The problem of the most visited site in random environment

by

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Summary. We prove that the process of the most visited site of Sinai's simple random walk in random environment is transient. The rate of escape is characterized via an integral criterion. Our method also applies to a class of recurrent diffusion processes with random potentials. It is interesting to note that the corresponding problem for the usual symmetric Bernoulli walk or for Brownian motion remains open.

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1. Introduction

The simple Random Walk in Random Environment (RWRE) is defined as follows: let $\Xi = \{\xi_j\}_{j\in\mathbb{Z}}$ be a sequence of independent and identically distributed random variables taking values in $(0,1)$. Define the RWRE $\{S_n\}_{n\ge 0}$ by $S_0 \stackrel{\text{def}}{=} 0$ and for $n \ge 1$ and $i \in \mathbb{Z}$,

$$\mathbb{P}\Big[S_{n+1} = i+1 \,\Big|\, S_n = i,\, \Xi\Big] = \xi_i, \quad \text{and} \quad \mathbb{P}\Big[S_{n+1} = i-1 \,\Big|\, S_n = i,\, \Xi\Big] = 1-\xi_i.$$

Note that Ξ and $\{S_n\}_{n\ge 0}$ are both random under $\mathbb{P}$, and that given Ξ (which is called the "environment"), $\{S_n\}_{n\ge 0}$ performs a nearest-neighbour random walk on the line. For notational simplification, we write throughout the paper

$$\eta_j \stackrel{\text{def}}{=} \log\Big(\frac{1-\xi_j}{\xi_j}\Big), \qquad j \in \mathbb{Z}. \tag{1.1}$$

The study of RWRE is motivated by modelisation of some random phenomena in physics and biology (see Hughes [19]). For recent progress, see for example [1], [3]–[5], [7], [9]–[10], [13]–[18], [20]–[21], [23]–[24], [28], [32]–[33], [35]–[37], [39], as well as the book of Révész [30].

We shall assume the following "usual" condition for the random environment:

$$\eta_0 \text{ is bounded almost surely, with } \mathbb{E}(\eta_0) = 0, \tag{1.2}$$

and write

$$\sigma^2 \stackrel{\text{def}}{=} \mathbb{E}(\eta_0^2). \tag{1.3}$$

It is worth noting that if $\sigma = 0$, $\{S_n\}_{n\ge 0}$ becomes the usual simple symmetric random walk (Bernoulli walk). By a slight abuse of notation, we keep using the terminology "RWRE" even in case $\sigma = 0$.

According to a general recurrence/transience criterion of Solomon [36], under (1.2), the random walk $\{S_n\}_{n\ge 0}$ is recurrent, i.e. it visits any given point infinitely often. An important result of Sinai [35] tells that if (1.2) holds, and if $\sigma > 0$ (which excludes the Bernoulli walk), then $S_n/(\log n)^2$ converges to a non-degenerate limiting distribution (the computation of this distribution is later independently achieved by Kesten [23] and Golosov [14]). This contrasts the case of the Bernoulli walk, for which the usual central limit theorem says that $S_n/\sqrt{n}$ converges to a Gaussian distribution.

The main concern of this paper is to study the favourite points. For $n \ge 0$ and $x \in \mathbb{Z}$, define

$$L(n,x) \stackrel{\text{def}}{=} \sum_{i=0}^{n} \mathbb{1}_{\{S_i = x\}}, \tag{1.4}$$

the number of visits of RWRE at x up to time n, which is also referred to as the local time of RWRE. Let

$$\mathbb{F}(n) \stackrel{\text{def}}{=} \Big\{x \in \mathbb{Z}_+ : L(n,x) = \max_{y\in\mathbb{Z}_+} L(n,y)\Big\},$$

which, following Erdos and Révész [11] and Bass and Griffin [2], is called the set of the **favourite sites** or the **most visited sites** (in $\mathbb{Z}_+$) of RWRE. Since $\mathbb{F}(n)$ is not necessarily a singleton, we consider

$$F(n) = \max_{x\in\mathbb{F}(n)} x, \tag{1.5}$$

the maximal favourite site (though all the results presented in the paper for $F(n)$ still hold if in (1.5), "max" is replaced say by "min").

Let us first recall two results of $F(n)$ for the Bernoulli walk.

