \big\langle \gamma\,,\, A(x,\nu(x))\Phi\big\rangle\, d\Sigma dt + \int_{\{t=0\}\times\mathcal{O}} \big\langle g\,,\, \Phi(0,x)\big\rangle\, dx\,.$$

This analysis of the two summands proves (2.1) so completes the construction of a solution with $u\in e^{-\mu t}L^\infty([0,\infty[\,;\,L^2(\mathcal{O}))$ and $u|_{\partial\mathcal{O}}\in L^2(]0,\infty[\times\partial\mathcal{O}))$.

2.2.2 Step 2. Uniqueness of solutions

We prove that two solutions must coincide by showing that their Laplace transforms are equal. This reduces to uniqueness for an elliptic problem. The solutions have just enough regularity to justify integration by parts in the energy method for the elliptic problem.

Laplace transformation Consider solutions satisfying

$$u\in L^\infty([0,\infty[\,;\,L^2(\mathcal{O}))\,,\qquad u\big|_{\partial\mathcal{O}} \;\in\; L^2([0,\infty[\,;\,L^2(\partial\mathcal{O}))\,.$$

The difference of two such solutions that have the same initial value has Laplace transform $\widetilde{u}(\tau)$ analytic in $\operatorname{Re}\tau>0$ with values in $L^2(\mathcal{O})$ and with $\widetilde{u}|_{\partial\mathcal{O}}$ analytic with values in $L^2(\partial\mathcal{O})$ and satisfying

$$\tau\widetilde{u}+G(x,\partial_x)\widetilde{u} \;=\; 0\,,\qquad \widetilde{u}|_{\partial\mathcal{O}}\in\mathcal{N}\,. \tag{2.2}$$

Uniqueness is therefore a consequence of the following uniqueness result for the transformed problem.

Theorem 2.4. *If $\operatorname{Re}\tau>0$ and $v\in L^2(\mathcal{O})$ satisfies*

$$\tau v+G(x,\partial_x)v=0,\quad v\big|_{\partial\mathcal{O}}\in\mathcal{N},\quad \text{and}\quad v|_{\partial\mathcal{O}}\in L^2(\partial\mathcal{O}), \tag{2.3}$$

then $v=0$.

Remark 2.1. **i.** *Since $\partial\mathcal{O}$ is lipschitzian the Sobolev spaces $H^s(\partial\mathcal{O})$ are well defined for $|s| \le 1$.* **ii.** *The existence of the trace $v|_{\partial\mathcal{O}} \in H^{-1/2}(\partial\mathcal{O})$ follows from the fact that v and Gv belong to $L^2(\mathcal{O})$ and $\partial\mathcal{O}$ is noncharacteristic for G.* **iii.** *The second condition in (2.3) makes sense for $v, Gv \in L^2(\mathcal{O})$. The third asserts an improved regularity that is true for the solutions constructed.*

At a formal level the result is immediate. If the integrations by parts were justified, Theorem 2.4 would follow for $\operatorname{Re}\tau > 0$ from

$$
\begin{aligned}
0 &= 2\operatorname{Re}\int_{\mathcal{O}} \big(\tau v + G(x,\partial)v\,,\, v\big)\,dx \\
&= 2\operatorname{Re}\tau\|v\|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}} (Zv,v)dx + \int_{\partial\mathcal{O}} \big(A(x,\nu(x))v,v\big)\,dx \\
&\ge 2\operatorname{Re}\tau\,\|v\|^2_{L^2(\mathcal{O})}\,.
\end{aligned}
\tag{2.4}
$$

In the last step the positivity of Z and the dissipativity of the boundary condition are used. The method is to prove that the integration by parts is justified. That is done in several steps.

Lopatinski's condition

Proposition 2.5. *Suppose that*

$$
A(\partial) \;:=\; \sum_{j=1}^{d} A_j\partial_j
$$

is an elliptic operator with hermitian constant coefficients. Suppose that $H := \{x : x\cdot\xi > 0\}$ is an open half space so $A(\nu) = -A(\xi/|\xi|)$. Suppose that $\mathcal{N}$ is a maximal strictly dissipative subspace for $A(\nu)$. Then $\mathcal{N}$ satisfies the coercivity condition of Lopatinski for the half space H.

Proof. A linear change of coordinates reduces to the case $H = \{x_1 > 0\}$. In that case, Lopatinski's condition is that for all $0 \ne \xi' \in \mathbb{R}^{d-1}_{x'}$, if $w(x_1)$ satisfies the ordinary differential boundary value problem

$$
A(\partial_1, i\xi')w(x_1) \;=\; 0, \quad w(0)\in\mathcal{N}, \quad \text{and} \quad \lim_{x_1\to\infty} w(x_1) \;=\; 0,
$$

then $w = 0$.

Under the above hypotheses, the function $u = e^{ix'\xi'}w(x_1)$ is a stationary solution of the hyperbolic equation $\big(\partial_t + A(\partial)\big)u = 0$.

Denote by $\mathbf{e}_j$ the standard basis for $\mathbb{C}^d$. For $j = 2,\dots,d$ choose nonzero real numbers α_j so that $\alpha_j\xi'_j = 2\pi$. Define for $j \ge 2$, $\mathbf{v}_j := \alpha_j\mathbf{e}_j$. Introduce the lattice $\mathcal{L} \in \mathbb{R}^{d-1}_{x'}$ consisting of vectors $\sum_j n_j\mathbf{v}_j$ with $n_j \in \mathbb{Z}$. The stationary solution $u(x) = e^{i\xi'x'}w(x_1)$ is then $\mathcal{L}$-periodic in x'.

The energy identity for $\mathcal{L}$-periodic solutions of $Lu = 0$ then asserts that

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^3_+/\mathcal{L})} \;+\; \int_{\{x_1=0\}/\mathcal{L}} (-A_1 u(t,0,x')\,,\, u(t,0,x'))\,dx' \;=\; 0\,.$$

For the stationary solution one finds

$$\int_{\{x_1=0\}/\mathcal{L}} (-A_1 u(0,x')\,,\, u(0,x'))\,dx' \;=\; 0\,.$$

The strict dissipativity of $\mathcal{N}$ implies that $u|_{x_1=0} = 0$. Therefore $w(0) = 0$. Uniqueness for the ordinary differential equation initial value problem implies that w is identically equal to zero. Therefore u is identically equal to zero. $\square$

Corollary 2.6. *If $1/2 \ge s > 0$, H is one of the half spaces $\{x_j > 0\}$, r as in Section 2.1.4,*

$$v \in L^2(H), \qquad Gv \in H^{s-1}(H), \qquad \text{and} \qquad \Pi_+(x)v\big|_{\partial H} \in L^2(\partial H),$$

then $v \in H^s(\{0 \le x_j \le r/2\})$.

Proof. Choose a locally finite cover of $\{0 \le x_j \le r/2\}$ by balls B^k of radius $3r/4$ with centers on ∂H. Denote by $\widetilde{B}^k$ the ball with the same center and radius $4r/5$. Use the boundary regularity estimate that follows from the Lopatinski condition,

$$\|v\|^2_{H^s(B^k\cap H)} \;\lesssim\; \|Gv\|^2_{H^{s-1}(\widetilde{B}^k\cap H)} \;+\; \|v\|^2_{L^2(\widetilde{B}^k\cap\partial H)}\,. \tag{2.5}$$

Summing on k yields the conclusion. $\square$

Sufficient regularity at the singular set S With ϕ from Example 2.1, define a cutoff function

$$\chi(x) \;:=\; \Pi_{j=1}^d \phi(x_j)\,.$$

Define $\chi_\epsilon(x) := \chi(x/\epsilon)$. With v from Theorem 2.4, define $v^\epsilon := \chi_\epsilon(x)v \in \cap_s H^s(\mathcal{O})$. For this function integration by parts is justified and yields

$$\begin{aligned} 0 \;&= 2\,\mathrm{Re} \int_{\mathcal{O}} \langle \tau v^\epsilon + G(x,\partial)v^\epsilon\,,\, v^\epsilon\rangle\,dx \\ &= 2\,\mathrm{Re}\,\tau\|v^\epsilon\|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}} (Zv^\epsilon, v^\epsilon)dx + \int_{\partial\mathcal{O}} \big(A(x,\nu(x))v^\epsilon, v^\epsilon\big)\,dx\,. \end{aligned} \tag{2.6}$$

The boundary term vanishes. Write

$$G\,v^\epsilon \;=\; G\,(\chi_\epsilon v) \;=\; \chi_\epsilon\,G\,v \;-\; [G\,,\,\chi_\epsilon]v\,.$$

Need to pass to the limit in (2.6). The Z term, and $\chi_\epsilon G v$ pose no problem. The commutator is a multiplication operator by matrices with coefficients bounded in magnitude by C/ϵ so

$$\Big|\int_{\mathcal{O}} \langle [G\,,\,\chi_\epsilon]v\,,\,v\rangle\;dx\Big| \;\le\; \frac{C}{\epsilon}\int_{\mathrm{dist}(x,S)<\epsilon} |v|^2\;dx\,.$$

To complete the proof of uniqueness it suffices to show that

$$\lim_{\epsilon\to 0}\;\frac{1}{\epsilon}\;\int_{\mathrm{dist}(x,S)<\epsilon}|v|^2\;dx\;=\;0\,. \tag{2.7}$$

Once established this implies that $\int\langle v^\epsilon\,,\,[G\,,\,\chi_\epsilon]v^\epsilon\rangle\,dx\to 0$. The convergence (2.7) is proved in the next two paragraphs.

Negligible sets for $H^{1/2}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^3)$. We prove that sets of codimension 1 are negligible for $H^{1/2}(\mathbb{R}^2)$ and those of codimension 2 are negligible for $H^1(\mathbb{R}^3)$. The sets are not negligible for $H^{1/2+\epsilon}(\mathbb{R}^2)$ and $H^{1+\epsilon}(\mathbb{R}^3)$ respectively. Lemma 2.9 is used in Part I and Lemma 2.10 in Parts I and II.

Lemma 2.7. *There is $C>0$ independent of ϵ so that the following hold.*
i. *If $|D|^{1/2}w\in L^2(\mathbb{R}^2)$ then $w\in L^4(\mathbb{R}^2)$ and if $w\neq 0$,*

$$\int_{|x|<\epsilon}|w|^2\,dx\;\le\;C\,\epsilon\;\frac{\Big(\int_{|x|<\epsilon}|w|^4\,dx\Big)^{1/2}}{\Big(\int_{\mathbb{R}^2}|w|^4\,dx\Big)^{1/2}}\;\int_{\mathbb{R}^2}|\xi|\,|\widehat{w}(\xi)|^2\,d\xi\,. \tag{2.8}$$

ii. *If $|D|w\in L^2(\mathbb{R}^3)$ then $w\in L^6(\mathbb{R}^3)$ and if $w\neq 0$,*

$$\int_{|x|<\epsilon}|w|^2\,dx\;\le\;C\,\epsilon^2\;\frac{\Big(\int_{|x|<\epsilon}|w|^6\,dx\Big)^{1/3}}{\Big(\int_{\mathbb{R}^3}|w|^6\,dx\Big)^{1/3}}\;\int_{\mathbb{R}^3}|\xi|^2\,|\widehat{w}(\xi)|^2\,d\xi\,. \tag{2.9}$$

Proof. Following [5], the space $\dot H^s(\mathbb{R}^d)$ is the set of tempered distributions with Fourier transforms in L^1_{loc} and

$$\|u\|^2_{\dot H^s(\mathbb{R}^d)}\;:=\;\int_{\mathbb{R}^d}|\xi|^{2s}|\hat u(\xi)|^2\,d\xi\;<\;\infty\,.$$

If $0<s<d/2$, then the space $\dot H^s(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$.
i. Using the previous result with $d=2$ and $s=1/2$ yields

$$\Big(\int_{\mathbb{R}^2}|w(x)|^4\,dx\Big)^{1/4}dx\;\lesssim\;\Big(\int|\xi|\,|\widehat{w}(\xi)|^2\,d\xi\Big)^{1/2}. \tag{2.10}$$

The Cauchy-Schwartz inequality yields

$$\int_{|x|<\epsilon} |w|^2\,dx \;\le\; \Big(\int_{|x|<\epsilon} \big(|w|^2\big)^2\,dx\Big)^{1/2}\Big(\int_{|x|<\epsilon} 1^2\,dx\Big)^{1/2}$$

$$= \sqrt{2\pi\epsilon^2}\,\Big(\int_{|x|<\epsilon} |w|^4\,dx\Big)^{1/2} = C\,\epsilon\,\frac{\Big(\int_{|x|<\epsilon} |w|^4\,dx\Big)^{1/2}}{\Big(\int_{\mathbb{R}^2} |w|^4\,dx\Big)^{1/2}}\,\Big(\int_{\mathbb{R}^2} |w|^4\,dx\Big)^{1/2}.$$

The proof of **i** is completed by (2.10).

ii. The case $d=3, s=1$ yields

$$\Big(\int_{\mathbb{R}^3} |w(x)|^6\,dx\Big)^{1/6} dx \;\lesssim\; \Big(\int_{\mathbb{R}^3} |\xi|^2\,|\widehat{w}(\xi)|^2\,d\xi\Big)^{1/2}. \tag{2.11}$$

Estimate using Hölder's inequality with exponents 3 and 3/2,

$$\int_{|x|<\epsilon} |w|^2\,dx \;\le\; \Big(\int_{|x|<\epsilon} \big(|w|^2\big)^3\,dx\Big)^{1/3}\Big(\int_{|x|<\epsilon} 1^{3/2}\,dx\Big)^{2/3} \;=$$

$$\Big(\frac{4\pi}{3}\Big)^{\frac{2}{3}}\epsilon^2\,\Big(\int_{|x|<\epsilon} |w|^6\,dx\Big)^{\frac{1}{3}} = C\,\epsilon^2\,\frac{\Big(\int_{|x|<\epsilon} |w|^6\,dx\Big)^{\frac{1}{3}}}{\Big(\int_{\mathbb{R}^3} |w|^6\,dx\Big)^{\frac{1}{3}}}\,\Big(\int_{\mathbb{R}^3} |w|^6\,dx\Big)^{\frac{1}{3}}.$$

The proof of **ii** is completed by (2.11). □

Lemma 2.8. i. *If $d \ge 2$ and $|D|^{1/2}w \in L^2(\mathbb{R}^d)$ then*

$$\frac{1}{\epsilon}\int_{|x_1,x_2|<\epsilon} |w|^2\,dx \lesssim \||D|^{1/2}w\|^2_{L^2(\mathbb{R}^d)}, \quad \text{and as } \epsilon \to 0, \quad \frac{1}{\epsilon}\int_{|x_1,x_2|<\epsilon} |w|^2\,dx \;\to\; 0.$$

ii. *If $d \ge 3$ and $\nabla w \in L^2(\mathbb{R}^d)$ then*

$$\frac{1}{\epsilon^2}\int_{|x_1,x_2,x_3|<\epsilon} |w|^2\,dx \lesssim \|\nabla w\|^2_{L^2(\mathbb{R}^d)}, \;\text{ and as } \epsilon \to 0, \;\frac{1}{\epsilon^2}\int_{|x_1,x_2,x_3|<\epsilon} |w|^2\,dx \;\to\; 0.$$

Proof. i. Denote by $\widehat{w}(\xi_1,\xi_2,x')$ the partial Fourier transform with $x' := (x_3, x_4, \ldots, x_d)$. Then

$$\int |\xi_1,\xi_2|\,|\widehat{w}(\xi_1,\xi_2,x')|^2\,d\xi_1 d\xi_2 dx' \;\le\; \big\|\,|D|^{1/2}w\,\big\|^2_{L^2(\mathbb{R}^d)}.$$

Define

$$0 \;\le\; f(\epsilon,x') \;:=\; \frac{\Big(\int_{|x_1,x_2|<\epsilon} |w(x_1,x_2,x')|^4\,dx_1dx_2\Big)^{1/2}}{\Big(\int_{\mathbb{R}^2} |w(x_1,x_2,x')|^4\,dx_1dx_2\Big)^{1/2}} \;\le\; 1.$$

The function f is decreasing in ϵ. Lebesgue's monotone convergence theorem implies that as $\epsilon \to 0$,

$$\int f^2(\epsilon, x')\, dx' \;=\; C(d) \int_{|x_1,x_2|<\epsilon} |w(x_1,x_2,x')|^4\, dx_1 dx_2 \quad\to\quad 0\,.$$

It follows that f tends to zero for almost all x'.

