FLUCTUATION EXPONENTS AND LARGE DEVIATIONS FOR DIRECTED POLYMERS IN A RANDOM ENVIRONMENT

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ABSTRACT. For the model of directed polymers in a gaussian random environment introduced by Imbrie and Spencer, we establish:

- a Large Deviations Principle for the end position of the polymer under the Gibbs measure;
- a scaling inequality between the volume exponent and the fluctuation exponent of the free energy;
- a relationship between the volume exponent and the rate function of the Large Deviations Principle.

1. INTRODUCTION

Let us first describe the model of directed polymers in a random environment introduced by Imbrie and Spencer [12]. The random media is a family $(g(k,x), k \geq 1, x \in \mathbb{Z}^d)$ of independent identically distributed random variables defined on a probability space $(\Omega^{(g)}, \mathcal{F}^{(g)}, \mathbf{P})$. Let (S_n) be the simple symmetric random walk in $\mathbb{Z}^d$. Let $\mathbb{P}_x$ be the probability measure of $(S_n)_{n\in\mathbb{N}}$ starting at $x \in \mathbb{Z}^d$ and let $\mathbb{E}_x$ be the corresponding expectation.

The object of our study is the Gibbs measure $\langle\cdot\rangle^{(n)}$, which is a random probability measure defined as follows: Let Ω_n be the set of nearest neighbor paths of length n:

$$\Omega_n \stackrel{\text{def}}{=} \left\{\gamma : [1,n] \to \mathbb{Z}^d, |\gamma(k) - \gamma(k-1)| = 1, k = 2, \dots, n\right\}.$$

For any measurable function $F : \Omega_n \to \mathbb{R}_+$,

$$\langle F(S)\rangle^{(n)} \stackrel{\text{def}}{=} \frac{1}{Z_n(\beta)} \mathbb{E}_0\Big(F(S)e^{\beta H_n(g,S)}\Big), \quad H_n(g,\gamma) \stackrel{\text{def}}{=} \sum_{i=1}^{n} g(i,\gamma(i)),$$

where here and in the sequel, $\beta > 0$ denotes some fixed positive constant and $Z_n(\beta)$ is the partition function:

$$Z_n(\beta) = Z_n(\beta; g) = \mathbb{E}_0\Big(e^{\beta H_n(g,S)}\Big). \tag{1.1}$$

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In other words, for a given realization $g(\omega)$ of the environment, the Gibbs measure gives to a polymer chain γ having an energy $H_n(g,\gamma)$ at temperature $T=\frac{1}{\beta}$, a weight proportional to $e^{\beta H_n(g,\gamma)}$.

Dating back to the pioneer work of Imbrie and Spencer [12], the situation in dimension $d\geq 3$ is well understood for small $\beta>0$. There exists some constant $\beta_0>0$ such that for all $0<\beta<\beta_0$, S_n is *diffusive* under $\langle\cdot\rangle^{(n)}$.

Theorem A (Imbrie and Spencer [12], Bolthausen [3], Sinai [21], Albeverio and Zhou [1]) *If $d\geq 3$, then there exists $\beta_0=\beta_0(d)>0$ such that for $0<\beta<\beta_0$,*

$$\frac{\langle |S_n|^2\rangle^{(n)}}{n}\to 1 \qquad \mathbf{P}\,a.s.. \tag{1.2}$$

It is believed in many physics papers (see e.g. [10, 13]) that for low dimension $d=1,2$ and β small enough $\langle |S_n|^2\rangle^{(n)}$ behaves as $n^{2\zeta}$ for some $\zeta>\frac{1}{2}$. However no rigorous proof has been given yet.

The aim of this paper is to establish some results when the random media is given by i.i.d. $\mathcal{N}(0,1)$ gaussian random variables. Our results will be available for all dimension $d\geq 1$ and $\beta>0$; however, the behavior of $\langle\cdot\rangle^{(n)}$ will be much different from small β to large β even in the same underlying d-dimensional lattice space.

Our first result is a large deviations principle for the end position of the polymer under the Gibbs measure. Denote by $p(\beta)$ the free energy which plays an important rôle: for $d\geq 1$ and $\beta>0$,

$$p(\beta)\stackrel{\text{def}}{=}\sup_{n\geq 1}\frac{1}{n}\mathbf{E}\log Z_n(\beta)=\lim_{n\to\infty}\frac{1}{n}\log Z_n(\beta)\in(0,\frac{\beta^2}{2}],\ \mathbf{P}\,a.s. \tag{1.3}$$

(the above limit exists and also holds in $L^1(\mathbf{P})$, cf. Section 2. Let $\Delta_d\stackrel{\text{def}}{=}\{x=(x_1,...,x_d)\in\mathbb{R}^d : |x_1|+...+|x_d|\leq 1\}$. Let $\overset{o}{\Delta}_d$ be the interior of Δ_d and $\partial\Delta_d$ its boundary. We have

Theorem 1.1. *For $d\geq 1$ and $\beta>0$, there exists a deterministic convex rate function $I_\beta:\Delta_d\to[0,\log(2d)+p(\beta)]$ such that $\mathbf{P}$-almost surely,*

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\langle\frac{S_n}{n}\in F\rangle^{(n)} &\leq -\inf_{\xi\in F}I_\beta(\xi), \quad \text{for } F \text{ closed}\subset\Delta_d,\\ \liminf_{n\to\infty}\frac{1}{n}\log\langle\frac{S_n}{n}\in G\rangle^{(n)} &\geq -\inf_{\xi\in G}I_\beta(\xi), \quad \text{for } G \text{ open}\subset\Delta_d.\end{aligned}$$

The function I_β is continuous in the interior of Δ_d and

$$\lim_{y\to x:\,y\in\overset{o}{\Delta}_d}I_\beta(y)=I_\beta(x), \qquad \forall\,x\in\partial\Delta_d. \tag{1.4}$$

We shall prove that the rate function I_β is exactly the *pointwise rate function* at least in the interior of Δ_d:

Theorem 1.2. *Let $d \ge 1$ and $\beta > 0$. For $\theta \in \overset{o}{\Delta}_d \cap \mathbb{Q}^d$, we have*

$$\frac{1}{n} \log \left\langle \mathbf{1}_{(S_n = n\theta)} \right\rangle^{(n)} \to -I_\beta(\theta), \qquad \mathbf{P} - a.s., \tag{1.5}$$

where we take the limit along n such that $\mathbb{P}_0(S_n = n\theta) > 0$. Moreover, $I_\beta(0) = 0$, $I_\beta(\xi_1, ..., \xi_d) = I_\beta(\pm\xi_{\sigma(1)}, ... \pm \xi_{\sigma(d)})$ for any permutation σ of $[1, ..., d]$, and for $e_1 = (1, 0, ..., 0) \in \mathbb{Z}^d$, we have

$$I_\beta(e_1) = \log(2d) + p(\beta). \tag{1.6}$$

Finally, there exists some positive constant $c_d > 0$ only depending on d such that

$$I_\beta(\xi) \ge \Big(c_d|\xi|^2 - (\frac{\beta^2}{2} - p(\beta))\Big)^+, \quad \xi \in \Delta_d. \tag{1.7}$$

For the sake of notational convenience, we have omitted (and shall omit) the dependence on d in I_β (and in other quantities).

Remark 1.3. *It remains open to characterize the zero set of I_β. In the convergence (1.5), we do not know what happens with a general boundary point $\theta \in \partial\Delta_d$. However, (1.6) shows that (1.5) still holds for those $2d$ boundary points.*

Our second result is a scaling inequality between the volume exponent and the fluctuation exponent of the free energy. Following Piza [18] we define the *volume exponent*

$$\zeta \overset{\text{def}}{=} \inf \Big\{ \alpha > 0 : \langle 1_{(\max_{k \le n} |S_k| \le n^\alpha)} \rangle^{(n)} \to 1 \text{ in } \mathbf{P} \text{ probability} \Big\},$$

and the *fluctuation exponent of the free energy*

$$\chi = \sup \Big\{ \alpha > 0 : \mathrm{Var}(\log Z_n) \ge n^{2\alpha} \text{ for all large } n \Big\},$$

with the convention here that $\sup \emptyset = 0$. We shall establish a scaling inequality similar to the one obtained by Piza [18] in a different framework. He works with a polymer model more related to oriented percolation, where furthermore the potentials g are assumed non positive.

Theorem 1.4. *For all $d \ge 1$ and $\beta \ge 0$,*

$$\chi \ge \frac{1 - d\zeta}{2}.$$

Remark 1.5. *If we believe the superdiffusivity of (S_n) under $\langle\cdot\rangle^{(n)}$, i.e. $\zeta \ge \frac{1}{2}$ (which is always unproven to our best knowledge), the above result makes sense only in the one-dimensional case.*

The large deviations result (Theorem 1.1) may seem at first sight highly uncorrelated to these two exponents ζ and χ. However, there is a close relationship between the volume exponent and the rate function I_β of the large deviations principle.

