CHAPTER 11

NONLINEAR EQUATIONS

58 AUTONOMOUS SYSTEMS. THE PHASE PLANE AND ITS PHENOMENA

There have been two major trends in the historical development of differential equations. The first and oldest is characterized by attempts to find explicit solutions, either in closed form—which is rarely possible—or in terms of power series. In the second, one abandons all hope of solving equations in any traditional sense, and instead concentrates on a search for qualitative information about the general behavior of solutions. We applied this point of view to linear equations in Chapter 4. The qualitative theory of nonlinear equations is totally different. It was founded by Poincaré around 1880, in connection with his work in celestial mechanics, and since that time has been the object of steadily increasing interest on the part of both pure and applied mathematicians.[1]

The theory of linear differential equations has been studied deeply and extensively for the past 200 years, and is a fairly complete and well-rounded body of knowledge. However, very little of a general

[1] See Appendix A for a general account of Poincaré's work in mathematics and science.

nature is known about nonlinear equations. Our purpose in this chapter is to survey some of the central ideas and methods of this subject, and also to demonstrate that it presents a wide variety of interesting and distinctive new phenomena that do not appear in the linear theory. The reader will be surprised to find that most of these phenomena can be treated quite easily without the aid of sophisticated mathematical machinery, and in fact require little more than elementary differential equations and two-dimensional vector algebra.

Why should one be interested in nonlinear differential equations? The basic reason is that many physical systems—and the equations that describe them—are simply nonlinear from the outset. The usual linearizations are approximating devices that are partly confessions of defeat in the face of the original nonlinear problems and partly expressions of the practical view that half a loaf is better than none. It should be added at once that there are many physical situations in which a linear approximation is valuable and adequate for most purposes. This does not alter the fact that in many other situations linearization is unjustified.[2]

It is quite easy to give simple examples of problems that are essentially nonlinear. For instance, if x is the angle of deviation of an undamped pendulum of length a whose bob has mass m, then we saw in Section 5 that its equation of motion is

$$\frac{d^2x}{dt^2} + \frac{g}{a}\sin x = 0; \tag{1}$$

and if there is present a damping force proportional to the velocity of the bob, then the equation becomes

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0. \tag{2}$$

In the usual linearization we replace $\sin x$ by x, which is reasonable for small oscillations but amounts to a gross distortion when x is large. An example of a different type can be found in the theory of the vacuum tube, which leads to the important *van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0. \tag{3}$$

[2] It has even been suggested by Einstein that since the basic equations of physics are nonlinear, all of mathematical physics will have to be done over again. If his crystal ball was clear on the day he said this, the mathematics of the future will certainly be very different from that of the past and present.

It will be seen later that each of these nonlinear equations has interesting properties not shared by the others.

Throughout this chapter we shall be concerned with second order nonlinear equations of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right), \tag{4}$$

which includes equations (1), (2), and (3) as special cases. If we imagine a simple dynamical system consisting of a particle of unit mass moving on the x-axis, and if $f(x, dx/dt)$ is the force acting on it, then (4) is the equation of motion. The values of x (position) and dx/dt (velocity), which at each instant characterize the state of the system, are called its *phases,* and the plane of the variables x and dx/dt is called the *phase plane.* If we introduce the variable $y = dx/dt,$ then (4) can be replaced by the equivalent system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = f(x,y). \end{cases} \tag{5}$$

We shall see that a good deal can be learned about the solutions of (4) by studying the solutions of (5). When t is regarded as a parameter, then in general a solution of (5) is a pair of functions $x(t)$ and $y(t)$ defining a curve in the xy-plane, which is simply the phase plane mentioned above. We shall be interested in the total picture formed by these curves in the phase plane.

More generally, we study systems of the form

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y), \end{cases} \tag{6}$$

where F and G are continuous and have continuous first partial derivatives throughout the plane. A system of this kind, in which the independent variable t does not appear in the functions F and G on the right, is said to be *autonomous.* We now turn to a closer examination of the solutions of such a system.

It follows from our assumptions and Theorem 54-A that if t_0 is any number and (x_0,y_0) is any point in the phase plane, then there exists a unique solution

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \tag{7}$$

of (6) such that $x(t_0) = x_0$ and $y(t_0) = y_0$. If $x(t)$ and $y(t)$ are not both constant functions, then (7) defines a curve in the phase plane called a *path* of the system.[3] It is clear that if (7) is a solution of (6), then

$$\begin{cases} x = x(t + c) \\ y = y(t + c) \end{cases} \tag{8}$$

is also a solution for any constant c. Thus each path is represented by many solutions, which differ from one another only by a translation of the parameter. Also, it is quite easy to prove (see Problem 2) that any path through the point (x_0,y_0) must correspond to a solution of the form (8). It follows from this that at most one path passes through each point of the phase plane. Furthermore, the direction of increasing t along a given path is the same for all solutions representing the path. A path is therefore a *directed curve*, and in our figures we shall use arrows to indicate the direction in which the path is traced out as t increases.

The above remarks show that in general the paths of (6) cover the entire phase plane and do not intersect one another. The only exceptions to this statement occur at points (x_0,y_0) where both F and G vanish:

$$F(x_0,y_0) = 0 \quad \text{and} \quad G(x_0,y_0) = 0.$$

These points are called *critical points*, and at such a point the unique solution guaranteed by Theorem 54-A is the constant solution $x = x_0$ and $y = y_0$. A constant solution does not define a path, and therefore no path goes through a critical point. In our work we will always assume that each critical point (x_0,y_0) is *isolated*, in the sense that there exists a circle centered on (x_0,y_0) that contains no other critical point.

In order to obtain a physical interpretation of critical points, let us consider the special autonomous system (5) arising from the dynamical equation (4). In this case a critical point is a point $(x_0,0)$ at which $y = 0$ and $f(x_0,0) = 0$; that is, it corresponds to a state of the particle's motion in which both the velocity dx/dt and the acceleration $dy/dt = d^2x/dt^2$ vanish. This means that the particle is at rest with no force acting on it, and is therefore in a state of equilibrium.[4] It is obvious that the states of equilibrium of a physical system are among its most important features, and this accounts in part for our interest in critical points.

The general autonomous system (6) does not necessarily arise from any dynamical equation of the form (4). What sort of physical meaning can be attached to the paths and critical points in this case? Here it is convenient to consider Fig. 66 and the two-dimensional vector field

[3] The terms *trajectory* and *characteristic* are used by some writers.

[4] For this reason, some writers use the term *equilibrium point* instead of critical point.

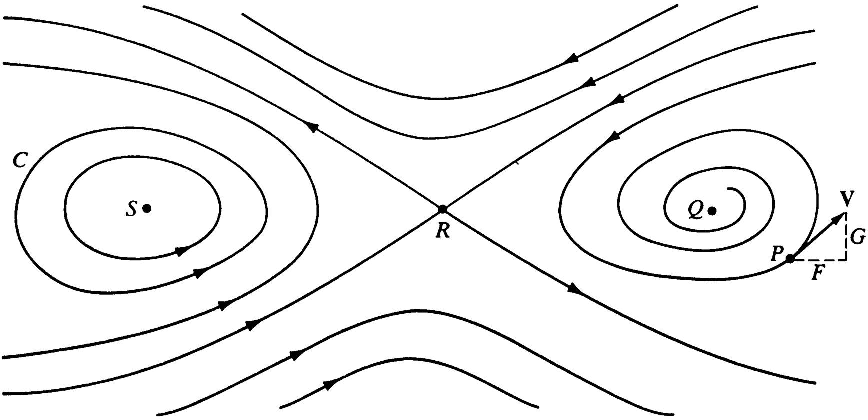

FIGURE 66

defined by

$$\mathbf{V}(x,y) = F(x,y)\mathbf{i} + G(x,y)\mathbf{j},$$

which at a typical point $P = (x,y)$ has horizontal component $F(x,y)$ and vertical component $G(x,y)$. Since $dx/dt = F$ and $dy/dt = G$, this vector is tangent to the path at P and points in the direction of increasing t. If we think of t as time, then **V** can be interpreted as the velocity vector of a particle moving along the path. We can also imagine that the entire phase plane is filled with particles, and that each path is the trail of a moving particle preceded and followed by many others on the same path and accompanied by yet others on nearby paths. This situation can be described as a two-dimensional *fluid motion*; and since the system (6) is autonomous, which means that the vector $\mathbf{V}(x,y)$ at a fixed point (x,y) does not change with time, the fluid motion is *stationary*. The paths are the trajectories of the moving particles, and the critical points Q, R, and S are points of zero velocity where the particles are at rest (i.e., stagnation points of the fluid motion).

The most striking features of the fluid motion illustrated in Fig. 66 are:

(a) the critical points;
(b) the arrangement of the paths near critical points;
(c) the stability or instability of critical points, that is, whether a particle near such a point remains near or wanders off into another part of the plane;
(d) closed paths (like C in the figure), which correspond to periodic solutions.

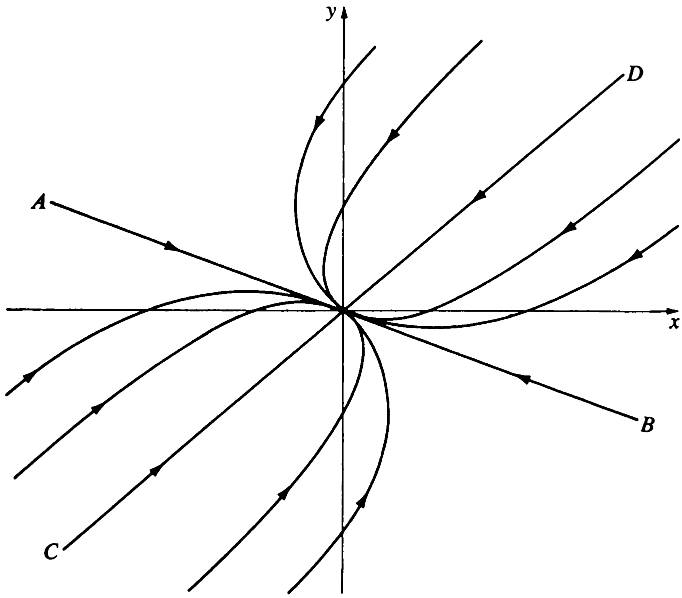

FIGURE 67

These features constitute a major part of the *phase portrait* (or overall picture of the paths) of the system (6). Since in general nonlinear equations and systems cannot be solved explicitly, the purpose of the qualitative theory discussed in this chapter is to discover as much as possible about the phase portrait directly from the functions F and G. To gain some insight into the sort of information we might hope to obtain, observe that if $x(t)$ is a periodic solution of the dynamical equation (4), then its derivative $y(t) = dx/dt$ is also periodic and the corresponding path of the system (5) is therefore closed. Conversely, if any path of (5) is closed, then (4) has a periodic solution. As a concrete example of the application of this idea, we point out that the van der Pol equation—which cannot be solved—can nevertheless be shown to have a unique periodic solution (if $\mu > 0$) by showing that its equivalent autonomous system has a unique closed path.

PROBLEMS

1. Derive equation (2) by applying Newton's second law of motion to the bob of the pendulum.
2. Let (x_0,y_0) be a point in the phase plane. If $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$ are solutions of (6) such that $x_1(t_1) = x_0$, $y_1(t_1) = y_0$ and $x_2(t_2) = x_0$, $y_2(t_2) = y_0$ for suitable t_1 and t_2, show that there exists a constant c such that

$$x_1(t + c) = x_2(t) \qquad \text{and} \qquad y_1(t + c) = y_2(t).$$

3. Describe the relation between the phase portraits of the systems

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -F(x,y) \\ \dfrac{dy}{dt} = -G(x,y). \end{cases}$$

4. Describe the phase portrait of each of the following systems:

(a) $\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 0; \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 2; \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 0; \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -y. \end{cases}$

5. The critical points and paths of equation (4) are by definition those of the equivalent system (5). Find the critical points of equations (1), (2), and (3).

6. Find the critical points of

(a) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - (x^3 + x^2 - 2x) = 0;$

(b) $\begin{cases} \dfrac{dx}{dt} = y^2 - 5x + 6 \\ \dfrac{dy}{dt} = x - y. \end{cases}$

7. Find all solutions of the nonautonomous system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = x + e^t, \end{cases}$$

and sketch (in the xy-plane) some of the curves defined by these solutions.

59 TYPES OF CRITICAL POINTS. STABILITY

Consider an autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y). \end{cases} \tag{1}$$

We assume, as usual, that the functions F and G are continuous and have continuous first partial derivatives throughout the xy-plane. The critical points of (1) can be found, at least in principle, by solving the simultaneous equations $F(x,y) = 0$ and $G(x,y) = 0$. There are four simple types of critical points that occur quite frequently, and our purpose in this section is to describe them in terms of the configurations of nearby paths. First, however, we need two definitions.

Let (x_0,y_0) be an isolated critical point of (1). If $C = [x(t),y(t)]$ is a path of (1), then we say that C *approaches* (x_0,y_0) as $t \to \infty$ if

$$\lim_{t\to\infty} x(t) = x_0 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = y_0.\text{[5]} \tag{2}$$

Geometrically, this means that if $P = (x,y)$ is a point that traces out C in accordance with the equations $x = x(t)$ and $y = y(t)$, then $P \to (x_0,y_0)$ as $t \to \infty$. If it is also true that

$$\lim_{t\to\infty} \frac{y(t) - y_0}{x(t) - x_0} \tag{3}$$

exists, or if the quotient in (3) becomes either positively or negatively infinite as $t \to \infty$, then we say that C *enters* the critical point (x_0,y_0) as $t \to \infty$. The quotient in (3) is the slope of the line joining (x_0,y_0) and the point P with coordinates $x(t)$ and $y(t)$, so the additional requirement means that this line approaches a definite direction as $t \to \infty$. In the above definitions, we may also consider limits as $t \to -\infty$. It is clear that these properties are properties of the path C, and do not depend on which solution is used to represent this path.

It is sometimes possible to find explicit solutions of the system (1), and these solutions can then be used to determine the paths. In most cases, however, to find the paths it is necessary to eliminate t between the two equations of the system, which yields

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}. \tag{4}$$

This first order equation gives the slope of the tangent to the path of (1) that passes through the point (x,y), provided that the functions F and G are not both zero at this point. In this case, of course, the point is a critical point and no path passes through it. The paths of (1) therefore coincide with the one-parameter family of integral curves of (4), and this

[5] It can be proved that if (2) is true for some solution $x(t)$, $y(t)$, then (x_0, y_0) is necessarily a critical point. See F. G. Tricomi, *Differential Equations*, p. 47, Blackie, Glasgow, 1961.

family can often be obtained by the methods of Chapter 2. It should be noted, however, that while the paths of (1) are directed curves, the integral curves of (4) have no direction associated with them. Each of these techniques for determining the paths will be illustrated in the examples below.

We now give geometric descriptions of the four main types of critical points. In each case we assume that the critical point under discussion is the origin $O = (0,0)$.

Nodes. A critical point like that in Fig. 67 is called a *node*. Such a point is approached and also entered by each path as $t \to \infty$ (or as $t \to -\infty$). For the node shown in Fig. 67, there are four half-line paths, AO, BO, CO, and DO, which together with the origin make up the lines AB and CD. All other paths resemble parts of parabolas, and as each of these paths approaches O its slope approaches that of the line AB.