Theorem A (Erdős and Révész [11], Bass and Griffin [2]). *Under* (1.2), *if* $\sigma = 0$,

$$\limsup_{n\to\infty} \frac{F(n)}{(2n\log\log n)^{1/2}} = 1, \qquad \text{a.s.} \tag{1.6}$$

Theorem B (Bass and Griffin [2]). *Under* (1.2), *if* $\sigma = 0$, *then with probability one,*

$$\liminf_{n\to\infty} \frac{(\log n)^a}{n^{1/2}} F(n) = \begin{cases} 0 & \text{if } a < 2, \\ \infty & \text{if } a > 11. \end{cases} \tag{1.7}$$

It is seen from (1.6) that $F(n)$ satisfies the same law of the iterated logarithm (LIL) as the Bernoulli walk. However, it is also proved by Erdős and Révész [11] that they have *different* upper functions, i.e. the usual Kolmogorov test (also referred to as the Erdős–Feller–Kolmogorov–Petrowsky or EFKP test, see Révész [30, p. 35]) does not apply to $F(n)$.

Theorem B is somewhat surprising, which a fortiori tells that $F(n)$ is transient. The exact rate of escape of $F(n)$ in (1.7) is unknown, and is believed to be a very challenging problem.

We now present the main results of the paper, concerning the behaviours of $F(n)$ when the environment is random.

Theorem 1.1. *Assuming* (1.2) *and* $\sigma > 0$,

$$\limsup_{n\to\infty} \frac{F(n)}{(\log n)^2 \log\log\log n} = \frac{8}{\pi^2\sigma^2}, \qquad \text{a.s.}$$

Theorem 1.2. *Assume* (1.2) *and* $\sigma > 0$. *For any non-decreasing sequence* $a_n > 1$,

$$\liminf_{n\to\infty} \frac{a_n}{(\log n)^2} F(n) = \begin{cases} 0 \\ \infty \end{cases}, \quad \text{a.s.} \quad \Longleftrightarrow \quad \sum_n \frac{\log a_n}{n\sqrt{a_n}\, \log n} \begin{cases} = \infty \\ < \infty \end{cases}.$$

In particular,

$$\liminf_{n\to\infty} \frac{(\log\log n)^a}{(\log n)^2} F(n) = \begin{cases} 0 & \text{if } a \le 2, \\ \infty & \text{otherwise.} \end{cases}$$

Remark 1.3. Theorem 1.1 is not deep. It merely confirms that in random environment $F(n)$ satisfies again the same LIL (see Section 6 for the exact statement) as the random walk, which is easily guessed in view of the corresponding result (i.e. Theorem A) for the Bernoulli walk. Theorem 1.2 tells that $F(n)$ is also transient in random environment. Usually, the presence of the random environment considerably complicates the situation, and the results obtained are often less complete than those for the Bernoulli walk. The problem of the escape rates of the most visited site is the **only** example we are aware of so far, which is solved in random environment but remains open for the Bernoulli walk.

The rest of the paper is as follows. In Section 2, we study some properties of the location of the minimum of one-dimensional Brownian motion. Section 3 is devoted to introduction of a continuous-time model in random environment. Some preliminary estimates are presented in Section 4, which will be used in Section 5 to prove Theorem 1.2. The proof of Theorem 1.1 is provided in Section 6. Finally, in Section 7, we give the corresponding results for a class of recurrent diffusion processes with random potentials, including the example of Brox's diffusion with Brownian potential.

Throughout the paper, we write indifferently $Z(t)$ and Z_t for any stochastic process Z. Since we only deal with (possibly random) indices r, n, t, $\cdots$ which ultimately go to infinity, our statements, sometimes without further mention, are to be understood for the situation when the appropriate index is sufficiently large. The usual symbol $a(x) \sim b(x)$ $(x \to x_0)$ denotes $\lim_{x\to x_0} a(x)/b(x) = 1$. We also adopt the abbreviation "i.o." for "infinitely often" (when the relevant index goes to infinity).

Unimportant finite positive constants will be denoted by c_i $(1 \le i \le 29)$.

2. Brownian motion

Let $\{W(t);\, t \ge 0\}$ be one-dimensional Brownian motion, with $W(0) = 0$. Define the processes of first hitting times for W: for $r > 0$,

$$H_r \stackrel{\text{def}}{=} \inf\Big\{t > 0 :\, W(t) > r\Big\}, \tag{2.1}$$

$$H_{-r} \stackrel{\text{def}}{=} \inf\Big\{t > 0 :\, W(t) < -r\Big\}. \tag{2.2}$$

Consider

$$\beta_r \overset{\text{def}}{=} \inf\Big\{t > 0 :\, W(t) = \inf_{0\le s\le H_r} W(s)\Big\}, \qquad r > 0, \tag{2.3}$$

which is the (first) location of the minimum of W over $[0, H_r]$.