Theorem 1.6. *Assume that for some constants* $\alpha \geq 1$ *and* $c > 0$*, the rate function satisfies:*

$$I_\beta(\theta) \geq c\,|\theta|^\alpha, \qquad \forall\, \theta \in \Delta_d\,. \tag{1.8}$$

Then, the volume exponent satisfies:

$$\zeta \leq 1 - \frac{1}{2\alpha}\,. \tag{1.9}$$

Using the lower bound of I_β in (1.7), we deduce from Theorem 1.6 the following corollary:

Corollary 1.7. *We have*

$$p(\beta) = \frac{\beta^2}{2} \quad \Longrightarrow \quad \zeta \leq \frac{3}{4} \tag{1.10}$$

$$\Longrightarrow \quad \chi \geq \frac{1}{8}, \qquad \textit{if } d = 1. \tag{1.11}$$

When $d \geq 3$ and $\beta > 0$ is small, we have $\zeta = 1/2$, $p(\beta) = \beta^2/2$ (cf. Theorem A) and $\chi = 0$ (by using e.g. Theorem 1.5 of [4]), therefore (1.10) does not give any effective bound in this situation; however, it seems interesting that some (rough) bounds on volume and fluctuation exponents can be obtained only in terms of free energy.

It is worthy noticing that in two related models, the exponent $3/4$ is universal for all $d \geq 1$ and $\beta > 0$. More precisely, when (S_n) is replaced by a discrete time d-dimensional Brownian motion and $g(\cdot, \cdot)$ by a gaussian fields, Petermann [17] showed $\zeta \geq 3/5$ in the one-dimensional case and Mejane [15] showed $\zeta \leq 3/4$ for all $d \geq 1$. Comets and Yoshida [6] recently gave another model with Brownian motion in a Poisson environment, they also showed that $\zeta \leq 3/4$. In both these models, the Girsanov transform for Brownian motion plays a crucial rôle which allows to obtain that $I_\beta(\theta) = |\theta|^2/2$ (see Theorem 2.4.4 in [6]). It would be very interesting to obtain an invariance principle between the Brownian motion model and the random walk model.

We are much inspired from Talagrand [23] for the use of concentration of measure and integration by parts. Furthermore, we would like to stress the fact that while in spin glasses the covariance structure of the energies of configurations is influenced by the exchangeability of the individuals spins, in the polymer model it is influenced by the Markov property of the underlying random walk.

This paper is organized as follows: In Section 2, we estimate the rate of convergence of n-th free energy (Propositions 2.3 and 2.4) by using the concentration inequality; The proof of Theorem 1.1 is given in Section 3 by using the subadditivity, whereas Theorem 1.2 is proven in Section 4 with Lemmas 4.1 and 4.3; In Section 5, we give a weak law of large numbers for the polymers measure with biased random walk; In Sections 6 and 7, we prove respectively Theorem 1.4 and Theorem 1.6.

Throughout this paper, c, c', c'' denote some unimportant positive constants whose values may change from one paragraphe to another.

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2. Notations and basic tools

For the sake of notational convenience, we omit sometimes the dependence of β in the partition functions. In addition to the partition function Z_n, we define the partition function starting from x:

$$Z_n(x) = Z_n(x; g) \stackrel{\text{def}}{=} \mathbb{E}_x\left(e^{\beta H_n(g,S)}\right), \qquad H_n(g, \gamma) = \sum_{i=1}^{n} g(i, \gamma(i)),$$

and the point to point partition function

$$Z_n(x, y) = Z_n(x, y; g) = \mathbb{E}_x\left(e^{\beta H_n(g,S)} 1_{(S_n=y)}\right).$$

This function is strictly positive if and only if there exists a nearest neighbor path of length n connecting x to y, i.e. $\mathbb{P}_x(S_n = y) = \mathbb{P}_0(S_n = y - x) > 0$. We shall denote this fact by $y - x \hookleftarrow n$. Observe that

$$x \hookleftarrow n \text{ if and only if } n - \sum_{1}^{d} x_j \equiv 0 \pmod 2 \text{ and } \sum_{1}^{d} |x_j| \le n,$$

with $x = (x_1, ..., x_d) \in \mathbb{Z}^d$. We shall also write $\sum_{x \hookleftarrow n}$ to mean that the sum is taken over those x such that $x \hookleftarrow n$. Let τ_n be the time shift of order n on the environment:

$$(\tau_n g)(k, x) = g(k + n, x) \qquad (x \in \mathbb{Z}^d, k \ge 1).$$

Our first tool is the

Lemma 2.1 (Markov Property). *For every integers n, m and every $x, y \in \mathbb{Z}^d$, we have:*

$$Z_{n+m}(x) = \sum_y Z_n(x, y; g) Z_m(y; \tau_n g) \tag{2.1}$$

$$Z_{n+m}(x, z) = \sum_y Z_n(x, y; g) Z_m(y, z; \tau_n g). \tag{2.2}$$

Proof: The two identities are proved similarly, with the help of the Markov property for the random walk S_n. Indeed,

$$H_{n+m}(g,S) = \sum_{i=1}^{n+m} g(i,S_i) = H_n(g,S) + H_m(\tau_n g, S_{n+\cdot})\,.$$

Therefore,

$$\begin{aligned} Z_{n+m}(x;g) &= \mathbb{E}_x\Big(e^{\beta H_n(g,S)} e^{\beta H_m(\tau_n g, S_{n+\cdot})}\Big) \\ &= \mathbb{E}_x\Big(e^{\beta H_n(g,S)} Z_m(S_n;\tau_n g)\Big) \\ &= \sum_y Z_n(x,y;g) Z_m(y;\tau_n g)\,. \end{aligned}$$

□

Let us recall the *concentration of measure property of Gaussian processes* (see Ibragimov and al. [11]).

Proposition 2.2. *Consider a function $F:\mathbb{R}^M \to \mathbb{R}$ and assume that its Lipschitz constant is at most A, i.e.*

$$|F(x) - F(y)| \le A||x-y|| \qquad (x,y \in \mathbb{R}^M)\,,$$

where $||x||$ denotes the euclidean norm of x. Then if $g = (g_1,\ldots,g_M)$ are i.i.d. $\mathcal{N}(0,1)$ we have

$$\mathbf{P}\Big(|F(g) - \mathbf{E}(F(g))| \ge u\Big) \le \exp(-\frac{u^2}{2A^2}) \qquad (u>0).$$

Following Talagrand [22], we apply this Proposition to partition functions. Define for n integer $p_n(\beta) = \mathbf{E}\big[\frac{1}{n}\log Z_n(\beta)\big]$. It is easy to establish the convergence of the free energy (1.3) (cf. [4]). Indeed, we deduce from Lemma 2.1 that the sequence $n \to p_n(\beta) = \frac{1}{n}\mathbf{E}\big[\log Z_n(\beta)\big]$ is superadditive so $p(\beta) = \lim p_n(\beta)$ is well defined. The concentration of measure (Proposition 2.3) implies that $\frac{1}{n}\log Z_n(\beta) - p_n(\beta) \to 0$ a.s and in L^1. See also Comets, Shiga and Yoshida [5] where they studied general environments g by using martingale deviations.

Proposition 2.3. *(i) For any $u>0$, and $x \hookleftarrow n$,*

$$\mathbf{P}\left(\left|\frac{1}{n}\log Z_n(\beta) - p_n(\beta)\right| \ge u\right) \le \exp(-\frac{nu^2}{2\beta^2}) \tag{2.3}$$

$$\mathbf{P}\left(|\log Z_n(0,x) - \mathbf{E}\left[\log Z_n(0,x)\right]| \ge u\right) \le \exp(-\frac{u^2}{2n\beta^2})\,. \tag{2.4}$$

(ii) Let $\nu > \frac{1}{2}$. Then almost surely there exists $n_0(\omega)$ such that for every $n \ge n_0$:

$$\big|\log Z_n(\beta) - np_n(\beta)\big| \le n^\nu \tag{2.5}$$

$$\big|\log Z_n(0,x) - \mathbf{E}\left[\log Z_n(0,x)\right]\big| \le n^\nu \qquad (x \in \mathbb{Z}^d, x \hookleftarrow n)\,. \tag{2.6}$$

(iii) There exists a constant $c = c(d,\beta) > 0$ such that:

$$\mathbf{E} \exp\Big(\frac{1}{\sqrt{n}}\big|\log Z_n(0,x) - \mathbf{E}(\log Z_n(0,x))\big|\Big) \le c \qquad (n \ge 1, x \hookleftarrow n).$$

Proof: (i) Fix n. Consider the set $\Xi = \{(i,x) : 1 \le i \le n,\ x \hookleftarrow i\}$ and let $M = \#\Xi$. We define a function $F : \mathbb{R}^M \to \mathbb{R}$ by

$$\begin{aligned} F(z) &= \log \mathbb{E}_0\Big[\exp(\beta \sum_{i=1}^n z_{i,S_i})\Big] \\ &= \log \mathbb{E}_0\Big[\exp(\beta \sum_{(i,x)\in\Xi} z_{i,x}\, \mathbf{1}_{(S_i = x)})\Big] \\ &= \log \mathbb{E}_0\Big[e^{\beta\, a^S . z}\Big], \end{aligned}$$

where $z = (z_{i,x}, (i,x) \in \Xi) \in \mathbb{R}^M$ and $a^\gamma \in \mathbb{R}^M$ is the vector with coordinates:

$$a^\gamma_{i,x} = \mathbf{1}_{(\gamma(i)=x)} \qquad ((i,x) \in \Xi)\,.$$

Cauchy-Schwarz' inequality yields that

$$|a^\gamma . z - a^\gamma . z'| \le ||a^\gamma||\, ||z - z'|| = \sqrt{n}\ ||z - z'||\,.$$

