Ray-Knight theorems related to a stochastic flow

by

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Summary. We study a stochastic flow of $\mathcal{C}^1$-homeomorphisms of $\mathbb{R}$. At certain stopping times, the spatial derivative of the flow is a diffusion in the space variable and its generator is given. This answers several questions posed in a previous study by Bass and Burdzy [3].

Keywords. Flow, Ray-Knight theorems, Jacobi processes.

1991 Mathematics Subject Classification. 60J55; 60J60.

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I. Introduction.

Let β_1 and β_2 be fixed real constants. Suppose B is Brownian motion on the real line issuing from 0. Associated with the equation

$$X_t(x) = x + B_t + \beta_1 \int_0^t ds\, 1_{(X_s(x)\le 0)} + \beta_2 \int_0^t ds\, 1_{(X_s(x)>0)}, \tag{1.1}$$

for $x \in \mathbb{R}$ and $t \ge 0$, is a stochastic flow. It has been shown by Bass and Burdzy [3], and we will also demonstrate below, that on the filtered probability space $\big(\Omega, (\mathcal{B}_t)_{t\ge 0}, \mathbb{P}\big)$, where $\{\mathcal{B}_t, t \ge 0\}$ denotes the natural filtration generated by B, there exists a random flow of homeomorphisms of the real line $\big(X_t, t \ge 0\big)$ such that for each x and t the above equation holds. For all t the map $x \to X_t(x)$ is increasing with continuous and strictly positive derivative which we denote as $DX_t(x)$. Because of the discontinuity in the drift term of equation (1.1) the map X_t cannot be expected to be $\mathcal{C}^2$ or better, see e.g. [6], [11] for some general references.

The main result of this paper is the description of the law of the map $X_T(\cdot)$ for a natural family of stopping times T. This description takes a form very similar to the classical Ray-Knight theorems concerning the occupation density of Brownian motion taken at certain stopping times. We will find that $x \to DX_T(x)$ is a diffusion process and determine its generator which is closely related to the hypergeometric equation.

Before proceeding let us clarify the elementary relationship existing between equation (1.1) and the presentation used by Bass and Burdzy [3]. For an arbitrary x put $\tilde{X}_t = X_t(x) - B_t$ and $\tilde{B}_t = -B_t$, then we have

$$\frac{d\tilde{X}_t}{dt} = \begin{cases} \beta_1 & \text{if} \quad \tilde{X}_t < \tilde{B}_t \\ \beta_2 & \text{if} \quad \tilde{X}_t > \tilde{B}_t \end{cases}, \qquad \tilde{X}_0 = x. \tag{1.2}$$

We have said that that we are going to describe the law of $(DX_T(x),\ x \in \mathbb{R})$. Such a result admits two other important interpretations by virtue of the following two results. The first completes a result due to Bass-Burdzy [3] – however they chose to take a different definition of local time leading to the appearance of a factor of 2.

Proposition 1.1. *With probability one there exists a bicontinuous process $(L_t^x; x \in \mathbb{R}, t \ge 0)$ such that for every x the process $(L_t^x; t \ge 0)$ is the semimartingale local time at zero for $(X_t(x), t \ge 0)$. Moreover*

$$DX_t(x) \equiv \frac{\partial X_t}{\partial x}(x) = \exp\Big((\beta_2 - \beta_1)L_t^x\Big). \tag{1.3}$$

Thus we will obtain descriptions of families of local times at various stopping times- but note that the family is indexed by the starting point not (as is usually the case) the level at which the local time accrued, see also [2] and [5]. However there is a natural way to relate these local times to the occupation measure of a single process. For any t the map X_t is bijective and so there is a unique (random) x such that $X_t(x) = 0$ – we denote this by $X_t^{-1}(0)$.

Proposition 1.2. *Almost surely, the process $(X_t^{-1}(0); t \geq 0)$ admits a bicontinuous occupation density $(\rho_t^x, t \geq 0, x \in \mathbb{R})$ determined by*

$$\int_0^t f(X_s^{-1}(0))ds = \int_{-\infty}^{\infty} dx\, f(x)\, \rho_t^x, \tag{1.4}$$

for any bounded test function f. In fact, we have $\rho_t^x = \kappa(L_t^x)$, for $x \in \mathbb{R}, t \geq 0$, and

$$\kappa(\ell) \stackrel{\text{def}}{=} \begin{cases} \frac{e^{(\beta_2-\beta_1)\ell}-1}{\beta_2-\beta_1}, & \text{if } \beta_1 \neq \beta_2, \\ \ell, & \text{if } \beta_1 = \beta_2, \end{cases} \qquad \ell \geq 0. \tag{1.5}$$

In this paper the process $\left(X_t^{-1}(0), t \geq 0\right)$ plays a natural role in the study of the flow X; more precisely, we shall study the filtration generated by its excursions below a varying level. The analysis is similar to the standard situation related to Brownian motion case (cf. [9, 14], see also [16, 17]). The process $\left(X_t^{-1}(0), t \geq 0\right)$ merits a more complete study, in particular, it is not expected to be a semimartingale (in contrast with the structure of a flow of smooth maps, see [11]).

Here, we consider two kinds of stopping times:

$$T(a) \stackrel{\text{def}}{=} \inf\{t > 0 : X_t(a) = 0\} = \inf\{t > 0 : X_t^{-1}(0) = a\}, \qquad a \in \mathbb{R}, \tag{1.6}$$

$$\tau_r(a) \stackrel{\text{def}}{=} \inf\{t > 0 : L_t^a > r\}, \qquad r \geq 0. \tag{1.7}$$

We shall describe the laws of the local times processes $\left(L_T^x, x \in \mathbb{R}\right)$ at times T of the form $T = T(a)$ and $T = \tau_r(a)$, for different values of the parameters β_1 and β_2. Recall (1.5) for $\kappa(\cdot)$.

Theorem 1.1 (Transient and Recurrent cases). *Let $\beta_1 \geq 0$ and fix $a < 0$. The process $\left(L_{T(a)}^{a+x}, x \geq 0\right)$ is a time inhomogeneous diffusion starting from 0 with infinitesimal generator:*

$$2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\left(\mathbb{1}_{(0 \leq x \leq a-)} - \beta_1\kappa(\ell)\right)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+. \tag{1.8}$$

The process $\left(L_{T(a)}^{a+x}, x \geq a^-\right)$ is absorbed at 0. If additionally $\beta_2 > 0$ (transient case), L_∞^0 is exponentially distributed with parameter β_2; conditioning on $\{L_\infty^0 = r\}$, $\left(L_\infty^x, x \geq 0\right)$ and

$\left(L_\infty^{-x}, x \geq 0\right)$ *are two independent diffusions starting from* r*, with infinitesimal generators respectively given by:*

$$(1.9) \qquad 2\kappa(\ell)\frac{d^2}{d\ell^2} - 2\beta_1\kappa(\ell)\frac{d}{d\ell}, \qquad 2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(1 - \beta_1\kappa(\ell)\Big)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+.$$

$\left(L_\infty^{x}, x \geq 0\right)$ *is absorbed at* 0.

Remark 1.1. By using a different method, Bass and Burdzy [3] have obtained (1.9) in the case $\beta_1 > \beta_2 > 0$. For every fixed $x > 0$,

L_∞^{-x} has the same law of L_∞^0, which is in fact the stationary distribution of the diffusion $\left(L_\infty^{-x}, x \geq 0\right)$. By transience we mean here $X_t(x) \to \infty$, as $t \to \infty$, for every $x \in \mathbb{R}$. The case that $X_t(x) \to -\infty$ as $t \to \infty$ (iff $\beta_1 < 0$ and $\beta_2 \leq 0$) follows by symmetry. The reason why we take $a < 0$ is to ensure $T(a) < \infty$, with probability one.

Theorem 1.2 (Recurrent case). *Suppose* $\beta_1 \geq 0 \geq \beta_2$*. Fix* $r > 0$ *and* $b \in \mathbb{R}$*. The two processes* $\left(L_{\tau_r(b)}^{b+x}, x \geq 0\right)$ *and* $\left(L_{\tau_r(b)}^{b-x}, x \geq 0\right)$ *are independent (inhomogeneous) diffusions, both starting from* r *and absorbed at* 0*, with respective infinitesimal generators:*

$$(1.10) \qquad 2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(\mathbb{1}_{(0\leq x\leq b^-)} - \beta_1\kappa(\ell)\Big)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+.$$

$$(1.11) \qquad 2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(\mathbb{1}_{(0\leq x\leq b^+)} + \beta_2\kappa(\ell)\Big)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+.$$

Remark 1.2. In the case $\beta_1 = \beta_2 \geq 0$, Theorems 1.1–2 give the classical Ray–Knight theorems for Brownian motion with a non–negative drift, see e.g. Norris et al. [14].