Lemma 2.1. *Let $0 < \theta < 2(\nu - 1)$. Almost surely for all large r and all $t \in [0, H_r]$,*

$$|t - \beta_r| \ge \frac{r^2}{(\log r)^\theta} \quad \Longrightarrow \quad W(t) \ge \inf_{0\le s\le H_r} W(s) + \frac{r}{(\log r)^\nu}.$$

Roughly, the lemma says that asymptotically, Brownian motion can realize a value which is close to its minimum only in a neighbourhood of the location of the minimum. This is intuitively clear. The proof is based on the following well-known path decomposition theorem, see Revuz and Yor [31, Proposition VI.3.13]:

Fact 2.2. *For any $r > 0$, the variable $|\inf_{0\le s\le H_r} W(s)|$ has density $r(x+r)^{-2}\mathbb{1}_{\{x>0\}}$. Moreover, given $|\inf_{0\le s\le H_r} W(s)| = x > 0$,*

$$\Big\{r - W(t);\, 0 \le t \le \beta_r\Big\} \quad \text{and} \quad \Big\{r - W(H_r - t);\, 0 \le t \le H_r - \beta_r\Big\}$$

are independent three-dimensional Bessel processes, the first starting from r and killed when hitting $x + r$ for the first time, the second starting from 0 and killed when hitting $x + r$.

Proof of Lemma 2.1. Fix $u > 0$ and $v > 0$, whose values will be chosen later. For $r > u$, define

$$E_{u,v}(r) \overset{\text{def}}{=} \Big\{\inf_{0\le t\le H_r,\ |t-\beta_r|\ge v} W(t) < \inf_{0\le s\le H_r} W(s) + u\Big\},$$

($\inf\emptyset \overset{\text{def}}{=} \infty$). By conditioning on $\inf_{0\le s\le H_r} W(s) = -x$, and using Fact 2.2,

$$\mathbb{P}(E_{u,v}(r)) = \int_0^\infty \mathrm{I}_{(2.4)} \times \mathrm{II}_{(2.4)} \frac{r\,\mathrm{d}x}{(r+x)^2}, \tag{2.4}$$

with

$$\mathrm{I}_{(2.4)} \overset{\text{def}}{=} \mathbb{P}_r\Big(\sup_{0\le t\le \tau(r+x)-v} R(t) > r + x - u,\ \tau(r+x) > v\Big),$$

$$\mathrm{II}_{(2.4)} \overset{\text{def}}{=} \mathbb{P}_0\Big(\sup_{0\le t\le \tau(r+x)-v} R(t) > r + x - u,\ \tau(r+x) > v\Big),$$

where $\{R(t);\, t \ge 0\}$ is a three-dimensional Bessel process, $\tau(s) \stackrel{\text{def}}{=} \inf\{t > 0 : R(t) > s\}$, and $\mathbb{P}_s$ denotes the probability under which R starts from s $(s \ge 0)$.

By the strong Markov property,

$$\mathrm{II}_{(2.4)} = \mathbb{P}_{r+x-u}\big(\tau(r+x) > v\big) = \mathbb{P}_{r+x-u}\Big(\sup_{0\le t\le v} R(t) < r+x\Big).$$

Under $\mathbb{P}_{r+x-u}$, the Bessel process R can be written as

$$R(t) = (r+x-u) + B(t) + \int_0^t \frac{\mathrm{d}s}{R(s)},$$

where B is standard Brownian motion. Therefore,

$$\mathrm{II}_{(2.4)} \le \mathbb{P}\Big(\sup_{0\le t\le v} B(t) < u\Big) \le \frac{u}{\sqrt{v}}.$$

Since $\mathrm{I}_{(2.4)} \le 1$, in view of (2.4), we have

$$\mathbb{P}(E_{u,v}(r)) \le \frac{u}{\sqrt{v}}, \qquad \text{for all } r > u. \tag{2.5}$$

Now fix $0 < \theta < 2(\nu - 1)$ and $k \ge 1$. Let $r_n \stackrel{\text{def}}{=} \mathrm{e}^n$ (for all $n \ge 1$). We choose $u \stackrel{\text{def}}{=} r_{k+1}/(\log r_k)^\nu$, $v \stackrel{\text{def}}{=} r_k^2/(\log r_{k+1})^\theta$, and define

$$\Lambda(k) \stackrel{\text{def}}{=} \inf\Big\{r \ge r_k : \omega \in E_{u,v}(r)\Big\}.$$

Clearly $H_{\Lambda(k)}$ is an $(\mathcal{F}_r)_{r\ge 0}$–stopping time, where $(\mathcal{F}_t)_{t\ge 0}$ is the natural filtration of W. On $\{\Lambda(k) < \infty\}$, we consider $\widehat{W}(t) \stackrel{\text{def}}{=} W(t + H_{\Lambda(k)}) - \Lambda(k)$ (for $t \ge 0$) which is Brownian motion independent of $\mathcal{F}_{H_{\Lambda(k)}}$. Define

$$G_k \stackrel{\text{def}}{=} \Big\{\widehat{W} \text{ hits } (r_{k+1} - r_k) \text{ before hitting } (-r_k)\Big\}.$$