Therefore, F has Lispschitz constant at most $A = \beta\sqrt{n}$, and we obtain the concentration of measure inequality (2.3). The inequality (2.4) is obtained in the same way.
(ii) The second part of the Proposition is proved by introducing the events:

$$A_n = \cup_{x \hookleftarrow n}\{|\log Z_n(0,x) - \mathbf{E}\left[\log Z_n(0,x)\right]| \ge n^\nu\} \\ \cup\ \{|\log Z_n(\beta) - np_n(\beta)| \ge n^\nu\}\,.$$

Then, $\mathbf{P}(A_n) \le c_d\, n^d \exp(-\frac{n^{2\nu-1}}{2\beta^2})$, $\sum_n \mathbf{P}(A_n) < +\infty$ and we conclude by Borel Cantelli's Lemma.
(iii) The third part is a straightforward consequence of the concentration inequality

$$\begin{aligned} &\mathbf{E} \exp\Big(\frac{1}{\sqrt{n}}\big|\log Z_n(0,x) - \mathbf{E}(\log Z_n(0,x))\big|\Big) \\ &= \int_0^\infty \mathbf{P}\left(|\log Z_n(0,x) - \mathbf{E}\left[\log Z_n(0,x)\right]| \ge \sqrt{n}\ \log v\right) dv \\ &\le 1 + \int_1^\infty \exp\left(-\frac{(\log v)^2}{2\beta^2}\right) dv \\ &= c(d,\beta) < +\infty\,. \end{aligned}$$

□

Proposition 2.4. *Fix $\nu > \frac{1}{2}$.*

(i) For all large n, we have

$$\mathbf{E}\big[\log Z_n(\beta)\big] \le n^\nu + \frac{1}{2}\mathbf{E}\left[\log Z_{2n}(0,0)\right].$$

(ii) For m large enough

$$\frac{1}{m}\mathbf{E}\left[\log Z_m(\beta)\right] \le p(\beta) \le \frac{1}{m}\mathbf{E}\left[\log Z_m(\beta)\right] + m^{\nu-1}$$

and almost surely, for m large enough

$$\frac{1}{m}\log Z_m(\beta) - m^{\nu-1} \le p(\beta) \le \frac{1}{m}\log Z_m(\beta) + m^{\nu-1}$$

(iii) Almost surely for all large even n,

$$0 \ge \log\langle 1_{(S_n=0)}\rangle^{(n)} \ge -n^\nu.$$

Proof: (i) The Markov Property (Lemma 2.1) implies that for $x \in \mathbb{Q}^d$ such that $nx \hookleftarrow n$,

$$Z_{2n}(0,0;g) \ge Z_n(0,nx;g)Z_n(nx,0;\tau_n g)\,.$$

Therefore,

(2.7)
$$\mathbf{E}\left[Z_{2n}(0,0)\right] \ge \mathbf{E}\left[\log Z_n(0,nx)\right] + \mathbf{E}\left[\log Z_n(nx,0)\right] = 2\mathbf{E}\left[\log Z_n(0,nx)\right].$$

Let $\frac{1}{2} < \nu' < \nu$ and let $\epsilon = n^{-\nu'}$. Then,

$$\begin{aligned}
\mathbf{E}\left[\log Z_n\right] &\le \frac{1}{\epsilon}\log \mathbf{E}\left[Z_n^\epsilon\right] && \text{(Jensens' inequality)}\\
&\le \frac{1}{\epsilon}\log \mathbf{E}\left[\sum_{x \hookleftarrow n} Z_n(0,x)^\epsilon\right] && \text{(since } 0<\epsilon<1)\\
&= \frac{1}{\epsilon}\log \mathbf{E}\left[\sum_{x \hookleftarrow n} e^{\epsilon(\log Z_n(0,x) - \mathbf{E}[\log Z_n(0,x)])} e^{\epsilon \mathbf{E}[\log Z_n(0,x)]}\right]\\
&\le \frac{1}{\epsilon}\log\left(c\sum_{x \hookleftarrow n} e^{\epsilon \mathbf{E}[\log Z_n(0,x)]}\right) && \text{(by Proposition 2.3)}\\
&\le \frac{1}{\epsilon}\log\left(c(2n+1)^d e^{\epsilon/2 \mathbf{E}[\log Z_{2n}(0,0)]}\right) && \text{(by (2.7))}\\
&\le n^\nu + \frac{1}{2}\mathbf{E}\left[\log Z_{2n}(0,0)\right],
\end{aligned}$$

for all large n.

(ii) The second claim in (ii) follows from the first one and from the concentration of measure property. We omit the parameter β. By construction $\frac{1}{m}\mathbf{E}\left[\log Z_m\right] \le p(\beta)$. Therefore, we only need to establish that for m large enough

$$p(\beta) \le \frac{1}{m}\mathbf{E}\left[\log Z_m\right] + m^{\nu-1}\,.$$

Fix a large m and a small $0 < \epsilon = \epsilon(m) < 1$ whose value will be determined later. Define

$$h_\epsilon(n) \stackrel{\text{def}}{=} \log \mathbf{E}\left[Z_n^\epsilon\right] \geq \epsilon \mathbf{E}\left[\log Z_n\right].$$

Observe that by the Markov property and the concavity of $x \to x^\epsilon$,

$$\begin{aligned} Z_{n+k}^\epsilon &= \Big(\sum_{x \leftarrow n} Z_n(0,x;g) Z_k(x;\tau_n g) \Big)^\epsilon \\ &\leq \sum_{x \leftarrow n} Z_n^\epsilon(0,x;g)\, Z_k^\epsilon(x;\tau_n g)\,. \end{aligned}$$

Hence, by independence,

$$\begin{aligned} h_\epsilon(n+k) &\leq \log \Big(\sum_{x \leftarrow n} \mathbf{E}\left[Z_n^\epsilon(0,x;g)\right] \mathbf{E}\left[Z_k^\epsilon(x;\tau_n g)\right] \Big) \\ &\leq \log \left((2n)^d \mathbf{E}\left[Z_n^\epsilon\right] \mathbf{E}\left[Z_k^\epsilon\right] \right) \\ &= h_\epsilon(n) + h_\epsilon(k) + d \log(2n)\,. \end{aligned}$$

Thanks to Hammersley's general subadditivity theorem [9], the following limit exists and satisfies: For all large m,

$$h(\epsilon) = \lim_{n \to +\infty} \frac{h_\epsilon(n)}{n} \leq \frac{h_\epsilon(m)}{m} + c \frac{\log m}{m}\,.$$

Recall that $h_\epsilon(n) \geq \epsilon \mathbf{E}\left[\log Z_n\right]$. Hence, $h(\epsilon) \geq \epsilon p(\beta)$. We now choose $\epsilon = m^{-\nu'}$ with $1/2 < \nu' < \nu$. We obtain that

$$\begin{aligned} h_\epsilon(m) &= \log \mathbf{E}\left[e^{\epsilon(\log Z_m - \mathbf{E}[\log Z_m])} \right] + \epsilon \mathbf{E}\left[\log Z_m\right] \\ &\leq c + \epsilon \mathbf{E}\left[\log Z_m\right] \qquad \text{(by Proposition 2.3)}\,. \end{aligned}$$

Therefore,

$$p(\beta) \leq \frac{h(\epsilon)}{\epsilon} \leq \frac{h_\epsilon(m)}{m\epsilon} + c\frac{\log m}{m\epsilon} \leq \frac{1}{m} \mathbf{E}\left[\log Z_m\right] + c' m^{\nu'-1} \log m\,,$$

proving the desired result since $\nu' < \nu$.

(iii) Combining (i) and (ii), we have that for all n even and large enough,

$$\frac{1}{n} \mathbf{E}\left[\log Z_n(0,0)\right] \leq p(\beta) \leq \frac{1}{n} \mathbf{E}\left[\log Z_n(0,0)\right] + n^{\nu-1}\,,$$

hence $\mathbf{E}\Big(\log \langle 1_{(S_n=0)} \rangle^{(n)} \Big) \geq -n^\nu$. This in view of the concentration property (Proposition 2.3) imply (iii). $\square$

We end this section by recalling the well known result (see e.g. Talagrand [23])

Lemma 2.5. *Let $(g(i))_{1 \leq i \leq N}$ be a family of $\mathcal{N}(0,1)$ random variables, not necessarily independent. Then,*

$$\mathbf{E}\left[\max_{1 \leq i \leq N} g(i) \right] \leq \sqrt{2 \log N}\,.$$

Proof: This short proof was given by M. Talagrand during the École d'été de Saint-Flour, but was not incorporated in the notes [23].
Let $g^* = \max_{1\leq i\leq N} g(i)$. Then for all real number x, by Jensen's inequality

$$Ne^{\frac{x^2}{2}} = \mathbf{E}\sum_{i=1}^{N} e^{xg(i)} \geq \mathbf{E}e^{xg^*} \geq \exp(x\mathbf{E}(g^*))\,.$$

Therefore, $\frac{x^2}{2} + \log N \geq x\mathbf{E}(g^*)$ and optimizing for $x = \sqrt{2\log N}$ yields the desired result. □

3. Large deviations principle: Proof of Theorem 1.1

The proof of Theorem 1.1, inspired from Varadhan [24], relies essentially on the sub-additivity. Fix $d \geq 1$ and $\beta > 0$. Firstly, we establish an auxiliary lemma:

Lemma 3.1. *For any $\lambda > 0$, there exists a continuous convex function $I_\beta^{(\lambda)} : \mathbb{R}^d \to \mathbb{R}_+$ such that almost surely, for any $\xi \in \Delta_d$ and any sequence $x_n \in \mathbb{R}^d$ satisfying $\frac{x_n}{n} \to \xi$, we have*

$$-\frac{1}{n}\log\langle e^{-\lambda|S_n - x_n|}\rangle^{(n)} \;\to\; I_\beta^{(\lambda)}(\xi).$$

The above limit also holds in L^1. Furthermore, we have

$$I_\beta^{(\lambda)}(\xi) \leq p(\beta) + \log(2d), \quad \xi \in \Delta_d,$$

where $p(\beta)$ is the free energy defined in (1.3).