In the bifurcation case ($\beta_1 < 0$ and $\beta_2 > 0$), Bass and Burdzy [3] showed there exists a unique (random) critical level $\xi \in \mathbb{R}$ such that with probability one,

$$(1.12) \qquad X_t(x) \begin{cases} \to \infty, & \text{if} \quad x > \xi \\ \to -\infty, & \text{if} \quad x < \xi \\ \text{hits 0 infinitely often}, & \text{if} \quad x = \xi \end{cases}, \qquad t \to \infty.$$

It turns out that with probability one, $\tau_r(\xi) < \infty$, for every $r > 0$. We have

Theorem 1.3 (Bifurcation). *Suppose* $\beta_1 < 0$ *and* $\beta_2 > 0$*. Fix* $r > 0$*. We have*

$$(1.13) \qquad \mathbb{P}\Big(\xi \in da\Big)/da = \frac{2\beta_1\beta_2}{\beta_1 - \beta_2} \exp\Big(-2\beta_2 a\mathbb{1}_{(a\geq 0)} - 2\beta_1 a\mathbb{1}_{(a<0)}\Big).$$

Conditioning on $\{\xi = a\}$, the processes $\left(L^{a+x}_{\tau_r(a)}, x \geq 0\right)$ and $\left(L^{a-x}_{\tau_r(a)}, x \geq 0\right)$ are two independent (inhomogeneous) diffusions starting from r and absorbed at 0, with respective infinitesimal generator given by:

$$2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(\mathbb{1}_{(0\leq x\leq a^-)} + \beta_1\kappa(\ell)\Big)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+. \tag{1.14}$$

$$2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(\mathbb{1}_{(0\leq x\leq a^+)} - \beta_2\kappa(\ell)\Big)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+. \tag{1.15}$$

Moreover, conditioning on $\{\xi = a\}$, $\left(L^{a+x}_{\infty}, x > 0\right)$ and $\left(L^{a-x}_{\infty}, x > 0\right)$ are two independent (inhomogeneous) diffusions, with infinitesimal generators given by (1.14) and (1.15) respectively.

Remark 1.3. Conditioning on $\{\xi = a\}$, $L^a_\infty = \infty$ and it is more convenient to consider $\left(1/DX_\infty(a + x), x \geq 0\right)$ and $\left(1/DX_\infty(a - x), x \geq 0\right)$ (recalling (1.3)), which are two independent Jacobi type diffusions, starting from 0, see (1.18) below. Theorem 1.3 could be obtained using a time reversal argument from Theorem 1.2, but this will not be presented in this paper.

Each of the above theorems can be expressed as a description of $x \in \mathbb{R} \to DX_T(x)$ as a diffusion. More precisely, for $x \to L^x_T$ associated with a generator

$$2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(\gamma_1 + \gamma_2\kappa(\ell)\Big)\frac{d}{d\ell}, \tag{1.16}$$

with $\gamma_1, \gamma_2 \in \mathbb{R}$ being two constants, elementary computations show that $x \to DX_T(x)$ given by (1.3) has generator

$$2(\beta_2 - \beta_1)\ell^2(\ell - 1)\frac{d^2}{d\ell^2} + 2\ell\Big((\gamma_2 + \beta_2 - \beta_1)(\ell - 1) + \gamma_1(\beta_2 - \beta_1)\Big)\frac{d}{d\ell}. \tag{1.17}$$

The range of $DX_T(x)$ is determined as follows: $DX_T(x) \equiv 1$ if $\beta_2 = \beta_1$; $DX_T(x) \in [1, \infty)$ if $\beta_2 > \beta_1$ and $DX_T(x) \in (0, 1]$ if $\beta_2 < \beta_1$.

Also, the process $x \to 1/DX_T(x)$ is a diffusion with generator

$$2(\beta_2 - \beta_1)\ell(1 - \ell)\frac{d^2}{d\ell^2} + 2\Big((\beta_2 - \beta_1 - \gamma_2)(1 - \ell) - \gamma_1(\beta_2 - \beta_1)\ell\Big)\frac{d}{d\ell}. \tag{1.18}$$

These latter processes, sometimes called Jacobi processes, are well-known and play an important role in many models of genetic frequencies, see for example [8]. They have also appeared in Ray-Knight type theorems before, see [21]. This is no strange coincidence – there is a connection between the flow we are considering here and the problem of describing the trajectory taken by Brownian motion conditional on knowing its occupation measure at some stopping time – see also Aldous [1]. This connection will be explored elsewhere.

The rest of this paper is organized as follows: In Section 2, we prove the existence of the flow of homeomorphisms associated with (1.1) and Propositions 1.1–2. In Section 3, using Tanaka's formula, we study the filtration generated by the excursions of $X_\cdot^{-1}(0)$ and prove Theorems 1.1 and 1.2, whereas Section 4 is devoted to the study of the bifurcation case, and to the proof of Theorem 1.3. In Section 5, we prove a result of the differentiability of one-dimensional flow with non-smooth coefficients. In Section 6, we consider a simple change of measure for the flow.

Acknowledgements. We are very grateful to Nathalie Eisenbaum, Jean Jacod and Marc Yor for helpful discussions. We thank an anonymous referee for her/his comments and suggestions.

2. Removal of drift and Tanaka's formula

Define

$$s(x) \stackrel{\text{def}}{=} \begin{cases} -\int_x^0 dy\, e^{-2\beta_1 y} & \text{if} \quad x \le 0 \\ \int_0^x dy\, e^{-2\beta_2 y} & \text{if} \quad x > 0 \end{cases}, \tag{2.1}$$

$$Z_t(y) \stackrel{\text{def}}{=} s(X_t(s^{-1}(y))), \qquad t \ge 0, \;\; y \in I \equiv (s(-\infty), s(\infty)), \tag{2.2}$$

where s^{-1} denotes the inverse of s, which is the scale function of $X_t(x)$. Therefore,

$$Z_t(y) = y + \int_0^t \sigma(Z_s(y))\, dB_s, \qquad y \in I,\ t \ge 0, \tag{2.3}$$

with

$$\sigma(z) \stackrel{\text{def}}{=} s'(s^{-1}(z)) = \begin{cases} 1 - 2\beta_1 z & \text{if} \quad z \le 0 \\ 1 - 2\beta_2 z & \text{if} \quad z > 0 \end{cases}, \qquad z \in I. \tag{2.4}$$

Observe that for each $y \in I$, the explosion time of (2.3) $\inf\{t > 0 : Z_t(y) \notin I\} = \infty$ with probability one, and that σ is a Lipschitz function. According to a continuity result due to Neveu (cf. Meyer [13, Theorem 1] and Uppman [20]), there exists a version of $(Z_t(y), t \ge 0, y \in I)$ such that with probability one, $(t, y) \in \mathbb{R}_+ \times I \to Z_t(y) \in I$ is continuous, and for all $y \in I$, the process $Z_\cdot(y)$ satisfies (2.3). We shall make use of this bicontinuous version of Z, which in view of (2.1)–(2.2), determines a bicontinuous version of X.

Lemma 2.1. *Fix $\beta_1, \beta_2 \in \mathbb{R}$. With probability one, we have*

$$Z_t(y_1) < Z_t(y_2), \qquad \forall\, t \geq 0, \qquad \forall\, y_1 < y_2, \qquad y_1, y_2 \in I. \tag{2.5}$$

$$\frac{\partial Z_t(y)}{\partial y} = \exp\Big(N_t^y - \frac{1}{2}\langle N^y\rangle_t\Big), \qquad \forall\, y \in I, \qquad t > 0, \tag{2.6}$$

where $N_t^y \overset{\text{def}}{=} -\int_0^t \big(2\beta_1 \mathbb{1}_{(Z_s(y)\leq 0)} + 2\beta_2 \mathbb{1}_{(Z_s(y)>0)}\big) dB_s$ *is bicontinuous in* $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$.

In view of (2.1)–(2.2) and Lemma 2.1, we deduce easily from (1.1) the surjectivity on $\mathbb{R}$ of $X_t(\cdot)$ for each $t \geq 0$. Hence, X is a $\mathcal{C}^1$-homeomorphism flow on $\mathbb{R}$.

Proof of Lemma 2.1. (2.5) follows from [22, Theorem 1.1]. The differentiability of $Z_t(\cdot)$ is well-known in the case that σ' is Hölder continuous (see e.g. Kunita [11, Theorem 3.1]), but here σ' has a discontinuity at 0. The condition on σ can be weakened because of the simple structure of $\mathbb{R}^1$; the proof of (2.6) is given in Section 5. □

Applying Tanaka's formula to $X_\cdot(x)$, we have

$$X_t^-(x) = x^- - \int_0^t \mathbb{1}_{(X_s(x)\leq 0)} dB_s - \beta_1 \int_0^t \mathbb{1}_{(X_s(x)\leq 0)} ds + \frac{1}{2} L_t^x, \tag{2.7}$$

$$X_t^+(x) = x^+ + \int_0^t \mathbb{1}_{(X_s(x)>0)} dB_s + \beta_2 \int_0^t \mathbb{1}_{(X_s(x)>0)} ds + \frac{1}{2} L_t^x, \tag{2.8}$$

where here, we can take (2.7) or (2.8) as a definition of local times $\big(L_t^x\big)$ appeared in Proposition 1.1. L_t^x is the (semimartingale) local time at 0 up to t of $X_\cdot(x)$.

Proof of Propositions 1.1 and 1.2. Recall (2.2) and let $y \overset{\text{def}}{=} s(x) \in I$. Using Lemma 2.1 and the fact that $\{Z_s(y) \leq 0\} = \{X_s(x) \leq 0\}$, we get

$$\begin{aligned} N_t^y - \frac{1}{2}\langle N^y\rangle_t &= -2\beta_1 \int_0^t \mathbb{1}_{(X_s(x)\leq 0)} dB_s - 2\beta_1^2 \int_0^t \mathbb{1}_{(X_s(x)\leq 0)} ds \\ &\quad - 2\beta_2 \int_0^t \mathbb{1}_{(X_s(x)>0)} dB_s - 2\beta_2^2 \int_0^t \mathbb{1}_{(X_s(x)>0)} ds \\ &= (\beta_2 - \beta_1) L_t^x + 2\beta_1\big(X_t^-(x) - x^-\big) - 2\beta_2\big(X_t^+(x) - x^+\big), \end{aligned} \tag{2.9}$$

which in view of (2.6) and (2.1) yields (1.3) and that $\big(L_t^x\big)$ is bicontinuous. Proposition 1.1 is proven.