The estimate of part **i** of Lemma 2.7 above implies that

$$\left(\frac{1}{\epsilon}\int_{|x_1,x_2|<\epsilon} |w|^2\, dx\right)^2 \lesssim \int_{\mathbb{R}^d} f^2(\epsilon, x')\, |\xi_1,\xi_2|\, |\widehat{w}(\xi_1,\xi_2,x')|^2\, d\xi_1 d\xi_2 dx'\,.$$

Part **i** follows from Lebesgue's dominated convergence theorem.

ii. Exactly analogous using part **ii** of Lemma 2.7. □

Lemma 2.9. *If $v \in H^{1/2}(\mathcal{O})$ then as $\epsilon \to 0$,*

$$\frac{1}{\epsilon}\int_{\mathrm{dist}(x,S)<\epsilon} |v|^2\, dx \quad\to\quad 0\,.$$

Proof. Extend v to an element of $H^{1/2}(\mathbb{R}^d)$. The set of points at distance less than ϵ from S is contained in a finite union of the cylinders $|x_i, x_j| < \epsilon$ with $1 \le i < j \le d$. Apply the preceding lemma to each cylinder. □

The conclusion of this lemma is exactly (2.7). *Therefore to complete the proof of uniqueness it suffices to show that $v \in H^{1/2}(\mathcal{O})$.*

Lemma 2.10. i. *If $w \in H^{1/2}(\mathbb{R}^d)$ is supported on a finite union of codimension one affine subspaces, then $w = 0$.*

ii. *If $w \in H^1(\mathbb{R}^d)$ is supported on a finite union of codimension two affine subspaces, then $w = 0$.*

Proof of ii. This case is somewhat harder. Part **i** is left to the reader. Denote by Γ the finite union. Construct a sequence of cutoff functions $\psi_\epsilon(x) \in C^\infty(\mathbb{R}^d)$ so that $\psi_\epsilon = 1$ at points x with $\mathrm{dist}(x,\Gamma) \ge \epsilon$, $\psi_\epsilon = 0$ at points x with $\mathrm{dist}(x,\Gamma) \le \epsilon/2$, and $\|\nabla\psi_\epsilon\|_{L^\infty(\mathbb{R}^d)} \le C/\epsilon$.

By hypothesis, $\psi_\epsilon w = 0$. The proof is completed by showing that for all $u \in H^1(\mathbb{R}^d)$, $\lim \|\psi_\epsilon u - u\|_{H^1} = 0$. For the dense set $u \in C_0^\infty(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ this is elementary.

The proof is completed by showing that uniformly in ϵ,

$$\|\psi_\epsilon u\|_{H^1(\mathbb{R}^d)} \;\lesssim\; \|u\|_{H^1(\mathbb{R}^d)}\,.$$

For this, estimate

$$\|\nabla(\psi_\epsilon u) - \psi_\epsilon \nabla u)\|^2_{H^1(\mathbb{R}^d)} \;\le\; \frac{C}{\epsilon^2}\int_{\mathrm{dist}(x,\Gamma)<\epsilon} |u|^2\; dx \;\lesssim\; \int_{\mathbb{R}^d} |\nabla u|^2\; dx$$

by part **ii** of Lemma 2.8. □

Proof that $v \in H^{1/2}(\mathcal{O})$

Lemma 2.11. *If*

$$w \in L^2(\mathcal{O}), \qquad Gw := f \in L^2(\mathcal{O}), \qquad \text{and,} \qquad w\big|_{\partial\mathcal{O}} \in L^2(\partial\mathcal{O}),$$

denote by $\underline{f} \in L^2(\mathbb{R}^d)$ and $\underline{w}$ the extension by zero of $f \in L^2(\mathcal{O})$ and $w \in L^2(\mathcal{O})$. Then in the sense of distributions on $\mathbb{R}^d$,

$$G\underline{w} \;=\; \underline{f} \;+\; w|_{\partial\mathcal{O}}\, d\Sigma\,. \tag{2.12}$$

Proof. If $x \in \mathcal{O}$ identity (2.12) holds in a neighborhood of x thanks to the equation $Gw = f$ in $\mathcal{O}$. If $x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}$ then both sides of (2.12) vanish on a neighborhood of x.

If x is a smooth point of the boundary w is smooth on a neighborhood of x thanks to the Lopatinski condition. Then (2.12) holds on a neighborhood of x.

Therefore, the difference between the left and right hand side of (2.12) is an element of $H^{-1}(\mathbb{R}^d)$ supported on the finite union of a codimension two affine subspaces of $\mathbb{R}^d$. The difference therefore vanishes by part **ii** of Lemma 2.10. □

Theorem 2.12. *If*

$$v \in L^2(\mathcal{O}), \qquad Gv \in L^2(\mathcal{O}), \qquad \text{and,} \qquad v\big|_{\partial\mathcal{O}} \in L^2(\partial\mathcal{O})$$

then $v \in H^{1/2}(\mathcal{O})$.

If $\mathcal{O}$ were a smoothly bounded set, this would follow from Lopatinski's condition. What is needed is to show the $H^{1/2}$ regularity in a neighborhood of the singular points of $\partial\mathcal{O}$.

Proof of Theorem 2.12. Choose r and $p(x,\xi)$ as in Section 2.1.4. By elliptic regularity one has

$$v \;\in\; H^{1/2}\Big(\Big\{\frac{r}{4} < \operatorname{dist}\big(x,\partial\mathcal{O}\big) < \frac{r}{2}\Big\}\Big)\,.$$

It suffices to show that

$$v \;\in\; H^{1/2}\Big(\Big\{\operatorname{dist}\big(x,\partial\mathcal{O}\big) \le \frac{r}{4}\Big\}\Big)\,.$$

Denote by $\underline{v}$ the extension by zero. Lemma 2.11 together with the fact that $p(x,D)$ is a parametrix imply that

$$G\Big(p(x,D)(f + v|_{\partial\mathcal{O}} d\Sigma)\Big) - G\underline{v} \;\in\; \cap_s H^s(\operatorname{dist}\big(x,\partial\mathcal{O}\big) \le \frac{3r}{4})\,.$$

Elliptic regularity on $\mathbb{R}^d$ implies that

$$p(x,D)(f+v|_{\partial\mathcal{O}}d\Sigma)-\underline{v}\ \in\ \cap_s H^s\Big(\mathrm{dist}\big(x,\partial\mathcal{O}\big)\le\frac{r}{2}\Big)\,.$$

Since p is of order -1, $p(x,D)f\in H^1(\mathbb{R}^d)$ so

$$p(x,D)(v|_{\partial\mathcal{O}}d\Sigma)-\underline{v}\ \in\ H^1\Big(\Big\{x\in\mathbb{R}^d:\mathrm{dist}\big(x,\partial\mathcal{O}\big)\le\frac{r}{2}\Big\}\Big)\,.$$

The proof is completed by showing that $p(x,D)(v|_{\partial\mathcal{O}}d\Sigma)\in H^{1/2}(\{x\in\mathcal{O}:\mathrm{dist}(x,\partial\mathcal{O})<r/2\})$.

Denote by $H_j:=\{x_j>0\}$. Decompose

$$v|_{\partial\mathcal{O}}\ =\ \sum g_j\,,\qquad g_j\ \in\ L^2(\partial H_j\cap\partial\mathcal{O})\,.$$

It is sufficient to show that $p(x,D)(g_j\,d\Sigma)\in H^{1/2}(\mathrm{dist}(x,\partial\mathcal{O})\le r/2$. We prove the stronger assertion that $p(x,D)(g_j\,d\Sigma)\in H^{1/2}(\{0\le x_j\le r/2\})$. The last assertion does not involve corners.

The trace inequality

$$\Big(\int_{\partial H_j}|\psi(0,x')|^2\,dx'\Big)^{1/2}\ \lesssim\ \Big(\int_{\mathbb{R}^d}|\nabla\psi(x)|^p\ dx\Big)^{1/p},\qquad p=\frac{2d}{d+1}<2,$$

together with $g\in L^2(\partial H_j)$ imply that

$$g_j\,d\Sigma\in\big(W^{1,p}(\mathbb{R}^d)\big)'=W^{-1,q}(\mathbb{R}^d),\qquad\frac{1}{q}+\frac{1}{p}=1,\qquad q>2\,.$$

Since p is order -1,

$$w_j\ :=\ p(x,D)\big(g_j\,d\Sigma\big)\ \in\ W^{0,q}(\mathbb{R}^d)\ =\ L^q(\mathbb{R}^d)\,.$$

The local regularity of w_j is better than L^2.

Since the distribution kernel of $p(x,D)$ is smooth off the diagonal and rapidly decaying with all derivatives as the distance to the diagonal grows this is sufficient to conclude that

$$w_j\ \in\ L^2(\mathbb{R}^d)\,.\tag{2.13}$$

In addition

$$Gw_j\ -\ g_j\,d\Sigma\ \in\ \cap_s H^s(\mathbb{R}^d)\,.$$

Since $g_j\,d\Sigma$ vanishes on $\mathcal{O}$ this implies that

$$G(w_j|_{H_j})\ \in\ \cap_s H^s(H_j)\,.\tag{2.14}$$

Proposition 2.3 implies that the trace of $w_j|_{H_j}$ at ∂H_j is square integrable,

$$w_j|_{\partial H_j} \in L^2(\partial H_j)\,.$$

In particular

$$\Pi_+(x) w_j|_{\partial H_j} \in L^2(\partial H)\,. \tag{2.15}$$

The three last numbered equations and Corollary 2.6 imply that $w_j|_{H_j} \in H^{1/2}(\{0 \le x_j \le r/2\})$. The proof is complete. □

This completes the verification of (2.7) and thereby completes the proof of uniqueness.

2.2.3 Step 3. Proof of the energy equality

With the $H^{1/2}(\mathcal{O})$ regularity in hand one can justify the integration by parts in the basic *a priori* estimate (2.4) for the operator $\Lambda + G$ with real $\Lambda \ge 0$,

$$\Lambda\ \|u\| \ \le\ \ \|(\Lambda + G)u\| \quad \text{provided} \quad u \in \mathcal{N} \text{ on } \partial\mathcal{O}\,.$$

In particular the family of maps $\Lambda(\Lambda + G)^{-1}$ is uniformly bounded from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ as $\Lambda \to \infty$. Therefore

$$(I + \epsilon G)^{-1} \ =\ \epsilon^{-1}(\epsilon^{-1} I + G)^{-1}$$

is uniformly bounded in $\mathrm{Hom}(L^2(\mathcal{O}))$. Define for $0 < \epsilon << 1$

$$g^\epsilon \ :=\ (I + \epsilon G)^{-1} g\,.$$

Lemma 2.13. *As $\epsilon \to 0$ one has for all $g \in L^2(\mathcal{O})$*

$$\big\|g^\epsilon - g\big\|_{L^2(\mathcal{O})} \quad \longrightarrow \quad 0\,.$$

Proof. From the uniform boundedness it is sufficient to prove the result for g in a dense subset of $L^2(\mathcal{O})$. Compute

$$I \ -\ (I + \epsilon G)^{-1} \ =\ \epsilon G (I + \epsilon G)^{-1}\,.$$

For g belonging to the dense subset $C_0^\infty(\mathcal{O})$,

$$\big\|g^\epsilon - g\big\| \ \le\ \epsilon\ \|(I + \epsilon G)^{-1}\|\ \|Gg\| \ \to\ 0$$

since the last two factors are bounded. □

Thanks to uniqueness we can define u^ϵ to be the only solution with initial value g^ϵ. The function $\partial_t u^\epsilon$ is the solution with initial value $Gg^\epsilon \in L^2(\mathcal{O})$. In particular $\partial_t u^\epsilon \in L^\infty([0,T]; L^2(\mathcal{O}))$. Therefore, $Gu^\epsilon \in L^\infty([0,T] : L^2(\mathcal{O}))$. Theorem 2.12 implies that $u^\epsilon \in L^\infty([0,T]; H^{1/2}(\mathcal{O}))$.

This suffices to justify the integration by parts in the energy identity for u^ϵ and also for $u^\epsilon - u^\delta$. The latter implies that

$$\sup_{0<t<T} \|u^\epsilon(t) - u^\delta(t)\| \;+\; \big\|(u^\epsilon - u^\delta)\big|_{\partial\mathcal{O}}\big\|_{L^2([0,T]\times\partial\mathcal{O})} \;\le\; C\,\|g^\epsilon - g^\delta\|\,.$$

Therefore u^ϵ is a Cauchy sequence in $C([0,T]:L^2(\mathcal{O}))$ and $u^\epsilon\big|_{\partial\mathcal{O}}$ is a Cauchy sequence in $L^2([0,T]\times\partial\mathcal{O})$.

It follows that the limit u is continuous with values in $L^2(\mathcal{O})$ with trace in $L^2([0,T]\times\partial\mathcal{O})$.

Passing to the limit in the energy identity for u^ϵ on $[t,T]\times\mathcal{O}$ yields the energy identity for u. $\square$

2.3 Maxwell and Euler, hidden ellipticity

This section proves existence and most importantly uniqueness for some problems with strictly dissipative corners for which G is not elliptic. A common feature is that the kernel of $G(\xi)$ has dimension independent of $\xi \neq 0$. In this situation it is always true that there is a partial differential operator $Q(D)$ with $QG = GQ = 0$ and so that G, Q is an overdetermined elliptic system, see [6]. Solutions of $\partial_t u + Gu = 0$ satisfy $\partial_t Qu = 0$ so one trivially knows the exact regularity of Qu for all time. The idea is to take advantage of the ellipticity of G, Q. This is what we call *hidden ellipticity*. We consider only the case where there is a Q of first order, a class of problems introduced by Majda in [2]. We do not propose a general strategy but treat three important examples; Maxwell's equations, the compressible Euler equations linearized about the stationary solution, and the wave equation written as a first order system. The first two yield to similar but different arguments. The Maxwell case is substantially harder. The last is equivalent to the linearized Euler equations.

2.3.1 Maxwell's equations

Taking the divergence of the dynamic Maxwell equations (1.2) yields

$$\partial_t\big(\operatorname{div}\epsilon(x)E \;-\; 4\pi\operatorname{div} j\big) \;=\; 0\,. \qquad \partial_t\big(\operatorname{div}\mu(x)B\big) = 0\,. \tag{2.16}$$

The continuity equation (1.3) implies

$$\partial_t\big(\operatorname{div}\epsilon E - 4\pi\rho\big) \;=\; 0\,. \tag{2.17}$$

The physical solutions satisfy (1.4) asserting that the time independent quantity in parentheses and $\operatorname{div}\mu B$ vanish identically.

In regions without currents and charges, equations (2.16) and (2.17) expresse hidden ellipticity with

$$Q \ := \ \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix}$$

Equations (1.2) have form $Lu = 0$ with

$$L \ = \ A_0(x)\,\partial_t \ + \ \sum_{j=1}^{3} A_j \partial_j\,, \qquad A_0(x) \ := \ \begin{pmatrix} \epsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix},$$

and 6×6 constant symmetric real matrices A_j for $j = 1, 2, 3$.

For a half space $\{x \cdot \nu > 0\}$ with unit vector ν, the nullspace of $\sum A_j \nu_j$ consists of the vectors E, B with both E and B parallel to ν. The range is the set of vectors E, B with both E and B tangent to the boundary. Introduce the tangential components

$$E_{tan} \ = \ E - (E \cdot \nu)\nu\,, \qquad B_{tan} \ = \ B - (B \cdot \nu)\nu\,.$$

For the halfspace $x_1 > 0$, the strictly dissipative subspaces $\mathcal{N}$ are those for which

$$(A_1 u\,,\, u) \geq c\Big(E_2^2 + E_3^2 + B_2^2 + B_3^2\Big), \qquad E_2^2 + E_3^2 + B_2^2 + B_3^2 \ = \ \big\|E_{tan}, B_{tan}\big\|^2\,.$$

Assumption 2.1. i. *In Part I, the strictly positive matrix valued permittivities are assumed to be infinitely differentiable with partial derivatives of all orders belonging to* $L^\infty(\mathbb{R}^3)$

ii. *For each* $1 \leq j \leq 3$*, suppose that* $\mathcal{N}_j$ *is a strictly dissipative subspace for* A_j*. That is for each* j*,* $\mathcal{N}_j$ *has dimension 4 and there is a constant* $c_j > 0$ *so that for all* $u \in \mathcal{N}_j$

$$\big(A_j u\,,\, u\big) \ \geq \ c_j\,\big\|E_{tan}, B_{tan}\big\|^2\,.$$

Theorem 2.14. *With Assumption 2.1, for each* $f, g \in L^2(\mathcal{O})$ *with* $\mathrm{div}\,\epsilon(x) f = \mathrm{div}\,\mu(x) g = 0$*, there is one and only one* $u = (E, B) \in L^\infty\big([0, \infty[\,;\, L^2(\mathcal{O})\big)$ *with*

$$\big\{E_{tan}, B_{tan}\big\}\big|_{\partial\mathcal{O}} \ \in \ L^2([0,\infty[\times\partial\mathcal{O}),$$

$$Lu = 0, \quad u(0) = (f, g), \quad \text{and,} \quad u|_{\{x_j=0\}\cap\{\partial\mathcal{O}\setminus S\}} \in \mathcal{N}_j \ \ \text{for} \ \ 1 \leq j \leq 3\,.$$

In addition, $\mathrm{div}\,\epsilon(x) E = \mathrm{div}\,\mu(x) B = 0$ *and for all* $0 \leq t < T < \infty$

$$\begin{aligned} \big\langle A_0(x)\,u(T,x))\,,\, u(T,x))\big\rangle + \int\limits_{[t,T]\times\partial\mathcal{O}} \big(A(\nu(x))u, u\big)\, dt d\Sigma \\ = \ \big\langle A_0(x)\,u(t,x))\,,\, u(t,x))\big\rangle\,. \end{aligned} \tag{2.18}$$

Proof. The existence of a solution satisfying all the conditions except the energy equality replaced by an inequality for the interval $[0, T]$ is proved as before. That is, the domain is smoothed and the boundary condition on the smoothed part is of the form

$$\Pi_- u \;=\; M(x)\, \Pi_{\geq 0} u \,.$$

A passage to the limit removes the smoothing.