Proof: Define

$$V_n^{(\lambda)}(x, y; g) = \log \mathbb{E}_x\Big[e^{\beta\sum_1^n g(i,S_i)}\, e^{-\lambda\,|S_n - y|}\Big], \qquad x \in \mathbb{Z}^d, y \in \mathbb{R}^d.$$

For $i \geq 0$ and $x \in \mathbb{Z}^d$, denote by $\tau_{i,x}$ the shift operator: $\tau_{i,x} \circ g(\cdot,\cdot) = g(i+\cdot, x+\cdot)$. Then $V_n^{(\lambda)}(x, y; g) = V_n^{(\lambda)}(0, y-x; \tau_{0,x} \circ g)$. Let $n, m \geq 1$ and $x \in \mathbb{Z}^d, y, z \in \mathbb{R}^d$. Since $|S_{n+m} - y| \leq |S_n - z| + |(S_{n+m} - S_n) - (y - z)|$, we have

$$\begin{aligned}
V_{n+m}^{(\lambda)}(x, y; g) &\geq \log \mathbb{E}_x\Big[e^{\beta\sum_1^{n+m} g(i,S_i)}\, e^{-\lambda|S_n - z|} e^{-\lambda|(S_{n+m}-S_n)-(y-z)|}\Big] \\
&= \log \mathbb{E}_x\Big[e^{\beta\sum_1^n g(i,S_i) - \lambda|S_n - z|}\, e^{V_m^{(\lambda)}(0,y-z;\tau_{n,S_n}\circ g)}\Big] \\
&= V_n^{(\lambda)}(x, z) + \log\sum_{u\in\mathbb{Z}^d}\sigma_n(u) e^{V_m^{(\lambda)}(0,y-z;\tau_{n,u}\circ g)} \\
&\geq V_n^{(\lambda)}(x, z) + \sum_{u\in\mathbb{Z}^d}\sigma_n(u)\, V_m^{(\lambda)}(0, y-z; \tau_{n,u}\circ g), \qquad (3.1)
\end{aligned}$$

by the concavity of the logarithmic function, and where

$$\sigma_n(u) \stackrel{\text{def}}{=} e^{-V_n^{(\lambda)}(x,y;g)}\,\mathbb{E}_x\Big[e^{\beta\sum_1^n g(i,S_i) - \lambda|S_n - z|}\, 1_{(S_n = u)}\Big]$$

satisfies $\sum_u \sigma_n(u) = 1$. Define

$$v_n^{(\lambda)}(y) = \mathbf{E}\Big(V_n^{(\lambda)}(0,y;g)\Big), \qquad y \in \mathbb{Z}^d.$$

Then $v_n^{(\lambda)}(y) = v_n^{(\lambda)}(-y)$ by symmetry. Since $\sigma_n(\cdot)$ is independent of $V_m^{(\lambda)}(0, y-z; \tau_{n,u} \circ g)$, we deduce from (3.1) that

$$v_{n+m}^{(\lambda)}(y) = \mathbf{E}\Big(V_{n+m}^{(\lambda)}(0,y;g)\Big) \ge v_n^{(\lambda)}(z) + v_m^{(\lambda)}(y-z), \qquad \forall y, z \in \mathbb{R}^d.$$

Observe the elementary relation:

$$\begin{aligned} |v_n^{(\lambda)}(y) - v_n^{(\lambda)}(z)| &\le \lambda\,|y-z|, \\ v_n^{(\lambda)}(y) &\le n\,p_n(\beta) \le np(\beta). \end{aligned}$$

Using the subadditivity theorem, we obtain a function $\phi_\lambda : \mathbb{R}^d \to (-\infty, p(\beta)]$ such that ϕ_λ is concave, λ-Lipschitz continuous and for any $\xi \in \mathbb{R}^d$

$$\lim_{n\to\infty, \frac{x_n}{n}\to\xi} \frac{v_n^{(\lambda)}(x_n)}{n} = \phi_\lambda(\xi).$$

Put

$$I_\beta^{(\lambda)}(\xi) \stackrel{\text{def}}{=} p(\beta) - \phi_\lambda(\xi) \ge 0, \qquad \xi \in \mathbb{R}^d,$$

gives the convergence in L^1 stated in the Lemma.

We shall prove the convergence a.s. by the concentration property of Gaussian measure. Firstly, since

$$\frac{1}{n}\Big| \log\langle e^{-\lambda|S_n - x_n|}\rangle^{(n)} - \log\langle e^{-\lambda|S_n - \tilde{x}_n|}\rangle^{(n)}\Big| \le \frac{\lambda\,|x_n - \tilde{x}_n|}{n}, \ \forall x_n, \tilde{x}_n \in \mathbb{R}^d,$$

it suffices to prove that for any fixed $\xi \in \mathbb{R}^d$,

$$-\frac{1}{n}\log\langle e^{-c|S_n - n\xi|}\rangle^{(n)} \to I_\beta^{(\lambda)}(\xi), \qquad a.s. \tag{3.2}$$

Following the same lines as in the proof of Proposition 2.3, we deduce from the concentration of Gaussian measure that almost surely, for all large n,

$$\Big|\log\langle e^{-\lambda|S_n - n\xi|}\rangle^{(n)} - \mathbf{E}\log\langle e^{-\lambda|S_n - n\xi|}\rangle^{(n)}\Big| = O(n^\nu),$$

for some $\nu > 1/2$ (the above estimate in fact holds uniformly on $\xi \in \Delta_d$). This yields (3.2). Finally, for any $\xi \in \Delta_d$, we take $x_n \in \mathbb{Z}^d$ such that $\frac{x_n}{n} \to \xi$ and $\mathbf{P}(S_n = x_n) > 0$. It follows from Jensen's inequality that

$$\begin{aligned} \langle e^{-\lambda|S_n - x_n|}\rangle^{(n)} &\ge \langle 1_{(S_n = x_n)}\rangle^{(n)} \\ &= \frac{1}{Z_n(\beta)}\mathbf{E}\Big(e^{\beta\sum_1^n g(i,S_i)}\Big|S_n = x_n\Big)\,\mathbf{P}\left(S_n = x_n\right) \\ &\ge \frac{\mathbf{P}\Big(S_n = x_n\Big)}{Z_n(\beta)}\exp\Big(\beta\mathbf{E}\big[\sum_1^n g(i,S_i)\big|S_n = x_n\big]\Big) \end{aligned}$$

which implies that for any $\lambda > 0$,

$$I_\beta^{(\lambda)}(\xi) \le p(\beta) + 2(d/2\pi)^{d/2}|\xi|^2 \le p(\beta) + \log(2d), \quad \xi \in \Delta_d,$$

proving the lemma. □

The function $I_\beta^{(\lambda)}$ is nondecreasing on λ and we let $\lambda \to \infty$, hence the limit function $I_\beta : \Delta_d \to [0, p(\beta) + \log(2d)]$ is convex and lower semi-continuous on Δ_d.

Proof of Theorem 1.1: The proof relies on an ω-by-ω argument.
Upper bound: It suffices to show that for any $\xi \in \Delta_d$ and small $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log\langle 1_{(|\frac{S_n}{n} - \xi| \le \epsilon)}\rangle^{(n)} \le -I_\infty(\xi) + o(1),$$

where $o(1)$ (possibly random) tends to 0 when $\epsilon \to 0$. Let $\delta > 0$. Take $\lambda > 0$ be sufficiently large such that $|I_\beta(\xi) - I_\beta^{(\lambda)}(\xi)| \le \delta$. Since the number of lattice points x_n such that $|\frac{x_n}{n} - \xi| \le \epsilon$ is of order n^d, we obtain:

$$\begin{aligned}
\langle 1_{(|\frac{S_n}{n} - \xi| \le \epsilon)}\rangle^{(n)} &= \sum_{x_n \leftrightarrow n:\, |\frac{x_n}{n} - \xi| \le \epsilon} \langle 1_{(S_n = x_n)}\rangle^{(n)} \\
&\le \sum_{x_n \leftrightarrow n:\, |\frac{x_n}{n} - \xi| \le \epsilon} \langle e^{-\lambda|S_n - x_n|}\rangle^{(n)} \\
&\le \frac{(2n+1)^d}{\epsilon^d}\, e^{\lambda n \epsilon} \langle e^{-\lambda|S_n - n\xi|}\rangle^{(n)},
\end{aligned}$$

which in view of Lemma 3.1 imply that

$$\limsup_{n\to\infty} \frac{1}{n} \log\langle 1_{(|\frac{S_n}{n} - \xi| \le \epsilon)}\rangle^{(n)} \le \lambda\epsilon - I_\beta^{(\lambda)}(\xi) \le -I_\beta(\xi) + 2\delta,$$

for $\epsilon \le \delta/\lambda$.
Lower bound: Let $\xi \in G$ such that $I_\beta(\xi) < \infty$. We shall bound below $\langle 1_{(|\frac{S_n}{n} - \xi| < \epsilon)}\rangle^{(n)}$. Let λ be sufficiently large such that $\lambda\epsilon > I_\beta(\xi) \ge I_\beta^{(\lambda)}(\xi)$ and $I_\beta^{(\lambda)}(\xi) = I_\beta(\xi) + o(1)$. Therefore by using again Lemma 3.1, we have

$$\langle 1_{(|\frac{S_n}{n} - \xi| < \epsilon)}\rangle^{(n)} \ge \langle e^{-\lambda|S_n - n\xi|}\rangle^{(n)} - e^{-\lambda\epsilon n} \ge e^{-I_\beta^{(\lambda)}(\xi)(1+o(1))n} \ge e^{-I_\beta(\xi)(1+o(1))n},$$

showing the lower bound.