To show Proposition 1.2, notice that for $x_2 > x_1 \in \mathbb{R}$, we deduce from (1.3), (1.1) together with the monotonicity that

$$\int_{x_1}^{x_2} dy \exp\Big((\beta_2 - \beta_1) L_t^y\Big) = X_t(x_2) - X_t(x_1)$$

$$= (x_2 - x_1) + (\beta_2 - \beta_1) \int_0^t ds\, \mathbb{1}_{(X_s(x_1)\leq 0 < X_s(x_2))} \tag{2.10}$$

$$= (x_2 - x_1) + (\beta_2 - \beta_1) \int_0^t ds \mathbb{1}_{(x_1 \leq X_s^{-1}(0) < x_2)}, \tag{2.11}$$

which in view of the continuity of (L_t^y), implies (1.4)–(1.5). □

3. Transient and Recurrent cases

Throughout this section, we suppose $\beta_1 \geq 0$. There are only two possibilities:

(3.1) Transient case: if $\beta_2 > 0$ (and $\beta_1 \geq 0$), then $I = (-\infty, \frac{1}{2\beta_2})$, and with probability one, for every $x \in \mathbb{R}$, we have $X_t(x) \to \infty$ as $t \to \infty$; hence $X_t^{-1}(0) \to -\infty$, a.s.

(3.2) Recurrent case: if $\beta_2 \leq 0$ (and $\beta_1 \geq 0$), then $I = \mathbb{R}$, and $X_t(x)$ is recurrent; hence with probability one, $X_t^{-1}(0) = 0$ infinitely often as $t \to \infty$.

Firstly, we study the filtration generated by the excursions of $X_\cdot^{-1}(0)$ below a varying level. For $t \geq 0$ and $x \in \mathbb{R}$, define

$$(3.3) \qquad A_t(x) \stackrel{\text{def}}{=} \int_0^t ds \mathbb{1}_{(X_s(x)>0)} \equiv \int_0^t ds \mathbb{1}_{(X_s^{-1}(0)<x)},$$

$$(3.4) \qquad \alpha_t(x) \stackrel{\text{def}}{=} \inf\{s > 0 : A_s(x) > t\},$$

$$(3.5) \qquad \mathcal{E}_x \stackrel{\text{def}}{=} \sigma\{X_{\alpha_t(x)}(x),\ t \geq 0\} = \sigma\{X^{-1}_{\alpha_t(x)}(0),\ t \geq 0\},$$

where the equality in (3.5) will be demonstrated below in the proof of Lemma 3.1 (cf. (3.9)). Notice that in the transient and recurrent cases (3.1) and (3.2), $\alpha_t(x)$ is almost surely finite, and $\mathcal{E}_x$ is well-defined. The following key lemma enables us to understand the martingales related to $(\mathcal{E}_x, x \in \mathbb{R})$:

Lemma 3.1. *$(\mathcal{E}_x, x \in \mathbb{R})$ is an increasing family of σ-fields. Furthermore, for every $x \in \mathbb{R}$ and $H \in L^2(\mathcal{E}_x)$, there exists a $(\mathcal{B}_t, t \geq 0)$-predictable process $(h_t, t \geq 0)$ such that*

$$(3.6) \qquad H = \mathbb{E}H + \int_0^\infty h_s \mathbb{1}_{(X_s(x)>0)}\, dB_s = \mathbb{E}H + \int_0^\infty h_s \mathbb{1}_{(X_s^{-1}(0)<x)}\, dB_s,$$

and $\mathbb{E}\int_0^\infty h_s^2 \mathbb{1}_{(X_s^{-1}(0)<x)} ds < \infty$, where $(\mathcal{B}_t, t \geq 0)$ denotes the natural filtration generated by $\{B_t, t \geq 0\}$.

Proof of Lemma 3.1. For notational convenience, we write in this proof $Y_t \equiv X_t^{-1}(0)$ for $t \geq 0$. (3.6) is folklore for the filtration generated by Brownian excursions below a varying level, and it follows easily from the use of Tanaka's formula and time-change (see e.g. [9] and [14] together with their references), here we prove (3.6) by using the same idea. Fix $x \in \mathbb{R}$. Using Dambis-Dubins-Schwarz' continuous martingale representation theorem (cf. [15, Theorem V.1.6]), we deduce from (2.8) that

$$(3.7) \qquad 0 \leq X_{\alpha_t(x)}(x) = X^+_{\alpha_t(x)}(x) = x^+ + \gamma_t(x) + \beta_2 t + \frac{1}{2} L^x_{\alpha_t(x)}, \qquad t \geq 0,$$

where $\gamma_t(x) \stackrel{\text{def}}{=} \int_0^{\alpha_t(x)} \mathbb{1}_{(X_s(x)>0)}\, dB_s$, for $t \geq 0$, is a Brownian motion. Applying Skorokhod's lemma (cf. [15, Lemma VI.2.1]) gives

$$\frac{1}{2} L^x_{\alpha_t(x)} = \sup_{0 \leq s \leq t} \Big(- x^+ - \gamma_s(x) - \beta_2 s \Big)^+, \qquad t \geq 0, \tag{3.8}$$

which implies that $\mathcal{E}_x \subset \sigma\{\gamma_t(x), t \geq 0\}$, and the reverse inclusion follows from (3.7), since $t \to L^x_{\alpha_t(x)}$ is the semimartingale local time at zero of $t \to X_{\alpha_t(x)}(x)$. Hence we have $\mathcal{E}_x = \sigma\{\gamma_t(x), t \geq 0\}$, and (3.6) follows from the representation theorem for Brownian filtration (cf. [15, Proposition V.3.2]) and from the time-change $\alpha_t(x)$.

The important fact to verify is that $\mathcal{E}_x$ is non-decreasing in $x \in \mathbb{R}$, but it suffices to show the equality in (3.5), i.e. (recalling $Y_. \equiv X_.^{-1}(0)$)

$$\mathcal{E}_x = \sigma\{Y_{\alpha_t(x)},\ t \geq 0\}, \tag{3.9}$$

since by time-change (remarking that $\alpha_t(x) = \inf\{s > 0 : \int_0^s \mathbb{1}_{(Y_u < x)} du > t\}$), the σ-fields of the RHS of (3.9) are always increasing in x.

To show (3.9), using Proposition 1.2, we have

$$\kappa(L^y_{\alpha_t(x)}) = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_0^t ds\, \mathbb{1}_{(y-\epsilon < Y_{\alpha_s(x)} \leq y)}, \qquad y \leq x. \tag{3.10}$$

Since $X_u(Y_u) = 0$, we deduce from (1.3) and (1.4) that

$$X_{\alpha_t(x)}(x) = X_{\alpha_t(x)}(x) - X_{\alpha_t(x)}(Y_{\alpha_t(x)}) = \int_{Y_{\alpha_t(x)}}^x dy \big(1 + (\beta_2 - \beta_1)\kappa(L^y_{\alpha_t(x)})\big), \tag{3.11}$$

showing that $\mathcal{E}_x \subset \sigma\{Y_{\alpha_t(x)},\ t \geq 0\}$. To prove the reverse relation, it suffices to show

$$\{Y_{\alpha_t(x)} < y\} = \{X_{\alpha_t(x)}(y) > 0\} \text{ is } \mathcal{E}_y\text{-measurable}, \qquad y \leq x. \tag{3.12}$$

To this end, applying (2.10) to $x_1 \stackrel{\text{def}}{=} y$, $x_2 \stackrel{\text{def}}{=} x$ and t replaced by $\alpha_t(x)$ gives

$$X_{\alpha_t(x)}(y) = y - x + X_{\alpha_t(x)}(x) + (\beta_1 - \beta_2) \int_0^t \mathbb{1}_{(X_{\alpha_u(x)}(y) \leq 0)}\, du, \qquad t \geq 0. \tag{3.13}$$

The process $t \to X_{\alpha_t(x)}(x)$ is a continuous semimartingale (cf. (3.7)–(3.8)), therefore using Zvonkin [24]'s method (cf. also [19]), the equation (3.13) has the pathwise uniqueness, and (3.12) follows, which completes the proof of (3.9), as desired. □

Recall (1.6)–(1.7). The following results (Lemmas 3.2 and 3.3) constitute the core of the proofs of Theorems 1.1 and 1.2.

Lemma 3.2 ($\beta_1 \geq 0$). *Fix $a < 0$. The process $x \in [a, \infty) \to \int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)} \, dB_s$ is an $\mathcal{E}_x$-adapted continuous martingale with increasing process $x \to \int_a^x \kappa(L^y_{T(a)}) \, dy$. Furthermore, in the transient case $\beta_2 > 0$, the same conclusion holds if we replace $T(a)$ by ∞.*

Remark 3.1. Except for the case $\beta_2 > \beta_1 = 0$, $\int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)} \, dB_s$ is a square-integrable martingale.

Proof of Lemma 3.2. Let $x \geq a$. Observe that $X_{T(a)}(x) \geq 0$. It follows from (2.7) that

$$(3.14) \quad 0 = X^-_{T(a)}(x) = x^- - \int_0^{T(a)} \mathbb{1}_{(X_s^{-1}(0) \geq x)} dB_s - \beta_1 \int_0^{T(a)} \mathbb{1}_{(X_s^{-1}(0) \geq x)} ds + \frac{1}{2} L^x_{T(a)}.$$

Using (3.14) for $L^x_{T(a)} - L^a_{T(a)}$ together with Proposition 1.2 gives

$$(3.15) \qquad L^x_{T(a)} = 2\big(a^- - x^-\big) - 2 \int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)} dB_s - 2\beta_1 \int_a^x \kappa(L^y_{T(a)}) dy.$$

Now, we show that $\int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)} \, dB_s$ and $L^x_{T(a)}$ are $\mathcal{E}_x$-measurable. In fact, in view of (3.15), (3.3) and Proposition 1.2, it suffices to show that $A_{T(a)}(x)$ is $\mathcal{E}_x$-measurable. Recall (3.4). Observe that for $t > 0$,

$$\{A_{T(a)}(x) \leq t\} = \{T(a) \leq \alpha_t(x)\} = \{\inf_{0 \leq s \leq \alpha_t(x)} X_s^{-1}(0) \leq a\} = \{\inf_{0 \leq s \leq t} X^{-1}_{\alpha_s(x)}(0) \leq a\},$$

which is $\mathcal{E}_x$-measurable by using (3.9), as desired.