Example 1. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -x + 2y. \end{cases} \tag{5}$$

It is clear that the origin is the only critical point, and the general solution can be found quite easily by the methods of Section 56:

$$\begin{cases} x = c_1 e^t \\ y = c_1 e^t + c_2 e^{2t}. \end{cases} \tag{6}$$

When $c_1 = 0$, we have $x = 0$ and $y = c_2 e^{2t}$. In this case the path (Fig. 68) is the positive y-axis when $c_2 > 0$, and the negative y-axis when $c_2 < 0$, and each path approaches and enters the origin as $t \to -\infty$. When $c_2 = 0$, we have $x = c_1 e^t$ and $y = c_1 e^t$. This path is the half-line $y = x$, $x > 0$, when $c_1 > 0$, and the half-line $y = x$, $x < 0$, when $c_1 < 0$, and again both paths approach and enter the origin as $t \to -\infty$. When both c_1 and c_2 are $\neq 0$, the paths lie on the parabolas $y = x + (c_2/c_1^2)x^2$, which go through the origin with slope 1. It should be understood that each of these paths consists of only part of a parabola, the part with $x > 0$ if $c_1 > 0$, and the part with $x < 0$ if $c_1 < 0$. Each of these paths also approaches and enters the origin as $t \to -\infty$; this can be seen at once from (6). If we proceed directly from (5) to the differential equation

$$\frac{dy}{dx} = \frac{-x + 2y}{x}, \tag{7}$$

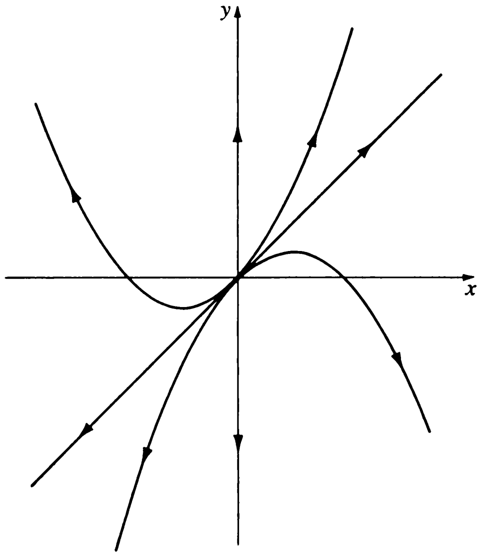

FIGURE 68

giving the slope of the tangent to the path through (x,y) [provided $(x,y) \neq (0,0)$], then on solving (7) as a homogeneous equation, we find that $y = x + cx^2$. This procedure yields the curves on which the paths lie (except those on the y axis), but gives no information about the manner in which the paths are traced out. It is clear from this discussion that the critical point (0,0) of the system (5) is a node.

Saddle points. A critical point like that in Fig. 69 is called a *saddle point*. It is approached and entered by two half-line paths AO and BO as $t \to \infty$, and these two paths lie on a line AB. It is also approached and entered by two half-line paths CO and DO at $t \to -\infty$, and these two paths lie on another line CD. Between the four half-line paths there are four regions, and each contains a family of paths resembling hyperbolas. These paths do not approach O as $t \to \infty$ or as $t \to -\infty$, but instead are asymptotic to one or another of the half-line paths as $t \to \infty$ and as $t \to -\infty$.

Centers. A *center* (sometimes called a *vortex*) is a critical point that is surrounded by a family of closed paths. It is not approached by any path as $t \to \infty$ or as $t \to -\infty$.

Example 2. The system

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x \end{cases} \tag{8}$$

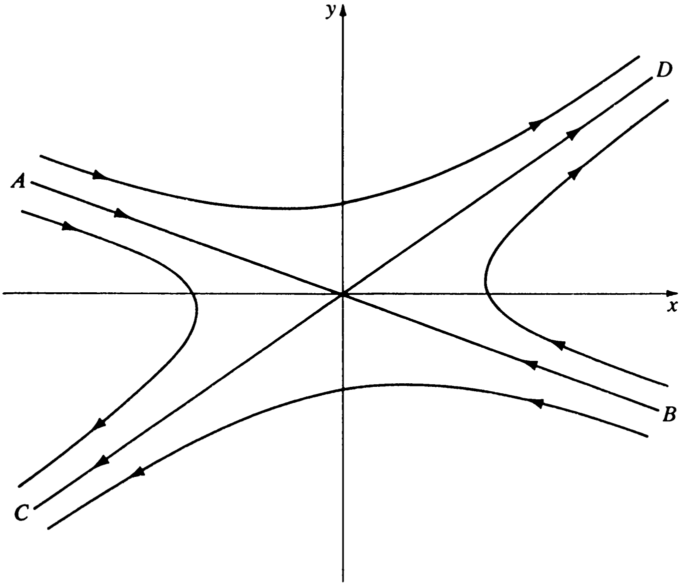

FIGURE 69

has the origin as its only critical point, and its general solution is

$$\begin{cases} x = -c_1 \sin t + c_2 \cos t \\ y = c_1 \cos t + c_2 \sin t. \end{cases} \tag{9}$$

The solution satisfying the conditions $x(0) = 1$ and $y(0) = 0$ is clearly

$$\begin{cases} x = \cos t \\ y = \sin t; \end{cases} \tag{10}$$

and the solution determined by $x(0) = 0$ and $y(0) = -1$ is

$$\begin{cases} x = \sin t = \cos\left(t - \dfrac{\pi}{2}\right) \\[2ex] y = -\cos t = \sin\left(t - \dfrac{\pi}{2}\right). \end{cases} \tag{11}$$

These two different solutions define the same path C (Fig. 70), which is evidently the circle $x^2 + y^2 = 1$. Both (10) and (11) show that this path is traced out in the counterclockwise direction. If we eliminate t between the equations of the system, we get

$$\frac{dy}{dx} = -\frac{x}{y},$$

whose general solution $x^2 + y^2 = c^2$ yields all the paths (but without their directions). It is obvious that the critical point (0,0) of the system (8) is a center.

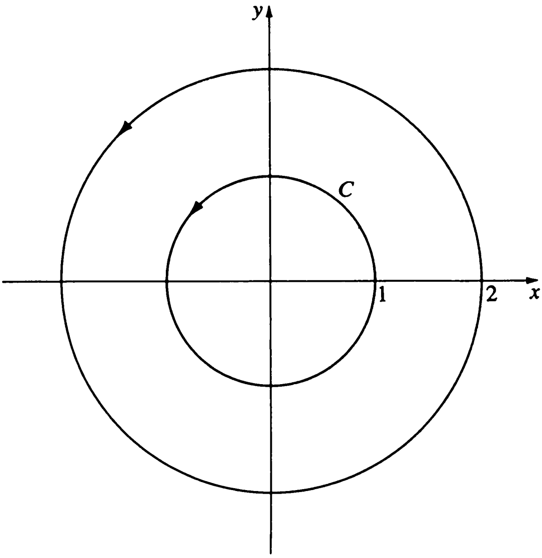

FIGURE 70

Spirals. A critical point like that in Fig. 71 is called a *spiral* (or sometimes a *focus*). Such a point is approached in a spiral-like manner by a family of paths that wind around it an infinite number of times as $t \to \infty$ (or as $t \to -\infty$). Note particularly that while the paths approach O, they do not enter it. That is, a point P moving along such a path approaches O as $t \to \infty$ (or as $t \to -\infty$), but the line OP does not approach any definite direction.

Example 3. If a is an arbitrary constant, then the system

$$\begin{cases} \dfrac{dx}{dt} = ax - y \\ \dfrac{dy}{dt} = x + ay \end{cases} \tag{12}$$

has the origin as its only critical point (why?). The differential equation of the paths,

$$\frac{dy}{dx} = \frac{x + ay}{ax - y}, \tag{13}$$

is most easily solved by introducing polar coordinates r and θ defined by $x = r\cos\theta$ and $y = r\sin\theta$. Since

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x},$$

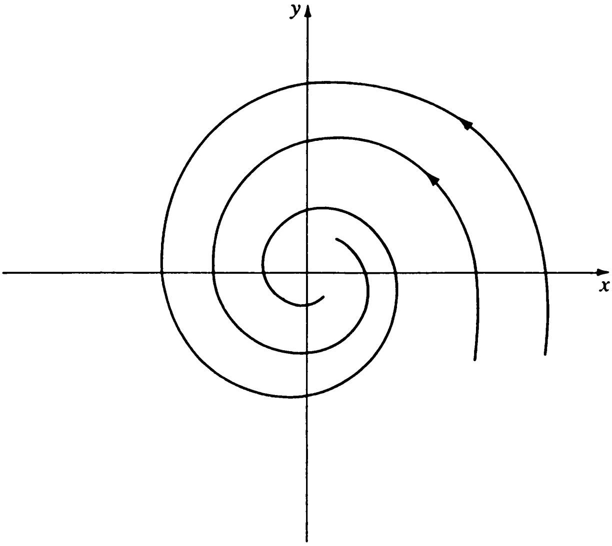

FIGURE 71

we see that

$$r\frac{dr}{dx} = x + y\frac{dy}{dx} \qquad \text{and} \qquad r^2\frac{d\theta}{dx} = x\frac{dy}{dx} - y.$$

With the aid of these equations, (13) can easily be written in the very simple form

$$\frac{dr}{d\theta} = ar,$$

so

$$r = ce^{a\theta} \tag{14}$$

is the polar equation of the paths. The two possible spiral configurations are shown in Fig. 72 and the direction in which these paths are traversed can be seen from the fact that $dx/dt = -y$ when $x = 0$. If $a = 0$, then (12) collapses to (8) and (14) becomes $r = c$, which is the polar equation of the family $x^2 + y^2 = c^2$ of all circles centered on the origin. This example therefore generalizes Example 2; and since the center shown in Fig. 70 stands on the borderline between the spirals of Fig. 72, a critical point that is a center is often called a *borderline case*. We will encounter other borderline cases in the next section.

We now introduce the concept of *stability* as it applies to the critical points of the system (1).

It was pointed out in the previous section that one of the most important questions in the study of a physical system is that of its steady states. However, a steady state has little physical significance unless it has a reasonable degree of permanence, i.e., unless it is stable. As a simple

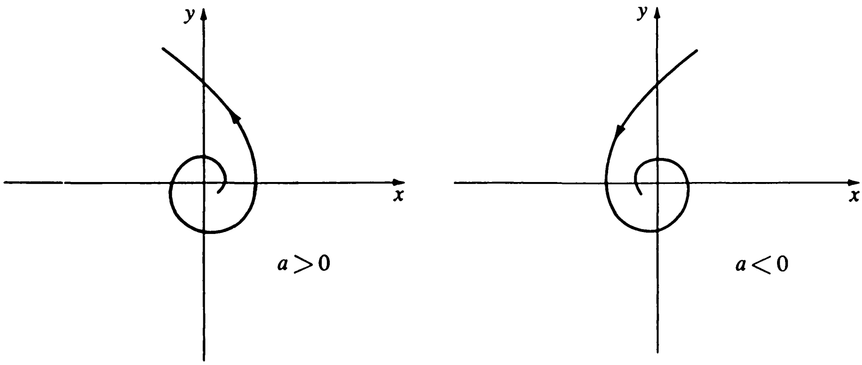

FIGURE 72

example, consider the pendulum of Fig. 73. There are two steady states possible here: when the bob is at rest at the highest point, and when the bob is at rest at the lowest point. The first state is clearly unstable, and the second is stable. We now recall that a steady state of a simple physical system corresponds to an equilibrium point (or critical point) in the phase plane. These considerations suggest in a general way that a small disturbance at an unstable equilibrium point leads to a larger and larger departure from this point, while the opposite is true at a stable equilibrium point.

We now formulate these intuitive ideas in a more precise way. Consider an isolated critical point of the system (1), and assume for the sake of convenience that this point is located at the origin $O = (0,0)$ of the phase plane. This critical point is said to be *stable* if for each positive number R there exists a positive number $r \leq R$ such that every path which is inside the circle $x^2 + y^2 = r^2$ for some $t = t_0$ remains inside the

FIGURE 73

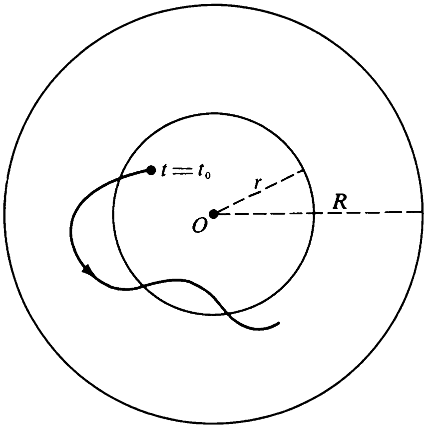

FIGURE 74

circle $x^2 + y^2 = R^2$ for all $t > t_0$ (Fig. 74). Loosely speaking, a critical point is stable if all paths that get sufficiently close to the point stay close to the point. Further, our critical point is said to be *asymptotically stable* if it is stable and there exists a circle $x^2 + y^2 = r_0^2$ such that every path which is inside this circle for some $t = t_0$ approaches the origin as $t \to \infty$. Finally, if our critical point is not stable, then it is called *unstable*.

As examples of these concepts, we point out that the node in Fig. 68, the saddle point in Fig. 69, and the spiral on the left in Fig. 72 are unstable, while the center in Fig. 70 is stable but not asymptotically stable. The node in Fig. 67, the spiral in Fig. 71, and the spiral on the right in Fig. 72 are asymptotically stable.

PROBLEMS

1. For each of the following nonlinear systems: (i) find the critical points; (ii) find the differential equation of the paths; (iii) solve this equation to find the paths; and (iv) sketch a few of the paths and show the direction of increasing t.

(a) $\begin{cases} \dfrac{dx}{dt} = y(x^2 + 1) \\ \dfrac{dy}{dt} = 2xy^2; \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = e^y \\ \dfrac{dy}{dt} = e^y \cos x; \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = y(x^2 + 1) \\ \dfrac{dy}{dt} = -x(x^2 + 1); \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = 2x^2y^2. \end{cases}$

2. Each of the following linear systems has the origin as an isolated critical point. (i) Find the general solution. (ii) Find the differential equation of the paths.

(iii) Solve the equation found in (ii) and sketch a few of the paths, showing the direction of increasing t. (iv) Discuss the stability of the critical point.

(a) $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y; \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -2y; \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = 4y \\ \dfrac{dy}{dt} = -x. \end{cases}$

3. Sketch the phase portrait of the equation $d^2x/dt^2 = 2x^3$, and show that it has an unstable isolated critical point at the origin.

60 CRITICAL POINTS AND STABILITY FOR LINEAR SYSTEMS

Our goal in this chapter is to learn as much as we can about nonlinear differential equations by studying the phase portraits of nonlinear autonomous systems of the form

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y). \end{cases}$$

One aspect of this is the problem of classifying the critical points of such a system with respect to their nature and stability. It will be seen in Section 62 that under suitable conditions this problem can be solved for a given nonlinear system by studying a related linear system. We therefore devote this section to a complete analysis of the critical points of linear autonomous systems.

We consider the system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y, \end{cases} \tag{1}$$

which has the origin (0,0) as an obvious critical point. We assume throughout this section that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \tag{2}$$

so that (0,0) is the only critical point. It was proved in Section 56 that (1) has a nontrivial solution of the form

$$\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$$

whenever m is a root of the quadratic equation

$$m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0, \tag{3}$$

which is called the *auxiliary equation* of the system. Observe that condition (2) implies that zero cannot be a root of (3).

Let m_1 and m_2 be the roots of (3). We shall prove that the nature of the critical point (0,0) of the system (1) is determined by the nature of the numbers m_1 and m_2. It is reasonable to expect that three possibilities will occur, according as m_1 and m_2 are real and distinct, real and equal, or conjugate complex. Unfortunately the situation is a little more complicated than this, and it is necessary to consider five cases, subdivided as follows.

Major cases:

Case A. The roots m_1 and m_2 are real, distinct, and of the same sign (node).

Case B. The roots m_1 and m_2 are real, distinct, and of opposite signs (saddle point).

Case C. The roots m_1 and m_2 are conjugate complex but not pure imaginary (spiral).

Borderline cases:

Case D. The roots m_1 and m_2 are real and equal (node).

Case E. The roots m_1 and m_2 are pure imaginary (center).

The reason for the distinction between the major cases and the borderline cases will become clear in Section 62. For the present it suffices to remark that while the borderline cases are of mathematical interest they have little significance for applications, because the circumstances defining them are unlikely to arise in physical problems. We now turn to the proofs of the assertions in parentheses.

Case A. If the roots m_1 and m_2 are real, distinct, and of the same sign, then the critical point (0,0) is a node.