The continuity of I_β in the interior of Δ_d follows from the convexity. It remains to show (1.4).
Since I_β is uniformly bounded below and above, we may repeat the same argument in the proof of Theorem 10.4 ([20], page 86) and prove that

$$\limsup_{y \to x:\, y \in \overset{\circ}{\Delta}_d} I_\beta(y) \le I_\beta(x),$$

which together with the lower-continuity of I_β yields (1.4). □

4. Pointwise rate function: Proof of Theorem 1.2

Before entering into the proof of Theorem 1.2, we need some preliminary lemmas (Lemma 4.1 is devoted to the proof of (1.6) and Lemma 4.3 to (1.5)).

Lemma 4.1. *There exists a constant $c = c(d) > 0$ such that for all $0 < \epsilon < 1/9$,*

$$\#\Omega_n^{(\epsilon)} \le e^{c\,n\,\epsilon\log(1/\epsilon)}, \tag{4.1}$$

where

$$\Omega_n^{(\epsilon)} \stackrel{\text{def}}{=} \Big\{\gamma \in \Omega_n : \gamma(0) = 0, |\gamma_i(n)| \ge (1-\epsilon)n, \exists i = 1, ..., d\Big\},$$

with $\gamma_i(n)$ denoting the i-th coordinate of $\gamma(n) \in \mathbb{Z}^d$. Consequently, almost surely for all small $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log\langle 1_{(|\frac{S_n}{n} - e_1| \le \epsilon)}\rangle^{(n)} \le c'\sqrt{\epsilon\log(1/\epsilon)} - \log(2d) - p(\beta), \tag{4.2}$$

for some constant $c' = c'(d) > 0$.

Proof: Let $S_n^1 = S_n \cdot e_1$ be the first coordinate of S_n. Then $(S_n^1)_{n\ge 0}$ is a symmetric random walk on $\mathbb{Z}$ with step distribution $\mathbf{P}\big(S_n^1 - S_{n-1}^1 = +1\big) = \mathbf{P}\big(S_n^1 - S_{n-1}^1 = -1\big) = 1/(2d)$ and $\mathbf{P}\big(S_n^1 - S_{n-1}^1 = 0\big) = 1 - 1/d$. The large deviations principle implies that

$$\log \mathbf{P}\Big(|S_n^1| \ge (1-\epsilon)n\Big) \sim -n \sup_{\lambda\in\mathbb{R}} \Big(\lambda(1-\epsilon) - \psi(\lambda)\Big) \stackrel{\text{def}}{=} -n\psi^*(1-\epsilon),$$

where $\psi(\lambda) = \log \mathbf{E} e^{\lambda S_1^1} = \log(1 + \frac{\cosh(\lambda)-1}{d})$. Elementary computations show that

$$\psi^*(1-\epsilon) = \log(2d) - (c'' + o(1))\,\epsilon\log(1/\epsilon), \qquad \epsilon \to 0,$$

for some constant $c'' > 0$. This implies (4.1) by symmetry. Finally,

$$\begin{aligned}
& \frac{1}{n}\mathbf{E}\Big(\log\langle 1_{(|\frac{S_n}{n} - e_1| \le \epsilon)}\rangle^{(n)}\Big) \\
= \; & \frac{1}{n}\mathbf{E}\Big(\log \mathbb{E}\big[1_{(|\frac{S_n}{n} - e_1| \le \epsilon)} e^{\beta H_n(g,S)}\big]\Big) - p_n(\beta) \\
\le \; & \frac{\beta}{n}\mathbf{E} \max_{\gamma\in\Omega_n^{(\epsilon)}} H_n(g,\gamma) + \frac{\log \#\Omega_n^{(\epsilon)}}{n} - \log(2d) - p_n(\beta) \\
\le \; & \beta\sqrt{2c\epsilon\log(1/\epsilon)} + c\epsilon\log(1/\epsilon) - \log(2d) - p_n(\beta),
\end{aligned}$$

by applying Lemma 2.5 to the gaussian family $\{\frac{H_n(g,\gamma)}{\sqrt{n}}, \gamma \in \Omega_n^{(\epsilon)}\}$. Thanks to the concentration inequality (2.6), we deduce from the Borel-Cantelli lemma that almost surely for all large n,

$$\Big|\log\langle 1_{(|\frac{S_n}{n} - e_1| \le \epsilon)}\rangle^{(n)} - \mathbf{E}\Big(\log\langle 1_{(|\frac{S_n}{n} - e_1| \le \epsilon)}\rangle^{(n)}\Big)\Big| = O(n^\nu),$$

for any $\nu > 1/2$, which completes the proof. □

The following lemma is an easy consequence of subadditivity:

Lemma 4.2. *For any* $\theta \in \overset{o}{\Delta}_d \cap \mathbb{Q}^d$,

$$a(\theta) \stackrel{\text{def}}{=} \sup_{n\theta \hookleftarrow n, n>1} \frac{1}{n}\mathbf{E}\Big(\log Z_n(0, n\theta)\Big) - p(\theta) = \lim_{n\theta \hookleftarrow n, n\to\infty} \frac{1}{n} \log \langle 1_{(S_n = n\theta)} \rangle^{(n)}$$

the above limit exists almost surely and in $L^1(\mathbf{P})$.

The function $a(\cdot)$ is auxiliary: In fact, according to Theorem 1.2, $a(\theta) = -I_\beta(\theta)$.
Proof: From the Markov Property (Lemma 2.1) we get that for $x \hookleftarrow n$ and $y \hookleftarrow m$,

$$Z_{n+m}(0, x+y; g) \geq Z_n(0, x; g) Z_m(x, x+y; \tau_n g) \qquad (x, y \in \mathbb{Z}^d)\,.$$

Taking the expectation of the logarithm, we get, since $\tau_n g$ is distributed as g,

$$\begin{aligned} \mathbf{E}\left[\log Z_{n+m}(0, x+y)\right] &\geq \mathbf{E}\left[\log Z_n(0, x)\right] + \mathbf{E}\left[\log Z_m(x, x+y)\right] \\ &= \mathbf{E}\left[\log Z_n(0, x)\right] + \mathbf{E}\left[\log Z_m(0, y)\right]. \end{aligned} \tag{4.3}$$

This shows that the sequence $n \to \mathbf{E}\log Z_n(0, n\theta)$ is super additive and the standard subadditivity theorem and the concentration inequality (2.6) yield that $\frac{1}{n}\log Z_n(0, n\theta)$ converges almost surely and in $L^1(\mathbf{P})$. The integrability is guaranteed by Lemma 2.5. This together with (1.3) complete the proof. □

Lemma 4.3. *For any* $\theta \in \overset{o}{\Delta}_d \cap \mathbb{Q}^d$, $\mathbf{P}$ *almost surely,*

$$\lim_{\epsilon\to 0} \liminf_{n\theta \hookleftarrow n, n\to\infty} \frac{1}{n} \log \langle 1_{(|S_n - n\theta| < \epsilon n)} \rangle^{(n)} = a(\theta),$$

where $a(\theta)$ *has been defined in Lemma 4.2.*

Proof: Pick a small $\epsilon > 0$ such that $3\epsilon d < 1 - |\theta|_{L^1}$, where $|\theta|_{L^1} \stackrel{\text{def}}{=} \sum_1^d |\theta_j| < 1$ for $\theta = (\theta_1, ..., \theta_d) \in \overset{o}{\Delta}_d \cap \mathbb{Q}^d$. Write $\theta = (\frac{p_1}{p}, ..., \frac{p_d}{p})$ where $p_1, ..., p_d \in \mathbb{Z}$ and p is an integer such that $p > \sum_1^d |p_j|$. We shall consider those $n \to \infty$ such that n/p is even. This choice ensures that $n\theta \hookleftarrow n$. Let

$$k = k(n) = 2p\Big\lfloor \frac{\epsilon d}{1 - |\theta|_{L^1}} \frac{n}{p} \Big\rceil \sim \frac{2\epsilon d}{1 - |\theta|_{L^1}} n,$$

where $\lfloor x \rceil$ denotes the integer part of x. For any $x_n \in \mathbb{Z}^d$ satisfying $x_n \hookleftarrow n$ and $|x_n - n\theta| \leq \epsilon n$, we define $\widetilde{x}_n \stackrel{\text{def}}{=} (k+n)\theta - x_n \in \mathbb{Z}^d$. Observe that

$$|\widetilde{x}_n|_{L^1} \leq k|\theta|_{L^1} + d|x_n - n\theta| < k,$$

by our choice of $k = k(n)$. Hence $\widetilde{x}_n \hookleftarrow k$. By the Markov property (Lemma 2.1), we get

$$\begin{aligned} Z_{n+k}(0,(n+k)\theta) &\geq \sum_{\widetilde{x}_n \hookleftarrow k} Z_k(0,\widetilde{x}_n)\, Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g) \\ &\geq \sum_{x_n \hookleftarrow n : |x_n - n\theta| \leq \epsilon n} Z_k(0,\widetilde{x}_n)\, Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g). \end{aligned} \tag{4.4}$$