The difficult part is uniqueness. The strategy is to show that for a solution with data equal to zero, the Laplace transform vanishes. The essential step is to show that the Laplace transform $\widehat{u} = \{\widehat{E}\,,\,\widehat{B}\}$ belongs to $H^{1/2}(\mathcal{O})$. That is done in the next subsubsection.

$E, B \in H^{1/2}(\mathcal{O})$. The tangential components $\{E_{tan}, B_{tan}\}\big|_{\partial\mathcal{O}}$ are square integrable from strict dissipativity. To take advantage of that, introduce potentials. The special structure of Maxwell's equations leads to a shorter proof of the $H^{1/2}$ regularity than in Section 2.2.2.

Lemma 2.15. *If $E \in L^2(\mathcal{O})$ satisfies*

$$E_{tan} \in L^2(\partial\mathcal{O}), \quad \operatorname{div} \epsilon(x) E = f \in H^{-1/2}(\mathcal{O}), \quad \operatorname{curl} E = h \in H^{-1/2}(\mathcal{O}),$$

then $E \in H^{1/2}(\mathcal{O})$ and

$$\|E\|_{H^{1/2}(\mathcal{O})} \;\lesssim\; \|E_{tan}\|_{L^2(\partial\mathcal{O})} \;+\; \|E\|_{L^2(\mathcal{O})} \;+\; \|\operatorname{div} E\,,\, \operatorname{curl} E\|_{H^{-1/2}(\mathcal{O})}\,.$$

Proof. • Reason locally. Introduce concentric balls B_1 of radius 1 and $B_{1/2}$ of radius $1/2$. The balls have centers either at distance > 1 to $\partial\mathcal{O}$ or are on a face of the boundary and distance > 1 to the singular points of the boundary, or are at a singular point and at distance > 1 to the corner $x = 0$ or are at the corner.

The result folllows on taking a locally finite cover of $\overline{\mathcal{O}}$ by balls B_1^k, a partition of unity ψ_k subordinate to the cover, and summing over k the local estimates

$$\begin{aligned}\|\psi_k E\|^2_{H^{1/2}(\mathcal{O})} \lesssim \|&\operatorname{div} \psi_k E\,,\, \operatorname{curl} \psi_k E\|_{H^{-1/2}(\mathcal{O})} \\ &+ \|\psi_k E\|_{L^2(\mathcal{O})} + \|\psi_k E_{tan}\|^2_{L^2(\partial\mathcal{O})}.\end{aligned}$$

In this way the proof of the estimate is reduced to the case of functions supported in the balls B_1.

• Consider E supported in one of the balls of radius one. If the ball is one whose center lies on $\partial\mathcal{O}$, choose an extension $\underline{h} \in H^{-1/2}(\mathbb{R}^3)$ of $\operatorname{curl} E$ with

$$\operatorname{supp} \underline{h} \subset B_1\,, \qquad \|\underline{h}\|_{H^{-1/2}(\mathbb{R}^3)} \lesssim \|h\|_{H^{-1/2}(\mathcal{O}\cap B_1)}\,, \qquad \int_{\mathbb{R}^3} \underline{h}\, dx = 0\,.$$

Thanks to the mean zero condition, explicit solution in Fourier constructs unique solution $\underline{E} \in H^{1/2}(\mathbb{R}^3)$ to

$$\mathrm{curl}\, \underline{E} \;=\; \underline{h}, \qquad \mathrm{div}\, \underline{E} \;=\; 0\,.$$

From $\underline{E} \in H^{1/2}(\mathbb{R}^3)$ and $\mathrm{curl}\, \underline{E} \in H^{-1/2}(\mathbb{R}^3)$ it follows that for any hyperplane H, $\underline{E}_{tan} \in L^2(H)$. Similarly from $\underline{E} \in H^{1/2}(\mathbb{R}^3)$ and $\mathrm{div}\, \epsilon \underline{E} \in H^{-1/2}(\mathbb{R}^3)$ it follows that $\underline{E}_{normal} \in L^2(H)$. Therefore,

$$\underline{E}\big|_{\partial\mathcal{O}} \;\in\; L^2(\partial\mathcal{O})\,.$$

• To complete the proof it suffices to show that $E - \underline{E} \in H^{1/2}(\mathcal{O} \cap B_1)$. Since $\mathrm{curl}\,(E - \underline{E}) = 0$ on $B_1 \cap \mathcal{O}$ there is a potential $\phi \in H^1(\mathcal{O} \cap B_1)$ so that

$$E - \underline{E} \;=\; \mathrm{grad}\, \phi \quad \text{in} \quad \overline{\mathcal{O}} \cap B_1\,.$$

The potential is unique up to an additive constant. On the lateral boundaries $B_1 \cap \partial\mathcal{O}$, $(\mathrm{grad}\, \phi)_{tan} = (E - \underline{E})_{tan}$ so

$$\big\|(\mathrm{grad}\, \phi)_{tan}\big\|_{L^2(B_1\cap\partial\mathcal{O})} \;=\; \big\|(E - \underline{E})_{tan}\big\|_{L^2(B_1\cap\partial\mathcal{O})}.$$

The potential is uniquely determined by imposing

$$\int_{B_{1/2}\cap\mathcal{O}} \phi\ dx \;=\; 0\,.$$

This is used with the inequality of Poincaré type

$$\big\|\phi\big\|_{H^1(B_1\cap\partial\mathcal{O})} \;\lesssim\; \big\|(\mathrm{grad}\, \phi)_{tan}\big\|_{L^2(B_1\cap\partial\mathcal{O})} \;+\; \Big|\int_{B_{1/2}\cap\mathcal{O}} \phi\, dx\Big|\,.$$

They yield

$$\|\phi\|_{H^1(B_1\cap\partial\mathcal{O})} \;\lesssim\; \|(E - \underline{E})_{tan}\|_{L^2(\partial\mathcal{O}\cap B_1)}\,. \tag{2.19}$$

On B_1 one has a scalar elliptic equation

$$\mathrm{div}\, \epsilon(x)\, \mathrm{grad}\, \phi \;=\; f + \mathrm{div}\, \epsilon(x)\underline{E} \;\in\; L^2(B_1)\,.$$

The regularity on the boundary from (2.19) together with elliptic regularity for the inhomogeneous Dirichlet problem [11] yields the borderline regularity $\phi \in H^{3/2}(\Omega \cap B_{1/2})$. In addition with constants independent of the balls,

$$\|\phi\|_{H^{3/2}(\mathcal{O}\cap B_{1/2})} \;\lesssim\; \|\phi\|_{H^1(\partial\mathcal{O}\cap B_1)} \;+\; \|f, h\|_{L^2(\partial\mathcal{O}\cap B_1)}\,.$$

This completes the proof of the Lemma 2.15. $\square$

Given Lemma 2.15, the proof of Theorem 2.14 is completed in the same way as the proof of Theorem 1.1. $\square$

2.3.2 Linearized Euler at velocity zero.

Consider the inviscid compressible Euler equations linearized about a state of constant density and velocity. Computing such a solution it is intelligent to make a galilean transformation moving at the background speed, thus reducing to the case of background speed equal to zero. It is that linearization that we study.

In non dimensionalized form, the linearized equations are

$$\partial_t u \;+\; \operatorname{grad} p \;=\; h\,, \qquad \partial_t p \;+\; \operatorname{div} u \;=\; 0\,. \tag{2.20}$$

We study the case $h = 0$. By Duhamel's principal that is sufficient. The unknown is a $d+1$ vctor (u,p). The system

$$\partial_t \begin{pmatrix} u_1 \\ \vdots \\ u_d \\ p \end{pmatrix} \;+\; \begin{pmatrix} 0 & 0 & \cdots & \partial_1 \\ 0 & 0 & \cdots & \partial_2 \\ \vdots & \vdots & & \vdots \\ \partial_1 & \partial_2 & \cdots & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_d \\ p \end{pmatrix} \;=\; 0$$

is clearly symmetric. This characteristic equation is $(\tau^2 \;-\; |\xi|^2)\,\tau^d \;=\; 0$. For a half plane with unit conormal ν the kernel of the normal matrix $A(\nu)$ consists of the states with $u\cdot\nu = 0 = p$. The nonzero spectral components satisfy

$$\big\|\Pi_+(u,p)\big\|^2 + \big\|\Pi_-(u,p)\big\|^2 \;=\; \big|u\cdot\nu\big|^2 + \big|p\big|^2\,.$$

The strictly dissipative subspaces are those on which

$$\Big(A(\nu)(u,p)\,,\,(u,p)\Big) \;\geq\; c\big(\big|u\cdot\nu\big|^2 + \big|p\big|^2\big), \qquad c>0\,.$$

Taking the curl of the first equation in (2.20) yields

$$\partial_t\big(\operatorname{curl} u\big) \;=\; \operatorname{curl} h\,. \tag{2.21}$$

This is an example of the hidden ellipticity with

$$Q \;:=\; \begin{pmatrix} \operatorname{curl} & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 2.16. *Suppose that for $1 \leq j \leq d$, $\mathcal{N}_j$ is a maximal strictly dissipative subspace for A_j in the Euler system, and $f \in L^2(\mathcal{O})$. Then there is a unique $u \in L^\infty(]0,\infty[\,;\,L^2(\mathcal{O}))$ so that for all $T>0$*

$$\big(u\cdot\nu\,,\,p\big)\big|_{]0,T[\times\partial\mathcal{O}} \;\in\; L^2(]0,T[\times\partial\mathcal{O})$$

satisfying (2.20), the boundary condition $u \in \mathcal{N}_j$ on $\big\{(\mathcal{O}\setminus\mathcal{S})\cap\{x_j=0\}\big\}$, and, the initial condition $(u(0),p(0)) = f$. In addition for any $0 \leq t_1 < t_2 \leq T$ the solution satisfies the energy identity

$$\big\|(u(t),p(t))\big\|^2\Big|_{t_1}^{t_2} \;+\; \int_{]t_1.t_2[\times\partial\mathcal{O}} \big(A(\nu)(u,p)\,,\,(u,p)\big)\;dt\,d\Sigma \;=\; 0\,.$$

Proof. The proof of existence is by rounding the corner and then passing to the limit. The hard part is uniqueness. Need to show that the only solution with $g = h = 0$ is the zero solution. This is proved by showing that the Laplace transform vanishes for all τ with $\mathrm{Re}\,\tau > 0$. Risking confusion the Laplace transform is written (u, p) and satisfies

$$\tau\, u \;+\; \mathrm{grad}\, p \;=\; 0, \qquad \tau\, p \;+\; \mathrm{div}\, u \;=\; 0, \tag{2.22}$$

together with

$$(u, p) \in L^2(\mathcal{O}), \quad (u \cdot \nu\,,\, p) \in L^2(\partial\mathcal{O}), \quad (u, p) \in \mathcal{N}_j \ \text{ on } \ \partial\mathcal{O} \setminus \mathcal{S}. \tag{2.23}$$

Multiplying the equation by (u, p) and integrating by parts would show that $u = p = 0$ because the boundary conditions are dissipative. To prove uniqueness it suffices to justify the integration by parts.

The first step is to take the curl of the first equation in (2.22) to find $\mathrm{curl}\, u = 0$. The integration by parts is justified by showing that u, p belongs to $H^{1/2}(\mathcal{O})$.

Lemma 2.17. *If* $\mathrm{Re}\,\tau > 0$ *and* (u, p) *satisfies* (2.22) *and* (2.23) *then* $(u, p) \in H^{1/2}(\mathcal{O})$.

Proof. Since $\mathrm{grad}\, p = -\tau u \in L^2$, one has $p \in H^1(\mathcal{O})$. It suffices to show that $u \in H^{1/2}(\mathcal{O})$.

Since $\mathrm{curl}\, u = 0$ it follows that there is a $\phi \in \mathcal{D}'(\mathcal{O})$ with $u = \mathrm{grad}\, \phi$ on $\mathcal{O}$. The potential ϕ is unique up to an additive constant. It belongs to the homogeneous Sobolev space $\dot{H}^1(\mathcal{O})$. Taking divergence yields

$$\Delta\, \phi \;=\; \mathrm{div}\ \mathrm{grad}\ \phi \;=\; \mathrm{div}\, u \;=\; -\,\tau\, p \;\in\; H^1(\mathcal{O})\,.$$

On $\partial\mathcal{O}$ one has

$$\frac{\partial \phi}{\partial \nu} \;=\; \nu \cdot \mathrm{grad}\, \phi \;=\; \nu \cdot u \;\in\; L^2(\partial\mathcal{O})\,.$$

Since $\mathrm{grad}\, \phi \in L^2(\mathcal{O})$, regularity for the Neumann problem in the borderline case (see [10]) implies

$$\mathrm{grad}\, \phi \;\in\; H^{1/2}(\mathcal{O}),$$

completing the proof of the Lemma. □

The lemma provides the regularity necessary to justify the integration by parts in the energy identity that yields uniqueness, completing the proof of Theorem 2.16. □

2.3.3 Wave equation written as a system

Reducing the wave equation to a first order differential system in dimension $d > 1$ introduces non physical stationary modes. As for Maxwell, the solutions of interest do not excite these modes.

Treat the case of the wave equation in dimension d. It is converted to a first order system for the $d+1$ derivatives $v_\mu := \partial_\mu u$, $\mu = 0, 1, \dots, d$ where $\partial_0 = \partial_t$. Write the variables as

$$v \ = \ \big(v_0, \mathbf{v}\big)\,.$$

The equations are

$$\partial_t v_j \ = \ \partial_j v_0\,, \quad j = 1, \dots, d\,, \qquad \partial_t v_0 \ = \ \operatorname{div} \mathbf{v}\,.$$

Up to a change of sign these are exactly the linearized Euler equations at velocity zero so are treated in the preceeding section.

2.4 Counterexamples

We construct examples, related to those of Osher [16] and Sarason-Smoller [20]. Our examples include nonuniqueness, ill posedness, and the failure of weak=strong in the sense of Friedrichs for problems that come from gluing boundary conditions at a corner where each condition is strictly dissipative for an $L^2(\mathbb{R}^2)$ scalar product. The L^2 scalar products appropriate for distinct faces of the boundary are distinct but equivalent norms. In Theorem 1.1 the scalar products on adjacent faces are identical.

2.4.1 Counterexamples from two transport equations

The examples are all in dimension $d = 2$ where

$$\mathcal{O} \ := \ \Big\{ x \in \mathbb{R}^2 \,:\, x_1 > 0, \ \text{and } x_2 > 0 \Big\}\,.$$

Introduce the shorthand $\mathbf{U} = (U_1, U_2)$ and $\mathbf{U}\partial_x := U_1 \partial_{x_1} + U_2 \partial_{x_2}$. Consider the pair of transport equations

$$(\partial_t \ + \ \mathbf{U}\partial_x) u \ = \ 0\,, \qquad (\partial_t \ + \ \mathbf{V}\partial_x) v \ = \ 0$$

for the unknown $(u(t,x), v(t,x))$. The vectors $\mathbf{U}$ and $\mathbf{V}$ satisfy

$$U_1 < 0 < U_2\,, \qquad \text{and,} \qquad V_2 < 0 < V_1\,.$$

The boundary segments of $\mathcal{O}$ are non characteristic. The vector field U (resp. V) points into the northwest (resp. southeast) quadrant. Every ray starting in $\mathcal{O}$ reaches the boundary at a point other than the origin after a finite amount of time. Broken ray paths suitably reflected at the boundary meet the boundary infinitely often and never at the origin.