Write $M_x \stackrel{\text{def}}{=} \int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)} \, dB_s$ in this proof. If we have proven that M_x is integrable, we would deduce easily from Lemma 3.1 that $(M_x, x \geq a)$ is a martingale. In fact, for $y > x \geq a$ and for every bounded $\mathcal{E}_x$-measurable variable H, it follows from (3.6) that

$$\mathbb{E}\Big(H\,(M_y - M_x)\Big) = \mathbb{E}\Big(\big(\mathbb{E}H + \int_0^\infty h_s \mathbb{1}_{(X_s^{-1}(0) < x)} \, dB_s\big) \int_0^{T(a)} \mathbb{1}_{(x \leq X_s^{-1}(0) < y)} \, dB_s\Big) = 0,$$

yielding the martingale property. Furthermore, $x \to M_x$ is continuous (see (3.15)), and its increasing process is given by $\langle M \rangle_x = \int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)} \, ds = \int_a^x \kappa(L^y_{T(a)}) dy$ by Proposition 1.2. This can be immediately obtained by using the following well-known method: Considering a sequence of subdivisons $\Delta_n = (x_0^{(n)} = x < x_1^{(n)} < \ldots < x_{j_n}^{(n)} = y)$, with $|\Delta_n| \stackrel{\text{def}}{=} \sup_{1 \leq i \leq j_n} |x_i^{(n)} - x_{i-1}^{(n)}| \to 0$ as $n \to \infty$. It is easy to see that

$$\begin{aligned}
&\sum_{\Delta_n} \Big(\big(M_{x_i^{(n)}} - M_{x_{i-1}^{(n)}}\big)^2 - \int_0^{T(a)} \mathbb{1}_{(x_{i-1}^{(n)} \leq X_s^{-1}(0) < x_i^{(n)})} \, ds\Big) \\
&= 2 \sum_{\Delta_n} \int_0^{T(a)} dB_s \, \mathbb{1}_{(x_{i-1}^{(n)} \leq X_s^{-1}(0) < x_i^{(n)})} \int_0^s dB_u \, \mathbb{1}_{(x_{i-1}^{(n)} \leq X_u^{-1}(0) < x_i^{(n)})} \\
&\xrightarrow{(p)} 0, \qquad\qquad |\Delta_n| \to 0,
\end{aligned}$$

as desired (cf. Bouleau [4]). Let us show that $\mathbb{E}|M_x| < \infty$. We distinguish two possible cases, $\beta_2 \geq \beta_1 \geq 0$ and $\beta_1 > \beta_2$. If $\beta_1 > \beta_2$, it follows from Proposition 1.2 that $\int_0^{T(a)} \mathbb{1}_{(a \leq X_s^{-1}(0) < x)}\, ds = \int_a^x \kappa(L^y_{T(a)}) dy \leq \frac{1}{\beta_1 - \beta_2}(x-a)$, implying in view of Doob's inequality that $\mathbb{E} M_x^2 < \infty$.

It remains to treat the case $\beta_2 \geq \beta_1 \geq 0$. We have from (2.8) that

$$\mathbb{E}X^+_{t\wedge T(a)}(a) = x^+ + \beta_2 \mathbb{E}\int_0^{t\wedge T(a)} \mathbb{1}_{(X_s^{-1}(0)<x)}\, ds + \frac{1}{2}\mathbb{E}L^x_{t\wedge T(a)}. \tag{3.16}$$

On the other hand, we deduce from (2.11) that

$$X_{t\wedge T(a)}(x) \leq x - a + (\beta_2 - \beta_1)\int_0^{t\wedge T(a)} \mathbb{1}_{(a\leq X_s^{-1}(0)<x)}\, ds, \tag{3.17}$$

since $X_{t\wedge T(a)}(a) \leq 0$. In view of (3.16) and (3.17), we have

$$\beta_1 \mathbb{E}\int_0^{t\wedge T(a)} \mathbb{1}_{(a\leq X_s^{-1}(0)<x)}\, ds + \frac{1}{2}\mathbb{E}L^x_{t\wedge T(a)} \leq a^- - x^-,$$

which yields, by letting $t \to \infty$ and using (3.15) that $\mathbb{E}\left| \int_0^{T(a)} \mathbb{1}_{(a\leq X_s^{-1}(0)<x)} dB_s \right| < \infty$, as desired. The case $\beta_2 > 0$ with $T(a)$ replaced by ∞ can be proven in the same way. □

Lemma 3.3 (recurrent case). *Assuming $\beta_1 \geq 0$ and $\beta_2 \leq 0$. Fix $b \in \mathbb{R}$ and $r > 0$. The process $x \in [b,\infty) \to \int_0^{\tau_r(b)} \mathbb{1}_{(b\leq X_s^{-1}(0)<x)}\, dB_s$ is an $\mathcal{E}_x$-adapted continuous square-integrable martingale with increasing process $x \to \int_b^x \kappa(L^y_{\tau_r(b)})\, dy$.*

Proof of Lemma 3.3. It is easy to see that $\int_0^{\tau_r(b)} \mathbb{1}_{(b\leq X_s^{-1}(0)<x)}\, dB_s$ is in L^2, in fact, this is well-known in the case $\beta_1 = \beta_2 = 0$ (i.e. the standard Brownian motion case), and in the case of $\beta_1 > \beta_2$, we have from Proposition 1.2 that $\int_0^{\tau_r(b)} \mathbb{1}_{(b\leq X_s^{-1}(0)<x)}\, ds \leq \frac{x-b}{\beta_1-\beta_2}$, as desired. The remaining claims can be proven in exactly the same way as in the proof of Lemma 3.2. □

Proof of Theorem 1.1. In view of Lemma 3.2 and (3.15), we have that

$$L^x_{T(a)} = 2\big(a^- - x^-\big) - 2\int_a^x \sqrt{\kappa(L^y_{T(a)})}\, dW_y - 2\beta_1 \int_a^x \kappa(L^y_{T(a)}) dy, \qquad x \geq a, \tag{3.18}$$

for some $\big(\mathcal{E}_x\big)$-Brownian motion W. This equation admits a unique solution in law (cf. [18, Corollary 10.1.2]). Hence (1.8) follows.

Now, we consider the transient case that $\beta_1 \geq 0$ and $\beta_2 > 0$. Let $x \geq a$ with an arbitrary but fixed $a < 0$. By applying (2.7) to $X_t^-(x) - X_t^-(a)$ and letting $t \to \infty$, we obtain

$$L^x_\infty = L^a_\infty + 2(a^- - x^-) - 2\int_0^\infty \mathbb{1}_{(a\leq X_s^{-1}(0)<x)}\, dB_s - 2\beta_1 \int_0^\infty \mathbb{1}_{(a\leq X_s^{-1}(0)<x)}\, ds,$$

which in veiw of Lemma 3.2 and Proposition 1.2 yield that

$$L^x_\infty = L^a_\infty + 2(a^- - x^-) - 2\int_a^x \sqrt{\kappa(L^y_\infty)}\, d\widehat{W}(y) - 2\beta_1 \int_a^x \sqrt{\kappa(L^y_\infty)}\, dy, \tag{3.19}$$

with some $(\mathcal{E}_x)$-Brownian motion $\widehat{W}$, hence $\widehat{W}$ is independent of $\mathcal{E}_a$. Since a is arbitrary, it follows from (3.19) that $\{L^x_\infty, x \in \mathbb{R}\}$ is a diffusion with generator given by (1.9).

Finally, it is elementary to verify that the exponentional distribution of parameter β_2 is the unique stationary distribution of the diffusion $(L^{-x}_\infty, x \geq 0)$. To end the proof of Theorem 1.1, it suffices to

show that for every fixed $a < 0$, L^a_∞ has the same distribution of L^0_∞. Let $\widetilde{X}_u(x) \stackrel{\text{def}}{=} X_{u+T(a)}(X^{-1}_{T(a)}(x)$ for $u \geq 0, x \in \mathbb{R}$, then $\widetilde{X}$ is a copy of the flow X. Observe that L^a_∞ equals the local time at zero of $(\widetilde{X}_u(0), 0 \leq u < \infty)$, hence $L^0_\infty \stackrel{\text{law}}{=} L^a_\infty$, completing the proof. □