Proof. We begin by assuming that m_1 and m_2 are both negative, and we choose the notation so that $m_1 < m_2 < 0$. By Section 56, the general solution of (1) in this case is

$$\begin{cases} x = c_1A_1e^{m_1t} + c_2A_2e^{m_2t} \\ y = c_1B_1e^{m_1t} + c_2B_2e^{m_2t}, \end{cases} \tag{4}$$

where the A's and B's are definite constants such that $B_1/A_1 \neq B_2/A_2$, and where the c's are arbitrary constants. When $c_2 = 0$, we obtain the solutions

$$\begin{cases} x = c_1A_1e^{m_1t} \\ y = c_1B_1e^{m_1t}, \end{cases} \tag{5}$$

and when $c_1 = 0$, we obtain the solutions

$$\begin{cases} x = c_2 A_2 e^{m_2 t} \\ y = c_2 B_2 e^{m_2 t}. \end{cases} \tag{6}$$

For any $c_1 > 0$, the solution (5) represents a path consisting of half of the line $A_1 y = B_1 x$ with slope B_1/A_1; and for any $c_1 < 0$, it represents a path consisting of the other half of this line (the half on the other side of the origin). Since $m_1 < 0$, both of these half-line paths approach (0,0) as $t \to \infty$; and since $y/x = B_1/A_1$, both enter (0,0) with slope B_1/A_1 (Fig. 75). In exactly the same way, the solutions (6) represent two half-line paths lying on the line $A_2 y = B_2 x$ with slope B_2/A_2. These two paths also approach (0,0) as $t \to \infty$, and enter it with slope B_2/A_2.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (4) represents curved paths. Since $m_1 < 0$ and $m_2 < 0$, these paths also approach (0,0) as $t \to \infty$. Furthermore, since $m_1 - m_2 < 0$ and

$$\frac{y}{x} = \frac{c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t}}{c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t}} = \frac{(c_1 B_1/c_2) e^{(m_1 - m_2)t} + B_2}{(c_1 A_1/c_2) e^{(m_1 - m_2)t} + A_2},$$

it is clear that $y/x \to B_2/A_2$ as $t \to \infty$, so all of these paths enter (0,0) with slope B_2/A_2. Figure 75 presents a qualitative picture of the situation. It is evident that our critical point is a node, and that it is asymptotically stable.

If m_1 and m_2 are both positive, and if we choose the notation so that $m_1 > m_2 > 0$, then the situation is exactly the same except that all the paths now approach and enter (0,0) as $t \to -\infty$. The picture of the paths

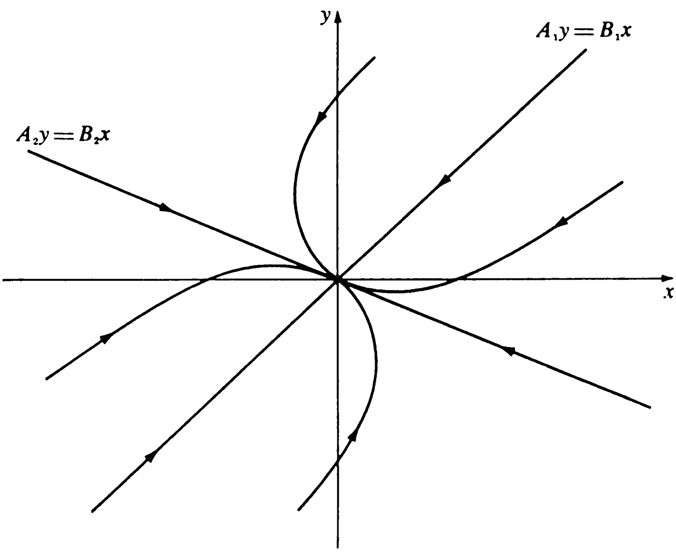

FIGURE 75

given in Fig. 75 is unchanged except that the arrows showing their directions are all reversed. We still have a node, but now it is unstable.

Case B. If the roots m_1 and m_2 are real, distinct, and of opposite signs, then the critical point (0,0) is a saddle point.

Proof. We may choose the notation so that $m_1 < 0 < m_2$. The general solution of (1) can still be written in the form (4), and again we have particular solutions of the forms (5) and (6). The two half-line paths represented by (5) still approach and enter (0,0) as $t \to \infty$, but this time the two half-line paths represented by (6) approach and enter (0,0) as $t \to -\infty$. If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (4) still represents curved paths, but since $m_1 < 0 < m_2$, none of these paths approaches (0,0) as $t \to \infty$ or $t \to -\infty$. Instead, as $t \to \infty$, each of these paths is asymptotic to one of the half-line paths represented by (6); and as $t \to -\infty$, each is asymptotic to one of the half-line paths represented by (5). Figure 76 gives a qualitative picture of this behavior. In this case the critical point is a saddle point, and it is obviously unstable.

Case C. If the roots m_1 and m_2 are conjugate complex but not pure imaginary, then the critical point (0,0) is a spiral.

Proof. In this case we can write m_1 and m_2 in the form $a \pm ib$ where a and b are nonzero real numbers. Also, for later use, we observe that the

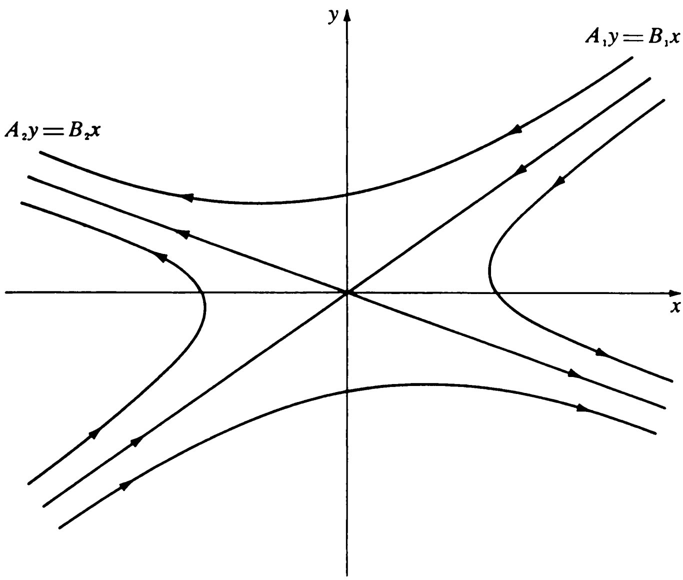

FIGURE 76

discriminant D of equation (3) is negative:

$$\begin{aligned} D &= (a_1 + b_2)^2 - 4(a_1 b_2 - a_2 b_1) \\ &= (a_1 - b_2)^2 + 4a_2 b_1 < 0. \end{aligned} \tag{7}$$

By Section 56, the general solution of (1) in this case is

$$\begin{cases} x = e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y = e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)], \end{cases} \tag{8}$$

where the A's and B's are definite constants and the c's are arbitrary constants.

Let us first assume that $a < 0$. Then it is clear from formulas (8) that $x \to 0$ and $y \to 0$ as $t \to \infty$, so all the paths approach (0,0) as $t \to \infty$. We now prove that the paths do not enter the point (0,0) as $t \to \infty$, but instead wind around it in a spiral-like manner. To accomplish this we introduce the polar cordinate θ and show that, along any path, $d\theta/dt$ is either positive for all t or negative for all t. We begin with the fact that $\theta = \tan^{-1}(y/x)$, so

$$\frac{d\theta}{dt} = \frac{x\, dy/dt - y\, dx/dt}{x^2 + y^2};$$

and by using equations (1) we obtain

$$\frac{d\theta}{dt} = \frac{a_2 x^2 + (b_2 - a_1)xy - b_1 y^2}{x^2 + y^2}. \tag{9}$$

Since we are interested only in solutions that represent paths, we assume that $x^2 + y^2 \neq 0$. Now (7) implies that a_2 and b_1 have opposite signs. We consider the case in which $a_2 > 0$ and $b_1 < 0$. When $y = 0$, (9) yields $d\theta/dt = a_2 > 0$. If $y \neq 0$, $d\theta/dt$ cannot be 0; for if it were, then (9) would imply that

$$a_2 x^2 + (b_2 - a_1)xy - b_1 y^2 = 0$$

or

$$a_2\left(\frac{x}{y}\right)^2 + (b_2 - a_1)\frac{x}{y} - b_1 = 0 \tag{10}$$

for some real number x/y—and this cannot be true because the discriminant of the quadratic equation (10) is D, which is negative by (7). This shows that $d\theta/dt$ is always positive when $a_2 > 0$, and in the same way we see that it is always negative when $a_2 < 0$. Since by (8), x and y change sign infinitely often as $t \to \infty$, all paths must spiral in to the origin (counterclockwise or clockwise according as $a_2 > 0$ or $a_2 < 0$). The critical point in this case is therefore a spiral, and it is asymptotically stable.

If $a > 0$, the situation is the same except that the paths approach (0,0) as $t \to -\infty$ and the critical point is unstable. Figure 72 illustrates the arrangement of the paths when $a_2 > 0$.

Case D. If the roots m_1 and m_2 are real and equal, then the critical point (0,0) is a node.

***Proof*.** We begin by assuming that $m_1 = m_2 = m < 0$. There are two subcases that require separate discussion: (i) $a_1 = b_2 \neq 0$ and $a_2 = b_1 = 0$; (ii) all other possibilities leading to a double root of equation (3).

We first consider the subcase (i), which is the situation described in the footnote in Section 56. If a denotes the common value of a_1 and b_2, then equation (3) becomes $m^2 - 2am + a^2 = 0$ and $m = a$. The system (1) is thus

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = ay, \end{cases}$$

and its general solution is

$$\begin{cases} x = c_1 e^{mt} \\ y = c_2 e^{mt}, \end{cases} \tag{11}$$

where c_1 and c_2 are arbitrary constants. The paths defined by (11) are half-lines of all possible slopes (Fig. 77), and since $m < 0$ we see that each path approaches and enters (0,0) as $t \to \infty$. The critical point is therefore a node, and it is asymptotically stable. If $m > 0$, we have the same situation except that the paths enter (0,0) as $t \to -\infty$, the arrows in Fig. 77 are reversed, and (0,0) is unstable.

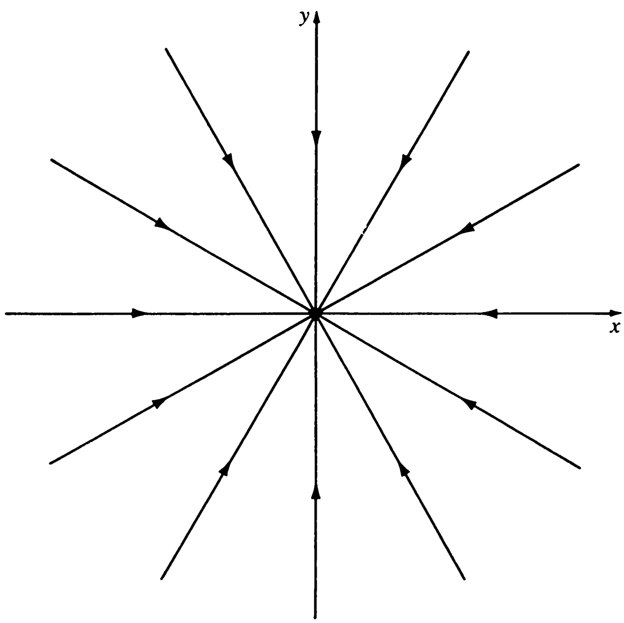

FIGURE 77

We now discuss subcase (ii). By formulas 56-(20) and Problem 56-(4), the general solution of (1) can be written in the form

$$\begin{cases} x = c_1Ae^{mt} + c_2(A_1 + At)e^{mt} \\ y = c_1Be^{mt} + c_2(B_1 + Bt)e^{mt}, \end{cases} \tag{12}$$

where the A's and B's are definite constants and the c's are arbitrary constants. When $c_2 = 0$, we obtain the solutions

$$\begin{cases} x = c_1Ae^{mt} \\ y = c_1Be^{mt}. \end{cases} \tag{13}$$

We know that these solutions represent two half-line paths lying on the line $Ay = Bx$ with slope B/A, and since $m < 0$ both paths approach (0,0) as $t \to \infty$ (Fig. 78). Also, since $y/x = B/A$, both paths enter (0,0) with slope B/A. If $c_2 \neq 0$, the solutions (12) represent curved paths, and since $m < 0$ it is clear from (12) that these paths approach (0,0) as $t \to \infty$. Furthermore, it follows from

$$\frac{y}{x} = \frac{c_1Be^{mt} + c_2(B_1 + Bt)e^{mt}}{c_1Ae^{mt} + c_2(A_1 + At)e^{mt}} = \frac{c_1B/c_2 + B_1 + Bt}{c_1A/c_2 + A_1 + At}$$

that $y/x \to B/A$ as $t \to \infty$, so these curved paths all enter (0,0) with slope B/A. We also observe that $y/x \to B/A$ as $t \to -\infty$. Figure 78 gives a qualitative picture of the arrangement of these paths. It is clear that (0,0) is a node that is asymptotically stable. If $m > 0$, the situation is unchanged

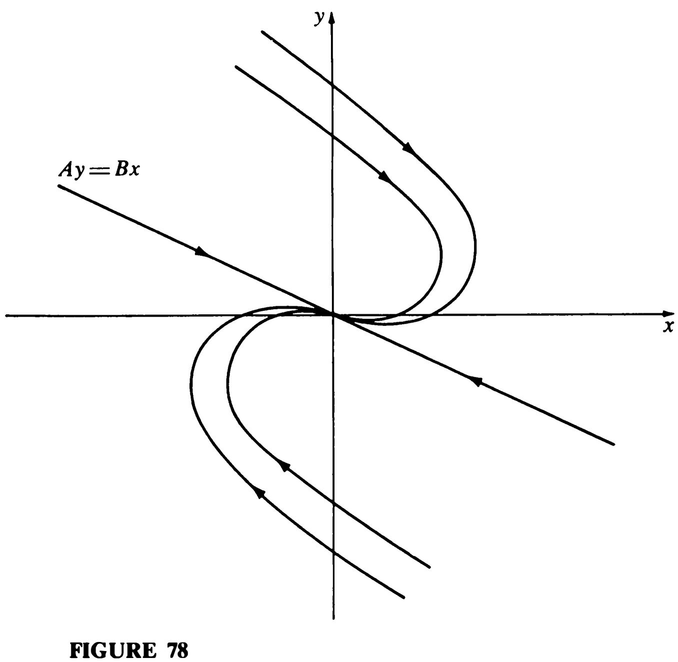

FIGURE 78

except that the directions of the paths are reversed and the critical point is unstable.

Case E. If the roots m_1 and m_2 are pure imaginary, then the critical point (0,0) is a center.

Proof. It suffices here to refer back to the discussion of Case C, for now m_1 and m_2 are of the form $a \pm ib$ with $a = 0$ and $b \neq 0$. The general solution of (1) is therefore given by (8) with the exponential factor missing, so $x(t)$ and $y(t)$ are periodic and each path is a closed curve surrounding the origin. As Fig. 79 suggests, these curves are actually ellipses; this can be proved (see Problem 5) by solving the differential equation of the paths,

$$\frac{dy}{dx} = \frac{a_2x + b_2y}{a_1x + b_1y}. \tag{14}$$

Our critical point (0,0) is evidently a center that is stable but not asymptotically stable.

In the above discussions we have made a number of statements about stability. It will be convenient to summarize this information as follows.