On the x_1 axis, a homogeneous boundary condition $u = \alpha v$ with $\alpha \in \mathbb{C}$ is imposed. On the x_2 axis one imposes $v = \beta u$ with $\beta \in \mathbb{C}$.

The scalars α and β are reflection coefficients. When $|\alpha| < 1$ the reflection on the x axis decreases amplitude. For $|\alpha| > 1$ the reflections amplify. The case $\alpha = 0$ corresponds to perfect absorption. For the L^2 norm

$$\int_\Omega M^2\,|u|^2 \;+\; |v|^2\;dx\,, \qquad M > 0, \tag{2.24}$$

the boundary condition on the x_1-axis is strictly dissipative when $M|\alpha| < 1$ and conservative when $M|\alpha| = 1$. Similarly for the x_2 axis, the norm is nonincreasing when $|\beta| \le M$. Osher [15] observes that there is a norm dissipated at both boundary segments when there is an $M > 0$ with

$$M|\alpha| \le 1, \qquad \text{and} \qquad |\beta| \le M\,.$$

This holds if and only if $|\alpha\beta| \le 1$. In this case it is easy to construct square integrable solutions for square integrable initial data.

Because the equation is so simple with no characteristics emerging from the origin one can prove uniqueness, characteristic by characteristic.

In the opposite case $|\alpha\beta| > 1$ the system can misbehave. Our ray tracing analysis is as in [20].

The case $\mathbf{U} = -\mathbf{V}$. When $\mathbf{U} = -\mathbf{V}$, the values of u are rigidly transported with speed $\mathbf{U}$ till they reach the x_2 axis where their value is multiplied by β

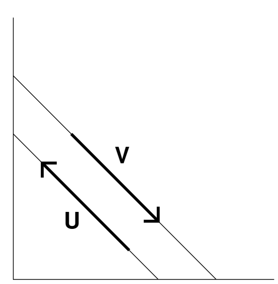

Figure 9: The case $\mathbf{U} = -\mathbf{V}$.

and fed to the V equation where they are transported to the x_1 axis *along the same ray traversed in the opposite direction.* Then they are multiplied by α and fed to the $\mathbf{U}$ equation. And so on.

When $|\alpha\beta| > 1$ one circuit with two reflections leads to an amplification. In the absence of corners, this would cause no problem at all.

For initial data supported in $x_1+x_2 > \delta$ the shortest circuit has length $2\delta\sqrt{2}$ and solutions are amplified by no more than $(\alpha\beta)^{(T/(2\delta\sqrt{2}))}$. One easily shows existence and uniqueness of solutions supported in $\{x_1 + x_2 > \delta\}$.

As one approaches the corner the amplification becomes more and more extreme. Data supported in $2^{-(n+1)} < x_1 + x_2 < 2^{-n}$ is amplified at time $t \sim 1$ by $|\alpha\beta|^{2^n}$. In order for a solution to have finite L^2 norm, the initial data must satisfy

$$\sum_{n\geq 0} |\alpha\beta|^{2^n} \int_{2^{-(n+1)}<x_1+x_2<2^{-n}} |f(x)|^2\, dx \;<\; \infty\,.$$

This is a dense set of data. For f vanishing outside a bounded subset of $\overline{\mathcal{O}}$, the corrsponding sums are finite for all $t > 0$ if and only if

$$\forall\ K > 0\,, \qquad \int_{\mathcal{O}} e^{K/|x|}\ |f(x)|^2\ dx \;<\; \infty\,.$$

Adjoint boundary spaces. For the general operator $L := \partial_t + G(x, \partial)$, denote by $L^\dagger$ the adjoint differential operator with respect to the $L^2([0, T] \times \Omega)$ scalar product. Since A_j has hermitian symmetric values,

$$L^\dagger w \;=\; -\,\partial_t w \;-\; \sum_j \partial_j(A_j(x)w) \;+\; B^* w\,.$$

For u and w in $C^1_{(0)}([0, T] \times \overline{\Omega})$ one has

$$(u(t), w(t))\big|_0^T + \int\limits_{[0,T]\times\Omega} (Lu, w) + (u, L^\dagger w)\ dtdx + \int\limits_{[0,T]\times\partial\Omega} (A(x, \nu(x))u\,,\ w)\ dtd\Sigma = 0.$$

If u satisfies a homogeneous boundary condition $u \in \mathcal{N}(x)$ then the boundary term does not appear if one requires that w satisfy the **adjoint boundary condition** $w \in \mathcal{N}^*(x)$ where

$$\mathcal{N}^*(x) := \big(A(x, \nu(x))\mathcal{N}\big)^\perp := \Big\{w \in \mathbb{C}^N : \forall u \in \mathcal{N}(x),\ \langle w\,,\ A(x, \nu(x))\, u\rangle = 0\Big\}.$$

When $u, w \in H^1([0, T] \times \Omega)$, $u(0) = w(T) = 0$, and $u \in \mathcal{N}$ and $w \in \mathcal{N}^*$ on $]0, T[\times\Omega$, one has

$$(Lu\,,\ w)_{]0,T[\times\Omega} \;+\; (u, L^\dagger w)_{]0,T[\times\Omega} \;=\; 0\,.$$

The adjoint Cauchy problem is solved backward in time.

Example 2.2. *For the pair of transport operators with unknown* $(u, v) \in \mathbb{C}^2$*,*

$$A_1 \;=\; \begin{pmatrix} U_1 & 0 \\ 0 & V_1 \end{pmatrix}.$$

For the boundary condition $u = \alpha v$ *on the* x_1*-axis,*

$$\mathcal{N} := (\alpha, 1)\mathbb{C}, \qquad A_1\mathcal{N} = (U_1\alpha, V_1)\mathbb{C}, \qquad \mathcal{N}^* = (V_1\,,\, -U_1\alpha)\mathbb{C}\,.$$

The adjoint operator $L^\dagger = -L$ *so the transport equations are unchanged. On the* x*-axis one has*

$$u \;=\; \frac{-V_1}{\alpha U_1}\; v$$

so the reflection coefficient for the adjoint problem is $-V_1/\alpha U_1$*.*

When $\mathbf{U} = -\mathbf{V}$ *this simplifies to* $1/\alpha$*. If* $|\alpha| > 1$ *the boundary condition for* L *is amplifying. The adjoint problem has reflection coefficient* $1/\alpha$ *with modulus* < 1*. The adjoint problem is solved with time running backward so this too is amplifying.*

Summary. For the case $\mathbf{U} = -\mathbf{V}$ the direct and adjoint problems glue together problems satisfying the uniform Kreiss-Sakamoto condition. When $|\alpha\beta| > 1$ there is existence only for a dense set of data that are small near the corner. Uniqueness is proved locally in each set $\{2^{-n} < x_1 + x_2 < 2^n\}\,,\; n > 0$.

Nonuniqueness for $\mathbf{U} = (-1, 1)$ **and** $\mathbf{V} = (1, -a)$**,** $1 < a$**.** For these $\mathbf{U}, \mathbf{V}$, an initial ray and its successive reflections are sketched Figure 10. Each cycle of two reflections brings one closer to the origin by a fixed factor

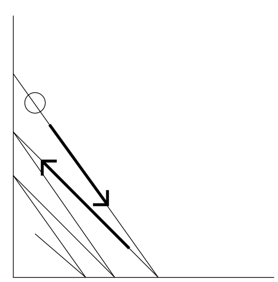

Figure 10: Multiply reflected rays.

< 1. A ray starting in the little disk in the figure approaches the origin in finite time with infinitely many reflections.

Suppose that $\alpha > 0$, $\beta > 0$, and $\alpha\beta < 1$. In each cycle there is decay. Suppose that at $t = 0$, u and v are nonnegative, not identically zero, and supported in the little disk. Consider the value of the solution transported along the broken ray starting at $t = 0$ at the center of the disk in the $\partial_t + \mathbf{V}\partial_x$ direction.

On the initial segment, v is constant. On the reflected ray, the value of u is α times the value of v on the incoming ray. In each cycle of two reflections the value of v is multiplied by $\alpha\beta$. Along the ray the value of v converges to 0 as t increases to T. The value of $|x|$ also converges to zero and the solution is $\leq C|x|$.

An entirely analogous argument shows that the initial values of u when traced forward in time yields wave approaching the origin where they are $O((|x|)$.

The reflecting waves constructed above have nonnegative components and move steadily toward the origin as t increases. Ray by ray they are absorbed at the origin. After the last ray starting in the disk arrives at the origin, say at time $\underline{T}$, there is nothing left. Define a candidate solution to be these values extended by zero at all points that are not reached by the broken rays starting at $t = 0$ in the disk.

Now play the candidate solution backward in time. This backward problem is amplifying in each cycle. The solution just constructed vanishes for large time. And as time decreases, a positive wave emerges from the corner.

It is not hard to show that the resulting candidate is indeed a weak solution of the boundary value problem. It is an example of non uniqueness.

The boundary conditions, though amplifying, both satisfy the uniform Kreiss-Sakamoto condition. Each is strictly dissipative with respect to a scalar product equivalent to that in L^2. The scalar products associated to the different boundary edges are different.

Summary. Gluing two boundary conditions together at a corner can yield nonuniqueness. A non zero wave can emerge from the corner. Even when each individual condition is strictly dissipative with respect to an L^2 scalar product of the form $\int M|u|^2 + |v|^2 \, dx$ with different M for the two conditions.

The existence and uniqueness theorem that we proved differs from this example in two respects. First the generator is elliptic. Second, the boundary conditions are strictly dissipative *with respect to the same scalar product.* The counterexample satisfies the hypothesis of square integrable traces on the boundaries.

2.4.2 Elliptic counterexamples.

The transport counterexample violates two hypotheses, ellipticity and strict dissipativity *with respect to a fixed scalar product*. We next sketch an elliptic version.

Let

$$G \ := \ A_1\partial_1 + A_2\partial_2\,, \qquad A_1 \ := \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad A_2 \ := \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

Modify the equations to be

$$(\partial_t \ + \ \mathbf{U}\partial_x + \mu G)u \ = \ 0\,, \qquad (\partial_t \ - \ \mathbf{U}\partial_x + \mu G)v \ = \ 0\,, \qquad 0 < \mu \ll 1,$$

where now both u and v are $\mathbb{C}^2$ valued. The characteristic variety is given by

$$\Big((\tau + \mathbf{U}\cdot\xi)^2 - \mu|\xi|^2\Big)\Big((\tau - \mathbf{U}\cdot\xi)^2 - \mu|\xi|^2\Big) \ = \ 0\,.$$

The boundary conditions are as before.

When $|\alpha\beta| > 1$, the problem is strongly explosive as one shows by geometric optics. The convenient choice is to take ξ parallel to $\mathbf{U}$ in which case the transport directions of geometric optics are parallel to $\mathbf{U}$. The leftward going wave will have a value $\tau_{\rm left}$ and the rightward waves a $\tau_{\rm right}$. The transport equations of geometric optics are exactly the equations of the Osher example. They are explosive.

Prove ill posedness as follows. Given a challenge number M, choose a ray and $T > 0$ so that the transport equation of geometric optics yields amplification $> M$ at time T. Choose a very small disk so that along this ray the translated disks and their successive reflections do not meet the corner. Then choose an initial amplitude function supported in the disk and consider the approximate solutions in the limit of wavelength tending to zero to find solutions whose L^2 norm is amplified by more than M.

Summary. Two strictly dissipative Kreiss well posed conditions glued together at a corner yield an ill posed problem. The problems are strictly dissipative for **different** but equivalent L^2 norm.

3 Part II. Trihedral Internal Maxwell Bérenger Corner

The set Ω comprised of eight octants $\mathcal{O}$ and $\mathcal{O}_\kappa$ is defined in Definition 1.3 page 8 and sketched in Figure 3 page 3.

The nonnegative absorptions $0 \le \sigma_j(x_j)$ are uniformly bounded, $\sigma_j \in L^\infty(\mathbb{R})$. The original absorption coefficients of Bérenger were chosen as heaviside functions. In dimension $d = 3$ the present paper is the first demonstrating that the Bérenger split Maxwell equations are well posed for absorptions less regular than C^2.

3.1 Splitting Maxwell

3.1.1 Vector calculus

Denote by $\mathbf{e}_1 := (1,0,0)$, $\mathbf{e}_2 := (0,1,0)$, and $\mathbf{e}_3 := (0,0,1)$ the standard basis elements of $\mathbb{C}^3$. Denote by $p_j : \mathbb{C}^3 \to \mathbb{C}^1$ the linear transformation $p_j(v_1, v_2, v_3) \ := \ v_j$. If π_j denotes the orthogonal projection on the j^{th} coordinate axis, then $\pi_j v = (p_j v)\mathbf{e}_j$. The 1×3 matrices of p_j are

$$p_1 := \ (1,0,0), \qquad p_2 \ := \ (0,1,0), \qquad p_3 \ := \ (0,0,1).$$

Then,

$$\operatorname{div} \ = \ (\partial_1, \partial_2, \partial_3) \ = \ \sum p_j\, \partial_j\,.$$

The C_j from (1.5), (1.6) satisfy

$$C_1^2 \ := \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_2^2 \ := \ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_3^2 \ := \ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_j^* = -C_j\,, \qquad \big(C_i C_j\big)^* \ = \ C_j C_i\,,$$

and,

$$C_1 C_2 \ = \ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Denote by $m_{i,j} \in \operatorname{Hom}(\mathbb{C}^3)$ the linear transformation whose matrix has a 1 in the i,j position and zeroes elsewhere. Then

$$C_1 C_2 \ = \ m_{2,1}\,.$$

More generally

$$\text{for } i \neq j, \qquad C_i C_j \ = \ m_{j,i}\,.$$

Expand

$$\operatorname{div}\operatorname{curl} \ = \ \sum_j p_j \partial_j \ \sum_k C_k \partial_k \ = \ \sum_{j,k} p_j\, C_k\, \partial_j \partial_k\,.$$

The identity $0 = \operatorname{div}\operatorname{curl}$ is equivalent to the matrix identity

$$p_j C_k \ + \ p_k C_j \ = \ 0 \quad \text{for all } \ j,k\,. \tag{3.1}$$

The Laplace transform of the split Bérenger Maxwell equations (1.7) is

$$\begin{aligned} \epsilon(\tau + \sigma_j(x_j))\widehat{E}^j \ &= \ C_j \partial_j \sum \widehat{B}^k \ , \\ \mu(\tau + \sigma_j(x_j))\widehat{B}^j \ &= \ -\, C_j \partial_j \sum \widehat{E}^k\,. \end{aligned} \qquad \text{for } \ j = 1,2,3. \tag{3.2}$$

Mulitply by $\tau/(\tau+\sigma_j)$ to find for $j=1,2,3$

$$\epsilon\,\tau\,\widehat{E}^j = \frac{\tau}{\tau+\sigma_j}\ C_j\partial_j \sum \widehat{B}^k, \qquad \mu\,\tau\,\widehat{B}^j = -\ \frac{\tau}{\tau+\sigma_j}\ C_j\partial_j \sum \widehat{E}^k\,. \tag{3.3}$$

Care must be taken to keep the factors in this order since the absorptions σ_j depend on x_j so do not commute with ∂_j.
Introduce the total fields defined in (1.8) to find on $\Omega\setminus\mathcal{O}$

$$\epsilon\,\tau\,\widehat{E}^j \ = \ \frac{\tau}{\tau\ +\ \sigma_j}\ C_j\partial_j\widehat{B}, \qquad \mu\,\tau\widehat{B}^j \ = \ -\frac{\tau}{\tau\ +\ \sigma_j}\ C_j\partial_j\widehat{E}\,. \tag{3.4}$$

Summing on j yields on $\Omega\setminus\mathcal{O}$

$$\epsilon\,\tau\,\widehat{E} \ = \ \sum_j \frac{\tau}{\tau+\sigma_j}\ C_j\partial_j\widehat{B}, \qquad \mu\,\tau\,\widehat{B} \ = \ -\sum_j \frac{\tau}{\tau+\sigma_j}\ C_j\partial_j\widehat{E}\,.$$

Including the Maxwell equation on $\mathcal{O}$ yields everywhere in Ω

$$\tau\,\epsilon\,\widehat{E} \ = \ \sum_j \frac{\tau}{\tau+\sigma_j} C_j\partial_j\,\widehat{B} + 4\pi\,\widehat{\mathbf{j}}, \quad \tau\,\mu\,\widehat{B} = -\sum_j \frac{\tau}{\tau+\sigma_j}\ C_j\partial_j\,\widehat{E}. \tag{3.5}$$

One can recover $\widehat{E}^j, \widehat{B}^j$ in $\mathcal{O}_\kappa$ from the values of $\widehat{E}, \widehat{B}$ in $\mathcal{O}_\kappa$ using (3.4).
This is the basic system of equations satisfied by $\widehat{E}, \widehat{B}$ that are, for each τ, functions of x. The next sections analyse the sytem of equations applied to arbitrary functions of x, returning to the Laplace transforms in 3.4.