Proof of Theorem 1.2. Consider first $x \geq b$. Since $X_{\tau_r(b)}(x) \geq X_{\tau_r(b)}(b) = 0$, we have from (2.7) that

$$0 = X^-_{\tau_r(b)}(x) = x^- - \int_0^{\tau_r(b)} 1\!\!1_{(X_s^{-1}(0) \geq x)} dB_s - \beta_1 \int_0^{\tau_r(b)} 1\!\!1_{(X_s^{-1}(0)\geq x)} ds + \frac{1}{2} L^x_{\tau_r(b)},$$

which, by considering $L^x_{\tau_r(b)} - L^b_{\tau_r(b)}$, implies that

$$L^x_{\tau_r(b)} = r + 2\big(b^- - x^-\big) - 2\int_0^{\tau_r(b)} 1\!\!1_{(b \leq X_s^{-1}(0) < x)} dB_s - 2\beta_1 \int_b^x \kappa(L^y_{\tau_r(b)}) dy, \tag{3.20}$$

whereas using Lemma 3.3

$$L^x_{\tau_r(b)} = r + 2\big(b^- - x^-\big) - 2\int_b^x \sqrt{\kappa(L^y_{\tau_r(b)})} d\tilde{W}_y - 2\beta_1 \int_b^x \kappa(L^y_{\tau_r(b)}) dy, \tag{3.21}$$

with some $\big(\mathcal{E}_x\big)$-Brownian motion $\tilde{W}$, independent of $\mathcal{E}_b$. ¿From this, (1.10) follows, and $\{L^x_{\tau_r(b)}, x \geq b\}$ is independent of $\{L^x_{\tau_r(b)}, x \leq b\}$. Finally, remark that (1.11) follows from (1.10) and the following symmetry property: precisely, let $\widehat{X}_t(x) \stackrel{\text{def}}{=} -X_t(-x), t \geq 0, x \in \mathbb{R}$ and define $\widehat{L}^x_t$, $\widehat{\tau}_r(a)$ related to $\widehat{X}$ in the same way as $L^x_t, \tau_r(a)$ are to X. Therefore, $\widehat{X}$ is a recurrent flow associated with $(-\beta_2, -\beta_1)$ in the same way as X is with (β_1, β_2). We have $\{\widehat{L}^x_{\widehat{\tau}_r(-b)}, x \geq -b\} = \{L^x_{\tau_r(b)}, x \leq b\}$, and the desired result follows. □

4. Bifurcation case

Throughout this section, we suppose that $\beta_1 < 0 < \beta_2$. The existence of the critical level ξ involved in (1.12) can be proven quickly as follows: Recall (2.1)–(2.3). Since

$s(-\infty) = 1/(2\beta_1) > -\infty$ and $s(\infty) = 1/(2\beta_2) < \infty$, we have that for every $x \in \mathbb{R}$, almost surely, either $X_t(x) \to \infty$ or $X_t(x) \to -\infty$. From this fact, together with the increasing property in x of $X_t(x)$, we deduce easily the existence and uniqueness of a critical level ξ verifying the two first properties in (1.12). The recurrence of $X_\cdot(\xi)$ is shown by (4.5) below.

We deal with this random level ξ by describing the flow conditional on $\{\xi = a\}$ with $a \in \mathbb{R}$ (in fact, this is equivalent to enlargement of filtration, see e.g. Yor [23] and the references therein for the latter). Using the scale function s defined in (2.1), we have

$$\begin{aligned} \mathbb{P}\Big(\xi < x\Big) &= \mathbb{P}\Big(\lim_{t\to\infty} X_t(x) = \infty\Big) \\ &= \frac{s(x) - s(-\infty)}{s(\infty) - s(-\infty)} \\ &= \begin{cases} 1 + \frac{\beta_1}{\beta_2 - \beta_1} e^{-2\beta_2 x}, & \text{if } x \geq 0 \\ \frac{\beta_2}{\beta_2 - \beta_1} e^{-2\beta_1 x}, & \text{if } x < 0 \end{cases} \stackrel{\text{def}}{=} h(x), \end{aligned} \tag{4.1}$$

yielding (1.13). For fixed $t \geq 0$, put $\widehat{X}_u(x) \stackrel{\text{def}}{=} X_{t+u}(X_t^{-1}(x))$ for $u \geq 0$, then $\widehat{X}$ is a copy of the flow X, and independent of $\mathcal{B}_t$. Consequently, we have

$$\mathbb{P}\Big(\xi < x \,\big|\, \mathcal{B}_t\Big) = \mathbb{P}\Big(\lim_{u\to\infty} \widehat{X}_u(X_t(x)) = \infty \,\big|\, \mathcal{B}_t\Big) = h(X_t(x)), \qquad t > 0,\ x \in \mathbb{R}.$$

Hence, we have from (1.3) that

$$\mathbb{P}\Big(\xi \in dx \,\big|\, \mathcal{B}_t\Big)/dx = h'(X_t(x))\, DX_t(x) = h'(X_t(x)) \exp\big((\beta_2 - \beta_1) L_t^x\big). \tag{4.2}$$

Fix $a \in \mathbb{R}$ and write $\mathbb{P}^a$ for the law of the flow X conditioned on $\{\xi = a\}$, which is given by the h-transform as follows:

$$\frac{d\mathbb{P}^a}{d\mathbb{P}}\Big|_{\mathcal{B}_t} \stackrel{\text{def}}{=} \frac{h'(X_t(a)) DX_t(a)}{h'(a)}, \qquad t \geq 0. \tag{4.3}$$

The following result describes the law of the conditioned flow X under $\mathbb{P}^a$:

Lemma 4.1. *Fix $a \in \mathbb{R}$. There exists a $(\mathbb{P}^a, \mathcal{B}_t, t \geq 0)$-Brownian motion $(\widehat{B}_t, t \geq 0)$ such that with probability one, for all $x \in \mathbb{R}$ and $t \geq 0$, we have*

$$\begin{aligned} X_t(x) = x + \widehat{B}_t &+ \beta_1 \int_0^t \mathbb{1}_{(X_s(x) \leq 0)} ds + \beta_2 \int_0^t \mathbb{1}_{(X_s(x) > 0)} ds \\ &- 2\beta_1 \int_0^t \mathbb{1}_{(X_s(a) \leq 0)} ds - 2\beta_2 \int_0^t \mathbb{1}_{(X_s(a) > 0)} ds. \end{aligned} \tag{4.4}$$

Consequently, we have under $\mathbb{P}^a$ that

$$(4.5)\qquad X_t(x)\begin{cases}\to\infty, & \text{if} \quad x>a\\ \to-\infty, & \text{if} \quad x<a\\ \text{recurrent}, & \text{if} \quad x=a\end{cases},\qquad t\to\infty,\qquad \text{a.s.}$$

In particular, under $\mathbb{P}^a$, $\lim_{t\to\infty} Y_t \to a$, a.s., and a is the recurrent point for Y.

Proof of Lemma 4.1. From (4.1)–(4.2) and (2.9) (with $x=a$), we get

$$\frac{h'(X_t(a))DX_t(a)}{h'(a)} = \exp\Big(-2\beta_1\int_0^t 1\!\!1_{(X_s(a)\le 0)}dB_s - 2\beta_1^2\int_0^t 1\!\!1_{(X_s(a)\le 0)}ds \\ - 2\beta_2\int_0^t 1\!\!1_{(X_s(a)>0)}dB_s - 2\beta_2^2\int_0^t 1\!\!1_{(X_s(a)>0)}ds\Big).$$

Girsanov's transform says that

$$(4.6)\qquad \widehat{B}_t \stackrel{\text{def}}{=} B_t + 2\beta_1\int_0^t 1\!\!1_{(X_s(a)\le 0)}ds + 2\beta_2\int_0^t 1\!\!1_{(X_s(a)>0)}ds,$$

is a $\big(\mathbb{P}^a,(\mathcal{B}_t)\big)$-Brownian motion, and (4.4) follows.

The convergence towards ∞ and towards $-\infty$ in (4.5) follows from the definition of ξ, and from the fact that $\mathbb{P}^a = \mathbb{P}\big(\,\cdot\,|\xi=a\big)$. By taking $x=a$ in (4.4), the recurrence of $X_t(a)$ follows. □

Fix $a\in\mathbb{R}$ and consider $x\ge a$ in the sequel. We shall work under $\mathbb{P}^a$. Define $A_t(x), \alpha_t(x), \mathcal{E}_x$ via (3.3)–(3.5). Remark that under $\mathbb{P}^a$, $\alpha_t(x)<\infty$, a.s. Using (4.6) and (2.8) by replacing t by $\alpha_t(x)$, we have

$$X_{\alpha_t(x)}(x) = X^+_{\alpha_t(x)}(x)$$

$$(4.7)\qquad = x^+ + W_t^{(x)} + \beta_2 t - 2\beta_1\int_0^t 1\!\!1_{(X_{\alpha_s(x)}(a)\le 0)}ds - 2\beta_2\int_0^t 1\!\!1_{(X_{\alpha_s(x)}(a)>0)}ds + \frac{1}{2}L^x_{\alpha_t(x)},$$

where $W_t^{(x)} \stackrel{\text{def}}{=} \int_0^{\alpha_t(x)} 1\!\!1_{(X_s(x)>0)}d\widehat{B}_s, t\ge 0$, is the Dambis-Dubins-Schuwarz Brownian motion associated with the continuous martingale $t\to\int_0^t 1\!\!1_{(X_s(x)>0)}d\widehat{B}_s$. Notice that $W^{(x)}$ is a $\mathbb{P}^a$–Brownian motion. The following result is the key to our analysis of local times under $\mathbb{P}^a$ (notice that (3.6) is not useful since B is a semimartingale, but not a martingale under $\mathbb{P}^a$):

Lemma 4.2. *For every $x\ge a$, we have*

$$(4.8)\qquad \mathcal{E}_x = \sigma\{W_t^{(x)}, t\ge 0\}.$$

Hence, for $H \in L^2(\mathbb{P}^a, \mathcal{E}_x)$, there exists some predictable process (h_s) such that

$$H = \mathbb{E}H + \int_0^\infty h_s \, \mathbb{1}_{(X_s(x)>0)} \, d\widehat{B}_s = \mathbb{E}H + \int_0^\infty h_s \, \mathbb{1}_{(X_s^{-1}(0)<x)} \, d\widehat{B}_s. \tag{4.9}$$

Proof of Lemma 4.2. For $x = a$, since $X_{\alpha_s(a)}(a) \le 0$, a.s., (4.8) follows immediately from the Skorokhod reflection lemma for (4.7) (cf. (3.7)–(3.8)). Consider $x > a$ and write for simplification in this proof

$$\xi_t \overset{\text{def}}{=} X_{\alpha_t(x)}(x) \ge 0, \qquad \eta_t \overset{\text{def}}{=} X_{\alpha_t(x)}(a) < \xi_t, \qquad t \ge 0. \tag{4.10}$$

By using (2.10), we have by time-change that

$$\xi_t = \eta_t + x - a + (\beta_2 - \beta_1) \int_0^t \mathbb{1}_{(\eta_s \le 0)} \, ds. \tag{4.11}$$

We rewrite (4.7) as follows:

$$0 \le \xi_t = x^+ + W_t^{(x)} - \beta_2 t + 2(\beta_2 - \beta_1) \int_0^t \mathbb{1}_{(\eta_s \le 0)} \, ds + \frac{1}{2} L_\xi(t), \tag{4.12}$$

where $L_\xi(t)$ denotes the (semimartingale) local time at 0 of ξ up to time t. Consider $(\xi_t, \eta_t)_{t \ge 0}$ as a solution of the system of equations (4.11) and (4.12). It suffices to show the pathwise uniqueness for (4.11)–(4.12), the proof is similar to the case of perturbed Brownian motion studied by Le Gall and Yor [12]. Denote by $(\mathcal{W}_t)$ the natural filtration generated by $W^{(x)}$.