Observe that by stationarity, $Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g)$ has the same law as $Z_n(0,x_n)$. It follows from (2.4) that for $\nu > 1/2$,

$$\begin{aligned} &\mathbf{P}\Big(\exists x_n : x_n \hookleftarrow n, \big|\log Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g) - \log Z_n(0,x_n)\big| > 2n^{\nu}\Big) \\ \leq &\sum_{x_n \hookleftarrow n} \mathbf{P}\Big(\big|\log Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g) - \mathbf{E}\log Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g)\big| > n^{\nu}\Big) \\ &\quad + \sum_{x_n \hookleftarrow n} \mathbf{P}\Big(\big|\log Z_n(0,x_n) - \mathbf{E}\log Z_n(0,x_n)\big| > n^{\nu}\Big) \\ \leq &\ 2(2d+1)^n e^{-n^{2\nu-1}/(2\beta^2)}, \end{aligned}$$

whose sum on n converges. The Borel-Cantelli lemma implies that almost surely for any $\frac{1}{2} < \nu < 1$ and all large n,

$$\max_{\forall x_n \hookleftarrow n} \big|\log Z_n(\widetilde{x}_n,(n+k)\theta;\tau_k g) - \log Z_n(0,x_n)\big| \leq 2n^{\nu}. \tag{4.5}$$

On the other hand, for any $y \hookleftarrow k$, Jensen's inequality implies that

$$\begin{aligned} \mathbf{E}\log Z_k(0,y) &= \mathbf{E}\Big[\log \mathbb{E}\Big(e^{\beta\sum_1^k g(i,S_i)}\big|S_k = y\Big)\Big] + \log \mathbb{P}(S_k = y) \\ &\geq \log \mathbb{P}(S_k = y) \\ &\geq -k\log(2d), \end{aligned}$$

which combined with (2.6) imply that almost surely for all large k,

$$\inf_{y \hookleftarrow k} \log Z_k(0,y) \geq -k(1+\log(2d)). \tag{4.6}$$

Now, we can complete the proof of Lemma 4.3 by an ω-by-ω argument. Almost surely, let n be large such that n/p is even. Injecting (4.5) and (4.6) into (4.4), we get

$$\begin{aligned} Z_{n+k}(0,(n+k)\theta) &\geq e^{-k(1+\log(2d))-2n^{\nu}} \sum_{x_n \hookleftarrow n : |x_n - n\theta| \leq \epsilon n} Z_n(0,x_n) \\ &= e^{-k(1+\log(2d))-2n^{\nu}}\, Z_n(\beta)\, \langle 1_{(|S_n - n\theta| \leq \epsilon n)}\rangle^{(n)}. \end{aligned}$$

Since $k = k(n) \sim \frac{2\epsilon d}{1-|\theta|_{L^1}} n$, we deduce from the above estimate and from Lemma 4.2 and (1.3) that

$$\begin{aligned} a(\theta) &= \lim_{\frac{n}{p} \text{ even}, n\to\infty} \frac{1}{n+k} \log\left(\frac{Z_{n+k}(0,(n+k)\theta)}{Z_{n+k}(\beta)}\right) \\ &\geq -c\,\epsilon + \liminf_{n\theta \hookleftarrow n, n\to\infty} \frac{1}{n+k} \log\langle 1_{(|S_n - n\theta| \leq \epsilon n)}\rangle^{(n)} \\ &\geq -c\,\epsilon + (1+c'\,\epsilon) \liminf_{n\theta \hookleftarrow n, n\to\infty} \frac{1}{n} \log\langle 1_{(|S_n - n\theta| \leq \epsilon n)}\rangle^{(n)}, \end{aligned}$$

where $c > 0$ and $c' > 0$ denote some constants depending on d and θ. Let $\epsilon \to 0$, we obtain that

$$\limsup_{\epsilon\to 0} \liminf_{n\theta \hookleftarrow n, n\to\infty} \frac{1}{n} \log\langle 1_{(|S_n - n\theta| \leq \epsilon n)}\rangle^{(n)} \leq a(\theta),$$

yielding Lemma 4.3 since $\langle 1_{(|S_n - n\theta| \leq \epsilon n)}\rangle^{(n)} \geq \langle 1_{(S_n = n\theta)}\rangle^{(n)}$. □

Combining Lemmas 4.3, 4.1 with Theorem 1.1, we immediately obtain Theorem 1.2:

Proof of Theorem 1.2: The proof is again an ω-by-ω argument. Let $\theta \in \Delta_d \cap \mathbb{Q}^d$ and let $n \to \infty$ with $n\theta \hookleftarrow n$. Since the single point set $\{\theta\}$ is closed, we deduce from the upper bound of Theorem 1.1 that $\mathbf{P}$ almost surely,

$$a(\theta) = \lim_{n\to\infty, n\theta \hookleftarrow n} \frac{1}{n} \log\langle 1_{(S_n = n\theta)}\rangle^{(n)} \leq -I_\beta(\theta),$$

bu using the notation $a(\theta)$ introduced in Lemma 4.2. To show the lower bound, we can assume that $I_\beta(\theta) < \infty$, because otherwise there is nothing to prove. Pick a small $\epsilon > 0$. Using the lower bound of Theorem 1.1, we have that $\mathbf{P}$ almost surely, for all large n,

$$\langle 1_{(|\frac{S_n}{n} - \theta| < \epsilon)}\rangle^{(n)} \geq e^{-(I_\beta(\theta) + o(1))n},$$

where $o(1) \to 0$ when $\epsilon \to 0$. This in view of Lemma 4.3 imply that $a(\theta) \geq -I_\beta(\theta)$, completing the proof of (1.5).

To show (1.7), thanks to the concavity of the logarithm, we have that for $\xi \in \mathbb{Q}^d$ and $n\xi \hookleftarrow n$,

$$\begin{aligned} \frac{1}{n}\mathbf{E}\left[\log Z_n(0, nz)\right] &\leq \frac{1}{n} \log \mathbf{E}\left[Z_n(0, n\xi)\right] \\ &= \frac{1}{n} \log\left(e^{n\frac{\beta^2}{2}} \mathbb{P}_0\left[S_n = n\xi\right]\right). \end{aligned}$$

We conclude from the local central limit theorem. Finally, (1.6) follows from (4.2) by letting $\epsilon \to 0$. According to Proposition 2.4 (iii), we let $n \to \infty$ and obtain that $I_\beta(0) = 0$. □

5. A weak law of large numbers

In this section, we present a law of large numbers for biased random walk. Firstly, in view of Theorem 1.1, Varadhan's lemma ([7], [8]) implies that

Proposition 5.1. *For every $\lambda \in \mathbb{R}^d$, we have the convergence*

$$\lim_{n\to+\infty} \frac{1}{n} \log \left\langle e^{\lambda\cdot S_n} \right\rangle^{(n)} = \phi_\beta(\lambda) \quad \mathbf{P}a.s. \text{ and in } L^1(\mathbf{P})\,.$$

with

$$\phi_\beta(\lambda) = \sup\{\lambda\xi - I_\beta(\xi) : \xi \in \Delta_d\}.$$

The function $\phi_\beta : \mathbb{R}^d \to \mathbb{R}$ is convex nonnegative, $\phi_\beta(0) = 0$, and for every permutation σ: $\phi_\beta(\lambda_1, \dots, \lambda_d) = \phi_\beta(\pm\lambda_{\sigma(1)}, \dots \pm \lambda_{\sigma(d)})$.

We can also directly prove the above convergence by using the subadditivity.

We now state a *Law of Large Numbers* in dimension $d = 1$. Say $(X_n)_{n\in\mathbb{N}}$ is a nearest neighbour random walk on $\mathbb{Z}$ with mean $a \in [-1, +1]$ if X_n is the partial sum of iid variables with common distribution $\mathbb{P}(X_1 = +1) = \frac{1+a}{2}$ and $\mathbb{P}(X_1 = -1) = \frac{1-a}{2}$. We define a polymer measure $\langle\cdot\rangle^{(n,a)}$ associated with (X_n) and $(g(i,x))$ in the same way as $\langle\cdot\rangle^{(n)}$ does to (S_n) and $(g(i,x))$.

Proposition 5.2. *Assume that the function ϕ_β is differentiable at $\lambda \in \mathbb{R}$. Then the walk (X_n) satisfies a law of large numbers under the polymer measure $\langle\cdot\rangle^{(n,\tanh\lambda)}$:*

$$\langle \frac{X_n}{n} \rangle^{(n,\tanh\lambda)} \to \phi'_\beta(\lambda) \qquad \textit{almost surely}\,.$$

The exact value of $\phi'_\beta(\lambda)$ is unknown. This very weak law of large numbers is not a surprise for the symmetric random walk, since we have then $\langle S_n\rangle^{(n)} \overset{\text{law}}{=} -\langle S_n\rangle^{(n)}$ and thus $\mathbf{E}\left[\langle S_n\rangle^{(n)}\right] = 0$.