Theorem A. *The critical point* (0,0) *of the linear system* (1) *is stable if and only if both roots of the auxiliary equation* (3) *have nonpositive real parts, and it is asymptotically stable if and only if both roots have negative real parts.*

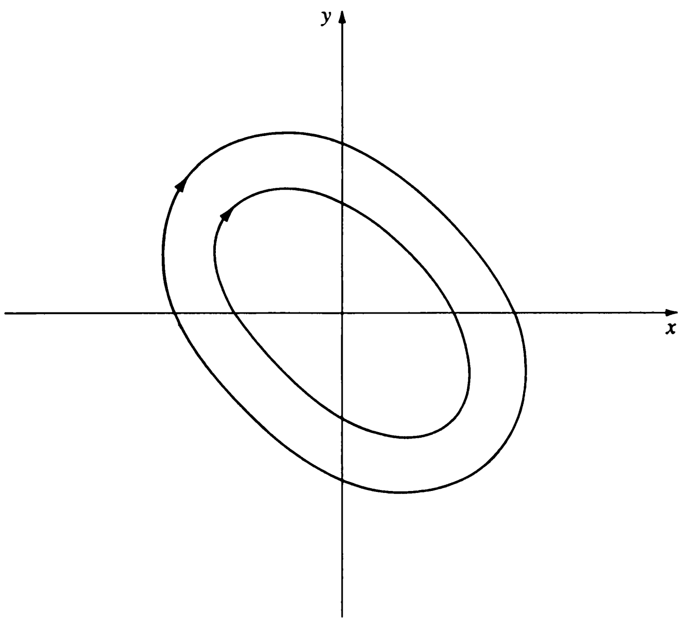

FIGURE 79

If we now write equation (3) in the form

$$(m - m_1)(m - m_2) = m^2 + pm + q = 0, \tag{15}$$

so that $p = -(m_1 + m_2)$ and $q = m_1 m_2$, then our five cases can be described just as readily in terms of the coefficients p and q as in terms of the roots m_1 and m_2. In fact, if we interpret these cases in the pq-plane, then we arrive at a striking diagram (Fig. 80) that displays at a glance the nature and stability properties of the critical point (0,0). The first thing to notice is that the p-axis $q = 0$ is excluded, since by condition (2) we know that $m_1 m_2 \neq 0$. In the light of what we have learned about our five cases, all of the information contained in the diagram follows directly from the fact that

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Thus, above the parabola $p^2 - 4q = 0$, we have $p^2 - 4q < 0$, so m_1 and m_2 are conjugate complex numbers that are pure imaginary if and only if $p = 0$; these are Cases C and E comprising the spirals and centers. Below the p-axis we have $q < 0$, which means that m_1 and m_2 are real, distinct, and have opposite signs; this yields the saddle points of Case B. And finally, the zone between these two regions (including the parabola but excluding the p-axis) is characterized by the relations $p^2 - 4q \geq 0$ and $q > 0$, so m_1 and m_2 are real and of the same sign; here we have the nodes of Cases A and D. Furthermore, it is clear that there is precisely

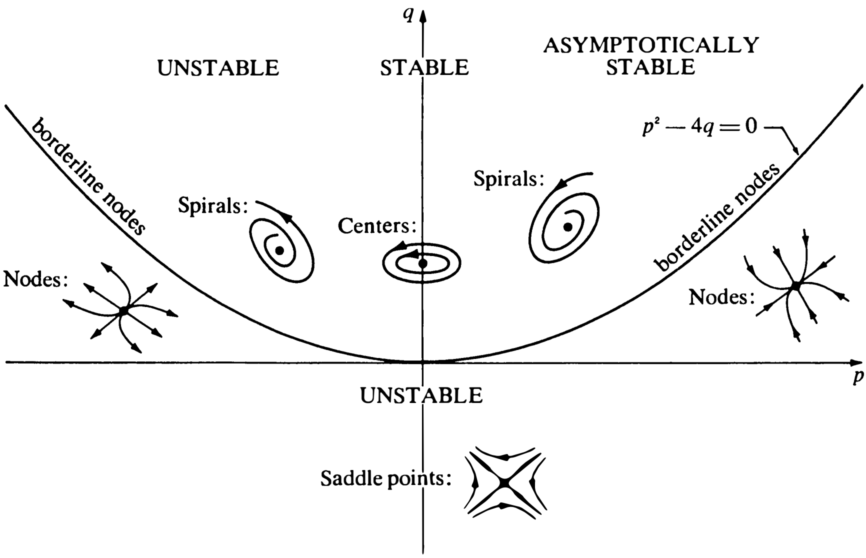

FIGURE 80

one region of asymptotic stability: the first quadrant. We state this formally as follows.

Theorem B. *The critical point* (0,0) *of the linear system* (1) *is asymptotically stable if and only if the coefficients* $p = -(a_1 + b_2)$ *and* $q = a_1b_2 - a_2b_1$ *of the auxiliary equation* (3) *are both positive.*

Finally, it should be emphasized that we have studied the paths of our linear system near a critical point by analyzing explicit solutions of the system. In the next two sections we enter more fully into the spirit of the subject by investigating similar problems for nonlinear systems, which in general cannot be solved explicitly.

PROBLEMS

1. Determine the nature and stability properties of the critical point (0,0) for each of the following linear autonomous systems:

(a) $\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 3y; \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = -x - 2y \\ \dfrac{dy}{dt} = 4x - 5y; \end{cases}$

(c) $\begin{cases} \dfrac{dx}{dt} = -3x + 4y \\ \dfrac{dy}{dt} = -2x + 3y; \end{cases}$

(d) $\begin{cases} \dfrac{dx}{dt} = 5x + 2y \\ \dfrac{dy}{dt} = -17x - 5y; \end{cases}$

(e) $\begin{cases} \dfrac{dx}{dt} = -4x - y \\ \dfrac{dy}{dt} = x - 2y; \end{cases}$

(f) $\begin{cases} \dfrac{dx}{dt} = 4x - 3y \\ \dfrac{dy}{dt} = 8x - 6y; \end{cases}$

(g) $\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y. \end{cases}$

2. If $a_1b_2 - a_2b_1 = 0$, show that the system (1) has infinitely many critical points, none of which are isolated.

3. (a) If $a_1b_2 - a_2b_1 \neq 0$, show that the system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + c_1 \\ \dfrac{dy}{dt} = a_2x + b_2y + c_2 \end{cases}$$

has a single isolated critical point (x_0, y_0).

(b) Show that the system in (a) can be written in the form of (1) by means of the change of variables $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$.

(c) Find the critical point of the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - 2y + 10 \\ \dfrac{dy}{dt} = 11x - 8y + 49, \end{cases}$$

write the system in the form of (1) by changing the variables, and determine the nature and stability properties of the critical point.

4. In Section 20 we studied the free vibrations of a mass attached to a spring by solving the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = 0,$$

where $b \geq 0$ and $a > 0$ are constants representing the viscosity of the medium and the stiffness of the spring, respectively. Consider the equivalent autonomous system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -a^2x - 2by, \end{cases} \qquad (*)$$

which has (0,0) as its only critical point.

(a) Find the auxiliary equation of (*). What are p and q?

(b) For each of the following four cases, describe the nature and stability properties of the critical point, and give a brief physical interpretation of the corresponding motion of the mass:

(i) $b = 0$; (iii) $b = a$;
(ii) $0 < b < a$; (iv) $b > a$.

5. Solve equation (14) under the hypotheses of Case E, and show that the result is a one-parameter family of ellipses surrounding the origin. *Hint*: Recall that if $Ax^2 + Bxy + Cy^2 = D$ is the equation of a real curve, then the curve is an ellipse if and only if the discriminant $B^2 - 4AC$ is negative.

61 STABILITY BY LIAPUNOV'S DIRECT METHOD

It is intuitively clear that if the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable. This idea was generalized by Liapunov[6] into a simple but powerful

[6] Alexander Mikhailovich Liapunov (1857–1918) was a Russian mathematician and mechanical engineer. He had the very rare merit of producing a doctoral dissertation of lasting value. This classic work was originally published in 1892 in Russian, but is now available in an English translation, *Stability of Motion,* Academic Press, New York, 1966. Liapunov died by violence in Odessa, which cannot be considered a surprising fate for a middle-class intellectual in the chaotic aftermath of the Russian Revolution.

method for studying stability problems in a broader context. We shall discuss Liapunov's method and some of its applications in this and the next section.

Consider an autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y), \end{cases} \tag{1}$$

and assume that this system has an isolated critical point, which as usual we take to be the origin (0,0).[7] Let $C = [x(t),y(t)]$ be a path of (1), and consider a function $E(x,y)$ that is continuous and has continuous first partial derivatives in a region containing this path. If a point (x,y) moves along the path in accordance with the equations $x = x(t)$ and $y = y(t)$, then $E(x,y)$ can be regarded as a function of t along C [we denote this function by $E(t)$] and its rate of change is

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G. \end{aligned} \tag{2}$$

This formula is at the heart of Liapunov's ideas, and in order to exploit it we need several definitions that specify the kinds of functions we shall be interested in.

Suppose that $E(x,y)$ is continuous and has continuous first partial derivatives in some region containing the origin. If E vanishes at the origin, so that $E(0,0) = 0$, then it is said to be *positive definite* if $E(x,y) > 0$ for $(x,y) \neq (0,0)$, and *negative definite* if $E(x,y) < 0$ for $(x,y) \neq (0,0)$. Similarly, E is called *positive semidefinite* if $E(0,0) = 0$ and $E(x,y) \geq 0$ for $(x,y) \neq (0,0)$, and *negative semidefinite* if $E(0,0) = 0$ and $E(x,y) \leq 0$ for $(x,y) \neq (0,0)$. It is clear that functions of the form $ax^{2m} + by^{2n}$, where a and b are positive constants and m and n are positive integers, are positive definite. Since $E(x,y)$ is negative definite if and only if $-E(x,y)$ is positive definite, functions of the form $ax^{2m} + by^{2n}$ with $a < 0$ and $b < 0$ are negative definite. The functions x^{2m}, y^{2m}, and $(x - y)^{2m}$ are not positive definite, but are nevertheless positive

[7] A critical point (x_0,y_0) can always be moved to the origin by a simple translation of coordinates $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$, so there is no loss of generality in assuming that it lies at the origin in the first place.

semidefinite. If $E(x,y)$ is positive definite, then $z = E(x,y)$ can be interpreted as the equation of a surface (Fig. 81) that resembles a paraboloid opening upward and tangent to the xy-plane at the origin.

A positive definite function $E(x,y)$ with the property that

$$\frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G \tag{3}$$

is negative semidefinite is called a *Liapunov function* for the system (1). By formula (2), the requirement that (3) be negative semidefinite means that $dE/dt \leq 0$—and therefore E is nonincreasing—along the paths of (1) near the origin. These functions generalize the concept of the total energy of a physical system. Their relevance for stability problems is made clear in the following theorem, which is Liapunov's basic discovery.

Theorem A. *If there exists a Liapunov function $E(x,y)$ for the system* (1), *then the critical point* (0,0) *is stable. Furthermore, if this function has the additional property that the function* (3) *is negative definite, then the critical point* (0,0) *is asymptotically stable.*

Proof. Let C_1 be a circle of radius $R > 0$ centered on the origin (Fig. 82), and assume also that C_1 is small enough to lie entirely in the domain of definition of the function E. Since $E(x,y)$ is continuous and positive

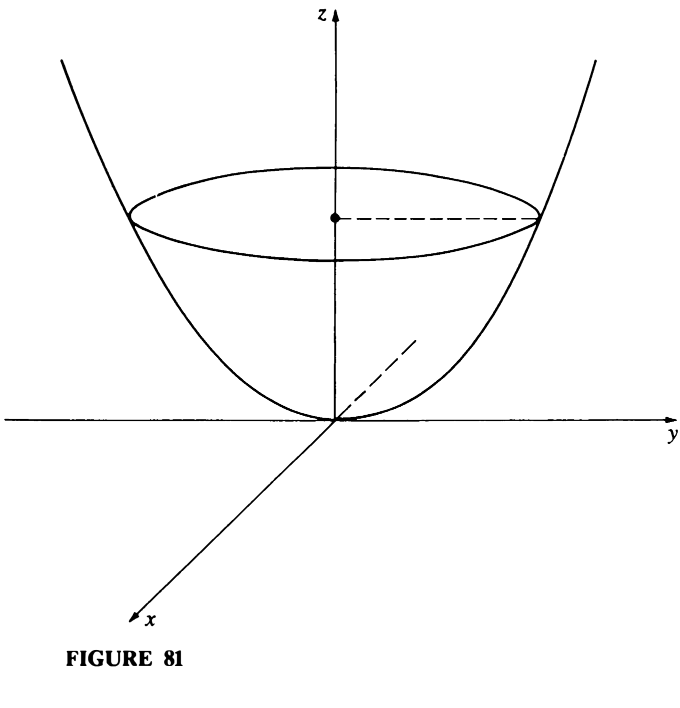

FIGURE 81

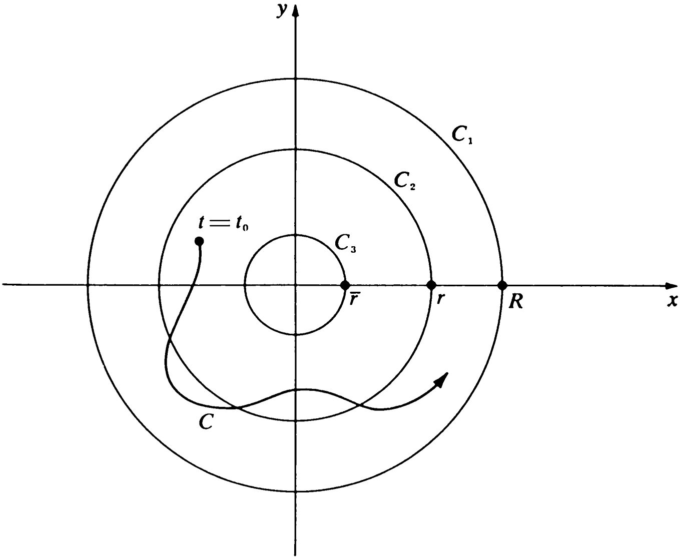

FIGURE 82

definite, it has a positive minimum m on C_1. Next, $E(x,y)$ is continuous at the origin and vanishes there, so we can find a positive number $r < R$ such that $E(x,y) < m$ whenever (x,y) is inside the circle C_2 of radius r. Now let $C = [x(t), y(t)]$ be any path which is inside C_2 for $t = t_0$. Then $E(t_0) < m$, and since (3) is negative semidefinite we have $dE/dt \leq 0$, which implies that $E(t) \leq E(t_0) < m$ for all $t > t_0$. It follows that the path C can never reach the circle C_1 for any $t > t_0$, so we have stability.

To prove the second part of the theorem, it suffices to show that under the additional hypothesis we also have $E(t) \to 0$, for since $E(x,y)$ is positive definite this will imply that the path C approaches the critical point (0,0). We begin by observing that since $dE/dt < 0$, it follows that $E(t)$ is a decreasing function; and since by hypothesis $E(t)$ is bounded below by 0, we conclude that $E(t)$ approaches some limit $L \geq 0$ as $t \to \infty$. To prove that $E(t) \to 0$ it suffices to show that $L = 0$, so we assume that $L > 0$ and deduce a contradiction. Choose a positive number $\bar{r} < r$ with the property that $E(x,y) < L$ whenever (x,y) is inside the circle C_3 with radius $\bar{r}$. Since the function (3) is continuous and negative definite, it has a negative maximum $-k$ in the ring consisting of the circles C_1 and C_3 and the region between them. This ring contains the entire path C for $t \geq t_0$, so the equation

$$E(t) = E(t_0) + \int_{t_0}^{t} \frac{dE}{dt}\,dt$$

yields the inequality

$$E(t) \leq E(t_0) - k(t - t_0) \tag{4}$$

for all $t \geq t_0$. However, the right side of (4) becomes negatively infinite as

$t \to \infty$, so $E(t) \to -\infty$ as $t \to \infty$. This contradicts the fact that $E(x,y) \geq 0$, so we conclude that $L = 0$ and the proof is complete.

Example 1. Consider the equation of motion of a mass m attached to a spring:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0. \tag{5}$$

Here $c \geq 0$ is a constant representing the viscosity of the medium through which the mass moves, and $k > 0$ is the spring constant. The autonomous system equivalent to (5) is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{k}{m}x - \dfrac{c}{m}y, \end{cases} \tag{6}$$

and its only critical point is (0,0). The kinetic energy of the mass is $my^2/2$, and the potential energy (or the energy stored in the spring) is

$$\int_0^x kx\,dx = \frac{1}{2}kx^2.$$

Thus the total energy of the system is

$$E(x,y) = \frac{1}{2}my^2 + \frac{1}{2}kx^2. \tag{7}$$

It is easy to see that (7) is positive definite; and since

$$\begin{aligned}\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= kxy + my\left(-\frac{k}{m}x - \frac{c}{m}y\right) \\ &= -cy^2 \leq 0,\end{aligned}$$

(7) is a Liapunov function for (6) and the critical point (0,0) is stable. We know from Problem 60-4 that when $c > 0$ this critical point is asymptotically stable, but the particular Liapunov function discussed here is not capable of detecting this fact.[8]

[8] It is known that both stability and asymptotic stability can always be detected by suitable Liapunov functions, but knowing in principle that such a function exists is a very different matter from actually finding one. For references on this point, see L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations,* p. 111, Academic Press, New York, 1963; or G. Sansone and R. Conti, *Non-Linear Differential Equations,* p. 481, Macmillan, New York, 1964.

Example 2. The system

$$\begin{cases} \dfrac{dx}{dt} = -2xy \\ \dfrac{dy}{dt} = x^2 - y^3 \end{cases} \tag{8}$$

has (0,0) as an isolated critical point. Let us try to prove stability by constructing a Liapunov function of the form $E(x,y) = ax^{2m} + by^{2n}$. It is clear that

$$\begin{aligned} \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= 2max^{2m-1}(-2xy) + 2nby^{2n-1}(x^2 - y^3) \\ &= (-4max^{2m}y + 2nbx^2y^{2n-1}) - 2nby^{2n+2}. \end{aligned}$$

We wish to make the expression in parentheses vanish, and inspection shows that this can be done by choosing $m = 1$, $n = 1$, $a = 1$, and $b = 2$. With these choices we have $E(x,y) = x^2 + 2y^2$ (which is positive definite) and $(\partial E/\partial x)F + (\partial E/\partial y)G = -4y^4$ (which is negative semidefinite). The critical point (0,0) of the system (8) is therefore stable.