Firstly, we suppose $a \ge 0$, hence $\xi_0 = x > 0$ and $\eta_0 = a$. We show that there exists a unique way to construct (ξ_t) and (η_t) from the path of $\widehat{B}$. Let $H_1(\xi) \overset{\text{def}}{=} \inf\{t \ge 0 : \xi_t = 0\}$. For $0 \le t < H_1(\xi)$, we have

$$\xi_t = x + W_t^{(x)} - \beta_2 t + 2(\beta_2 - \beta_1) \int_0^t \mathbb{1}_{(\eta_s \le 0)} \, ds, \tag{4.13}$$

$$\begin{aligned} \eta_t &= a - x + \xi_t - (\beta_2 - \beta_1) \int_0^t \mathbb{1}_{(\eta_s \le 0)} \, ds \\ &= a + W_t^{(x)} - \beta_2 t + (\beta_2 - \beta_1) \int_0^t \mathbb{1}_{(\eta_s \le 0)} \, ds. \end{aligned} \tag{4.14}$$

It is clear that the system of equations (4.13)–(4.14) admits pathwise uniqueness. In fact, by using Zvonkin's method for (4.14), η is the unique strong solution of (4.14), which also shows the measurability of ξ_t with respect to $\mathcal{W}_t$, as desired. Therefore $H_1(\xi)$ is a $(\mathcal{W}_t)$ stopping time, and $\sigma\{\eta_t, \xi_t, t \le H_1(\xi)\} \subset \mathcal{W}_{H_1(\xi)}$. If $H_1(\xi)$ is infinite, we have shown the unique way to construct ξ and η, as desired. On $\{H_1(\xi) < \infty\}$, since $\xi_{H_1(\xi)} = 0$, we have $\eta_{H_1(\xi)} < 0$, and we define $H_1(\eta) \stackrel{\text{def}}{=} \inf\{t > H_1(\xi) : \eta_t = 0\}$.

Let $\tilde{\eta}_t \stackrel{\text{def}}{=} \eta_{t+H_1(\xi)} - \eta_{H_1(\xi)}$, $\tilde{\xi}_t \stackrel{\text{def}}{=} \xi_{t+H_1(\xi)} - \xi_{H_1(\xi)} = \xi_{t+H_1(\xi)}$, and $\tilde{W}_t \stackrel{\text{def}}{=} W^{(x)}_{t+H_1(\xi)} - W^{(x)}_{H_1(\xi)}$ for $t \geq 0$. $\tilde{W}$ is a Brownian motion independent of $\mathcal{W}_{H_1(\xi)}$. Write $(\tilde{\mathcal{W}}_t)$ for the natural filtration generated by $\tilde{W}$. Notice that for $0 \leq t < H_1(\eta) - H_1(\xi)$, we have

$$(4.15) \qquad \begin{cases} \tilde{\eta}_t = \tilde{\xi}_t - (\beta_2 - \beta_1)t, \\ \tilde{\xi}_t = \tilde{W}_t + (\beta_2 - 2\beta_1)t + \frac{1}{2}L_{\tilde{\xi}}(t), \end{cases}$$

where $L_{\tilde{\xi}}(t)$ denotes the local time at 0 of $\tilde{\xi}$. Hence by the Skorokhod reflecting lemma, $\tilde{\xi}_t = \tilde{W}_t + (\beta_2 - 2\beta_1)t + \sup_{0\leq s\leq t}\big(-\tilde{W}_s - (\beta_2 - 2\beta_1)s\big)$ for $t < H_1(\eta) - H_1(\xi)$; and $H_1(\eta) - H_1(\xi) = \inf\{t > 0 : \tilde{\eta}_t = -\eta_{H_1(\xi)}\} = \inf\{t > 0 : \tilde{\xi}_t - (\beta_2 - \beta_1)t = -\eta_{H_1(\xi)}\}$ hence is a $\tilde{\mathcal{W}}$ stopping time. Therefore, we have shown that $\big(\tilde{\eta}_t, \tilde{\xi}_t, 0 \leq t \leq H_1(\eta) - H_1(\xi), H_1(\eta) - H_1(\xi)\big)$ are $\tilde{\mathcal{W}}_{H_1(\eta)-H_1(\xi)}$ -measurable.

Notice $H_1(\eta) - H_1(\xi) < \infty$, a.s. and $\xi_{H_1(\eta)} > 0$, a.s. We iterate this procedure by defining $H_n(\xi) \stackrel{\text{def}}{=} \inf\{t \geq H_{n-1}(\eta) : \xi_t = 0\}$, and $H_n(\eta) \stackrel{\text{def}}{=} \inf\{t \geq H_n(\xi) : \eta_t = 0\}$, for $n \geq 2$ (with conventions that $\inf \emptyset \equiv \infty$ and $H_n(\eta) \equiv \infty$ if $H_n(\xi) = \infty$). Therefore we see that $H_n(\eta)$ is a $\mathcal{W}$-stopping time, and $\sigma\{\xi_t, \eta_t, t \leq H_n(\eta); H_n(\eta)\} \subset \mathcal{W}_{H_n(\eta)}$.

The only thing left to verify is that $\lim_{n\to\infty} H_n(\eta) = \infty$, a.s. But this is easy, for on $\{H_n(\eta) < \infty\}$, we have $H_n(\eta) \geq \sum_{k=1}^n \big(H_k(\eta) - H_k(\xi)\big)$, the sum of n iid positive variables, hence $H_n(\eta) \to \infty$, a.s., as desired.

The case $a < 0$ can be treated in the same way by considering firstly the return time at 0 of η. The details are omitted. □

Proof of Theorem 1.3. Let $x \geq a$. Using (3.20) with $b = a$, and (4.6), we have

$$(4.16) \qquad L^x_{\tau_r(a)} = r + 2\big(a^- - x^-\big) - 2\int_0^{\tau_r(a)} \mathbb{1}_{(a\leq Y_s<x)} d\widehat{B}_s + 2\beta_1 \int_a^x \kappa(L^y_{\tau_r(a)})dy.$$

In view of Lemma 4.2 and (3.5), we see that $\int_0^{\tau_r(a)} \mathbb{1}_{(a\leq Y_s<x)} d\widehat{B}_s$ is a continuous (local) $\big(\mathbb{P}^a, (\mathcal{E}_x)\big)$-martingale, with increasing process $x \to \int_a^x \kappa(L^y_{\tau_r(a)})\, dy$. It follows that $\{L^{a+y}_{\tau_r(a)}, y \geq 0\}$ is a diffusion with generator given by (1.14), and independent of $\mathcal{E}_a$. Hence $\{L^{a+y}_{\tau_r(a)}, y \geq 0\}$ is independent of $\{L^{a-y}_{\tau_r(a)}, y \geq 0\}$. The results about $\{L^{a-y}_{\tau_r(a)}, y \geq 0\}$ follow from symmetry as done in the proof of Theorem 1.2. The process $\{L^{a+y}_\infty, y \in \mathbb{R}\}$ can be treated in the same way. □

5. Differentiability of a real-valued flow

Let $I \subset \mathbb{R}$ be an open interval (finite or not). Consider the following equation

$$Z_t(x) = x + \int_0^t \sigma(Z_s(x))\, dB_s, \qquad x,\, Z_t(x) \in I, \tag{5.1}$$

where $\sigma : I \to (0, \infty)$ is a globally Lipschitz continuous function which piecewisely belongs to $\mathcal{C}^{1,\alpha}$, i.e. there exist j numbers $a_1 < a_2 < \dots < a_j$ in I such that on $I \backslash \{a_i, 1 \le i \le j\}$, σ' exists and is uniformly Hölder continuous of order $0 < \alpha \le 1$. We suppose that σ satisfies Feller's test of non-explosion (cf. Karatzas and Shreve [10, Theorem 5.5.29]) such that for every $x \in I$, $\inf\{t > 0 : Z_t(x) \notin I\} = \infty$ with probability one.

Lemma 5.1. *With probability one, we have*

$$\frac{\partial Z_t}{\partial x}(x) = \exp\Big(M_t(x) - \frac{1}{2}\langle M(x)\rangle_t\Big), \qquad t > 0,\ x \in I, \tag{5.2}$$

where $M_t(x) \stackrel{\text{def}}{=} \int_0^t \sigma'(Z_s(x))\, \mathbb{1}_{(Z_s(x) \neq a_i, 1 \le i \le j)}\, dB_s$, *for* $(t, x) \in \mathbb{R}_+ \times I$, *admits a bicontinuous version.*

Before the proof, we would like to say that (5.2) is well-known if σ is smooth (see e.g. Kunita [11]). In fact, we suspect that (5.2) itself, or something stronger still, is also well-known. Unfortunately, we were not able to find it ourselves in the literature.