3.1.2 Tilde operators

Introduce operators so that equations (3.5), in the domains $\mathcal{O}_\kappa$ resemble Maxwell's equations.

Definition 3.1. *In each of the eight octants define differential operators*

$$\begin{aligned}
\widetilde{\mathrm{div}} \ &:= \sum_j \frac{\tau}{\tau+\sigma_j}\ p_j\ \partial_j\,, \qquad \widetilde{\mathrm{curl}} \ := \sum_j \ \frac{\tau}{\tau+\sigma_j}\ C_j\partial_j\,,\\
\widetilde{\mathrm{grad}}\,\phi \ &:= \ \Big(\frac{\tau}{\tau+\sigma_1}\partial_1\phi\,,\ \frac{\tau}{\tau+\sigma_2}\partial_2\phi\,,\ \frac{\tau}{\tau+\sigma_3}\partial_3\phi\Big)\,,\\
\widetilde{\Delta} \ &:= \ \widetilde{\mathrm{div}}\,\widetilde{\mathrm{grad}} \ = \ \sum_j \frac{\tau}{\tau+\sigma_j}\ \partial_j\ \Big(\frac{\tau}{\tau+\sigma_j}\ \partial_j\ \phi\Big)\,.
\end{aligned}$$

Lemma 3.1. *The tilde operators satisfy in each octant*

$$\widetilde{\mathrm{div}}\ \widetilde{\mathrm{curl}} \ = \ 0, \qquad \widetilde{\mathrm{grad}}\ \widetilde{\mathrm{div}} \ - \ \widetilde{\mathrm{curl}}\ \widetilde{\mathrm{curl}} \ = \ \begin{pmatrix} \widetilde{\Delta} & 0 & 0\\ 0 & \widetilde{\Delta} & 0\\ 0 & 0 & \widetilde{\Delta}\end{pmatrix}\,.$$

Proof. For the first identity compute

$$\begin{aligned}\widetilde{\operatorname{div}}\,\widetilde{\operatorname{curl}} &= \sum_k \frac{\tau}{\tau+\sigma_k}\,p_k\partial_k\Big(\sum_j \frac{\tau}{\tau+\sigma_j}\,C_j\partial_j\Big)\\ &= \sum_{j,k} \frac{\tau}{\tau+\sigma_k}\,p_k\partial_k\,\Big(\frac{\tau}{\tau+\sigma_j}\,C_j\partial_j\Big)\,.\end{aligned}$$

Expand

$$\partial_k\,\Big(\frac{\tau}{\tau+\sigma_j}\,C_j\partial_j\Big) \;=\; \partial_k\,\Big(\frac{\tau}{\tau+\sigma_j}\Big)\,C_j\partial_j \;+\; \frac{\tau}{\tau+\sigma_j}\,C_j\,\partial_k\partial_j.$$

The first term vanishes except when $k=j$. The sum of the resulting contributions to the divergence vanishes since it is equal to

$$\sum_j \frac{\tau}{\tau+\sigma_j}\,\partial_j\Big(\frac{\tau}{\tau+\sigma_j}\Big)\,p_jC_j\partial_j$$

and $p_jC_j = 0$ by (3.1). The sum of the remaining contributions to the divergence is equal to

$$\begin{aligned}\sum_{j,k}\frac{\tau}{\tau+\sigma_k}\,\frac{\tau}{\tau+\sigma_j}\,p_k\,C_j\partial_k\partial_j &= \sum_j\,\Big(\frac{\tau}{\tau+\sigma_j}\,\Big)^2 p_j\,C_j\partial_j^2\\ &\quad+\sum_{j\neq k}\frac{\tau}{\tau+\sigma_k}\,\frac{\tau}{\tau+\sigma_j}\,(p_k\,C_j+p_j\,C_k)\partial_k\partial_j.\end{aligned}$$

This vanishes since $p_jC_j=0$ and for $k\neq j$, $p_kC_j+p_jC_k=0$ completing the proof of the first identity.

For the second identity compute the first component of the left hand side applied to B to find

$$\begin{aligned}&\frac{\tau}{\tau+\sigma_1}\partial_1\left(\frac{\tau}{\tau+\sigma_1}\partial_1B_1+\frac{\tau}{\tau+\sigma_2}\partial_2B_2+\frac{\tau}{\tau+\sigma_3}\partial_3B_3\right)\\ &\qquad-\frac{\tau}{\tau+\sigma_2}\partial_2\left(\frac{\tau}{\tau+\sigma_1}\partial_1B_2-\frac{\tau}{\tau+\sigma_2}\partial_2B_1\right)\\ &\qquad\qquad+\frac{\tau}{\tau+\sigma_3}\partial_3\left(\frac{\tau}{\tau+\sigma_3}\partial_3B_1-\frac{\tau}{\tau+\sigma_1}\partial_1B_3\right)\,.\end{aligned}$$

Rearrange to find

$$\begin{aligned}&\frac{\tau}{\tau+\sigma_1}\partial_1\left(\frac{\tau}{\tau+\sigma_1}\partial_1B_1\right)+\frac{\tau}{\tau+\sigma_2}\partial_2\left(\frac{\tau}{\tau+\sigma_2}\partial_2B_1\right)+\frac{\tau}{\tau+\sigma_3}\partial_3\left(\frac{\tau}{\tau+\sigma_3}\partial_3B_1\right)\\ &\qquad+\frac{\tau}{\tau+\sigma_1}\partial_1\left(\frac{\tau}{\tau+\sigma_2}\partial_2B_2\right)-\frac{\tau}{\tau+\sigma_2}\partial_2\left(\frac{\tau}{\tau+\sigma_1}\partial_1B_2\right)\\ &\qquad\qquad+\frac{\tau}{\tau+\sigma_1}\partial_1\left(\frac{\tau}{\tau+\sigma_3}\partial_3B_3\right)-\frac{\tau}{\tau+\sigma_3}\partial_3\left(\frac{\tau}{\tau+\sigma_1}\partial_1B_3\right)\,.\end{aligned}$$

Since σ_j depends on x_j only, the second and third lines vanish. The first line is the tilde laplacian of B_1. The other components are similar. □

The permittivities are scalar outside K. In particular wherever there are boundaries. A partition of unity serves to separate the regions where the permitivities are not scalar from the rest.

Definition 3.2. *For*

$$U \ := \ (E, B) \qquad \text{define} \quad \widetilde{\operatorname{div}}\, U \ := \ (\widetilde{\operatorname{div}}\, E, \widetilde{\operatorname{div}}\, B),$$

and

$$LU \ := \ \Big(\epsilon\tau E - \widetilde{\operatorname{curl}}\, B\,,\ \mu\tau B + \widetilde{\operatorname{curl}}\, E\Big)\,.$$

Remark 3.1. *In $\mathcal{O}$ the absorptions vanish and this is Maxwell equations.*

3.2 Estimates in $\overline{\mathcal{O}}$ and in $\mathcal{B}$

The function U is split as $U_1 + U_2$ with U_1 supported in $\overline{\mathcal{O}}$ and U_2 supported where the permittivites ϵ, μ are scalar that is in $\mathcal{B} := \mathbb{R}^3 \setminus K$.

The estimates for U_1 are elementary estimates for symmetric hyperbolic systems that are true in much greater generality [19]. The key point is that they take place where there absorptions vanish identically. The statement and sketch of proof follow.

Proposition 3.2. *There are constants $C, M > 0$ so that if $\operatorname{Re}\tau > M$, $U = (E, B) \in H^1(\mathbb{R}^3)$, $\operatorname{supp} U \subset \overline{\mathcal{O}}$, and, $LU \in H^1(\mathbb{R}^3)$ then*

$$\operatorname{Re}\tau\, \|U\|_{H^1(\mathbb{R}^3)} \ \le \ C\, \|LU\|_{H^1(\mathbb{R}^3)}\,. \tag{3.6}$$

Remark 3.2. *Estimates (3.6) together with $\tau\,(\operatorname{div}\epsilon E\,,\, \operatorname{div}\mu B) = \operatorname{div} LU$ valid in $\mathcal{O}$ yield*

$$\|\tau\,(\operatorname{div}\epsilon E\,,\, \operatorname{div}\mu B)\|_{L^2(\mathbb{R}^3)} = \big\|\operatorname{div} LU\big\|_{L^2(\mathbb{R}^3)}. \tag{3.7}$$

The differential equation yields the additional estimate

$$\|\tau\, U\|_{L^2(\mathbb{R}^3)} \ \lesssim \ \|\nabla U\|_{L^2(\mathbb{R}^3)} \ + \ \|LU\|_{L^2(\mathbb{R}^3)}\,, \quad s = 0, 1\,. \tag{3.8}$$

Remark 3.3. i. *The factor τ is the Laplace transform of ∂_t and is on an even footing with ∇_x.*

Sketch of proof of Proposition 3.2. Taking the scalar product of the first equation with E and the second with B, and integrating over $\mathbb{R}^3$ yields after an integration by parts

$$\operatorname{Re}\tau \int \langle \epsilon E, E\rangle \ + \ \langle \mu B, B\rangle \ dx \ = \ \operatorname{Re} \int \langle U\,,\, LU\rangle \ dx\,.$$

Using the Cauchy-Schwartz inequality on the right yields

$$\operatorname{Re}\tau\,\|U\|_{L^2(\mathbb{R}^3)} \;\lesssim\; \|LU\|_{L^2(\mathbb{R}^3)}$$

Assuming that $U \in H^2(\mathbb{R}^3)$, differentiating the equations with respect to x_j repeating the argument and summing on j yields the derivative estimate

$$\operatorname{Re}\tau\,\|U\|_{H^1(\mathbb{R}^3)} \;\lesssim\; \|LU\|_{H^1(\mathbb{R}^3)}\,.$$

For $U \in H^1$ with $LU \in H^1$ the proof is completed by applying the estimate valid for H^2 functions to $U^\epsilon := j_\epsilon * U$ a Friedrichs mollification of U with j_ϵ supported in $\mathcal{O}$ to find

$$\operatorname{Re}\tau\,\|U^\epsilon\|_{H^1(\mathbb{R}^3)} \;\lesssim\; \|LU^\epsilon\|_{H^1(\mathbb{R}^3)}\,.$$

The norm on the left hand side converges to $\|U\|_{H^1(\mathbb{R}^3)}$. Friedrichs' lemma ([9] Lemma 17.1.5) implies that the norm on the right converges to $\|LU\|_{H^1(\mathbb{R}^3)}$. ☐

The second main estimate concerns functions supported in the region where the permittivities are scalar and includes the regions where the absorptions are non zero. The estimate is weaker and much harder to prove. The proof uses a well adapted complex Helmholtz equation. When $LU \in H^1(\mathbb{R}^3)$, $\widetilde{\operatorname{div}}\,LU$ is a well defined element of $L^2(\mathbb{R}^3)$. The hypothesis $\widetilde{\operatorname{div}}\,LU \in H^1(\mathbb{R}^3)$ in the next proposition means that this element of $L^2(\mathbb{R}^3)$ belongs to $H^1(\mathbb{R}^3)$.

Proposition 3.3. *There are constants $C, M > 0$ so that for $\operatorname{Re}\tau > M$ and $U \in H^1(\mathbb{R}^3)$ with $LU \in H^1(\mathbb{R}^3)$, $\widetilde{\operatorname{div}}\,LU \in H^1(\mathbb{R}^3)$ and $\operatorname{supp} U$ contained in $\overline{\mathcal{B}}$, one has*

$$\begin{aligned}&\operatorname{Re}\tau\,\|U\|_{L^2(\mathbb{R}^3)} + \|\nabla U\|_{L^2(\mathbb{R}^3)}\\ &\qquad\le C\Big(\|LU\|_{H^1(\mathbb{R}^3)} + \|\tau LU\|_{L^2(\mathbb{R}^3)} + \|\frac{1}{\tau}\,\widetilde{\operatorname{div}}\,LU\|_{H^1(\mathbb{R}^3)}\Big).\end{aligned} \tag{3.9}$$

In addition,

$$\|\tau\,\widetilde{\operatorname{div}}\,U\|_{H^1(\mathbb{R}^3)} \;\le\; \|\widetilde{\operatorname{div}}\,LU\|_{H^1(\mathbb{R}^3)}\,, \tag{3.10}$$

and

$$\|\tau\,U\|_{L^2(\mathbb{R}^3)} \;\lesssim\; \|\nabla U\|_{L^2(\mathbb{R}^3)} \;+\; \|LU\|_{L^2(\mathbb{R}^3)}\,. \tag{3.11}$$

Remark 3.4. i. *Estimate (3.9) is unbalanced because the $\|\widetilde{\operatorname{div}}\,LU\|_{H^1}$ is a second derivative estimate on LU and there is no second derivative on the left. However, (3.10) shows that the second derivatives $\|\widetilde{\operatorname{div}}\,U\|_{H^1}$ are also bounded by the righthand side. This balances the $\|\widetilde{\operatorname{div}}\,LU\|_{H^1}$ term.*

ii. *Similarly, (3.11) shows that one can add $\|\tau U\|_{L^2}$ to the left hand side of (3.9) balancing the $\|\tau LU\|_{L^2}$ term.*

iii. *The estimates depend only on the L^∞ norms of the absorptions. In particular they are uniform on families of absorptions that are bounded in L^∞.*

The proof of this Proposition occupies the next subsubsections.

3.2.1 A well adapted Helmholtz operator

This section makes a delicate selection of a Helmholtz operator for L acting on functions U with $\operatorname{supp} U \subset \mathcal{B}$ where the permittivities are scalar. This domain includes all the points where the absorptions are non zero. The result is an equation that not only holds in Ω but on the whole of $\mathcal{B}$. A typical selection would yield a second order operator that when applied to the unknowns would produce delta functions on the boundary surfaces of Ω where U and ∇U may be discontinuous. The good choice does not. The first step is to identify quantities that are continuous across the smooth parts of $\partial\Omega$.

Consider the Laplace transformed Maxwell equations,

$$LU \ := \ \big(\epsilon\tau E - \widetilde{\operatorname{curl}}\, B\,,\ \mu\tau B + \widetilde{\operatorname{curl}}\, E\big) \ = (\Phi, \Psi). \tag{3.12}$$

The transmission conditions demand that the tangential components of $\sum E^k$ and $\sum B^k$ are continuous across the smooth parts of the interfaces $\{x_j = 0\,,\ x_k \neq 0 \text{ for } k \neq j\}$

The face $x_1 = 0$ of $\partial\Omega$ consists of the two dimensional smooth stratum where $x_2 \neq 0 \neq x_3$, one dimensional edges where exactly two coordinates vanish, and the origin. At the regular part $\{x_1 = 0, x_2 \neq 0 \neq x_3\}$, $B \times n$ and $E \times n$ are continuous across the boundary thanks to the transmission condition. This means that the tangential components E_2, E_3, B_2, B_3 are continuous across that part of the boundary. Using square brackets, $[\ \cdot\ ]$, to denote the jump,

$$\Big[E_2, E_3, B_2, B_3\Big] \ = \ 0 \quad \text{across} \quad \{x_1 = 0,\ x_2 \neq 0 \neq x_3\}\,.$$

Taking tangential derivatives implies that

$$j \geq 2 \ \Rightarrow\ \Big[\partial_j E_2, \partial_j E_3, \partial_j B_2, \partial_j B_3\Big] \ = \ 0 \quad \text{across} \quad \{x_1 = 0,\ x_2 \neq 0 \neq x_3\}\,.$$

On $\{x_1 = 0,\ x_2 \neq 0 \neq x_3\}$ the factors σ_2 and σ_3 are smooth. Furthermore the coefficients ϵ and μ are C^2 and scalar inside $\mathcal{B}$. For $k \geq 2$ the function σ_k is smooth across $\{x_1 = 0,\ x_2 \neq 0 \neq x_3\}$ so

$$j, k \geq 2 \ \Rightarrow\ \Big[\frac{\tau}{\tau + \sigma_k}\partial_j\{\epsilon E_j, \mu B_j\}\Big] = 0 \quad \text{across} \quad \{x_1 = 0,\ x_2 \neq 0 \neq x_3\}. \tag{3.13}$$