It is clear from this example that in complicated situations it may be very difficult indeed to construct suitable Liapunov functions. The following result is sometimes helpful in this connection.

Theorem B *The function $E(x,y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$, and is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$.*

Proof. If $y = 0$, we have $E(x,0) = ax^2$, so $E(x,0) > 0$ for $x \neq 0$ if and only if $a > 0$. If $y \neq 0$, we have

$$E(x,y) = y^2\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right];$$

and when $a > 0$ the bracketed polynomial in x/y (which is positive for large x/y) is positive for all x/y if and only if $b^2 - 4ac < 0$. This proves the first part of the theorem, and the second part follows at once by considering the function $-E(x,y)$.

PROBLEMS

1. Determine whether each of the following functions is positive definite, negative definite, or neither:
   (a) $x^2 - xy - y^2$;
   (b) $2x^2 - 3xy + 3y^2$;
   (c) $-2x^2 + 3xy - y^2$;
   (d) $-x^2 - 4xy - 5y^2$.
2. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

3. Show that (0,0) is an asymptotically stable critical point for each of the following systems:

(a) $$\begin{cases} \dfrac{dx}{dt} = -3x^3 - y \\ \dfrac{dy}{dt} = x^5 - 2y^3; \end{cases}$$ (b) $$\begin{cases} \dfrac{dx}{dt} = -2x + xy^3 \\ \dfrac{dy}{dt} = -x^2y^2 - y^3. \end{cases}$$

4. Prove that the critical point (0,0) of the system (1) is unstable if there exists a function $E(x,y)$ with the following properties:
   (a) $E(x,y)$ is continuous and has continuous first partial derivatives in some region containing the origin;
   (b) $E(0,0) = 0$;
   (c) every circle centered on (0,0) contains at least one point where $E(x,y)$ is positive;
   (d) $(\partial E/\partial x)F + (\partial E/\partial y)G$ is positive definite.
5. Show that (0,0) is an unstable critical point for the system

$$\begin{cases} \dfrac{dx}{dt} = 2xy + x^3 \\ \dfrac{dy}{dt} = -x^2 + y^5. \end{cases}$$

6. Assume that $f(x)$ is a function such that $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$ [that is, $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x < 0$].
   (a) Show that

$$E(x,y) = \frac{1}{2}y^2 + \int_0^x f(x)\,dx$$

   is positive definite.
   (b) Show that the equation

$$\frac{d^2x}{dt^2} + f(x) = 0$$

   has $x = 0$, $y = dx/dt = 0$ as a stable critical point.
   (c) If $g(x) \geq 0$ in some neighborhood of the origin, show that the equation

$$\frac{d^2x}{dt^2} + g(x)\frac{dx}{dt} + f(x) = 0$$

   has $x = 0$, $y = dx/dt = 0$ as a stable critical point.

62 SIMPLE CRITICAL POINTS OF NONLINEAR SYSTEMS

Consider an autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \tag{1}$$

with an isolated critical point at (0,0). If $F(x,y)$ and $G(x,y)$ can be expanded in power series in x and y, then (1) takes the form

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + c_1x^2 + d_1xy + e_1y^2 + \cdots \\ \dfrac{dy}{dt} = a_2x + b_2y + c_2x^2 + d_2xy + e_2y^2 + \cdots . \end{cases} \tag{2}$$

When $|x|$ and $|y|$ are small—that is, when (x,y) is close to the origin—the terms of second degree and higher are very small. It is therefore natural to discard these nonlinear terms and conjecture that the qualitative behavior of the paths of (2) near the critical point (0,0) is similar to that of the paths of the related linear system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y. \end{cases} \tag{3}$$

We shall see that in general this is actually the case. The process of replacing (2) by the linear system (3) is usually called *linearization.*

More generally, we shall consider systems of the form

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + f(x,y) \\ \dfrac{dy}{dt} = a_2x + b_2y + g(x,y). \end{cases} \tag{4}$$

It will be assumed that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \tag{5}$$

so that the related linear system (3) has (0,0) as an isolated critical point; that $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives for all (x,y); and that as $(x,y) \to (0,0)$ we have

$$\lim \frac{f(x,y)}{\sqrt{x^2 + y^2}} = 0 \qquad \text{and} \qquad \lim \frac{g(x,y)}{\sqrt{x^2 + y^2}} = 0. \tag{6}$$

Observe that conditions (6) imply that $f(0,0) = 0$ and $g(0,0) = 0$, so (0,0) is a critical point of (4); also, it is not difficult to prove that this critical point is isolated (see Problem 1). With the restrictions listed above, (0,0) is said to be a *simple critical point* of the system (4).

Example 1. In the case of the system

$$\begin{cases} \dfrac{dx}{dt} = -2x + 3y + xy \\ \dfrac{dy}{dt} = -x + y - 2xy^2 \end{cases} \tag{7}$$

we have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} = 1 \neq 0,$$

so (5) is satisfied. Furthermore, by using polar coordinates we see that

$$\frac{|f(x,y)|}{\sqrt{x^2+y^2}} = \frac{|r^2 \sin\theta \cos\theta|}{r} \le r$$

and

$$\frac{|g(x,y)|}{\sqrt{x^2+y^2}} = \frac{|2r^3 \sin^2\theta \cos\theta|}{r} \le 2r^2,$$

so $f(x,y)/r$ and $g(x,y)/r \to 0$ as $(x,y) \to (0,0)$ (or as $r \to 0$). This shows that conditions (6) are also satisfied, so (0,0) is a simple critical point of the system (7).

The main facts about the nature of simple critical points are given in the following theorem of Poincaré, which we state without proof.[9]

Theorem A. *Let* (0,0) *be a simple critical point of the nonlinear system* (4), *and consider the related linear system* (3). *If the critical point* (0,0) *of* (3) *falls under any one of the three major cases described in Section* 60, *then the critical point* (0,0) *of* (4) *is of the same type.*

As an illustration, we examine the nonlinear system (7) of Example 1, whose related linear system is

$$\begin{cases} \dfrac{dx}{dt} = -2x + 3y \\ \dfrac{dy}{dt} = -x + y. \end{cases} \tag{8}$$

The auxiliary equation of (8) is $m^2 + m + 1 = 0$, with roots

$$m_1, m_2 = \frac{-1 \pm \sqrt{3}\,i}{2}.$$

[9] Detailed treatments can be found in W. Hurewicz, *Lectures on Ordinary Differential Equations,* pp. 86–98, MIT, Cambridge, Mass., 1958; L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations,* pp. 157–163, Academic Press, New York, 1963; or F. G. Tricomi, *Differential Equations,* pp. 53–72, Blackie, Glasgow, 1961.

Since these roots are conjugate complex but not pure imaginary, we have Case C and the critical point (0,0) of the linear system (8) is a spiral. By Theorem A, the critical point (0,0) of the nonlinear system (7) is also a spiral.

It should be understood that while the type of the critical point (0,0) is the same for (4) as it is for (3) in the cases covered by the theorem, the actual appearance of the paths may be somewhat different. For example, Fig. 76 shows a typical saddle point for a linear system, whereas Fig. 83 suggests how a nonlinear saddle point might look. A certain amount of distortion is clearly present in the latter, but nevertheless the qualitative features of the two configurations are the same.

It is natural to wonder about the two borderline cases, which are not mentioned in Theorem A. The facts are these: if the related linear system (3) has a borderline node at the origin (Case D), then the nonlinear system (4) can have either a node or a spiral; and if (3) has a center at the origin (Case E), then (4) can have either a center or a spiral. For example, (0,0) is a critical point for each of the nonlinear systems

$$\begin{cases} \dfrac{dx}{dt} = -y - x^2 \\ \dfrac{dy}{dt} = x \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \dfrac{dy}{dt} = x. \end{cases} \tag{9}$$

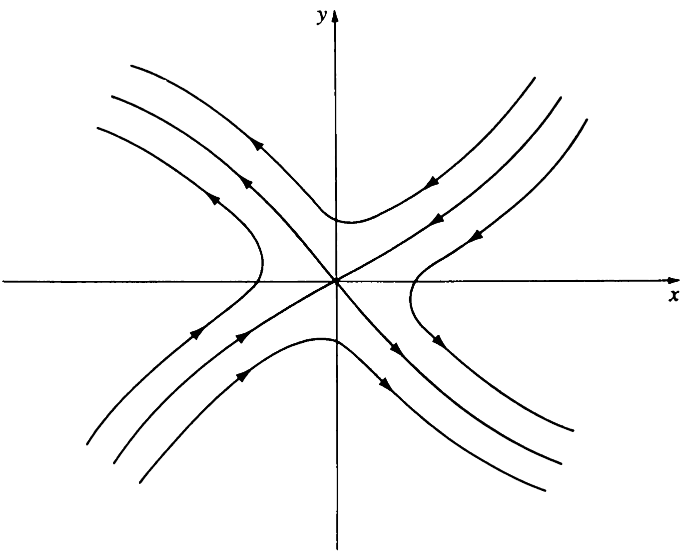

FIGURE 83

In each case the related linear system is

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x. \end{cases} \tag{10}$$

It is easy to see that (0,0) is a center for (10). However, it can be shown that while (0,0) is a center for the first system of (9), it is a spiral for the second.[10]

We have already encountered a considerable variety of configurations at critical points of linear systems, and the above remarks show that no new phenomena appear at simple critical points of nonlinear systems. What about critical points that are not simple? The possibilities here can best be appreciated by examining a nonlinear system of the form (2). If the linear terms in (2) do not determine the pattern of the paths near the origin, then we must consider the second degree terms; if these fail to determine the pattern, then the third degree terms must be taken into account, and so on. This suggests that in addition to the linear configurations, a great many others can arise, of infinite variety and staggering complexity. Several are shown in Fig. 84. It is perhaps surprising to realize that such involved patterns as these can occur in connection with systems of rather simple appearance. For example, the three figures in the upper row show the arrangement of the paths of

$$\begin{cases} \dfrac{dx}{dt} = 2xy \\ \dfrac{dy}{dt} = y^2 - x^2, \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = x^3 - 2xy^2 \\ \dfrac{dy}{dt} = 2x^2y - y^3, \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = x - 4y\sqrt{|xy|} \\ \dfrac{dy}{dt} = -y + 4x\sqrt{|xy|}. \end{cases}$$

In the first case, this can be seen at once by looking at Fig. 3 and equation 3-(8).

We now discuss the question of stability for a simple critical point. The main result here is due to Liapunov: if (3) is asymptotically stable at the origin, then (4) is also. We state this formally as follows.

Theorem B. *Let* (0,0) *be a simple critical point of the nonlinear system* (4), *and consider the related linear system* (3). *If the critical point* (0,0) *of* (3) *is asymptotically stable, then the critical point* (0,0) *of* (4) *is also asymptotically stable.*

[10] See Hurewicz, *op. cit.*, p. 99.

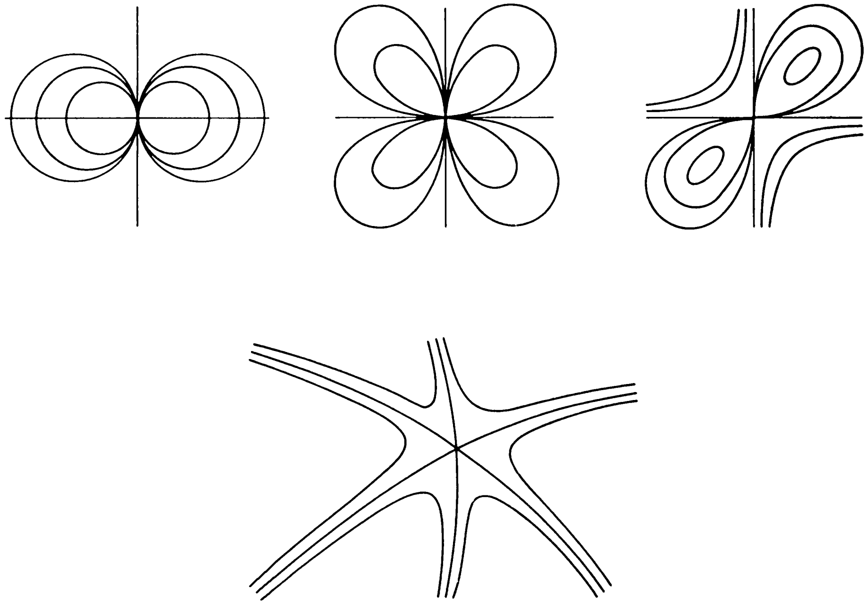

FIGURE 84

Proof. By Theorem 61-A, it suffices to construct a suitable Liapunov function for the system (4), and this is what we do.

Theorem 60–B tells us that the coefficients of the linear system (3) satisfy the conditions

$$p = -(a_1 + b_2) > 0 \quad \text{and} \quad q = a_1b_2 - a_2b_1 > 0. \tag{11}$$

Now define

$$E(x,y) = \frac{1}{2}(ax^2 + 2bxy + cy^2)$$

by putting

$$a = \frac{a_2^2 + b_2^2 + (a_1b_2 - a_2b_1)}{D},$$

$$b = -\frac{a_1a_2 + b_1b_2}{D},$$

and

$$c = \frac{a_1^2 + b_1^2 + (a_1b_2 - a_2b_1)}{D},$$

where

$$D = pq = -(a_1 + b_2)(a_1b_2 - a_2b_1).$$

By (11), we see that $D > 0$ and $a > 0$. Also, an easy calculation shows that

$$\begin{aligned} D^2(ac - b^2) &= (a_2^2 + b_2^2)(a_1^2 + b_1^2) \\ &\quad + (a_2^2 + b_2^2 + a_1^2 + b_1^2)(a_1 b_2 - a_2 b_1) \\ &\quad + (a_1 b_2 - a_2 b_1)^2 - (a_1 a_2 + b_1 b_2)^2 \\ &= (a_2^2 + b_2^2 + a_1^2 + b_1^2)(a_1 b_2 - a_2 b_1) \\ &\quad + 2(a_1 b_2 - a_2 b_1)^2 \\ &> 0, \end{aligned}$$

so $b^2 - ac < 0$. Thus, by Theorem 61-B, we know that the function $E(x,y)$ is positive definite. Furthermore, another calculation (whose details we leave to the reader) yields

$$\frac{\partial E}{\partial x}(a_1 x + b_1 y) + \frac{\partial E}{\partial y}(a_2 x + b_2 y) = -(x^2 + y^2). \tag{12}$$

This function is clearly negative definite, so $E(x,y)$ is a Liapunov function for the linear system (3).[11]

We next prove that $E(x,y)$ is also a Liapunov function for the nonlinear system (4). If F and G are defined by

$$F(x,y) = a_1 x + b_1 y + f(x,y)$$

and

$$G(x,y) = a_2 x + b_2 y + g(x,y),$$

then since E is known to be positive definite, it suffices to show that

$$\frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G \tag{13}$$

is negative definite. If we use (12), then (13) becomes

$$-(x^2 + y^2) + (ax + by)f(x,y) + (bx + cy)g(x,y);$$

and by introducing polar coordinates we can write this as

$$-r^2 + r[(a \cos\theta + b \sin\theta)f(x,y) + (b \cos\theta + c \sin\theta)g(x,y)].$$

Denote the largest of the numbers $|a|$, $|b|$, $|c|$ by K. Our assumption (6) now implies that

$$|f(x,y)| < \frac{r}{6K} \qquad \text{and} \qquad |g(x,y)| < \frac{r}{6K}$$

for all sufficiently small $r > 0$, so

$$\frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G < -r^2 + \frac{4Kr^2}{6K} = -\frac{r^2}{3} < 0$$

[11] The reason for the definitions of a, b, and c can now be understood: we want (12) to be true.

for these r's. Thus $E(x,y)$ is a positive definite function with the property that (13) is negative definite. Theorem 61-A now implies that (0,0) is an asymptotically stable critical point of (4), and the proof is complete.

To illustrate this theorem, we again consider the nonlinear system (7) of Example 1, whose related linear system is (8). For (8) we have $p = 1 > 0$ and $q = 1 > 0$, so the critical point (0,0) is asymptotically stable, both for the linear system (8) and for the nonlinear system (7).