Proof: The function ϕ_β is the limit of the convex C^1 functions $f_n(\lambda) = \frac{1}{n}\log\left\langle e^{\lambda S_n}\right\rangle^{(n)}$. Therefore, since ϕ_β is differentiable at λ (see Lemma 5.3), almost surely

$$\phi'_\beta(\lambda) = \lim f'_n(\lambda) = \lim \frac{1}{n} \frac{\left\langle S_n e^{\lambda S_n}\right\rangle^{(n)}}{\left\langle e^{\lambda S_n}\right\rangle^{(n)}}\,.$$

Define $\psi(\lambda) = \log\cosh(\lambda)$. Then $M_n^\lambda = e^{\lambda S_n - n\psi(\lambda)}$ is a martingale and under the new probability

$$\mathbb{E}_x^{(\lambda)}\left[f(S_k, k\le n)\right] \overset{\text{def}}{=} \mathbb{E}_0\Big[f(S_k, k\le n)e^{\lambda S_n - n\psi(\lambda)}\Big]$$

the nearest neighbour random walk S has mean $\tanh(\lambda)$. Let us denote by $S^{(\lambda)}$ this walk; then almost surely

$$\phi'_\beta(\lambda) = \lim_{n\to\infty} \frac{1}{n} \big\langle S_n^{(\lambda)} \big\rangle^{(n)} ,$$

which is the desired result. □

Lemma 5.3. *Assume the sequence f_n of real valued, differentiable, convex functions converge on an open interval I to a function f. Then f is convex, and for every x such that f is differentiable at x, $f_n'(x)$ converges to $f'(x)$.*

Proof: See Theorem 25.7 ([20], pp. 248). Although we only assume the differentiability of f at one point, the argument in pages 249-250 still works. □

6. A scaling inequality involving the volume and the fluctuation exponents: Proof of Theorem 1.4

The following lemma is elementary, but nevertheless gives useful lower bounds on the variance:

Lemma 6.1. *([16], Lemma 2) Let $X \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ and assume that $\mathcal{G}_1, ..., \mathcal{G}_k, ...$ is a sequence of independent sub-σ-fields of $\mathcal{F}$. We have*

$$Var(X) \geq \sum_{k=1}^{\infty} Var\Big(\mathbf{E}\left[X \,|\, \mathcal{G}_k\right]\Big).$$

Lemma 6.2. *Denote by $g \stackrel{\text{law}}{=} \mathcal{N}(0,1)$ a standard real-valued Gaussian variable. For any $\beta > 0$, there exists some constant $c_\beta > 0$ such that for all $u, v \geq 0$, we have*

$$Cov\Big(\log(u+e^{\beta g}), \log(v+e^{\beta g})\Big) \geq c_\beta \max\Big(0,\ 1_{(u\leq 1, v\leq 1)},\ \frac{1}{uv} 1_{(u>1,v>1)}\Big).$$

Proof: Denote by $f(u,v)$ the above covariance. Since $f(u,v) = \mathrm{Cov}\big(\log(1+\frac{1}{u}e^{\beta g}), \log(1+\frac{1}{v}\, e^{\beta g})\big)$, we get $\lim_{u\to\infty} f(u,v) = 0$. Let $\widetilde{g}$ be an independent copy of g, we have

$$
\begin{aligned}
\frac{\partial^2 f}{\partial u \partial v}(u,v) &= \mathrm{Cov}\Big(\frac{1}{u+e^{\beta g}}, \frac{1}{v+e^{\beta g}}\Big) \\
&= \mathbf{E}\left[\frac{1}{(u+e^{\beta g})(v+e^{\beta g})} - \frac{1}{(u+e^{\beta g})(v+e^{\beta \widetilde{g}})}\right] \\
&= \mathbf{E}\left[\frac{e^{\beta \widetilde{g}} - e^{\beta g}}{(u+e^{\beta g})(v+e^{\beta g})(v+e^{\beta \widetilde{g}})}\right] \\
&= \mathbf{E}\left[\frac{(e^{\beta \widetilde{g}} - e^{\beta g})(u+e^{\beta \widetilde{g}})}{(u+e^{\beta g})(u+e^{\beta \widetilde{g}})(v+e^{\beta g})(v+e^{\beta \widetilde{g}})}\right] \\
&= \mathbf{E}\left[\frac{(e^{\beta \widetilde{g}} - e^{\beta g})e^{\beta \widetilde{g}}}{(u+e^{\beta g})(u+e^{\beta \widetilde{g}})(v+e^{\beta g})(v+e^{\beta \widetilde{g}})}\right] \\
&= \frac{1}{2}\mathbf{E}\left[\frac{(e^{\beta \widetilde{g}} - e^{\beta g})^2}{(u+e^{\beta g})(u+e^{\beta \widetilde{g}})(v+e^{\beta g})(v+e^{\beta \widetilde{g}})}\right], \qquad (6.1)
\end{aligned}
$$

where the last equality is obtained by interchanging g and $\widetilde{g}$. Remark that for $u, v \geq 0$, $\lim_{v\to\infty} \frac{\partial f}{\partial u}(u,v) = 0$, hence

$$f(x,y) = \int_x^\infty du \int_y^\infty dv \frac{\partial^2 f}{\partial u \partial v}(u,v) \geq 0.$$

Going back to (6.1), we remark that

$$\inf_{0\leq u\leq 2, 0\leq v\leq 2} \frac{\partial^2 f}{\partial u \partial v}(u,v) = c \in (0,\infty),$$

and for all $u \geq \frac{1}{2}, v \geq \frac{1}{2}$,

$$\frac{\partial^2 f}{\partial u \partial v}(u,v) = \frac{1}{2u^2v^2}\mathbf{E}\left[\frac{(e^{\beta \widetilde{g}} - e^{\beta g})^2}{(1+\frac{1}{u}e^{\beta g})(1+\frac{1}{u}e^{\beta \widetilde{g}})(1+\frac{1}{v}e^{\beta g})(1+\frac{1}{v}e^{\beta \widetilde{g}})}\right] \geq \frac{c'}{u^2v^2}.$$

From the above estimates, the desired conclusion follows. □

Proof of Theorem 1.4: Applying Lemma 6.1 to $\big\{\sigma(g(j,x)), 1 \leq j \leq n, x \in \mathbb{Z}^d\big\}$, we get

$$\mathrm{Var}\Big(\log Z_n(\beta)\Big) \geq \sum_{j=1}^n \sum_{x \leftrightarrow j} \mathrm{Var}\Big(\mathbf{E}\left[\log Z_n(\beta) \,|\, g(j,x)\right]\Big).$$

Fix $j \leq n$ and $x \in \mathbb{Z}^d$ such that $x \leftrightarrow j$. Denote by $D_n(g,S) \stackrel{\text{def}}{=} e^{\beta \sum_1^n g(i,S_i)}$. We have

$$Z_n(\beta) = \mathbb{E}_0\Big[D_n(g,S)1_{(S_j\neq x)}\Big] + \mathbb{E}_0\Big[D_{j-1}(g,S)1_{(S_j=x)}\Big] e^{\beta g(j,x)}\, Z_{n-j}(x, g\circ\theta_j)$$

It follows that

$$\log Z_n(\beta) = \log\Big(Y + e^{\beta g(j,x)}\Big) + \log\Big[Z_{n-j}(x, g\circ\theta_j)\mathbb{E}_0\big[D_{j-1}(g,S)1_{(S_j=x)}\big]\Big],$$

where $\mathbb{P}_0\big[S_j = x\big] > 0$ and

$$Y \stackrel{\text{def}}{=} \frac{\mathbb{E}_0\big[D_n(g,S)1_{(S_j\neq x)}\big]}{Z_{n-j}(x, g\circ\theta_j)\mathbb{E}_0\big[D_{j-1}(g,S)1_{(S_j=x)}\big]}$$

is independent of $g(j,x)$. Since $Z_{n-j}(x, g\circ\theta_j)\mathbb{E}_0\big[D_{j-1}(g,S)1_{(S_j=x)}\big]$ is also independent of $g(j,x)$, we get

$$\begin{aligned}
&\mathrm{Var}\Big(\mathbf{E}\left[\log Z_n(\beta)\,|\,g(j,x)\right]\Big) = \mathrm{Var}\Big(\mathbf{E}\left[\log(Y + e^{\beta g(j,x)})\,|\,g(j,x)\right]\Big)\\
&\quad = \mathbf{E}\left[\int_{y>0} \mathbf{P}\left(Y\in dy\right)\Big[\log(y+e^{\beta g(j,x)}) - \mathbf{E}\left[\log(y+e^{\beta g(j,x)})\right]\Big]\right]^2\\
&\quad = \int\!\!\int \mathbf{P}\left(Y\in dy_1\right)\mathbf{P}\left(Y\in dy_2\right)\mathrm{Cov}\Big(\log(y_1+e^{\beta g(j,x)}), \log(y_2+e^{\beta g(j,x)})\Big)\\
&\quad \geq c_\beta \max\Big(\Big(\mathbf{P}\left(Y\leq 1\right)\Big)^2, \Big(\mathbf{E}\left[Y^{-1}1_{(Y>1)}\right]\Big)^2\Big)\\
&\quad \geq \frac{c_\beta}{4}\Big(\mathbf{E}\big[1\wedge\frac{1}{Y}\big]\Big)^2,
\end{aligned}$$

where the first inequality is due to Lemma 6.2. Recalling the definition of Y, we remark that

$$\frac{1}{Y} = e^{-\beta g(j,x)}\,\frac{\langle 1_{(S_j=x)}\rangle^{(n)}}{1-\langle 1_{(S_j=x)}\rangle^{(n)}} \geq e^{-\beta g(j,x)}\,\langle 1_{(S_j=x)}\rangle^{(n)},$$

which in view of Lemma 6.1 imply that

$$\mathrm{Var}(\log Z_n) \geq \frac{c_\beta}{4}\sum_{j\leq n, x\in\mathbb{Z}^d}\Big(\mathbf{E}\left[1\wedge\left(e^{-\beta g(j,x)}\,\langle 1_{(S_j=x)}\rangle^{(n)}\right)\right]\Big)^2.$$