The case $j = k$ is important below. The source terms (Φ, Ψ) are compactly supported in $\overline{\omega}$. Thus in a neighborhood of $\partial\Omega$,

$$\begin{aligned} \frac{\tau}{\tau+\sigma_1}\,\partial_1(\epsilon E_1) \ + \ \frac{\tau}{\tau+\sigma_2}\,\partial_2(\epsilon E_2) \ + \ \frac{\tau}{\tau+\sigma_3}\,\partial_3(\epsilon E_3) \ &= \ 0\,, \\ \frac{\tau}{\tau+\sigma_1}\,\partial_1(\mu B_1) \ + \ \frac{\tau}{\tau+\sigma_2}\,\partial_2(\mu B_2) \ + \ \frac{\tau}{\tau+\sigma_3}\,\partial_3(\mu B_3) \ &= \ 0\,. \end{aligned} \tag{3.14}$$

In particular the first terms on the left have well defined traces in $H^{-1/2}(\{x_1 = 0\} \setminus S)$. Equation (3.13) and (3.14) yield

$$\Big[\frac{\tau}{\tau+\sigma_1}\partial_1\{\epsilon E_1\,,\,\mu B_1\}\Big] \;=\; 0 \quad \text{across} \quad \{x_1=0,\ x_2\neq 0\neq x_3\}\,. \tag{3.15}$$

For the second and third components of $\partial_1 E, \partial_1 B$ analyse (3.12). The second and third lines express

$$\frac{\tau}{\tau+\sigma_1}\big\{\partial_1 B_2\,,\,\partial_1 B_3\big\} \qquad \Big(\text{resp.}\quad \frac{\tau}{\tau+\sigma_1}\big\{\partial_1 E_2\,,\,\partial_1 E_3\big\}\Big) \tag{3.16}$$

as sums of functions continuous across the smooth parts of the boundary face $x_1 = 0$ and their tangential derivatives. Therefore

$$\Big[\frac{\tau}{\tau+\sigma_1}\,\partial_1\big\{E_2, E_3, B_2, B_3\big\}\Big] = 0 \;\text{ across }\; \{x_1 = 0, x_2\neq 0\neq x_3\}. \tag{3.17}$$

Together with (3.15), this yields

$$\Big[\frac{\tau}{\tau+\sigma_1}\;\partial_1\{\epsilon E, \mu B\}\Big] \;=\; 0 \quad \text{across} \quad \big\{x_1 = 0\,,\ x_2\neq 0\neq x_3\big\}\,.$$

By symmetry,

$$\Big[\frac{\tau}{\tau+\sigma_j}\;\partial_j\{\epsilon E, \mu B\}\Big] \;=\; 0 \quad \text{across} \quad \big\{x_j = 0\,,\ x_k\neq 0\ \text{ for } k\neq j\big\}\,. \tag{3.18}$$

Next derive a Helmholtz equation that is not well adapted. Multiplying (3.12) by $\tau\mu$ yields

$$\begin{aligned}
\epsilon\mu\,\tau^2 E = \;& \mu\tau(\Phi + \widetilde{\operatorname{curl}}\ B) \;=\; \mu\tau\Phi + \widetilde{\operatorname{curl}}\ (\mu\tau B) - \widetilde{\operatorname{grad}}\,\mu\wedge(\tau B)\\
=\;& \mu\tau\Phi + \widetilde{\operatorname{curl}}\ (\Psi - \widetilde{\operatorname{curl}}\,E) - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge(\Psi - \widetilde{\operatorname{curl}}\,E)\\
=\;& -\widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}}\,E + \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\widetilde{\operatorname{curl}}\,E + \mu\tau\Phi + \widetilde{\operatorname{curl}}\ \Psi - \frac{\widetilde{\operatorname{grad}}\,\mu}{\mu}\wedge\Psi.
\end{aligned}$$

Expand $\widetilde{\operatorname{curl}}\ \widetilde{\operatorname{curl}}\ E = -\widetilde{\operatorname{div}}\,\widetilde{\operatorname{grad}}\ E + \widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\ E$. Introduce $\epsilon\,E$ in the previous quantities to find

$$\begin{aligned}
\widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\ E \;=\; \widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\ \frac{\epsilon E}{\epsilon} \;&=\; \widetilde{\operatorname{grad}}\,(\widetilde{\operatorname{grad}}\,(\frac{1}{\epsilon})\cdot\epsilon E + \frac{1}{\epsilon}\widetilde{\operatorname{div}}\,(\epsilon E))\\
&=\; -\widetilde{\operatorname{grad}}\,(\frac{\widetilde{\operatorname{grad}}\,\epsilon}{\epsilon}\;\cdot E) + \widetilde{\operatorname{grad}}\,(\frac{1}{\epsilon\tau}\widetilde{\operatorname{div}}\ \Phi)\,,
\end{aligned}$$

$$\widetilde{\operatorname{div}}\,\widetilde{\operatorname{grad}}\ E_j \;=\; \widetilde{\operatorname{div}}\,\widetilde{\operatorname{grad}}\ \frac{\epsilon E_j}{\epsilon} \;=\; \widetilde{\operatorname{div}}\,(\frac{1}{\epsilon}\widetilde{\operatorname{grad}}\,(\epsilon E_j)) - \widetilde{\operatorname{div}}\,(\frac{\widetilde{\operatorname{grad}}\,\epsilon}{\epsilon}\;E_j)\,,$$

$$
\begin{aligned}
\epsilon\mu\,\tau^2 E\ &= -\widetilde{\mathrm{curl}}\,\widetilde{\mathrm{curl}}\,E + \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\widetilde{\mathrm{curl}}\,E + \mu\tau\Phi + \widetilde{\mathrm{curl}}\ \Psi - \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\Psi \\
&= \ \widetilde{\mathrm{div}}\,(\frac{1}{\epsilon}\widetilde{\mathrm{grad}}\,(\epsilon\{E_j\})) - \widetilde{\mathrm{div}}\,(\frac{\widetilde{\mathrm{grad}}\,\epsilon}{\epsilon}\,\{E_j\}) \\
&\quad + \widetilde{\mathrm{grad}}\,(\frac{\widetilde{\mathrm{grad}}\,\epsilon}{\epsilon}\cdot E) - \widetilde{\mathrm{grad}}\,(\frac{1}{\epsilon\tau}\widetilde{\mathrm{div}}\ \Phi) \\
&\quad + \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\widetilde{\mathrm{curl}}\,E + \mu\tau\Phi + \widetilde{\mathrm{curl}}\ \Psi - \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\Psi\,.
\end{aligned}
$$

This gives the system of wave equations, scalar in its prinicipal part,

$$
\epsilon\mu\,\tau^2 E\ -\ \widetilde{\mathrm{div}}\,(\frac{1}{\epsilon}\widetilde{\mathrm{grad}}\,(\epsilon E))\ +\ \mathcal{L}_E E\ =\ \Phi_E, \tag{3.19}
$$

with

$$
\mathcal{L}_E E = \{\widetilde{\mathrm{div}}\,(\frac{\widetilde{\mathrm{grad}}\,\epsilon}{\epsilon}\,E_j)\}_j - \widetilde{\mathrm{grad}}\,(\frac{\widetilde{\mathrm{grad}}\,\epsilon}{\epsilon}\cdot E) - \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\widetilde{\mathrm{curl}}\,E, \tag{3.20}
$$

$$
\Phi_E = -\widetilde{\mathrm{grad}}\,(\frac{1}{\epsilon\tau}\widetilde{\mathrm{div}}\ \Phi) + \mu\tau\Phi + \widetilde{\mathrm{curl}}\ \Psi - \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\Psi. \tag{3.21}
$$

The magnetic field satisfies the similar equation

$$
\epsilon\mu\,\tau^2 B - \widetilde{\mathrm{div}}\,(\frac{1}{\mu}\widetilde{\mathrm{grad}}\,(\mu B)) + \mathcal{L}_B B = \Phi_B, \tag{3.22}
$$

with

$$
\mathcal{L}_B B = \{\widetilde{\mathrm{div}}\,(\frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\,B_j)\} - \widetilde{\mathrm{grad}}\,(\frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\cdot B) - \frac{\widetilde{\mathrm{grad}}\,\epsilon}{\epsilon}\wedge\widetilde{\mathrm{curl}}\,B, \tag{3.23}
$$

$$
\Phi_B = -\widetilde{\mathrm{grad}}\,(\frac{1}{\mu\tau}\widetilde{\mathrm{div}}\ \Psi) + \epsilon\tau\Psi - \widetilde{\mathrm{curl}}\ \Phi + \frac{\widetilde{\mathrm{grad}}\,\epsilon}{\epsilon}\wedge\Phi. \tag{3.24}
$$

Equations (3.19) and (3.22) hold in each of the octants. If one computes for instance $\mathcal{L}_B B$ in all of $\mathbb{R}^3$ there will be delta functions that appear at the boundaries because of jumps in the first derivative of B. The well adapted equation will have ∂_j derivatives applied only to quantities that are continuous across $x_j = 0$. The continuity is guaranteed by the vanishing jumps that come from the transmission condition.

Multiplying the tilde Helmholtz equation (3.19) by $(\tau+\sigma_1)(\tau+\sigma_2)(\tau+\sigma_3)/\tau^3$ yields the equivalent equation in the octants

$$
P_E(\tau, x, \partial_1, \partial_2\,, \partial_3)\,E\ =\ \frac{(\tau+\sigma_1)(\tau+\sigma_2)(\tau+\sigma_3)}{\tau^3}\,\Phi_E\,. \tag{3.25}
$$

The reduced wave operator P is defined by

$$
P_E := \epsilon\mu\,\frac{\prod_j(\tau+\sigma_j(x_j))}{\tau} - p_E\ + \ell_E\,.
$$

The elliptic opertor p_E is defined, with the understanding that all indices are taken modulo 3, by

$$p_E(E) := \sum_j \partial_j \, \frac{1}{\epsilon} \frac{(\tau+\sigma_{j+1})(\tau+\sigma_{j+2})}{\tau(\tau+\sigma_j)} \, \partial_j(\epsilon E)\,. \tag{3.26}$$

The j^{th} component of the lower order term becomes, with $k = j+1$ and $l = j+2$ modulo 3,

$$\begin{aligned}(\ell_E E)_j =& \partial_k \left(\frac{(\tau+\sigma_j)(\tau+\sigma_l)}{\tau(\tau+\sigma_k)} \frac{\partial_k \epsilon}{\epsilon} E_j \right) + \partial_l \left(\frac{(\tau+\sigma_j)(\tau+\sigma_k)}{\tau(\tau+\sigma_l)} \frac{\partial_l \epsilon}{\epsilon} E_j \right) \\ &- \partial_j \left(\frac{(\tau+\sigma_l)}{\tau} \frac{\partial_k \epsilon}{\epsilon} E_k \right) - \partial_j \left(\frac{(\tau+\sigma_k)}{\tau} \frac{\partial_l \epsilon}{\epsilon} E_l \right) \\ &- \frac{(\tau+\sigma_l)(\tau+\sigma_j)}{\tau^2} \, \frac{\partial_k \mu}{\mu} \, E_l + \frac{(\tau+\sigma_k)(\tau+\sigma_j)}{\tau^2} \, \frac{\partial_l \mu}{\mu} \, E_k.\end{aligned} \tag{3.27}$$

The operators acting on B are defined similarly. Our well-adapted operator acting on $U = (E, B)$ is defined by

$$\begin{aligned} P \;&:=\; \epsilon\mu \, \frac{\prod_j (\tau+\sigma_j(x_j))}{\tau} - p + \ell, \quad \text{with} \\ p(E,B) &= (p_E\, E, p_B\, B), \qquad \ell(E,B) = (\ell_E\, E, \ell_B\, B).\end{aligned} \tag{3.28}$$

Proposition 3.4. **i.** *p maps $H^1(\mathbb{R}^3)$ to $H^{-1}(\mathbb{R}^3)$.*
ii. *If $u = (E, B) \in \mathcal{D}'(\Omega)$ satisfies (3.5,3.12) with data as in Proposition 3.3 supported in $\overline{\omega}$, then*

$$P\,u \;-\; \frac{(\tau+\sigma_1(x_1))\,(\tau+\sigma_2(x_2))(\tau+\sigma_3(x_3))}{\tau^2} \; \Phi \;=\; 0 \qquad \text{on} \quad \Omega \tag{3.29}$$

with $\Phi = (\Phi_E, \Phi_B) \in L^2(\mathbb{R}^3)$.
iii. *If $u = (E, B) \in H^1(\mathbb{R}^3)$ satisfies (3.29) on Ω then it satisfies (3.29) on $\mathbb{R}^3$. In this case, (E, B) satisfies the Laplace transformed Bérenger system.*

Remark 3.5. *The first part of property* **iii** *distinguishes the well adapted Helmholtz operator.*

Proof. Part **i.** and **ii** were proved before the proposition.
Part **iii.** Since (3.29) holds in Ω, the support of the lefthand side in (3.29) is contained in $\mathbb{R}^3 \setminus \Omega = \partial\Omega$. With the notations in (3.26), $E \in H^1(\mathbb{R}^3)$ implies that for all j,

$$F_j \;:=\; \frac{(\tau+\sigma_{j+1})(\tau+\sigma_{j+2})}{\tau(\tau+\sigma_j)} \, \partial_j(\epsilon E) \;\in\; L^2(\mathbb{R}^3)\,.$$

Denote by $\chi_\pm$ the characteristic function of $\{\pm x_j \geq 0\}$. Then $F_j = \chi_+ F_j + \chi_- F_j$.

Denote the face $\{x_j = 0\}$ by $\mathcal{F}$. The singular points in $\mathcal{F}$ are the edges, $\mathcal{G} = \{x, x_j = 0,\ x_{j+1}x_{j+2} = 0\}$. Lemma 2.10 shows that $\mathcal{G}$ is a negligible set for $H^{1/2}(\mathcal{F}) \sim H^{1/2}(\mathbb{R}^2)$. This implies that the open set $C_0^\infty(\mathbb{R}^2 \setminus \mathcal{G})$ is dense in $H^{1/2}(\mathbb{R}^2)$. Therefore the most restrictive definition of $H^{1/2}(\mathbb{R}^2 \setminus \mathcal{G})$, namely this closure, is identical to the least restrictive definition, namely that the restriction to each component of $\mathbb{R}^2 \setminus \mathcal{G}$ has an extension to an element of $H^{1/2}(\mathbb{R}^2)$. Both are equal to the intermediate space defined as restrictions to $\mathbb{R}^2 \setminus \mathcal{G}$ of elements of $H^{1/2}(\mathbb{R}^2)$. Since each element of $H^{1/2}(\mathbb{R}^2 \setminus \mathcal{G})$ is the restriction of exactly one element of $H^{1/2}(\mathbb{R}^2)$, the space $H^{1/2}(\mathbb{R}^2 \setminus \mathcal{G})$ is naturally isomorphic to $H^{1/2}(\mathbb{R}^2)$. An analogous argument shows that $H^{-1/2}(\mathbb{R}^2 \setminus G) = H^{-1/2}(\mathbb{R}^2)$.

Equation (3.14) implies that $F_j \in C(\mathbb{R}_{x_j}\,;\, H^{-1/2}(\mathcal{F} \setminus \mathcal{G}))$. In particular, $\chi_\pm \partial_j F$ makes sense as a piecewise continuous function with values in $H^{-1/2}(\mathcal{F} \setminus \mathcal{G})$. Equation (3.14) implies that $\partial_j F_j \in C(\mathbb{R}_{x_j}; H^{-3/2}(\mathbb{R}^2 \setminus \mathcal{G}))$. The formula for the distribution derivative of piecewise smooth function of x_j implies that on $\mathbb{R}^3 \setminus \mathcal{S}$

$$\partial_j \frac{\chi_+ F_j}{\epsilon} = \chi_+ \partial_j \frac{F_j}{\epsilon} + \frac{F_j}{\epsilon}\bigg|_{x_j=0+} \delta(x_j), \quad \partial_j \frac{\chi_- F_j}{\epsilon} = \chi_- \partial_j \frac{F_j}{\epsilon} - \frac{F_j}{\epsilon}\bigg|_{x_j=0-} \delta(x_j).$$

Define

$$Q \;:=\; \frac{1}{\epsilon}\, F_j\bigg|_{x_j=0+} - \frac{1}{\epsilon}\, F_j\bigg|_{x_j=0-} \;\in\; H^{-1/2}(\mathbb{R}^2 \setminus \mathcal{G}) \;=\; H^{-1/2}(\mathbb{R}^2)\,.$$

Equation (3.15) asserting continuity across $\{x_j = 0\} \setminus S$ yields

$$\text{supp } Q \;\subset\; S \cap \{x_j = 0\} \;=\; \mathcal{G}.$$

Lemma 2.10 shows that $\mathcal{G}$ is negligible for $H^{-1/2}(\mathbb{R}^2)$. Therefore $Q = 0$.

