Resolution of linear systems. Exercise 2

Exercice 1 : Permutation matrices.

Let σ a permutation of$\{1, \cdots, n\}$. Let $\mathcal{E} = (e_1, \ldots, e_n)$ a basis of $E = \mathbb{R}^n$. Consider the automorphisme of E defined by $f_\sigma(e_j) = e_{\sigma(j)}$. The permutation matrix P_σ is the matrix of f_σ in $\mathcal{E}$.

1) Show that $(P_\sigma)_{ij} = \delta_{i\sigma(j)}$. Deduce that $P_\sigma P_\tau = P_{\sigma\circ\tau}$, $(P_\sigma)^{-1} = P_{\sigma^{-1}}$ and $(P_\sigma)^T = P_{\sigma^{-1}}$.

2) Show that multiply A on the left by P_σ amount to make the permutation σ^{-1} on the lines of A.

3) Show that multiply A on the right by P_σ amount to make the permutation σ on the columns of A.

Exercice 2 : Block matrices.

1) Show that the product of two lower block triangular matrices (resp. upper) is a lower block triangular matrix (resp. upper). **2)** The aim is to compute the determinant of $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ split in blocks. The blocks A_{11} et A_{22} are square.

a) Compute the determinant of

$$A_1 = \begin{pmatrix} A_{11} & 0 \\ 0 & I \end{pmatrix}, \quad \text{and } A_2 = \begin{pmatrix} I & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Deduce the determinant of

$$A_3 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

b) Compute the determinant of

$$A_4 = \begin{pmatrix} I & A_{12} \\ 0 & I \end{pmatrix}$$

and the produce of the two block matrices

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \text{ et } \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}.$$

Deduce the determinant of

$$A_5 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

c) Compute the produce of the two block matrices

$$\begin{pmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{pmatrix} \text{ et } \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

Deduce the determinant of A.

3) Compute the determinant of the block matrix

$$\begin{pmatrix} A_{11} & A_{12} & & A_{1n} \\ 0 & A_{22} & & A_{2n} \\ 0 & 0 & \ddots & \\ 0 & 0 & & A_{nn} \end{pmatrix}$$

Exercice 3 : Irreducible matrices.

Let $A = (a_{ij})_{1 \le i,j \le n}$. A is said to be reducible if there exists a permutation matrix P such that

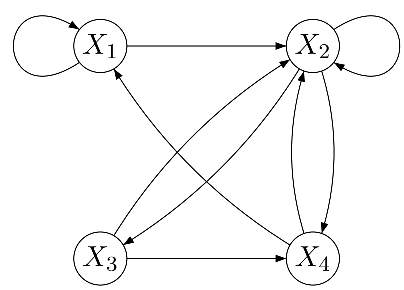
$${}^tPAP = B = \begin{bmatrix} B^{(11)} & B^{(12)} \\ 0 & B^{(22)} \end{bmatrix}$$

where $B^{(11)}$ and $B^{(22)}$ are square matrices of order p et $n-p$ respectively.
1) Show that A is reducible if and only if there exists a partition of $\{1, .., n\}$ into I et J such that $a_{ij} = 0$ for i in I and j in J.

A is said to be irreducible if it is not reducible.

Define the *graph* associated to A as the set of points X_i, for $1 \le i \le n$. The points X_i and X_j are connected by an oriented *arc* if $a_{ij} \neq 0$.

$$A = \begin{pmatrix} 3 & 3 & 0 & 0 \\ 0 & 2 & 4 & 3 \\ 0 & 2 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{pmatrix}$$

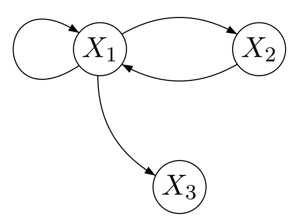


To each edge X_i can be associated the set of its neighbours,

$$V(X_i) = \{X_j,\ j \neq i,\ \text{such that } X_iX_j \text{ is an arc}\}.$$

A *path* is a sequence of arcs. The graph is said to be *strongly connected* if any group of two points is connected by a path.

$$A = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



2) Show that a matrix is irreducible if and only if its graph is strongly connected.

Exercice 4 : Diagonally dominant matrices.

1) Prove the Gerschgörin-Hadamard theorem : any eigenvalue λ of A belongs to the union of disks D_k defined by

$$|z - a_{kk}| \leq \Lambda_k = \sum_{\substack{1 \leq j \leq n \\ j \neq k}} |a_{kj}|$$

2) Show that if A is irreducible and if an eigenvalue λ belongs to the boundary of the union of disks D_k, then λ belongs to all circles.

A is said to be *diagonally dominant* if

$$\forall i, 1 \leq i \leq n, |a_{ii}| \geq \Lambda_i$$

A is said to be *strict diagonally dominant* if

$$\forall i, 1 \leq i \leq n, |a_{ii}| > \Lambda_i$$

A is said to be *strong diagonally dominant* if diagonally dominant and moreover

$$\exists i, 1 \leq i \leq n, |a_{ii}| > \Lambda_i$$

3) Prove that if A est à strict diagonally dominant, it is invertible.

4) Prove that if A is strong diagonally dominant and irreducible, it is invertible.

5) Prove that if A is either strict diagonally dominant, or strong diagonally dominant and irreducible, and if all the diagonal entries are strictly positive real numbers, then the real parts of the eigenvalues are strictly positive.

6) Let A strong diagonally dominant and irreducible matrix. Show that the Jacobi algorithm converges.
7) Let A a strong diagonally dominant and irreducible matrix. Show that the relaxation algorithm converges for $\omega \in]0, 1]$.

Exercice 5 : Discretization of Laplace equation in dimension 1.