Example 2. We know from Section 58 that the equation of motion for the damped vibrations of a pendulum is

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0,$$

where c is a positive constant. The equivalent nonlinear system is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{g}{a}\sin x - \dfrac{c}{m}y. \end{cases} \tag{14}$$

Let us now write (14) in the form

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{g}{a}x - \dfrac{c}{m}y + \dfrac{g}{a}(x - \sin x). \end{cases} \tag{15}$$

It is easy to see that

$$\frac{x - \sin x}{\sqrt{x^2 + y^2}} \to 0$$

as $(x,y) \to (0,0)$, for if $x \neq 0$, we have

$$\frac{|x - \sin x|}{\sqrt{x^2 + y^2}} \leq \frac{|x - \sin x|}{|x|} = \left|1 - \frac{\sin x}{x}\right| \to 0;$$

and since (0,0) is evidently an isolated critical point of the related linear system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{g}{a}x - \dfrac{c}{m}y, \end{cases} \tag{16}$$

it follows that (0,0) is a simple critical point of (15). Inspection shows ($p = c/m > 0$ and $q = g/a > 0$) that (0,0) is an asymptotically stable critical point of (16), so by Theorem B it is also an asymptotically stable critical point of (15). This reflects the obvious physical fact that if the pendulum is slightly disturbed, then the resulting motion will die out with the passage of time.

PROBLEMS

1. Prove that if (0,0) is a simple critical point of (4), then it is necessarily isolated. *Hint*: Write conditions (6) in the form $f(x,y)/r = \epsilon_1 \to 0$ and $g(x,y)/r = \epsilon_2 \to 0$, and in the light of (5) use polar coordinates to deduce a contradiction from the assumption that the right sides of (4) both vanish at points arbitrarily close to the origin but different from it.
2. Sketch the family of curves whose polar equation is $r = a \sin 2\theta$ (see Fig. 84), and express the differential equation of this family in the form $dy/dx = G(x,y)/F(x,y)$.
3. If (0,0) is a simple critical point of (4) and $q = a_1b_2 - a_2b_1 < 0$, then Theorem A implies that (0,0) is a saddle point of (4) and is therefore unstable. Prove that if $p = -(a_1 + b_2) < 0$ and $q = a_1b_2 - a_2b_1 > 0$, then (0,0) is an unstable critical point of (4). *Hint*: Adapt the proof of Theorem B to show that there exists a positive definite function $E(x,y)$ such that

$$\frac{\partial E}{\partial x}(a_1x + b_1y) + \frac{\partial E}{\partial y}(a_2x + b_2y) = x^2 + y^2,$$

and apply Problem 61-4. (Observe that these facts together with Theorem B demonstrate that all the information in Fig. 80 about asymptotic stability and instability carries over directly to nonlinear systems with simple critical points from their related linear systems.)
4. Show that (0,0) is an asymptotically stable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \dfrac{dy}{dt} = x - y^3, \end{cases}$$

but is an unstable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y + x^3 \\ \dfrac{dy}{dt} = x + y^3. \end{cases}$$

How are these facts related to the parenthetical remark in Problem 3?
5. Verify that (0,0) is a simple critical point for each of the following systems, and determine its nature and stability properties:

(a) $\begin{cases} \dfrac{dx}{dt} = x + y - 2xy \\ \dfrac{dy}{dt} = -2x + y + 3y^2; \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -x - y - 3x^2y \\ \dfrac{dy}{dt} = -2x - 4y + y \sin x. \end{cases}$

6. The van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - \mu(x^2 - 1)y. \end{cases}$$

Investigate the stability properties of the critical point (0,0) for the cases $\mu > 0$ and $\mu < 0$.

63 NONLINEAR MECHANICS. CONSERVATIVE SYSTEMS

It is well known that energy is dissipated in the action of any real dynamical system, usually through some form of friction. However, in certain situations this dissipation is so slow that it can be neglected over relatively short periods of time. In such cases we assume the law of conservation of energy, namely, that the sum of the kinetic energy and the potential energy is constant. A system of this kind is said to be *conservative*. Thus the rotating earth can be considered a conservative system over short intervals of time involving only a few centuries, but if we want to study its behavior throughout millions of years we must take into account the dissipation of energy by tidal friction.

The simplest conservative system consists of a mass m attached to a spring and moving in a straight line through a vacuum. If x denotes the displacement of m from its equilibrium position, and the restoring force exerted on m by the spring is $-kx$ where $k > 0$, then we know that the equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

A spring of this kind is called a *linear spring* because the restoring force is a linear function of x. If m moves through a resisting medium, and the resistance (or damping force) exerted on m is $-c(dx/dt)$ where $c > 0$, then the equation of motion of this nonconservative system is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

Here we have *linear damping* because the damping force is a linear function of dx/dt. By analogy, if f and g are arbitrary functions with the property that $f(0) = 0$ and $g(0) = 0$, then the more general equation

$$m\frac{d^2x}{dt^2} + g\left(\frac{dx}{dt}\right) + f(x) = 0 \qquad (1)$$

can be interpreted as the equation of motion of a mass m under the action of a *restoring force* $-f(x)$ and a *damping force* $-g(dx/dt)$. In general these forces are nonlinear, and equation (1) can be regarded as the basic equation of nonlinear mechanics. In this section we shall briefly consider the special case of a nonlinear conservative system described by the equation

$$m\frac{d^2x}{dt^2} + f(x) = 0, \tag{2}$$

in which the damping force is zero and there is consequently no dissipation of energy.[12]

Equation (2) is equivalent to the autonomous system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{f(x)}{m}. \end{cases} \tag{3}$$

If we eliminate dt, we obtain the differential equation of the paths of (3) in the phase plane,

$$\frac{dy}{dx} = -\frac{f(x)}{my}, \tag{4}$$

and this can be written in the form

$$my\,dy = -f(x)\,dx. \tag{5}$$

If $x = x_0$ and $y = y_0$ when $t = t_0$, then integrating (5) from t_0 to t yields

$$\frac{1}{2}my^2 - \frac{1}{2}my_0^2 = -\int_{x_0}^{x} f(x)\,dx$$

or

$$\frac{1}{2}my^2 + \int_0^x f(x)\,dx = \frac{1}{2}my_0^2 + \int_0^{x_0} f(x)\,dx. \tag{6}$$

To interpret this result, we observe that $\frac{1}{2}my^2 = \frac{1}{2}m(dx/dt)^2$ is the kinetic energy of the dynamical system and

$$V(x) = \int_0^x f(x)\,dx \tag{7}$$

[12] Extensive discussions of (1), with applications to a variety of physical problems, can be found in J. J. Stoker, *Nonlinear Vibrations*, Interscience-Wiley, New York, 1950; and in A. A. Andronow and C. E. Chaikin, *Theory of Oscillations*, Princeton University Press, Princeton, N.J., 1949.

is its potential energy. Equation (6) therefore expresses the law of conservation of energy,

$$\frac{1}{2}my^2 + V(x) = E, \tag{8}$$

where $E = \frac{1}{2}my_0^2 + V(x_0)$ is the constant total energy of the system. It is clear that (8) is the equation of the paths of (3), since we obtained it by solving (4). The particular path determined by specifying a value of E is a curve of constant energy in the phase plane. The critical points of the system (3) are the points $(x_c,0)$ where the x_c are the roots of the equation $f(x) = 0$. As we pointed out in Section 58, these are the equilibrium points of the dynamical system described by (2). It is evident from (4) that the paths cross the x-axis at right angles and are horizontal when they cross the lines $x = x_c$. Equation (8) also shows that the paths are symmetric with respect to the x-axis.

If we write (8) in the form

$$y = \pm\sqrt{\frac{2}{m}[E - V(x)]}, \tag{9}$$

then the paths can be constructed by the following easy steps. First, establish an xz-plane with the z-axis on the same vertical line as the y-axis of the phase plane (Fig. 85). Next, draw the graph of $z = V(x)$ and several horizontal lines $z = E$ in the xz-plane (one such line is shown in the figure), and observe the geometric meaning of the difference $E - V(x)$. Finally, for each x, multiply $E - V(x)$ as obtained in the preceding step by $2/m$ and use formula (9) to plot the corresponding values of y in the phase plane directly below. Note that since $dx/dt = y$, the positive direction along any path is to the right above the x-axis and to the left below this axis.

Example 1. We saw in Section 58 that the equation of motion of an undamped pendulum is

$$\frac{d^2x}{dt^2} + k\sin x = 0, \tag{10}$$

where k is a positive constant. Since this equation is of the form (2), it can be interpreted as describing the undamped rectilinear motion of a unit mass under the influence of a nonlinear spring whose restoring force is $-k\sin x$. The autonomous system equivalent to (10) is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -k\sin x, \end{cases} \tag{11}$$

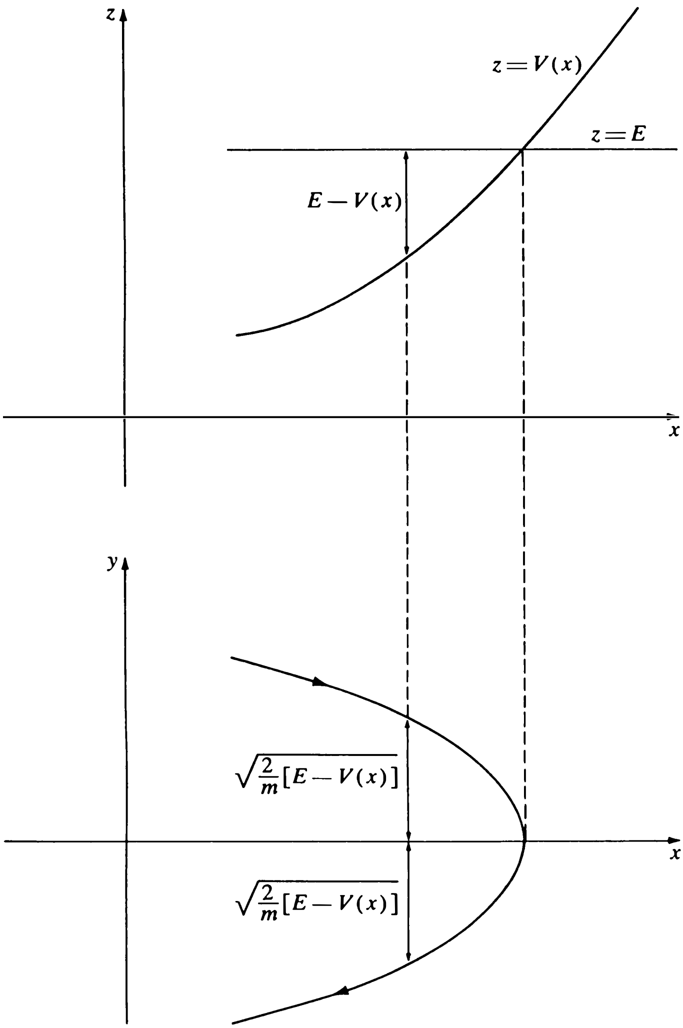

FIGURE 85

and its critical points are (0,0), $(\pm\pi,0)$, $(\pm 2\pi,0)$, ... The differential equation of the paths is

$$\frac{dy}{dx} = -\frac{k \sin x}{y},$$

and by separating variables and integrating, we see that the equation of the family of paths is

$$\frac{1}{2}y^2 + (k - k\cos x) = E.$$

This is evidently of the form (8), where $m = 1$ and

$$V(x) = \int_0^x f(x)\,dx = k - k\cos x$$

is the potential energy. We now construct the paths by first drawing the graph of $z = V(x)$ and several lines $z = E$ in the xz-plane (Fig. 86, where $z = E = 2k$ is the only line shown). From this we read off the values $E - V(x)$ and sketch the paths in the phase plane directly below by using $y = \pm\sqrt{2[E - V(x)]}$. It is clear from this phase portrait that if the total energy E is between 0 and $2k$, then the corresponding paths are closed and equation (10) has periodic solutions. On the other hand, if $E > 2k$, then the path is not closed and the corresponding solution of (10) is not periodic. The value $E = 2k$ separates the two types of motion, and for this reason a path corresponding to $E = 2k$ is called a *separatrix*. The wavy paths outside the separatrices correspond to whirling motions of the pendulum, and the closed paths inside to oscillatory motions. It is evident that the critical points are alternately unstable saddle points and stable but not asymptotically stable centers. For the sake of contrast, it is interesting to consider the

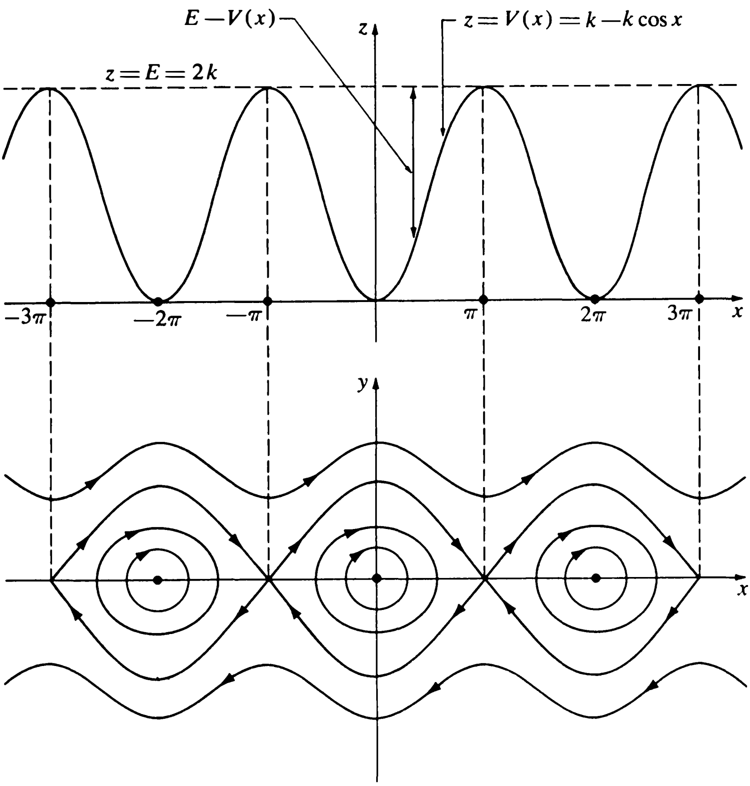

FIGURE 86

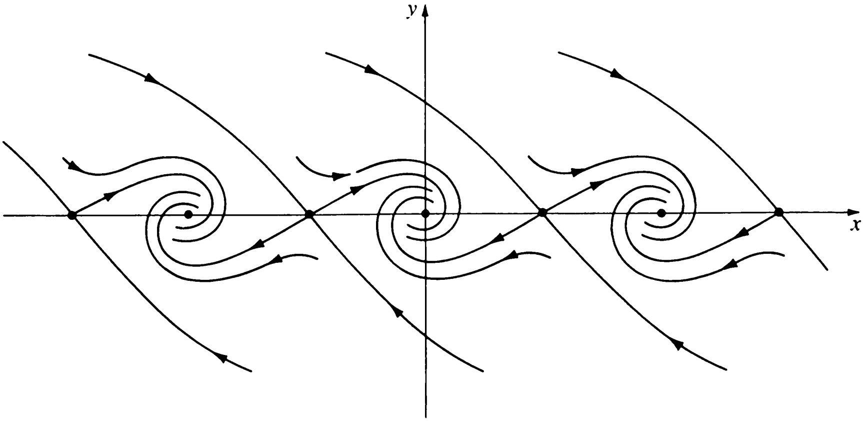

FIGURE 87

effect of transforming this conservative dynamical system into a nonconservative system by introducing a linear damping force. The equation of motion then takes the form

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k\sin x = 0, \qquad c > 0,$$

and the configuration of the paths is suggested in Fig. 87. We find that the centers in Fig. 86 become asymptotically stable spirals, and also that every path—except the separatrices entering the saddle points as $t \to \infty$—ultimately winds into one of these spirals.

PROBLEMS

1. If $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$, show that the paths of

$$\frac{d^2x}{dt^2} + f(x) = 0$$

are closed curves surrounding the origin in the phase plane; that is, show that the critical point $x = 0$, $y = dx/dt = 0$ is a stable but not asymptotically stable center. Describe this critical point with respect to its nature and stability if $f(0) = 0$ and $xf(x) < 0$ for $x \neq 0$.

2. Most actual springs are not linear. A nonlinear spring is called *hard* or *soft* according as the magnitude of the restoring force increases more rapidly or less rapidly than a linear function of the displacement. The equation

$$\frac{d^2x}{dt^2} + kx + \alpha x^3 = 0, \qquad k > 0,$$

describes the motion of a hard spring if $\alpha > 0$ and a soft spring if $\alpha < 0$. Sketch the paths in each case.

3. Find the equation of the paths of

$$\frac{d^2x}{dt^2} - x + 2x^3 = 0,$$

and sketch these paths in the phase plane. Locate the critical points and determine the nature of each.