Pick $\zeta' > \zeta$. Define $M_n(g) \stackrel{\text{def}}{=} \max_{j\leq n,|x|\leq n} g(j,x)$. Cauchy-Schwarz' inequality implies that

$$\begin{aligned}
&\sum_{j\leq n,|x|\leq n^{\zeta'}}\Big(\mathbf{E}\left[1\wedge\left(e^{-\beta g(j,x)}\,\langle 1_{(S_j=x)}\rangle^{(n)}\right)\right]\Big)^2\\
&\qquad\geq \frac{\Big(\sum_{j\leq n,|x|\leq n^{\zeta'}}\mathbf{E}\left[1\wedge\left(e^{-\beta g(j,x)}\,\langle 1_{(S_j=x)}\rangle^{(n)}\right)\right]\Big)^2}{\sum_{j\leq n,|x|\leq n^{\zeta'}} 1}\\
&\qquad\geq n^{-1-\zeta' d}\Big(\sum_{j\leq n,|x|\leq n^{\zeta'}}\mathbf{E}\left[1\wedge\left(e^{-M_n(g)}\,\langle 1_{(S_j=x)}\rangle^{(n)}\right)\right]\Big)^2\\
&\qquad\geq n^{-1-\zeta' d}\Big(\sum_{j\leq n,|x|\leq n^{\zeta'}}\mathbf{E}\left[\left(e^{-M_n(g)}\,1_{(M_n(g)\geq 0)}\,\langle 1_{(S_j=x)}\rangle^{(n)}\right)\right]\Big)^2\\
&\qquad= n^{-1-\zeta' d}\Big(\sum_{j\leq n}\mathbf{E}\left[e^{-M_n(g)}\,1_{(M_n(g)\geq 0)}\,\langle 1_{(|S_j|\leq n^{\zeta'})}\rangle^{(n)}\right]\Big)^2.
\end{aligned}$$

It follows from the standard extreme value theory [19] that if

$$N_n \stackrel{\text{def}}{=} |\{(j,x) : j \le n, |x| \le n\}| = (n+1)(2n+1)^d \asymp n^{d+1}$$

then,

$$\frac{M_n(g)}{\sqrt{2\log N_n}} \to 1, \qquad \text{almost surely.}$$

We introduce the event

$$A_n = \left\{ \frac{1}{2}\sqrt{2\log N_n} \le M_n \le 2\sqrt{2\log N_n} \right\},$$

hence $\mathbf{P}(A_n) \to 1$. Let

$$X_n = \left\langle \mathbf{1}_{(\sup_{k\le n}|S_k| \le n^{\zeta'})} \right\rangle^{(n)} \in (0,1).$$

then $X_n \to 1$ in probability thanks to the definition of ζ. Therefore, $X_n \to 1$ in $L^1(\mathbf{P})$, and $\mathbf{E}[X_n 1_{A_n}] \to 1$. So, there exists n_0 such that for $n \ge n_0$,

$$\mathbf{E}[X_n 1_{A_n}] \ge \frac{1}{2}.$$

Consequently, for $n \ge n_0$ and $j \le n$,

$$\begin{aligned}
\mathbf{E}\left[e^{-M_n(g)} \mathbf{1}_{(M_n(g)\ge 0)} \left\langle \mathbf{1}_{(|S_j| \le n^{\zeta'})} \right\rangle^{(n)} \right] &\ge \mathbf{E}\left[e^{-M_n(g)} X_n 1_{A_n} \right] \\
&\ge \exp(-2\sqrt{2\log N_n}) \mathbf{E}[X_n 1_{A_n}] \\
&\ge \frac{1}{2}\exp(-2\sqrt{2\log N_n}).
\end{aligned}$$

Hence

$$\mathrm{Var}(\log Z_n) \ge \frac{c_\beta}{16} n^{1-\zeta' d} e^{-4\sqrt{2\log N_n}},$$

which implies that $2\chi \ge 1 - \zeta' d$ for all $\zeta' > \zeta$, ending the proof. □

7. A relationship between the volume exponent and the shape of the rate function: Proof of Theorem 1.6

Proof of Theorem 1.6: Fix $\nu > \frac{1}{2}$ and γ such that $1 + \alpha(\gamma - 1) > \nu$. Using the concentration of measure inequality as in the proof of Proposition 2.3, it is easy to show that almost surely for n large enough, for all $k \le n$ and $z \leftrightarrow k$:

$$\left| \log \left\langle \mathbf{1}_{(S_k = z)} \right\rangle^{(n)} - \mathbf{E}\left[\log \left\langle \mathbf{1}_{(S_k=z)} \right\rangle^{(n)} \right] \right| \le 2n^\nu.$$

Assume that k is even, so that $0 \leftrightarrow k$. Then, Markov property implies

$$\left\langle \mathbf{1}_{(S_k=z)} \right\rangle^{(n)} = \frac{Z_k(0,z) Z_{n-k}(z;\tau_k g)}{\sum_x Z_k(0,x) Z_{n-k}(x;\tau_k g)} \le \frac{Z_k(0,z)}{Z_k(0,0)} \frac{Z_{n-k}(z;\tau_k g)}{Z_{n-k}(0;\tau_k g)}.$$

Since $Z_{n-k}(z;\tau_k g) \overset{\text{law}}{=} Z_{n-k}(0;\tau_k g)$, we obtain

$$\mathbf{E}\left[\log \left\langle \mathbf{1}_{(S_k=z)} \right\rangle^{(n)}\right] \le \mathbf{E}\Big(\log Z_k(0,z)\Big) - \mathbf{E}\Big(\log Z_k(0,0)\Big).$$

By Lemma 4.2 (recalling $a(\theta) = -I_\beta(\theta)$), we have $\mathbf{E}\Big(\log Z_k(0,z)\Big) \le -kI_\beta(\frac{z}{k}) + k\,p(\beta)$. On the other hand, by means of Proposition 2.4 (i) and (ii), $\mathbf{E}\Big(\log Z_k(0,0)\Big) \ge kp(\beta) - k^\nu$ for all large even k. It follows that

$$\mathbf{E}\left[\log \left\langle \mathbf{1}_{(S_k=z)} \right\rangle^{(n)}\right] \le -kI_\beta(z/k) + k^\nu \, .$$

Hence, if $|z| \ge n^\gamma$, we get almost surely for large enough n,

$$\begin{aligned} \log \left\langle \mathbf{1}_{(S_k=z)} \right\rangle^{(n)} &\le -kI_\beta(z/k) + k^\nu + 2n^\nu \\ &\le -c\,k\,|z/k|^\alpha + 3n^\nu && \text{(by assumption on } I_\beta) \\ &\le -c\,k^{1-\alpha} n^{\gamma\alpha} + 3n^\nu && \text{(since } |z| \ge n^\gamma) \\ &\le -c\;n^{1-\alpha+\gamma\alpha} + 3n^\nu && \text{(since } k \le n \text{ and } \alpha > 1) \\ &\le -c'\,n^{1-\alpha+\gamma\alpha} \, . \end{aligned}$$

Thus we have proven that for k even,

$$\left\langle \mathbf{1}_{(|S_k| \ge n^\gamma)} \right\rangle^{(n)} \le c''\,n^d\,e^{-c' n^{1-\alpha+\gamma\alpha}} \, .$$

If k is odd, then

$$\left\langle \mathbf{1}_{(|S_k| \ge n^\gamma)} \right\rangle^{(n)} \le \left\langle \mathbf{1}_{(|S_{k-1}| \ge n^\gamma - 1)} \right\rangle^{(n)}$$

and we obtain the same type of upper bound.

It turns out that

$$\left\langle \mathbf{1}_{(\exists k \le n, |S_k| \ge n^\gamma)} \right\rangle^{(n)} \le \sum_{k \le n} \left\langle \mathbf{1}_{(|S_k| \ge n^\gamma)} \right\rangle^{(n)} \le c''\,n^{d+1} e^{-c'\,n^{1-\alpha+\gamma\alpha}} \to 0$$

Therefore, $\mathbf{P}$ almost surely,

$$\left\langle \mathbf{1}_{(\max_{k \le n} |S_k| \le n^\gamma)} \right\rangle^{(n)} \to 1 \, ,$$

hence $\gamma \ge \zeta$. We conclude by letting $\nu \downarrow \frac{1}{2}$ and $\gamma \downarrow 1 - \frac{1}{2\alpha}$. □

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