Consider

$$\begin{cases} -u'' = f \text{ on }]a, b[, \\ u(a) = 0, \\ u(b) = 0. \end{cases} \tag{1}$$

with f continuous on $]a, b[$.

1) Show that if u is sufficiently regular,

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2) \tag{2}$$

The segment $[a,b]$ is divided into n intervalls with length $h = (b-a)/n$.

2) Using (2), write the linear system ldeduce from (1) whose unknowns u_i are approximations of $u(a+ih)$ for $1 \leq i \leq n-1$. The matrix of the system is denoted by A.

3) Using Exercice II, show that A is symmetric definite positive.

4) Show the maximum principle : If all f_i are ≤ 0 , then the u_i are ≤ 0, and the maximum is reached on the border : $i = 1$ ou $n-1$.

5) Let a et b be two real numbers. For $n \geq 0$, denote by Δ_n the tridiagonal determinant

$$\Delta_n = \begin{vmatrix} a & -b & 0 & & \\ -b & a & & & \\ \ddots & & \ddots & & \\ & & -b & a & -b \\ & & 0 & -b & a \end{vmatrix}$$

Write a two-level recursion relation on the Δ_n.

6) Let $P_n(\lambda)$ the caracteristic polynomial of A. By the change of variables

$$\lambda + 2 = -2\cos\theta,$$

show that $P_n(\lambda) = \frac{\sin(n+1)\theta}{\sin\theta}$. Deduce that the eigenvalues of A are $\lambda_k = \frac{4}{h^2} sin^2(\frac{k\pi}{2n})$ with eigenvectors $u^{(k)}$ given by $u_j^{(k)} = sin(\frac{k\pi j}{n})$.

7) Compute the condition number of A.

8) Study the convergence of the relaxation algorithms.

Exercice 6 : Discretisation of the Laplace equation in dimension 2.

Consider the boundary value problem on $]0,1[\times]0,1[$

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f \text{ in }]0,1[\times]0,1[, \\ u = 0 \text{ on the boundary of the rectangle} \end{cases} \tag{3}$$

The horizontal interval $[0,1]$ is divided into $M+1$ intervals $[x_i, x_{i+1}]$, $x_i = a + ih$, $0 \leq i \leq M+1$, with $h = 1/(M+1)$. The vertical interval $[0,1]$ is also divided into $M+1$ intervalles $[y_j, y_{j+1}]$, $y_j = c + jh$, $0 \leq j \leq M+1$. This provides a mesh in the two directions x, y. A point of the mesh is (x_i, y_j). $u_{i,j}$ denotes an approximation of $u(x_i, y_j)$.

The Poisson equation (3) is replaced by

$$-(\Delta_h u)_{i,j} = -\frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{1}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = f_{i,j}, \quad 1 \leq i \leq M, 1 \leq j \leq M \tag{4}$$

To write this in matrix form, it is necessary to store the unknowns as a vector, and therefore to number the mesh points. It can be done in increasing j order, and for any j in increasing i (Figure 3).

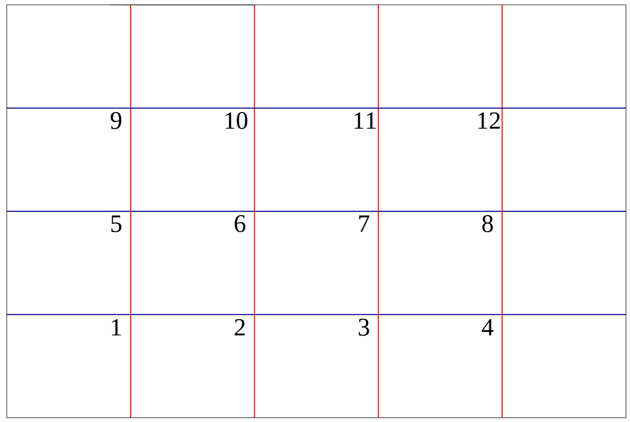


FIGURE 1 – Numbering by line

Suppose only the unknown values are stored. Then the point (x_i, y_j) has for number $i + (j-1)M$. A vector of all unknowns X is created :

$$Z = (u_{1,1}, u_{2,1}, u_{M,1}), (u_{1,2}, u_{2,2}, u_{M,2}), \cdots (u_{1,M}, u_{2,M}, u_{M,M})$$

with $Z_{i+(j-1)*M} = u_{i,j}$.

1) If the equations are numbered the same way (equation $\#k$ is the equation at point k), show that the matrix is block-diagonal :

$$A = \frac{1}{h^2} \begin{pmatrix} B & -C & & 0_M & \\ -C & B & -C & & \\ & \ddots & \ddots & \ddots & \\ & & -C & B & -C \\ & 0_M & & -C & B \end{pmatrix} \tag{5}$$

$$C = I_M, \quad B = \begin{pmatrix} 4 & -1 & & 0 & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & 0 & & -1 & 4 \end{pmatrix}$$

The righthand side is $b_{i+(j-1)*M} = f_{i,j}$, and the system takes the form $AZ = b$.

2) Show that A is invertible, with strictly positive eigenvalues.

3) Show that the eigenvalues of A are $\lambda_{pq} = \frac{4}{h^2}(sin^2(\frac{p\pi}{2M}) + sin^2(\frac{q\pi}{2M}))$. Compute the condition number of A.

4) Write the matrix A for a chessboard numbering

18	9	19	10	20
6	16	7	17	8
13	4	14	5	15
1	11	2	12	3

FIGURE 2 – Chessboard numbering

5) Write the matrix A for a diagonale numbering

10	14	17	19	20
6	9	13	16	18
3	5	8	12	15
1	2	4	7	11

FIGURE 3 – Diagonale numbering