4. Since by equation (7) we have $dV/dx = f(x)$, the critical points of (3) are the points on the x-axis in the phase plane at which $V'(x) = 0$. In terms of the curve $z = V(x)$—if this curve is smooth and well behaved—there are three possibilities: maxima, minima, and points of inflection. Sketch all three possibilities, and determine the type of critical point associated with each (a critical point of the third type is called a *cusp*).

64 PERIODIC SOLUTIONS. THE POINCARÉ–BENDIXSON THEOREM

Consider a nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \tag{1}$$

in which the functions $F(x,y)$ and $G(x,y)$ are continuous and have continuous first partial derivatives throughout the phase plane. Our work so far has told us practically nothing about the paths of (1) except in the neighborhood of certain types of critical points. However, in many problems we are much more interested in the global properties of paths than we are in these local properties. *Global properties* of paths are those that describe their behavior over large regions of the phase plane, and in general they are very difficult to establish.

The central problem of the global theory is that of determining whether (1) has closed paths. As we remarked in Section 58, this problem is important because of its close connection with the issue of whether (1) has periodic solutions. A solution $x(t)$ and $y(t)$ of (1) is said to be *periodic* if neither function is constant, if both are defined for all t, and if there exists a number $T > 0$ such that $x(t + T) = x(t)$ and $y(t + T) = y(t)$ for all t. The smallest T with this property is called the *period* of the solution.[13] It is evident that each periodic solution of (1) defines a closed path that is traversed once as t increases from t_0 to $t_0 + T$ for any t_0.

[13] Every periodic solution has a period in this sense. Why?

Conversely, it is easy to see that if $C = [x(t), y(t)]$ is a closed path of (1), then $x(t)$, $y(t)$ is a periodic solution. Accordingly, the search for periodic solutions of (1) reduces to a search for closed paths.

We know from Section 60 that a linear system has closed paths if and only if the roots of the auxiliary equation are pure imaginary, and in this case every path is closed. Thus, for a linear system, either every path is closed or else no path is closed. On the other hand, a nonlinear system can perfectly well have a closed path that is isolated, in the sense that no other closed paths are near to it. The following is a well-known example of such a system:

$$\begin{cases} \dfrac{dx}{dt} = -y + x(1 - x^2 - y^2) \\[2ex] \dfrac{dy}{dt} = x + y(1 - x^2 - y^2). \end{cases} \tag{2}$$

To solve this system we introduce polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. If we differentiate the relations $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$, we obtain the useful formulas

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \qquad \text{and} \qquad x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}. \tag{3}$$

On multiplying the first equation of (2) by x and the second by y, and adding, we find that

$$r\frac{dr}{dt} = r^2(1 - r^2). \tag{4}$$

Similarly, if we multiply the second by x and the first by y, and subtract, we get

$$r^2\frac{d\theta}{dt} = r^2. \tag{5}$$

The system (2) has a single critical point at $r = 0$. Since we are concerned only with finding the paths, we may assume that $r > 0$. In this case, (4) and (5) show that (2) becomes

$$\begin{cases} \dfrac{dr}{dt} = r(1 - r^2) \\[2ex] \dfrac{d\theta}{dt} = 1. \end{cases} \tag{6}$$

These equations are easy to solve separately, and the general solution of the system (6) is found to be

$$\begin{cases} r = \dfrac{1}{\sqrt{1 + ce^{-2t}}} \\ \theta = t + t_0. \end{cases} \tag{7}$$

The corresponding general solution of (2) is

$$\begin{cases} x = \dfrac{\cos (t + t_0)}{\sqrt{1 + ce^{-2t}}} \\ y = \dfrac{\sin (t + t_0)}{\sqrt{1 + ce^{-2t}}}. \end{cases} \tag{8}$$

Let us analyze (7) geometrically (Fig. 88). If $c = 0$, we have the solutions $r = 1$ and $\theta = t + t_0$, which trace out the closed circular path $x^2 + y^2 = 1$ in the counterclockwise direction. If $c < 0$, it is clear that $r > 1$ and that $r \to 1$ as $t \to \infty$. Also, if $c > 0$, we see that $r < 1$, and again $r \to 1$ as $t \to \infty$. These observations show that there exists a single closed path ($r = 1$) which all other paths approach spirally from the outside or the inside as $t \to \infty$.

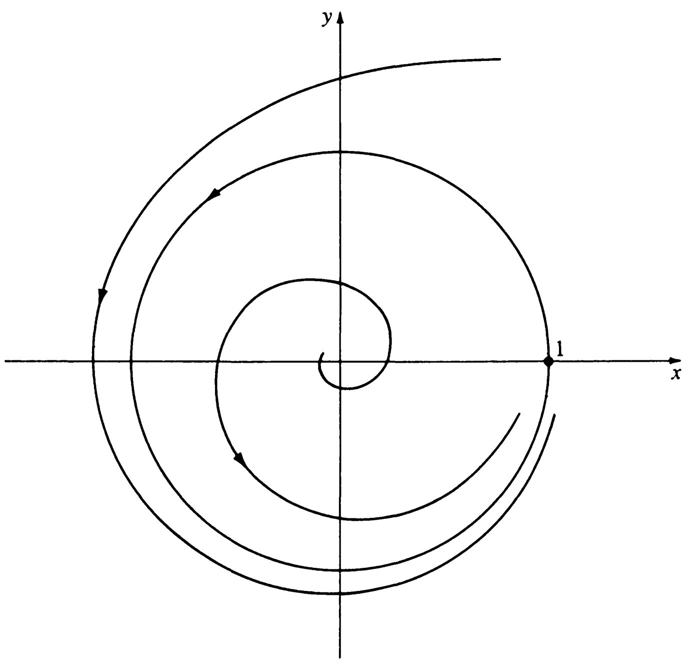

FIGURE 88

In the above discussion we have shown that the system (2) has a closed path by actually finding such a path. In general, of course, we cannot hope to be able to do this. What we need are tests that make it possible for us to conclude that certain regions of the phase plane do or do not contain closed paths. Our first test is given in the following theorem of Poincaré. A proof is sketched in Problem 1.

Theorem A. *A closed path of the system* (1) *necessarily surrounds at least one critical point of this system.*

This result gives a negative criterion of rather limited value: a system without critical points in a given region cannot have closed paths in that region.

Our next theorem provides another negative criterion, and is due to Bendixson.[14]

Theorem B. *If* $\partial F/\partial x + \partial G/\partial y$ *is always positive or always negative in a certain region of the phase plane, then the system* (1) *cannot have closed paths in that region.*

Proof. Assume that the region contains a closed path $C = [x(t),y(t)]$ with interior R. Then Green's theorem and our hypothesis yield

$$\int_C (F\,dy - G\,dx) = \iint_R \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dx\,dy \neq 0.$$

However, along C we have $dx = F\,dt$ and $dy = G\,dt$, so

$$\int_C (F\,dy - G\,dx) = \int_0^T (FG - GF)\,dt = 0.$$

This contradiction shows that our initial assumption is false, so the region under consideration cannot contain any closed path.

These theorems are sometimes useful, but what we really want are positive criteria giving sufficient conditions for the existence of closed paths of (1). One of the few general theorems of this kind is the classical *Poincaré–Bendixson theorem,* which we now state without proof.[15]

[14] Ivar Otto Bendixson (1861–1935) was a Swedish mathematician who published one important memoir in 1901 supplementing some of Poincaré's earlier work. He served as professor (and later as president) at the University of Stockholm, and was an energetic long-time member of the Stockholm City Council.

[15] For details, see Hurewicz, *loc. cit.*, pp. 102–111, or Cesari, *loc. cit.*, pp. 163–167.

Theorem C. *Let R be a bounded region of the phase plane together with its boundary, and assume that R does not contain any critical points of the system* (1). *If $C = [x(t),y(t)]$ is a path of* (1) *that lies in R for some t_0 and remains in R for all $t \geq t_0$, then C is either itself a closed path or it spirals toward a closed path as $t \to \infty$. Thus in either case the system* (1) *has a closed path in R.*

In order to understand this statement, let us consider the situation suggested in Fig. 89. Here R consists of the two dashed curves together with the ring-shaped region between them. Suppose that the vector

$$\mathbf{V}(x,y) = F(x,y)\mathbf{i} + G(x,y)\mathbf{j}$$

points *into* R at every boundary point. Then every path C through a boundary point (at $t = t_0$) must enter R and can never leave it, and under these circumstances the theorem asserts that C must spiral toward a closed path C_0. We have chosen a ring-shaped region R to illustrate the theorem because a closed path like C_0 must surround a critical point (P in the figure) and R must exclude all critical points.

The system (2) provides a simple application of these ideas. It is clear that (2) has a critical point at (0,0), and also that the region R between the circles $r = \frac{1}{2}$ and $r = 2$ contains no critical points. In our earlier analysis we found that

$$\frac{dr}{dt} = r(1 - r^2) \qquad \text{for } r > 0.$$

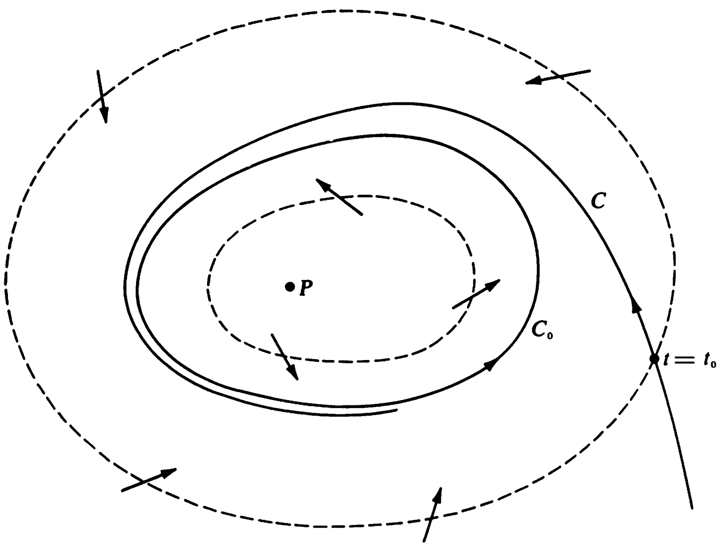

FIGURE 89

This shows that $dr/dt > 0$ on the inner circle and $dr/dt < 0$ on the outer circle, so the vector $\mathbf{V}$ points into R at all boundary points. Thus any path through a boundary point will enter R and remain in R as $t \to \infty$, and by the Poincaré–Bendixson theorem we know that R contains a closed path C_0. We have already seen that the circle $r = 1$ is the closed path whose existence is guaranteed in this way.

The Poincaré–Bendixson theorem is quite satisfying from a theoretical point of view, but in general it is rather difficult to apply. A more practical criterion has been developed that assures the existence of closed paths for equations of the form

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \tag{9}$$

which is called *Liénard's equation.*[16] When we speak of a closed path for such an equation, we of course mean a closed path of the equivalent system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -g(x) - f(x)y; \end{cases} \tag{10}$$

and as we know, a closed path of (10) corresponds to a periodic solution of (9). The fundamental statement about the closed paths of (9) is the following theorem.

Theorem D. (Liénard's Theorem.) *Let the functions $f(x)$ and $g(x)$ satisfy the following conditions*: (i) *both are continuous and have continuous derivatives for all x*; (ii) *$g(x)$ is an odd function such that $g(x) > 0$ for $x > 0$, and $f(x)$ is an even function*; *and* (iii) *the odd function $F(x) = \int_0^x f(x)\,dx$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$. Then equation* (9) *has a unique closed path surrounding the origin in the phase plane, and this path is approached spirally by every other path as $t \to \infty$.*

For the benefit of the skeptical and tenacious reader who is rightly reluctant to accept unsupported assertions, a proof of this theorem is

[16] Alfred Liénard (1869–1958) was a French scientist who spent most of his career teaching applied physics at the School of Mines in Paris, of which he became director in 1929. His physical research was mainly in the areas of electricity and magnetism, elasticity, and hydrodynamics. From time to time he worked on mathematical problems arising from his other scientific investigations, and in 1933 was elected president of the French Mathematical Society. He was an unassuming bachelor whose life was devoted entirely to his work and his students.

given in Appendix B. An intuitive understanding of the role of the hypotheses can be gained by thinking of (9) in terms of the ideas of the previous section. From this point of view, equation (9) is the equation of motion of a unit mass attached to a spring and subject to the dual influence of a restoring force $-g(x)$ and a damping force $-f(x)\,dx/dt$. The assumption about $g(x)$ amounts to saying that the spring acts as we would expect, and tends to diminish the magnitude of any displacement. On the other hand, the assumptions about $f(x)$—roughly, that $f(x)$ is negative for small $|x|$ and positive for large $|x|$—mean that the motion is intensified for small $|x|$ and retarded for large $|x|$, and therefore tends to settle down into a steady oscillation. This rather peculiar behavior of $f(x)$ can also be expressed by saying that the physical system absorbs energy when $|x|$ is small and dissipates it when $|x|$ is large.

The main application of Liénard's theorem is to the van der Pol[17] equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0, \tag{11}$$

where μ is assumed to be a positive constant for physical reasons. Here $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, so condition (i) is clearly satisfied. It is equally clear that condition (ii) is true. Since

$$F(x) = \mu\left(\frac{1}{3}x^3 - x\right) = \frac{1}{3}\mu x(x^2 - 3),$$

we see that $F(x)$ has a single positive zero at $x = \sqrt{3}$, is negative for $0 < x < \sqrt{3}$, is positive for $x > \sqrt{3}$, and that $F(x) \to \infty$ as $x \to \infty$. Finally, $F'(x) = \mu(x^2 - 1)$ is positive for $x > 1$, so $F(x)$ is certainly nondecreasing (in fact, increasing) for $x > \sqrt{3}$. Accordingly, all the conditions of the theorem are met, and we conclude that equation (11) has a unique closed path (periodic solution) that is approached spirally (asymptotically) by every other path (nontrivial solution).

PROBLEMS

1. A proof of Theorem A can be built on the following geometric ideas (Fig. 90). Let C be a simple closed curve (not necessarily a path) in the phase plane, and assume that C does not pass through any critical point of the system (1). If

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[17] Balthasar van der Pol (1889–1959), a Dutch scientist specializing in the theoretical aspects of radioengineering, initiated the study of equation (11) in the 1920s, and thereby stimulated Liénard and others to investigate the mathematical theory of self-sustained oscillations in nonlinear mechanics.

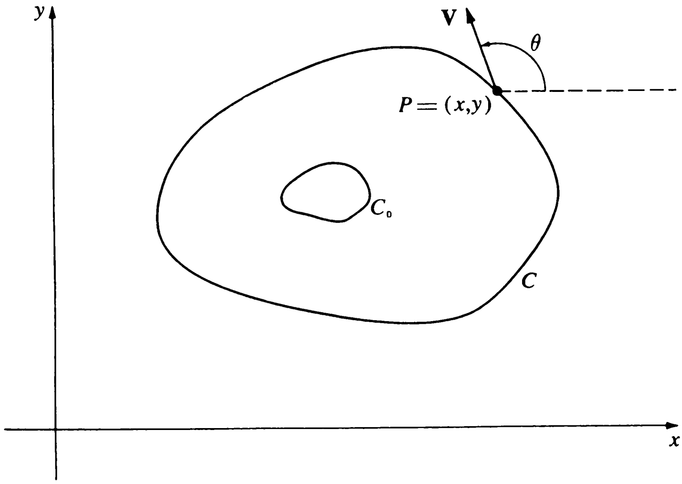

FIGURE 90

$P = (x,y)$ is a point on C, then

$$\mathbf{V}(x,y) = F(x,y)\mathbf{i} + G(x,y)\mathbf{j}$$

is a nonzero vector, and therefore has a definite direction given by the angle θ. If P moves once around C in the counterclockwise direction, the angle θ changes by an amount $\Delta\theta = 2\pi n$, where n is a positive integer, zero, or a negative integer. This integer n is called the *index* of C. If C shrinks continuously to a smaller simple closed curve C_0 without passing over any critical point, then its index varies continuously; and since the index is an integer, it cannot change.

(a) If C is a path of (1), show that its index is 1.
(b) If C is a path of (1) that contains no critical points, show that a small C_0 has index 0, and from this infer Theorem A.