Proof of Lemma 5.1. Denote by $U_t(x)$ the exponential term on the RHS of (5.2). We follow the method in Dellacherie et al. [7, pp. 369], and prove that with probability one

$$(t, x) \in \mathbb{R}_+ \times I \to U_t(x) \qquad \text{is continuous,} \tag{5.3}$$

$$\int_I h(x) U_t(x)\, dx = -\int_I h'(x)\, Z_t(x)\, dx, \tag{5.4}$$

for all $h \in C_c^\infty(I)$ smooth and with compact support in I, and the desired differentiability of $Z_t(\cdot)$ follows. For notational simplification, we consider the case $j = 1$ and $a_1 = 0$, the general case can be done in the same way without supplementary difficulties.

Firstly, we show that for $T > 0$ and $p > 2/\alpha$, there exists a constant $C_{p,T} > 0$ such that for all $|x|, |y| \le T$

$$\mathbb{E} \sup_{0 \le t \le T} \big|M_t(x) - M_t(y)\big|^p \le C_{p,T}\Big(|x - y|^{p/4} + |x - y|^{\alpha p}\Big). \tag{5.5}$$

Observe that $\big|\sqrt{\langle M.(x)\rangle_t} - \sqrt{\langle M.(y)\rangle_t}\big| \le \sqrt{\langle M.(x) - M.(y)\rangle_t}$, from this and Burkholder-Davis-Gundy (BDG)'s inequality, (5.5) yields in fact the same estimate for $\big|\sqrt{\langle M.(x)\rangle_t} - \sqrt{\langle M.(y)\rangle_t}\big|$ as for $\big|M_t(x) - M_t(y)\big|$ in (5.5) (possibly with a larger constant $C(p, T)$). By applying Kolmogorov's lemma to the application: $x \to (M_.(x), \sqrt{\langle M(x)\rangle_.})$, we obtain the bicontinuity (5.3).

Assuming $x < y$, we have from the monotonicity $Z_t(x) \leq Z_t(y)$ that

$$
\begin{aligned}
M_t(x) - M_t(y) &= \int_0^t \left[\mathbb{1}_{(Z_s(y)<0)} + \mathbb{1}_{(Z_s(x)>0)}\right]\left(\sigma'(Z_s(x)) - \sigma'(Z_s(y))\right) dB_s + \\
&\qquad \int_0^t \mathbb{1}_{(Z_s(x)<0<Z_s(y))}\left(\sigma'(Z_s(x)) - \sigma'(Z_s(y))\right) dB_s \\
&\equiv \Delta_1(t) + \Delta_2(t), \qquad (5.6)
\end{aligned}
$$

with obvious notation. Observe that for $\Delta_1(t)$, $Z_s(x)$ and $Z_s(y)$ are in the same intervals of Hölder continuity of σ', it may be shown by using BDG's inequality that

$$
\mathbb{E} \sup_{0\leq t\leq T} \left|\Delta_1(t)\right|^p \leq C_1(p)\, \mathbb{E}\Big(\int_0^T ds \left|Z_s(x) - Z_s(y)\right|^{2\alpha}\Big)^{p/2} \leq C_2(p,T)\, |x-y|^{p\alpha/2}, \qquad (5.7)
$$

where the last estimate follows from Hölder's inequality and the known estimate that $\mathbb{E}|Z_t(x) - Z_t(y)|^{p\alpha} = \mathcal{O}(|x-y|^{p\alpha})$ (this follows from the fact that σ is Lipschitz, cf. Kunita [11]). For $\Delta_2(t)$, we bound σ' by a constant and get

$$
\begin{aligned}
\mathbb{E} \sup_{0\leq t\leq T} \left|\Delta_2(t)\right|^p &\leq C_3(p)\, \mathbb{E}\Big(\int_0^t ds\, \mathbb{1}_{(Z_s(x)\leq 0<Z_s(y))}\Big)^{p/2} \\
&\leq C_3(p)\, \mathbb{E}\Big(\int_0^t ds \mathbb{1}_{(-\delta\leq Z_s(x)\leq 0)}\Big)^{p/2} + C_3(p)\, \mathbb{E}\Big(\int_0^t ds \mathbb{1}_{(|Z_s(y)-Z_s(x)|\geq\delta)}\Big)^{p/2} \\
&\leq C_4(p)\, \delta^{p/2}\, \mathbb{E} \sup_{z\in I} L_{Z(x)}^{p/2}(t,z) + C_3(p)\, \mathbb{E}\Big(\int_0^t ds \frac{|Z_s(x) - Z_s(y)|}{\delta}\Big)^{p/2}, \qquad (5.8)
\end{aligned}
$$

where $0 < \delta < \delta_0$ with δ_0 fixed but sufficiently small such that $\inf_{|x|\leq\delta_0} |\sigma(x)| > 0$, and $L_{Z(y)}(t,x)$ denotes the local time associated the continuous martingale $Z_.(x)$. Using a BDG-type inequality for the local times (cf. [15, Theorem XI.2.4]) shows that the first term in (5.8) is bounded by $C_5(p,T)\delta^{p/2}$, whereas the second term is bounded as in (5.7) by $C_5(p,T)|x-y|^{p/2}\,\delta^{-p/2}$. Hence by taking $\delta = \min(\delta_0, |x-y|^{1/2})$, we get the desired estimate in (5.5) for the term $M_t(x) - M_t(y)$.

Now, we prove (5.4) by approximating of σ by $\sigma_n \in C^{1,\alpha}(I)$, and such that $\sigma_n(0) = \sigma(0)$, σ_n' is bounded on I, and $\sigma_n'(x) = \sigma'(x)$ for $|x| \geq 1/n$. Consider the flow $Z^{(n)}$ associated with σ_n and driven by the same Brownian motion B as in (5.1). Define

$$
U_t^{(n)}(x) \stackrel{\text{def}}{=} \exp\Big(\int_0^t \sigma_n'(Z_s^{(n)}(x))\, dB_s - \frac{1}{2}\int_0^t \sigma_n'^2(Z_s^{(n)}(x))\, ds\Big),
$$

which is the derivative of $Z^{(n)}$ (cf. [11]). Therefore (5.4) holds for $(Z^{(n)}, U^{(n)})$ in lieu of (Z,U). All we need to show is that as $n \to \infty$

$$
\mathbb{E}\int_I |h'(x)|\,|Z_t^{(n)}(x) - Z_t(x)|dx \to 0, \qquad (5.9)
$$

$$
\mathbb{E}\int_I |h(x)|\,|U_t^{(n)}(x) - U_t(x)|dx \to 0. \qquad (5.10)
$$

But it is standard (e.g. by using Gronwall's inequality) to obtain that for a compact $K \subset I$,

$$\sup_{x\in K} \mathbb{E} \sup_{0\le s\le t} \left(Z_s^{(n)}(x) - Z_s(x)\right)^2 \to 0, \qquad n \to \infty,$$

implying (5.9). Finally, since $\sigma_n'(x) = \sigma'(x)$ for $|x| \ge 1/n$, it is easy to obtain that for fixed x, $\mathbb{E}\int_0^t \left(\sigma_n'(Z_s^{(n)}(x)) - \sigma'(Z_s(x))\mathbb{1}_{(Z_s(x)\neq 0)}\right)^2 ds \to 0$. Hence $U_t^{(n)}(x) \xrightarrow{(\mathrm{p})} U_t(x)$, and the family $\{U_t^{(n)}(x), n \ge 1, x \in K\}$ is uniformly integrable, since

$$\sup_{x\in K,\, n\ge 1} \mathbb{E}\Big(\sup_{0\le s\le t} \Big[|U_s^{(n)}(x)| + |U_s(x)|\Big]\Big)^2 < \infty,$$

hence (5.10) follows, and which ends the whole proof. □

6. An example of Girsanov's transform

We shall consider in this section the recurrent case of (1.1), i.e. $\beta_1 \ge 0 \ge \beta_2$, to ensure the finiteness of the stopping times $T(a)$ and $\tau_r(b)$ defined in (1.6) and (1.7). We also exclude the Brownian case by assuming $\beta_1 > \beta_2$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\int_{\mathbb{R}} \phi^2(z)dz < \infty$. Define a new probability $\mathbb{Q}$ via:

$$\text{(6.1)} \qquad \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{B}_t} \stackrel{\text{def}}{=} \exp\Big(\int_0^t \phi(X_s^{-1}(0))dB_s - \frac{1}{2}\int_0^t \phi^2(X_s^{-1}(0))ds\Big) \stackrel{\text{def}}{=} \mathbb{D}_t, \qquad t > 0,$$

where $(\mathcal{B}_t)_{t\ge 0}$ denotes the natural filtration generated by B. Observe that from Proposition 1.2, $\int_0^t \phi^2(X_s^{-1}(0))ds = \int_{\mathbb{R}} \phi^2(z)\rho_t^z dz \le \frac{1}{\beta_1-\beta_2}\int_{\mathbb{R}}\phi^2(z)dz < \infty$, so that $t \to \mathbb{D}_t$ is a uniformly integrable martingale, and $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{B}_\infty$. Girsanov's transform tells us that there is a $(\mathbb{Q}, (\mathcal{B}_t)_{t\ge 0})$-Brownian motion $(\widehat{B}_t, t \ge 0)$ such that

$$\text{(6.2)} \qquad X_t(x) = x + \widehat{B}_t + \beta_1 \int_0^t \mathbb{1}_{(X_s(x)\le 0)}ds + \beta_2 \int_0^t \mathbb{1}_{(X_s(x)>0)}ds + \int_0^t \phi(X_s^{-1}(0))ds,$$

for $t \ge 0, x \in \mathbb{R}$. We remark that any flow satisfying (6.2) has the same law. The goal of this section is to give a Ray-Knight theorem for the law of $(L_T^x, x \in \mathbb{R})$ under $\mathbb{Q}$. We only state the result for $T = T(a)$, the case of $T = \tau_r(b)$ can be considered similarly.