Since $Q = 0$, adding the $\chi_\pm$ identities above yields

$$\partial_j \frac{F_j}{\epsilon} \;=\; \chi_+ \partial_j \frac{F_j}{\epsilon} \;+\; \chi_- \partial_j \frac{F_j}{\epsilon} \quad \text{on} \quad \mathbb{R}^3 \setminus \mathcal{S}\,.$$

Summing on j shows that the differential equation in (3.29), originally known to hold on Ω, in fact holds on the larger set $\mathbb{R}^3 \setminus \mathcal{S}$ that includes the smooth points of $\partial\Omega$. Therefore the left hand side of (3.29) is an element of $H^{-1}(\mathbb{R}^3)$ with support in $\mathcal{S}$. Lemma 2.10 proves that it vanishes identically. □

3.2.2 Estimates for the well adapted Helmholtz operator (3.28)

Proposition 3.5. *If $\sigma_j \in L^\infty(\mathbb{R})$ for $1 \leq j \leq 3$, then there are constants C, M so that for all $u \in H^1(\mathbb{R}^3)$ with $Pu := f \in L^2(\mathbb{R}^3)$, and τ with* $\operatorname{Re}\tau >$

M

$$\mathrm{Re}\,\tau\ \|u\|_{L^2(\mathbb{R}^3)}\ +\ \|\nabla u\|_{L^2(\mathbb{R}^3)}\ \le\ C\ \|Pu\|_{L^2(\mathbb{R}^3)}\,. \tag{3.30}$$

Proof. It suffices to treat the equation for E.

Main Step I. Consider first the case $\epsilon = \mu = 1$ that has all the essential difficulties and is much easier to read. In this case, the first order term ℓ vanishes, and we treat the equation

$$\frac{\prod(\tau+\sigma_j)}{\tau}\,u - pu = f, \quad pu = \sum_j\ \partial_j\ \frac{\prod_{k\neq j}(\tau+\sigma_k)}{\tau(\tau+\sigma_j)}\,\partial_j u\,.$$

The estimate is proved by considering the real and imaginary parts of the identities

$$\big(u, f\big)\ =\ \Big(\frac{\prod(\tau+\sigma_j)}{\tau}u\,,\,u\Big) - \big(pu, u\big), \tag{3.31}$$

$$-\big(pu, u\big)\ =\ \int\Big(\sum_{j=1}^{3}\ \frac{\prod_{k\neq j}(\tau+\sigma_k)}{\tau(\tau+\sigma_j)}\ |\partial_j u|^2\Big)\ dx\,. \tag{3.32}$$

• We can see from (3.32) that the term $\big(pu, u\big)$ is not far from its unperturbed value. Indeed,

$$\frac{(\tau+\sigma_1)(\tau+\sigma_2)}{\tau(\tau+\sigma_3)} - 1\ =\ \frac{(\sigma_1+\sigma_2-\sigma_3)\tau + \sigma_1\sigma_2}{\tau(\tau+\sigma_3)}\,.$$

Since the σ_j are uniformly bounded, there is a constant C_1 so that for $\mathrm{Re}\,\tau$ sufficiently large,

$$\left|\frac{(\tau+\sigma_1)(\tau+\sigma_2)}{\tau(\tau+\sigma_3)} - 1\right|\ \le\ \frac{C_1}{|\tau|}\,,$$

and therefore,

$$\Big|\big(pu, u\big)\ +\ \|\nabla u\|^2\Big|\ \ \le\ C_1\,\frac{\|\nabla u\|^2}{|\tau|}\,. \tag{3.33}$$

In particular for $\mathrm{Re}\tau$ sufficiently large,

$$|\mathrm{Im}\,\big(pu, u\big)| \le C_1\frac{\|\nabla u\|^2}{|\tau|}\,,\quad \mathrm{Re}\,\big(pu, u\big) + \|\nabla u\|^2 \le C_1\frac{\|\nabla u\|^2}{|\tau|}\,. \tag{3.34}$$

• Expand the zero order term

$$\begin{aligned}\frac{\prod(\tau+\sigma_j)}{\tau} &= \frac{\overline{\tau}\prod(\tau+\sigma_j)}{|\tau|^2}\\ &= \frac{\overline{\tau}(\tau^3 + \sum\sigma_j\tau^2 + \sum\sigma_i\sigma_j\tau + \sigma_1\sigma_2\sigma_3)}{|\tau|^2}\\ &= \frac{|\tau|^2\tau^2 + \sum\sigma_j|\tau|^2\tau + \sum\sigma_i\sigma_j|\tau|^2 + \sigma_1\sigma_2\sigma_3\overline{\tau}}{|\tau|^2}\,.\end{aligned}$$

Extract the imaginary part

$$\operatorname{Im} \frac{\prod(\tau+\sigma_j)}{\tau} \;=\; \frac{|\tau|^2 \operatorname{Im}\tau^2 + \sum \sigma_j |\tau|^2 \operatorname{Im}\tau - \sigma_1\sigma_2\sigma_3 \operatorname{Im}\tau}{|\tau|^2}$$
$$\;=\; (2|\tau|^2 \operatorname{Re}\tau + \sum \sigma_j |\tau|^2 - \sigma_1\sigma_2\sigma_3)\frac{\operatorname{Im}\tau}{|\tau|^2}\,.$$

Therefore for $\operatorname{Re}\tau$ sufficiently large,

$$\left|\operatorname{Im} \frac{\prod(\tau+\sigma_j)}{\tau}\right| \;\geq\; \frac{3}{2}\, \operatorname{Re}\tau|\, \operatorname{Im}\tau|\,. \tag{3.35}$$

The real part satisfies

$$\operatorname{Re} \frac{\prod(\tau+\sigma_j)}{\tau} = \frac{|\tau|^2 \operatorname{Re}\tau^2 + \sum \sigma_j |\tau|^2 \operatorname{Re}\tau + \sum \sigma_i\sigma_j |\tau|^2 + \sigma_1\sigma_2\sigma_3 \operatorname{Re}\tau}{|\tau|^2}\,.$$

Therefore for $\operatorname{Re}\tau$ sufficiently large,

$$\operatorname{Re} \frac{\prod(\tau+\sigma_j)}{\tau} \;\geq\; \operatorname{Re}\tau^2 \;=\; (\operatorname{Re}\tau)^2 - (\operatorname{Im}\tau)^2\,. \tag{3.36}$$

• Next use identity (3.31). The imaginary part of (3.31) yields

$$\operatorname{Im}\big(\frac{\prod(\tau+\sigma_j)}{\tau} u\,,\, u\big) = \operatorname{Im}\big(pu, u\big) + \operatorname{Im}\big(f, u\big).$$

Insert (3.35) and (3.34) to find

$$\frac{3}{2}\, \operatorname{Re}\tau|\, \operatorname{Im}\tau|\, \|u\|^2 \;\leq\; \|f\|\, \|u\| \;+\; \frac{C_1}{|\tau|}\, \|\nabla u\|^2.$$

This is used to estimate $\operatorname{Im}\tau\|u\|$. Multiply by $\frac{2}{3}|\operatorname{Im}\tau|/\operatorname{Re}\tau$ to find

$$|\operatorname{Im}\tau|^2\, \|u\|^2 \;\leq\; \frac{2}{3}\frac{|\operatorname{Im}\tau|}{\operatorname{Re}\tau}\, \|u\|\|f\| \;+\; \frac{2C_1}{3}\frac{|\operatorname{Im}\tau|}{|\tau|\operatorname{Re}\,\tau}\, \|\nabla u\|^2\,. \tag{3.37}$$

The information in the real part of (3.31) yields,

$$\operatorname{Re}\Big(\frac{\prod(\tau+\sigma_j)}{c^2\tau} u\,,\, u\Big) \;=\; \operatorname{Re}\big(pu, u\big) \;+\; \operatorname{Re}\big(f, u\big)\,.$$

Insert (3.36) and (3.34) to find

$$(\operatorname{Re}\tau)^2\|u\|^2 \;+\; \|\nabla u\|^2 \;\leq\; |(f,u)| \;+\; (\operatorname{Im}\tau)^2\|u\|^2 \;+\; \frac{C_1\,\|\nabla u\|^2}{|\tau|}\,.$$

On the right use the Cauchy-Schwartz inequality for the first term, and insert (3.37) in the second term to find

$$(\operatorname{Re}\tau)^2\|u\|^2 \;+\; \|\nabla u\|^2 \;\leq\; \|f\|\, \|u\|\, \Big(1+\frac{|2\operatorname{Im}\tau|}{3\operatorname{Re}\tau}\Big) \;+\; \Big(1+\frac{|2\operatorname{Im}\tau|}{3\operatorname{Re}\tau}\Big)\frac{C_1\,\|\nabla u\|^2}{|\tau|}\,.$$

Pick $\alpha \in (0,1)$. For $\operatorname{Re}\tau$ sufficiently large one has

$$\Big(1+\frac{2|\operatorname{Im}\tau|}{3\operatorname{Re}\tau}\Big)\ \frac{\max(C_1,1)}{|\tau|}\ \le\ \alpha\,.$$

Then

$$(\operatorname{Re}\tau)^2\ \|u\|^2\ +\ \frac{\|\nabla u\|^2}{2}\ \le\ \alpha\left(\|\nabla u\|^2+\|\tau u\|\,\|f\|\right)\,. \tag{3.38}$$

• Next estimate τu using the equation $\frac{\prod(\tau+\sigma_j)}{\tau}u - pu = f$, isolating

$$\tau^2 u = pu + f - ((\sigma_1+\sigma_2+\sigma_3)\tau + (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \frac{\sigma_1\sigma_2\sigma_3}{\tau})u. \tag{3.39}$$

Choose a constant C_3 so that for $\operatorname{Re}\tau$ sufficiently large,

$$|(\sigma_1+\sigma_2+\sigma_3)\tau + (\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \sigma_1\sigma_2\sigma_3/\tau| \le C_3|\tau|\,. \tag{3.40}$$

Multiply (3.39) by $\bar u$ and use (3.40) and (3.33) to find

$$\|\tau u\|^2\ \le\ \frac{C_1}{|\tau|}\,\|\nabla u\|^2\ +\ C_3\,\|\tau u\|\,\|u\| + \|u\|\,\|f\|\,.$$

Therefore by Cauchy-Schwarz and elementary Young's inequality, there is a C_4 so that $\operatorname{Re}\tau$ sufficiently large

$$\|\tau u\|^2\ \le\ C_4\Big(\|\nabla u\|^2 + \|u\|^2 + \|f\|^2\Big)\,. \tag{3.41}$$

Using also elementary Young's inequality with coefficient $\beta \in (0,1)$ in (3.38) yields

$$(\operatorname{Re}\tau)^2\,\|u\|^2\ +\ \frac{\|\nabla u\|^2}{2}\ \le\ \alpha\Big(\|\nabla u\|^2 + \frac{\beta}{2}\|\tau u\|^2 + \frac{1}{2\beta}\|f\|^2)\Big)$$

Inserting estimate (3.41) in the righthand side yields

$$(\operatorname{Re}\tau)^2\,\|u\|^2 + \frac{\|\nabla u\|^2}{2}\ \le\ \alpha\Big((1+\frac{\beta C_4}{2})\,\|\nabla u\|^2 + \frac{\beta C_4}{2}\|u\|^2 + (\frac{\beta C_4}{2}+\frac{1}{2\beta})\|f\|^2\Big).$$

Choose α and β so that $\alpha(1+\frac{\beta C_4}{2}) = \frac14$ to obtain with a constant C,

$$(\operatorname{Re}\tau)^2\|u\|^2\ +\ \frac14\|\nabla u\|^2\ \le\ \frac14\|u\|^2 + C\|f\|^2.$$

This gives (3.30) for $\operatorname{Re}\tau$ sufficiently large.

Step II. Endgame. Return now to the problem with variable coefficients ϵ and μ. Since they are $\mathcal{C}^2$, bounded and bounded below in $\mathbb{R}^3$, they intervene only by multiplicative constants in the zero and first order terms. The zero order term is dominated by $C\|u\|^2$ that is absorbed in the $\mathrm{Re}\,\tau\|u\|^2$ term for $\mathrm{Re}\,\tau$ large.

For the first order term $(E, \ell_E E)$, Cauchy Schwarz and Young's inequalities yield an upper bound $\alpha\|\nabla E\|^2 + C\|E\|^2/\alpha$. The first term is absorbed in the gradient term $\|\nabla E\|^2$ by choosing α, and C/α is absorbed in $\mathrm{Re}\,\tau$. $\square$

Proof of Proposition 3.3. E satisfies

$$P_E E \;=\; \Phi_E \;:=\; -\widetilde{\mathrm{grad}}\,(\frac{1}{\epsilon\tau}\widetilde{\mathrm{div}}\;\Phi) + \mu\tau\Phi + \widetilde{\mathrm{curl}}\;\Psi - \frac{\widetilde{\mathrm{grad}}\,\mu}{\mu}\wedge\Psi,$$

with $(\Phi, \Psi) = LU$. The estimate

$$\|\Phi_E\| \;\lesssim\; \|\frac{1}{\tau}\widetilde{\mathrm{div}}\;\Phi\|_{H^1} + \|\tau\Phi\|_{L^2} + \|\Psi\|_{H^1}\,,$$

with a similar estimate for the B equation, gives

$$\|f\| \;\lesssim\; \|LU\|_{H^1} + \|\tau LU\|_{L^2} + \|\frac{1}{\tau}\,\widetilde{\mathrm{div}}\;LU\|_{H^1}\,. \qquad \square$$

3.3 Estimates for Laplace transformed Bérenger

The proof of the estimates in Theorem 1.2 proceeds by estimating the Laplace transform. The Laplace transform is split into two pieces U_1 and U_2 that are estimated using the Propositions of Section 3.2. If $U(\tau, x)$ is the Laplace transform then

$$L\,U \;=\; (4\pi\widehat{\mathbf{j}}\,,\,0)\,.$$

Definition 3.3. *Choose $\phi_1(x) \in C_0^\infty(\mathbb{R}^3)$ supported in $\mathcal{O}$ with $\phi_1 = 1$ on a neighborhood of $\overline{\omega}$. Define $\phi_2 \;:=\; 1 - \phi_1$. Define $U_j := \phi_j U$.*

Proposition 3.6. *There are positive constants C, M so that if $\mathrm{Re}\,\tau > M$, $U, LU \in H^1(\mathbb{R}^3)$, $\mathrm{supp}\,LU \subset \overline{\omega}$ then*

$$\mathrm{Re}\,\tau\;\|U_1\|_{H^1} \;+\; \mathrm{Re}\,\tau\;\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\le\; C\,\|LU\|_{H^1}\,. \tag{3.42}$$

A weaker estimate, as strong as the Helmholtz estimates is

$$\mathrm{Re}\,\tau\;\|U\|_{L^2} \;+\; \|\nabla_x U\|_{L^2} \;\lesssim\; \|LU\|_{H^1}\,. \tag{3.43}$$

Remark 3.6. *The field of interest in the computation is U_1. U_1 satisfies estimates exactly as strong as the estimates for Maxwell's equations. The estimate for U_2 is weaker but without loss of derivatives.*

Proof. Compute a system of equations satisfied by the pair U_1, U_2. Define

$$M_j(\tau, x) := [L, \phi_j] = \sum_k A_k \partial_k \phi_j = \begin{pmatrix} 0 & [\widetilde{\mathrm{curl}}\,,\phi_j] \\ -[\widetilde{\mathrm{curl}}\,,\phi_j] & 0 \end{pmatrix}.$$

For each $\tau \in \mathbb{C}$, the M_j are smooth matrix valued functions,

$$M_j(\tau, \cdot\,) \;\in\; C_0^\infty(\mathcal{O})\,, \qquad \operatorname{supp} M_j \cap \overline{\omega} = \emptyset\,. \tag{3.44}$$

Compute

$$LU_j = L(\phi_j U) = \phi_j LU + [L, \phi_j]U \;=\; \phi_j LU + M_j U\,. \tag{3.45}$$

3.3.1 Estimate for U_2

For our problem with $\mathbf{j}$ supported in $\overline{\omega}$, $\phi_2 F = 0$. The equation for U_2 is

$$LU_2 \;=\; M_2 U\,.$$

Since $M_2\,U$ is supported in $\mathcal{O}$ where the absorptions vanish so $\widetilde{\operatorname{div}} = \operatorname{div}$, Proposition 3.3 implies that

$$\begin{aligned}\operatorname{Re}\tau\,\|U_2\|_{L^2} + \|\nabla_x U_2\|_{L^2} \;&\lesssim\; \|LU_2\|_{H^1} + \|\tau\, LU_2\|_{L^2} + \big\|\frac{1}{\tau}\operatorname{div} LU_2\big\|_{H^1}\\ &\lesssim\; \|M_2\,U\|_{H^1} + \|\tau\, M_2\,U\|_{L^2} + \big\|\frac{1}{\tau}\,M_2\,U\big\|_{H^2}\,.\end{aligned} \tag{3.46}$$

On $\mathcal{O}\setminus\overline{\omega}$, ϵ and μ are scalar and $LU = 0$. It follows that

$$\big(\epsilon\mu\,\tau^2 \;-\; \Delta \;+\; q(x, \partial_x)\big)U \;=\; 0\,, \qquad \text{on} \quad \mathcal{O}\setminus\overline{\omega} \tag{3.47}$$

with q a system of partial differential operator of degree 1. The terms of degree one have C^1 coefficients with bounded derivatives and the term of order zero has bounded coefficient thanks to the hypothesis ϵ, μ have derivatives up to order two continuous and bounded.