2. Consider the nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = 4x + 4y - x(x^2 + y^2) \\ \dfrac{dy}{dt} = -4x + 4y - y(x^2 + y^2). \end{cases}$$

(a) Transform the system into polar coordinate form.
(b) Apply the Poincaré–Bendixson theorem to show that there is a closed path between the circles $r = 1$ and $r = 3$.
(c) Find the general nonconstant solution $x = x(t)$ and $y = y(t)$ of the original system, and use this to find a periodic solution corresponding to the closed path whose existence was established in (b).
(d) Sketch the closed path and at least two other paths in the phase plane.

3. Show that the nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = 3x - y - xe^{x^2+y^2} \\ \dfrac{dy}{dt} = x + 3y - ye^{x^2+y^2} \end{cases}$$

has a periodic solution.

4. In each of the following cases use a theorem of this section to determine whether or not the given differential equation has a periodic solution:

(a) $\dfrac{d^2x}{dt^2} + (5x^4 - 9x^2)\dfrac{dx}{dt} + x^5 = 0;$

(b) $\dfrac{d^2x}{dt^2} - (x^2 + 1)\dfrac{dx}{dt} + x^5 = 0;$

(c) $\dfrac{d^2x}{dt^2} - \left(\dfrac{dx}{dt}\right)^2 - (1 + x^2) = 0;$

(d) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \left(\dfrac{dx}{dt}\right)^5 - 3x^3 = 0;$

(e) $\dfrac{d^2x}{dt^2} + x^6\dfrac{dx}{dt} - x^2\dfrac{dx}{dt} + x = 0.$

5. Show that any differential equation of the form

$$a\frac{d^2x}{dt^2} + b(x^2 - 1)\frac{dx}{dt} + cx = 0 \quad (a, b, c > 0)$$

can be transformed into the van der Pol equation by a change of the independent variable.

APPENDIX A. POINCARÉ

Jules Henri Poincaré (1854–1912) was universally recognized at the beginning of the twentieth century as the greatest mathematician of his generation. He began his academic career at Caen in 1879, but only two years later he was appointed to a professorship at the Sorbonne. He remained there for the rest of his life, lecturing on a different subject each year. In his lectures—which were edited and published by his students—he treated with great originality and mastery of technique virtually all known fields of pure and applied mathematics, and many that were not known until he discovered them. Altogether he produced more than 30 technical books on mathematical physics and celestial mechanics, half a dozen books of a more popular nature, and almost 500 research papers on mathematics. He was a quick, powerful, and restless thinker, not given to lingering over details, and was described by one of his contemporaries as "a conquerer, not a colonist." He also had the advantage of a prodigious memory, and habitually did his mathematics in his head as he paced back and forth in his study, writing it down only after it was complete in his mind. He was elected to the Academy of Sciences at the very early age of thirty-two. The academician who proposed him for membership said that "his work is above ordinary praise, and reminds us inevitably of what Jacobi wrote of Abel—that he had settled questions which, before him, were unimagined."

Poincaré's first great achievement in mathematics was in analysis. He generalized the idea of the periodicity of a function by creating his theory of automorphic functions. The elementary trigonometric and exponential functions are singly periodic, and the elliptic functions are doubly periodic. Poincaré's automorphic functions constitute a vast generalization of these, for they are invariant under a countably infinite group of linear fractional transformations and include the rich theory of elliptic functions as a detail. He applied them to solve linear differential equations with algebraic coefficients, and also showed how they can be used to uniformize algebraic curves, that is, to express the coordinates of any point on such a curve by means of single-valued functions $x(t)$ and $y(t)$ of a single parameter t. In the 1880s and 1890s automorphic functions developed into an extensive branch of mathematics, involving (in addition to analysis) group theory, number theory, algebraic geometry, and non-Euclidean geometry.

Another focal point of his thought can be found in his researches into celestial mechanics (*Les Méthodes Nouvelle de la Mécanique Céleste,* three volumes, 1892–1899). In the course of this work he developed his theory of asymptotic expansions (which kindled interest in divergent series), studied the stability of orbits, and initiated the qualitative theory of nonlinear differential equations. His celebrated investigations into the evolution of celestial bodies led him to study the equilibrium shapes of a rotating mass of fluid held together by gravitational attraction, and he discovered the pear-shaped figures that played an important role in the later work of Sir G. H. Darwin (Charles' son).[18] In Poincaré's summary of these discoveries, he writes: "Let us imagine a rotating fluid body contracting by cooling, but slowly enough to remain homogeneous and for the rotation to be the same in all its parts. At first very approximately a sphere, the figure of this mass will become an ellipsoid of revolution which will flatten more and more, then, at a certain moment, it will be transformed into an ellipsoid with three unequal axes. Later, the figure will cease to be an ellipsoid and will become pear-shaped until at last the mass, hollowing out more and more at its 'waist,' will separate into two distinct and unequal bodies." These ideas have gained additional interest

[18] See G. H. Darwin, *The Tides,* chap. XVIII, Houghton Mifflin, Boston, 1899.

in our own time; for with the aid of artificial satellites, geophysicists have recently found that the earth itself is slightly pear-shaped.

Many of the problems he encountered in this period were the seeds of new ways of thinking, which have grown and flourished in twentieth-century mathematics. We have already mentioned divergent series and nonlinear differential equations. In addition, his attempts to master the qualitative nature of curves and surfaces in higher dimensional spaces resulted in his famous memoir *Analysis situs* (1895), which most experts agree marks the beginning of the modern era in algebraic topology. Also, in his study of periodic orbits he founded the subject of topological (or qualitative) dynamics. The type of mathematical problem that arises here is illustrated by a theorem he conjectured in 1912 but did not live to prove: if a one-to-one continuous transformation carries the ring bounded by two concentric circles into itself in such a way as to preserve areas and to move the points of the inner circle clockwise and those of the outer circle counterclockwise, then at least two points must remain fixed. This theorem has important applications to the classical problem of three bodies (and also to the motion of a billiard ball on a convex billiard table). A proof was found in 1913 by Birkhoff, a young American mathematician.[19] Another remarkable discovery in this field, now known as the Poincaré recurrence theorem, relates to the long-range behavior of conservative dynamical systems. This result seemed to demonstrate the futility of contemporary efforts to deduce the second law of thermodynamics from classical mechanics, and the ensuing controversy was the historical source of modern ergodic theory.

One of the most striking of Poincaré's many contributions to mathematical physics was his famous paper of 1906 on the dynamics of the electron. He had been thinking about the foundations of physics for many years, and independently of Einstein had obtained many of the results of the special theory of relativity.[20] The main difference was that Einstein's treatment was based on elemental ideas relating to light signals, while Poincaré's was founded on the theory of electromagnetism and was therefore limited in its applicability to phenomena associated with this theory. Poincaré had a high regard for Einstein's abilities, and in 1911 recommended him for his first academic position.[21]

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[19] See G. D. Birkhoff, *Dynamical Systems,* chap. VI, American Mathematical Society Colloquium Publications, vol. IX, Providence, R.I., 1927.

[20] A discussion of the historical background is given by Charles Scribner, Jr., "Henri Poincaré and the Principle of Relativity," *Am. J. Phys.*, vol. 32, p. 672 (1964).

[21] See M. Lincoln Schuster (ed.), *A Treasury of the World's Great Letters,* p. 453, Simon and Schuster, New York, 1940.

In 1902 he turned as a side interest to writing and lecturing for a wider public, in an effort to share with nonspecialists his enthusiasm for the meaning and human importance of mathematics and science. These lighter works have been collected in four books, *La Science et l'Hypothèse* (1903), *La Valeur de la Science* (1904), *Science et Méthode* (1908), and *Dernières Pensées* (1913).[22] They are clear, witty, profound, and altogether delightful, and show him to be a master of French prose at its best. In the most famous of these essays, the one on mathematical discovery, he looked into himself and analyzed his own mental processes, and in so doing provided the rest of us with some rare glimpses into the mind of a genius at work. As Jourdain wrote in his obituary, "One of the many reasons for which he will live is that he made it possible for us to understand him as well as to admire him."

At the present time mathematical knowledge is said to be doubling every 10 years or so, though some remain skeptical about the permanent value of this accumulation. It is generally believed to be impossible now for any human being to understand thoroughly more than one or two of the four main subdivisions of mathematics—analysis, algebra, geometry, and number theory—to say nothing of mathematical physics as well. Poincaré had creative command of the whole of mathematics as it existed in his day, and he was probably the last man who will ever be in this position.

APPENDIX B. PROOF OF LIÉNARD'S THEOREM

Consider Liénard's equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \qquad (1)$$

and assume that $f(x)$ and $g(x)$ satisfy the following conditions: (i) $f(x)$ and $g(x)$ are continuous and have continuous derivatives; (ii) $g(x)$ is an odd function such that $g(x) > 0$ for $x > 0$, and $f(x)$ is an even function; and (iii) the odd function $F(x) = \int_0^x f(x)\,dx$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$. We shall prove that equation (1) has a unique closed path surrounding the origin in the phase plane, and that this path is approached spirally by every other path as $t \to \infty$.

[22] All have been published in English translation by Dover Publications, New York.

The system equivalent to (1) in the phase plane is

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -g(x) - f(x)y. \end{cases} \tag{2}$$

By condition (i), the basic theorem on the existence and uniqueness of solutions holds. It follows from condition (ii) that $g(0) = 0$ and $g(x) \neq 0$ for $x \neq 0$, so the origin is the only critical point. Also, we know that any closed path must surround the origin. The fact that

$$\begin{aligned} \frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} &= \frac{d}{dt}\left[\frac{dx}{dt} + \int_0^x f(x)\,dx\right] \\ &= \frac{d}{dt}[y + F(x)] \end{aligned}$$

suggests introducing a new variable,

$$z = y + F(x).$$

With this notation, equation (1) is equivalent to the system

$$\begin{cases} \dfrac{dx}{dt} = z - F(x) \\ \dfrac{dz}{dt} = -g(x) \end{cases} \tag{3}$$

in the xz-plane. Again we see that the existence and uniqueness theorem holds, that the origin is the only critical point, and that any closed path must surround the origin. The one-to-one correspondence $(x,y) \leftrightarrow (x,z)$ between the points of the two planes is continuous both ways, so closed paths correspond to closed paths and the configurations of the paths in the two planes are qualitatively similar. The differential equation of the paths of (3) is

$$\frac{dz}{dx} = \frac{-g(x)}{z - F(x)}. \tag{4}$$

These paths are easier to analyze than their corresponding paths in the phase plane, for the following reasons.

First, since both $g(x)$ and $F(x)$ are odd, equations (3) and (4) are unchanged when x and z are replaced by $-x$ and $-z$. This means that any curve symmetric to a path with respect to the origin is also a path. Thus if we know the paths in the right half-plane ($x > 0$), those in the left half-plane ($x < 0$) can be obtained at once by reflection through the origin.

Second, equation (4) shows that the paths become horizontal only as they cross the z-axis, and become vertical only as they cross the curve $z = F(x)$. Also, an inspection of the signs of the right sides of equations (3) shows that all paths are directed to the right above the curve $z = F(x)$ and to the left below this curve, and move downward or upward according as $x > 0$ or $x < 0$. These remarks mean that the curve $z = F(x)$, the z-axis, and the vertical line through any point Q on the right half of the curve $z = F(x)$ can be crossed only in the directions indicated by the arrows in Fig. 91. Suppose that the solution of (3) defining the path C through Q is so chosen that the point Q corresponds to the value $t = 0$ of the parameter. Then as t increases into positive values, a point on C with coordinates $x(t)$ and $y(t)$ moves down and to the left until it crosses the z-axis at a point R; and as t decreases into negative values, the point on C rises to the left until it crosses the z-axis at a point P. It will be convenient to let b be the abscissa of Q and to denote the path C by C_b.

It is easy to see from the symmetry property that when the path C_b is continued beyond P and R into the left half of the plane, the result will be a closed path if and only if the distances OP and OR are equal. To show that there is a unique closed path, it therefore suffices to show that there is a unique value of b with the property that $OP = OR$.

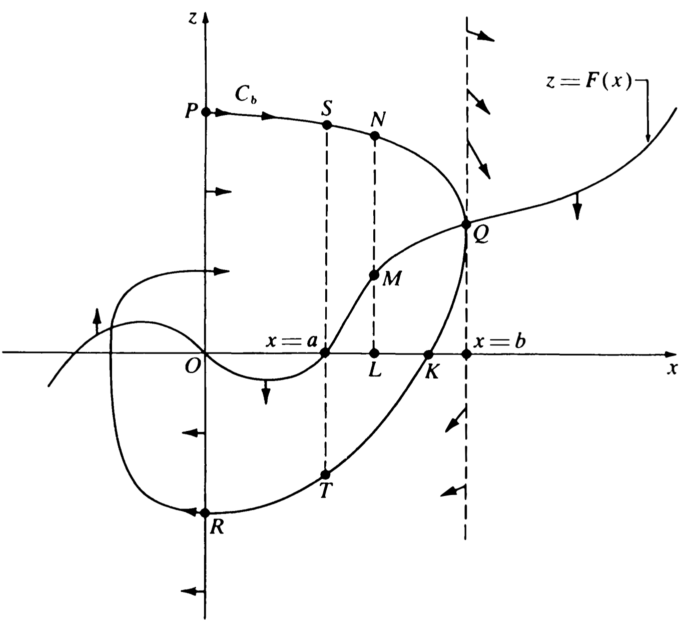

FIGURE 91

To prove this, we introduce

$$G(x) = \int_0^x g(x)\,dx$$

and consider the function

$$E(x,z) = \frac{1}{2}z^2 + G(x),$$

which reduces to $z^2/2$ on the z-axis. Along any path we have

$$\begin{aligned}\frac{dE}{dt} &= g(x)\frac{dx}{dt} + z\frac{dz}{dt}\\ &= -[z - F(x)]\frac{dz}{dt} + z\frac{dz}{dt}\\ &= F(x)\frac{dz}{dt},\end{aligned}$$

so

$$dE = F\,dz.$$

If we compute the line integral of $F\,dz$ along the path C_b from P to R, we obtain

$$I(b) = \int_{PR} F\,dz = \int_{PR} dE = E_R - E_P = \frac{1}{2}(OR^2 - OP^2),$$

so it suffices to show that there is a unique b such that $I(b) = 0$.

If $b \leq a$, then F and dz are negative, so $I(b) > 0$ and C_b cannot be closed. Suppose now that $b > a$, as in Fig. 91. We split $I(b)$ into two parts,

$$I_1(b) = \int_{PS} F\,dz + \int_{TR} F\,dz \quad \text{and} \quad I_2(b) = \int_{ST} F\,dz,$$

so that

$$I(b) = I_1(b) + I_2(b).$$

Since F and dz are negative as C_b is traversed from P to S and from T to R, it is clear that $I_1(b) > 0$. On the other hand, if we go from S to T along C_b we have $F > 0$ and $dz < 0$, so $I_2(b) < 0$. Our immediate purpose is to show that $I(b)$ is a decreasing function of b by separately considering $I_1(b)$ and $I_2(b)$. First, we note that equation (4) enables us to write

$$F\,dz = F\frac{dz}{dx}dx = \frac{-g(x)F(x)}{z - F(x)}dx.$$

The effect of increasing b is to raise the arc PS and to lower the arc TR, which decreases the magnitude of $[-g(x)F(x)]/[z - F(x)]$ for a given x between 0 and a. Since the limits of integration for $I_1(b)$ are fixed, the result is a decrease in $I_1(b)$. Furthermore, since $F(x)$ is positive and nondecreasing to the right of a, we see that an increase in b gives rise to an increase in the positive number $-I_2(b)$, and hence to a decrease in $I_2(b)$. Thus $I(b) = I_1(b) + I_2(b)$ is a decreasing function for $b \geq a$. We now show that $I_2(b) \to -\infty$ as $b \to \infty$. If L in Fig. 91 is fixed and K is to the right of L, then

$$I_2(b) = \int_{ST} F\,dz < \int_{NK} f\,dz \leq -(LM) \cdot (LN);$$

and since $LN \to \infty$ as $b \to \infty$, we have $I_2(b) \to -\infty$.

Accordingly, $I(b)$ is a decreasing continuous function of b for $b \geq a$, $I(a) > 0$, and $I(b) \to -\infty$ as $b \to \infty$. It follows that $I(b) = 0$ for one and only one $b = b_0$, so there is one and only one closed path C_{b_0}.

Finally, we observe that $OR > OP$ for $b < b_0$; and from this and the symmetry we conclude that paths inside C_{b_0} spiral out to C_{b_0}. Similarly, the fact that $OR < OP$ for $b > b_0$ implies that paths outside C_{b_0} spiral in to C_{b_0}.