Proposition 6.1. *Assuming that $\beta_1 \ge 0 \ge \beta_2$ ($\beta_1 \neq \beta_2$) and $\int_{\mathbb{R}} \phi^2(z)dz < \infty$. Fix $a < 0$. Under $\mathbb{Q}$, the process $(L_{T(a)}^{a+x}, x \ge 0)$ is a time inhomogeneous diffusion starting from 0 with infinitesimal generator:*

$$\text{(6.3)} \qquad 2\kappa(\ell)\frac{d^2}{d\ell^2} + 2\Big(\mathbb{1}_{(0\le x\le a^-)} - [\beta_1 + \phi(x+a)]\,\kappa(\ell)\Big)\frac{d}{d\ell}, \qquad \ell \in \mathbb{R}_+.$$

The process $\left(L^{a+x}_{T(a)}, x \geq a^-\right)$ *is absorbed at* 0.

The proof relies on the following observation (recalling that $z \to L^z_{T(a)}$ is a continuous semimartingale):

Lemma 6.1. *With probability one, we have*

$$\int_0^{T(a)} \phi(X_s^{-1}(0))dB_s = \int_a^0 \phi(z)dz - \beta_1 \int_0^{T(a)} \phi(X_s^{-1}(0))ds - \frac{1}{2}\int_{\mathbb{R}} \phi(z)d_z L^z_{T(a)}. \tag{6.4}$$

Proof of Lemma 6.1. It suffices to prove (6.4) for ϕ a step function with compact support of form: $\phi(z) = \sum_{i=1}^n \lambda_i \mathbb{1}_{(y_i \leq z < y_{i+1})}$ with $n \geq 1$, $\lambda_i \in \mathbb{R}$, and $-\infty < y_1 < y_2 < ... < y_{n+1} < \infty$. The LHS of (6.4) equals
(6.5)

$$\sum_{i=1}^n \lambda_i \int_0^{T(a)} \mathbb{1}_{(X_s(y_i)\leq 0 < X_s(y_{i+1}))} dB_s = \sum_{i=1}^n \lambda_i \int_0^{T(a)} \left[\mathbb{1}_{(X_s(y_i)\leq 0)} - \mathbb{1}_{(X_s(y_{i+1})<0)}\right] dB_s.$$

By using (2.8), we have

$$\begin{aligned}\int_0^{T(a)} \left[\mathbb{1}_{(X_s(y_i)\leq 0)} - \mathbb{1}_{(X_s(y_{i+1})<0)}\right] dB_s &= \left(y_i^- - X^-_{T(a)}(y_i)\right) - \left(y^-_{i+1} - X^-_{T(a)}(y_{i+1})\right) \\ &\quad - \beta_1 \int_0^{T(a)} ds \mathbb{1}_{(y_i \leq X_s^{-1}(0) < y_{i+1})} ds + \frac{1}{2}\left(L^{y_i}_{T(a)} - L^{y_{i+1}}_{T(a)}\right),\end{aligned}$$

which in view of (1.3) yields the formula (6.4) for this kind of ϕ. □

Proof of Proposition 6.1. The idea is that used by Norris et al. [14]. Firstly, we have from the fact that $t \to \mathbb{D}_t$ is a uniformly integrable martingale that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{B}_{T(a)}} = \mathbb{D}_{T(a)}. \tag{6.6}$$

By using Lemma 6.1 and the equation (3.18) for $L^{\cdot}_{T(a)}$, we have

$$\mathbb{D}_{T(a)} = \exp\Big(\int_a^\infty \phi(z)\sqrt{\rho^z_{T(a)}}\, dW_z - \frac{1}{2}\int_a^\infty \phi^2(z)\rho^z_{T(a)}\, dz\Big),$$

where we recall that $(W_x)_{x\geq a}$ is a $\left(\mathcal{E}_x, x \geq a\right)$-Brownian motion. This in view of (6.6) allows us to obtain the following formula of change of probability

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{E}_x} = \exp\Big(\int_a^x \phi(z)\sqrt{\rho^z_{T(a)}}\, dW_z - \frac{1}{2}\int_a^x \phi^2(z)\rho^z_{T(a)}\, dz\Big), \qquad x \geq a. \tag{6.7}$$

Recall that $\rho^z_{T(a)} = \kappa(L^z_{T(a)})$. Applying Girsanov's transform with (6.7) to (3.18) gives that

$$L^x_{T(a)} = 2(a^- - x^-) - 2\int_a^x \sqrt{\kappa(L^y_{T(a)})}\, d\widehat{W}_y - 2\int_a^x [\beta_1 + \phi(y)]\, \kappa(L^y_{T(a)}) dy, \qquad x \geq a,$$

with a $(\mathbb{Q}, \mathcal{E}_x, x \geq a)$-Brownian motion $\widehat{W}$. This completes the proof of (6.3). □

References

[1] Aldous, D.: Brownian excursion conditioned on its local time. *Elect. Comm. Probab.* 3 (1998) 79–90.

[2] Barlow, M.T., Burdzy, K., Kaspi, H. and Mandelbaum, A.: Variably skewed Brownian motion. (preprint, 1999)

[3] Bass, R.F. and Burdzy, K.: Stochastic bifurcation models. *Ann. Probab.* 27 (1999) 50–108.

[4] Bouleau, N.: Sur la variation quadratique de certaines mesures vectorielles. *Z. Wahrsch. verw. Gebiete* 61 (1982) 283–290.

[5] Burdzy, K. and Chen, Z.Q.: Local time flow related to skew Brownian motion. (preprint, 1999)

[6] Carverhill, A. and Elworthy, K.D.: Flows of stochastic dynamical systems: the functional analytic approach. *Z. Wahrsch. verw. Gebiete* 65 (1984) 245–268.

[7] Dellacherie, C., Maisonneuve, B. and Meyer, P.A.: *Probabilités et Potentiel: Processus de Markov (fin) Compléments de Calcul Stochastique.* Hermann. Paris, 1992.

[8] Ethier, S.N. and Kurtz, T.G.: *Markov Processes: Characterization and Convergence.* Wiley. New York, 1986.

[9] Jeulin, Th.: Ray-Knight's theorem on Brownian local times and Tanaka's formula. *Sem. Stoch. Proc. 1983 (Gainesville, Fla.)* 131–142. Birkhäuser Boston, 1984.

[10] Karatzas, I. and Shreve, S.E.: *Brownian Motion and Stochastic Calculus.* (2nd Edition) Springer, New York, Berlin. 1991.

[11] Kunita, H.: Stochastic differential equations and stochastic flows of diffeomorphisms. *École d'été de Probabilités de Saint-Flour XII (1982) Lect. Notes Math.* 1097 (1984) 143–303.

[12] Le Gall, J.F. and Yor, M.: Enlacements du mouvement brownien autour des courbes de l'espace. *Trans. Amer. Math. Soc.* 317 (1990) 687-722.

[13] Meyer, P.A.: Flot d'une équation differentielle stochastique. *Séminaire de Probabilités XV* (Eds: J. Azéma and M. Yor) pp. 103–117. Lect. Notes Math. 850 Springer, Berlin. 1981.

[14] Norris, J.R., Rogers, L.C.G. and Williams, D.: Self-avoiding random walk: a Brownian motion model with local time drift. *Probab. Th. Rel. Fields* 74 (1987) 271–287.

[15] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion.* (3rd edition) Springer. New York, Berlin. 1998.

[16] Rogers, L.C.G. and Walsh, J.B.: Local time and stochastic area integrals. *Ann. Probab.* 19 (1991) 457-482.

[17] Rogers, L.C.G. and Walsh, J.B.: The intrinsic local-time sheet of Brownian motion. *Probab. Th. Rel. Fields* 88 (1991) 363–379.

[18] Stroock, D.W. and Varadhan, S.R.S: *Multidimensional Diffusion Processes.* Springer–Verlag, New York. 1979.

[19] Stroock, D.W. and Yor, M.: Some remarkable martingales. *Séminaire de Probabilités XV* (Eds: J. Azéma and M. Yor) pp. 590–603. Lect. Notes Math. 850 Springer, Berlin. 1981.

[20] Uppman, A.: Sur le flot d'une équation differentielle stochastique. *Séminaire de Probabilités XV* (Eds: J. Azéma and M. Yor) pp. 268–284. Lect. Notes Math. 920 Springer, Berlin. 1982.

[21] Warren, J. and Yor, M.: The Brownian burglar: Conditioning Brownian motion by its local time process. *Sém. Probab. XXXII* (Eds: J. Azéma, M. Émery, M. Ledoux and M. Yor) pp. 328–342. Lect. Notes Math. 1686 Springer-Verlag Berlin. 1998.

[22] Yamada, T. and Ogura, Y.: On the strong comparison theorems for solutions of stochastic differential equations. *Z. Wahrsch. verw. Gebiete* 56 (1981) 3–19.

[23] Yor, M.: *Some Aspects of Brownian Motion. Part II (Some Recent Martingale Problems).* ETH Zürich, Birkhäuser, 1997.

[24] Zvonkin, A.K.: A transformation of the phase space of a process that removes the drift. *Math. USSR Sbornik* 2 (1974) 129–149.