Equation (3.47) is a homogeneous elliptic equation that holds on a neighborhood of $\operatorname{supp} M_2$. The elliptic regularity theorem for the laplacian is used to estimate the H^2 norm of $M_2 U$.

Define a finite sequence of scalar cutoff functions $\chi_j \in C_0^\infty(\mathcal{B})$. The first is chosen so that χ_1 is identically equal to one on a neighborhood of $\operatorname{supp} M_2$. The succeeding cutoffs are chosen so that $\operatorname{supp}\chi_{j+1}$ is identically equal to one on a neighborhood of $\operatorname{supp}\chi_j$. Elliptic regularity yields

$$\|\chi_1 U\|_{H^2} \;\lesssim\; \|\chi_2 \Delta U\|_{L^2} \;+\; \|\chi_2 U\|_{L^2}\,.$$

Using (3.47) yields

$$\|\chi_2 \Delta U\|_{L^2} \;\lesssim\; \|\tau^2\,\chi_2 U\|_{L^2} \;+\; \|\chi_3 U\|_{H^1}\,.$$

Using these, estimate

$$\big\|\frac{1}{\tau}\,M_2\,U\big\|_{H^2} \;\lesssim\; \|\frac{1}{\tau}\,\chi_1 U\|_{H^2} \;\lesssim\; \|\tau\,\chi_2 U\|_{L^2} \;+\; \|\frac{1}{\tau}\,\chi_3 U\|_{H^1}\,.$$

Inject in (3.46) to find

$$\operatorname{Re}\tau\,\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \|\tau\,\chi_2 U\|_{L^2} \;+\; \|\chi_3 U\|_{H^1}\,. \tag{3.48}$$

On the right express τU_2 in the support of χ_2 hence inside $\mathcal{O}$ using the equation to estimate

$$\|\tau\chi_2 U\|_{L^2} \;\lesssim\; \|\chi_2\nabla U\|_{L^2} \;+\; \|\chi_1 U\|_{L^2} \;\lesssim\; \|\chi_3 U\|_{H^1}\,.$$

Therefore (3.48) yields

$$\operatorname{Re}\tau\,\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \|\chi_3 U\|_{H^1}\,. \tag{3.49}$$

3.3.2 Estimate for U_1 and U_2

The equation for U_1 has source term $\phi_2 F$. The cutoff ϕ_2 was chosen to be identically equal to one a neighborhood of the support of the source term $\mathbf{j}$ leading to

$$LU_1 \;=\; M_1 U \;+\; 4\pi\widehat{\mathbf{j}}\,.$$

Proposition 3.2 yields

$$\operatorname{Re}\tau\;\|U_1\|_{H^1} \;\lesssim\; \|M_1 U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1} \;\lesssim\; \|\chi_1 U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,.$$

This easy derivation depended only on the fact that the cutoff function ϕ_1 was supported in $\mathcal{O}$. Exactly the same argument yields

$$\operatorname{Re}\tau\;\|\chi_j U\|_{H^1} \;\lesssim\; \|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,.$$

Combining the last two yields

$$\operatorname{Re}\tau\;\|U_1\|_{H^1} \;\lesssim\; \frac{1}{\operatorname{Re}\tau}\;\|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,. \tag{3.50}$$

This improves (3.49) to

$$\operatorname{Re}\tau\;\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \frac{1}{\operatorname{Re}\tau}\Big(\|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\Big)\,. \tag{3.51}$$

Summing yields

$$\operatorname{Re}\tau\;\|U_1\|_{H^1} \;+\; \operatorname{Re}\tau\;\|U_2\|_{L^2} \;+\; \|\nabla_x U_2\|_{L^2} \;\lesssim\; \frac{1}{\operatorname{Re}\tau}\;\|U\|_{H^1} \;+\; \|\widehat{\mathbf{j}}\|_{H^1}\,.$$

For $\operatorname{Re}\tau$ large this proves estimate (3.42), completing the proof of Proposition 3.6. □

3.4 Existence and uniqueness proofs

3.4.1 Laplace Transform, Paley-Wiener, and Plancherel.

The Laplace transform of a distribution F supported in $t \geq 0$ with $e^{-\lambda t} F \in L^1(\mathbb{R})$ for $\lambda > M$, is holomorphic in $\mathrm{Re}\, \tau > M$, given by

$$\widehat{F}(\tau) \ := \ \int e^{-\tau t} \ F(t) \ dt \,.$$

Our functions F take values in a Hilbert space H.

Theorem 3.7. *The Laplace transforms of functions $F \in e^{Mt}\, L^2(\mathbb{R}\,;\, H)$ supported in $t \geq 0$ are exactly the functions $G(\tau)$ holomorphic in $\mathrm{Re}\, \tau > M$ with values in H and so that*

$$\sup_{\lambda > M} \int_{\mathrm{Re}\, \tau = \lambda} \big\| \widehat{F}(\tau) \big\|_H^2 \ |d\tau| \ < \ \infty \,.$$

In this case $\widehat{F}(\tau)$ has trace at $\mathrm{Re}\, \tau = M$ that is square integrable and

$$\int e^{-2Mt} \ \|F(t)\|_H^2 \ dt \ = \ \sup_{\lambda > M} \int_{\mathrm{Re}\, \tau = \lambda} \big\| \widehat{F}(\tau) \big\|_H^2 \ |d\tau| \ = \ \int_{\mathrm{Re}\, \tau = M} \big\| \widehat{F}(\tau) \big\|_H^2 \ |d\tau| \,.$$

3.4.2 Estimates of Theorem 1.2

The *a priori* estimates corresponding to Theorem 1.2 are now immediate.

• The estimate

$$\int e^{-2\lambda t} \big(\lambda \|U\|_{L^2}^2 \ + \ \|\nabla_x U\|_{L^2}^2 \big) \ dt \ \lesssim \ \text{right hand side of (1.9)}$$

follows by combining the Plancherel identity of the preceding section with the estimate (3.43).

• The remaining estimate in (1.9) follows from Plancherel and the estimate for ∇U_1 in (3.42).

• The estimate for U_t comes from expressing U_t in terms of U_x and LU.

• The fact that $E^j_j = B^j_j = 0$ follows from (3.3) and the fact that the $j^{\rm th}$ row of the C_j vanishes.

• The estimate for $\partial_t \{E^j, B^j\}$ follows from the equation that shows that $|\tau \widehat{E}^j, \tau \widehat{B}^j| \leq |\nabla_x U|$.

3.4.3 Proof of Theorem 1.2

I. Proof of existence for smooth sources j and smooth absorptions $\sigma_j(x_j)$.

For $\sigma_j \in W^{2,\infty}(\mathbb{R})$ and $\mathbf{j} \in e^{\lambda t} H^2(\mathbb{R}\times\mathbb{R}^3)$ with support in $t \geq 0$ existence is proved in [7] following [17]. The method is an elaboration of [14]. A Hilbert space norm bounded below with the L^2 norm of E, B, E^j, B^j and above by the H^2 norm of these quantities is constructed so that for solutions the square of norm, denoted $\mathcal{E}(U)$ satisfies $d\mathcal{E}(U)/dt \lesssim \mathcal{E}(U) + \mathcal{E}(\mathbf{j})$. Similar estimates hold for the semidiscrete scheme that one finds by using the Yee discretization of the x derivatives. Passing to the limit of decreasing mesh size yields a solutions in $e^{\lambda t} L^2(\mathbb{R}\times\mathbb{R}^3)$.

We need solutions with total field U belonging to $e^{\lambda t} H^1(\mathbb{R}\times\mathbb{R}^3)$.

Since the equation is time translation invariant, if j and $\partial_t j$ belong to $e^{\lambda t} H^2(\mathbb{R}\times\mathbb{R}^3)$ one finds a total field U and U_t in $e^{\lambda t} L^2(\mathbb{R}\times\mathbb{R}^3)$. The Laplace transformed equations satisfy

$$\tau\,\epsilon\widehat{E} \;=\; \widetilde{\mathrm{curl}}\,\widehat{B} \;+\; 4\pi\widehat{\mathbf{j}}\,, \qquad \tau\,\epsilon\widehat{B} \;=\; -\widetilde{\mathrm{curl}}\,\widehat{E}\,.$$

Taking divergence yields

$$\tau\,\widetilde{\mathrm{div}}\,\epsilon\widehat{E} \;=\; 4\pi\,\widetilde{\mathrm{div}}\,\widehat{\mathbf{j}}\,, \qquad \tau\,\widetilde{\mathrm{div}}\,\mu\widehat{B} \;=\; 0\,.$$

This bounds

$$\widetilde{\mathrm{div}}\,\epsilon\widehat{E}\,, \quad \widetilde{\mathrm{curl}}\,\widehat{E}\,, \quad \widetilde{\mathrm{div}}\,\mu\widehat{B} \quad \text{and} \quad \widetilde{\mathrm{curl}}\,\widehat{B}\,.$$

Lemma 3.8. *The overdetermined system* $F \mapsto \widetilde{\mathrm{div}}\,\epsilon F\,,\ \widetilde{\mathrm{curl}}\,F$ *is for each* $\mathrm{Re}\,\tau > 0$ *is an overdetermined elliptic system. Similarly* $G \mapsto \widetilde{\mathrm{div}}\,\mu G\,,\ \widetilde{\mathrm{curl}}\,G$*. The ellipticity is uniform in* x*.*

Proof. Verify the Lopatinski condition for the system with permittivity ϵ. For $\underline{x}$ fixed the plane wave with real ξ, $e^{i\xi x}\,\mathbf{e}$, is a solution of the frozen $\widetilde{\mathrm{div}}\,,\widetilde{\mathrm{curl}}$ problem if and only if

$$e^{i\xi x}\,\underline{\mathbf{e}}, \qquad \underline{\mathbf{e}}_j := \frac{\tau}{\tau + \sigma_j(\underline{x})}\mathbf{e}_j$$

is a plane wave solution of the overdetermined system $H \mapsto \mathrm{div}\,\epsilon H, \mathrm{curl}\,H$. For the latter the necessary and sufficient conditions are

$$\xi\cdot\epsilon\underline{\mathbf{e}} = 0\,, \quad \text{and} \quad \xi\wedge\underline{\mathbf{e}} = 0\,.$$

The second implies that $\xi \parallel \underline{\mathbf{e}}$. Without loss of generality we can take $\underline{\mathbf{e}}$ real. The first condition then yields $\underline{\mathbf{e}} = 0$ because ϵ is positive definite. $\square$

Elliptic regularity implies that

$$\|\nabla_x\widehat{E}\| \;\lesssim\; \|\widetilde{\mathrm{div}}\,\epsilon\widehat{E}\| \;+\; \|\widetilde{\mathrm{curl}}\,\widehat{E}\| \;+\; \|\widehat{E}\|.$$

This together with the Plancherel Theorem implies that the total field $U \in e^{\lambda t} H^1(\mathbb{R}\times\mathbb{R}^3)$. This completes the proof of **I.** $\square$

II. Proof of existence. Choose sequences $\sigma_j^n \in W^{2,\infty}(\mathbb{R})$ with support in $]-\infty, 0]$, and $\sigma_j^n \to \sigma_j$ in $L^\infty(\mathbb{R})$ weak star as $n \to \infty$.

Choose $\mathbf{j}^n \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ with support in $\{t \geq 0\}$ and $\mathbf{j}^n \to \mathbf{j}$ in $e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$.

Denote by $E^n, B^n, E^{n,j}, B^{n,j}$ the solution of the Bérenger split problem with absorptions σ_j^n and current $\mathbf{j}^n$ constructed in **I.**.

The estimates of Theorem 1.2, proved in Section 3.4.2 apply to this sequence. The constants C, M associated to the family of absorptions $\{\sigma_j^n\}$ can be chosen uniformly. They depend only on the L^∞ norms of the absorptions, see part **iii** of Remark 3.4

Extracting a subsequence one may suppose that the total fields

$$U^n \quad \rightharpoonup \quad U \quad \text{weakly in} \quad e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)\,,$$

and the split fields satisfy for all κ, j,

$$U^{n,j} \quad \rightharpoonup \quad U^j \quad \text{weakly in} \quad e^{\lambda t} L^2(\mathbb{R} \times \mathcal{O}_\kappa)\,.$$

The limits are the desired solutions. This completes the proof of existence.

III. Proof of uniqueness. For a solution with vanishing data, the Laplace transform $\widehat{U}$ of the total field satisfies a homogeneous equation for which uniqueness is proved in Proposition 3.6. This completes the proof of uniqueness and thererfore of Theorem 1.2. $\square$

3.5 Numerical study of the loss of derivatives

The simulations in this section do **not** concern corners. We have proved that there is essentially no loss of derivatives for the Bérenger split Maxwell equations even in the presence of corners while it was commonly believed that there was loss *even without corners*. The simulations below show that for the split equations with neither boundaries nor absorptions there is loss for data whose divergence is non vanishing and no loss for divergence free data.

The simulations treat the 2-D transverse electric Maxwell system in $\mathbb{R}^2_{x,y}$, with light speed equal to one

$$\partial_t E_x = \partial_y B_z\,, \quad \partial_t E_y = -\partial_x B_z\,, \quad \partial_t B_z = -\partial_x E_y + \partial_y E_x. \tag{3.52}$$

In the Bérenger split system, the third equation is replaced by

$$\partial_t B_{zx} = -\partial_x E_y - \sigma(x) B_{zx}, \quad \partial_t B_{zy} = \partial_y E_x, \quad B_z := B_{zx} + B_{zy}. \tag{3.53}$$

Following Bérenger [4] and discussed in [7] §3.3, only the magnetic field is split in the x direction. The total magnetic field is the sum of the split fields.

The computation concerns the model with absorption $\sigma = 0$. It reveals the role of the splitting alone on the loss of derivatives. The field B_z in the Bérenger model is equal to the B_z of the Maxwell model. This is also true for the discrete models.

The equations are solved in the rectangle $[-1,1] \times [-1,1]$, with perfect conductor boundary conditions $n \wedge E = 0$ on the boundary. The equations are discretized by the the Yee scheme, see for instance [7]. The mesh sizes are $dx = dy = 10^{-3}$, $dt = 7.0711e-04$ just inside the CFL limit. The system is run for 40 time steps.

The magnetic field is zero at $t = 0$. Solutions are computed with initial electric field with frequency $\omega = 5 \times 2^n$ with $0 \le n \le 5$. The initial data has the form

$$a(x,y)\, e^{2\pi i\, \omega\, \mathbf{v}\cdot(x,y)}\,, \qquad \mathbf{v} \;=\; \frac{1}{\sqrt{2}}\,\big(1,-1\big)\,.$$

For the Maxwell system we measure the discrete L^2 norm in space and time of (E, B_z), for the Bérenger system the norm of (E, B_{zx}, B_{zy}, B). Both are normalized by the total norm of (E, B_z) at initial time.

In a first set of experiments, the initial electric field, is divergence free. The divergence of the finite difference approximations remains below 10^{-10}, and the norm of the solution is given in Table 1.

Frequency	5	10	20	40	80	160
Maxwell	0.0852	0.1269	0.1132	0.1162	0.1226	0.1266
Berenger	0.0444	0.0642	0.0568	0.0581	0.0613	0.0633

Table 1: L^2 norm as a function of the frequency. Divergence equal 0

In the second set, the data are almost the same as before, except for a change of sign in E_y. The modified electric field is not divergence free, and the norm of the solution is given in Table 2.

Frequency	5	10	20	40	80	160
Maxwell	0.1702	0.1702	0.1703	0.1703	0.1703	0.1703
Berenger	0.1835	0.2121	0.3012	0.5247	1.0036	1.9546

Table 2: L^2 norm as a function of the frequency. Divergence not 0

For divergence free initial electric field, the norm of the solution is constant as a function of the frequency. In this second case, the norm of the solution of the Maxwell system is constant, while that of the Bérenger system grows linearly with the frequency. The L^2 norm of the solution grows like the H^1 norm of the data, illustrating a loss of one derivative.

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