Observe that

$$\Big(\{r_k < \Lambda(k) \le r_{k+1}\} \cap G_k \cap \{\inf_{0\le s\le H_{r_k}} W(s) < -u\}\Big) \subset E_{u,v}(r_{k+1}),$$

which implies

$$\begin{aligned}
\mathbb{P}\big(E_{u,v}(r_{k+1})\big) &\ge \mathbb{P}\Big(r_k < \Lambda(k) \le r_{k+1}\Big)\, \mathbb{P}(H_{r_{k+1}-r_k} < H_{-r_k}) \\
&\qquad - \mathbb{P}\Big(\inf_{0\le s\le H_{r_k}} W(s) \ge -u\Big) \\
&= \mathbb{P}\Big(r_k < \Lambda(k) \le r_{k+1}\Big)\frac{r_k}{r_{k+1}} - \frac{u}{u + r_k}.
\end{aligned}$$

Since $\mathbb{P}(\Lambda(k) = r_k) = \mathbb{P}(E_{u,v}(r_k))$, it follows from (2.5) that

$$\begin{aligned}\mathbb{P}\Big(r_k \le \Lambda(k) \le r_{k+1}\Big) &\le \mathbb{P}\big(E_{u,v}(r_k)\big) + \frac{r_{k+1}}{r_k}\left(\mathbb{P}\big(E_{u,v}(r_{k+1})\big) + \frac{u}{u+r_k}\right)\\ &\le \frac{r_k + r_{k+1}}{r_k}\,\frac{u}{\sqrt{v}} + \frac{u\,r_{k+1}}{r_k(u+r_k)},\end{aligned}$$

which yields $\sum_k \mathbb{P}(r_k \le \Lambda(k) \le r_{k+1}) < \infty$. Lemma 2.1 is proved by an application of the Borel–Cantelli lemma. □

Lemma 2.3. *Let β_1 be as in (2.3),*

$$\mathbb{P}\big(\beta_1 < \lambda\big) \sim \sqrt{\frac{2\lambda}{\pi}} \qquad \lambda \to 0^+. \tag{2.6}$$

As a consequence, for all $0 < \lambda \le 1$,

$$c_1\sqrt{\lambda} \le \mathbb{P}\big(\beta_1 < \lambda\big) \le c_2\sqrt{\lambda}.$$

Proof. We again apply the path decomposition theorem in Fact 2.2, to see the Laplace transform of β_1: for all $u > 0$,

$$\mathbb{E}\big(\mathrm{e}^{-u\beta_1}\big) = \int_0^\infty \mathbb{E}_1\left(\mathrm{e}^{-u\tau(1+x)}\right)\frac{\mathrm{d}x}{(1+x)^2},$$

where, as before, R denotes a three-dimensional Bessel process, starting from 1 under $\mathbb{P}_1$ ($\mathbb{E}_1$ standing for the associated expectation), and $\tau(1+x) \stackrel{\text{def}}{=} \inf\{t > 0 : R(t) > 1 + x\}$. According to Kent [22],

$$\mathbb{E}_1\left(\mathrm{e}^{-u\tau(1+x)}\right) = \frac{(1+x)\sinh\sqrt{2u}}{\sinh((1+x)\sqrt{2u}\,)},$$

which implies

$$\mathbb{E}\big(\mathrm{e}^{-u\beta_1}\big) = \int_{\sqrt{2u}}^\infty \frac{\sinh\sqrt{2u}}{y\sinh y}\,\mathrm{d}y \sim \frac{1}{\sqrt{2u}}, \qquad u \to \infty.$$

This yields (2.6) by means of a Tauberian theorem, see for example Feller [12, p. 445]. □

Define, for $r > 0$,

$$U_r \stackrel{\text{def}}{=} \Big|\inf_{0\le s\le H_r} W(s)\Big| + r, \tag{2.7}$$

which is the range of W over $[0, H_r]$. Let $\{\widetilde{U}_r;\, r > 0\}$ be a process having the same law as $\{U_r;\, r > 0\}$, independent of $\{W(s);\, s \ge 0\}$. Define,

$$U_r^* \stackrel{\text{def}}{=} \max\big(U_r,\, \widetilde{U}_r\big), \qquad r > 0. \tag{2.8}$$

Lemma 2.4. *For any $\theta > 1$, there exist positive constants c_3, c_4 and c_5, depending on θ, such that for all $0 < \varepsilon \le 1/2$,*

$$\mathbb{P}\big(\,\beta_1 < \varepsilon (U_\theta^*)^2\big) \le c_3 \sqrt{\varepsilon}\, \log\Big(\frac{1}{\varepsilon}\Big), \tag{2.9}$$

$$\mathbb{P}\big(\,\beta_1 < \varepsilon\, \widetilde{U}_1^2,\, \widetilde{U}_1 < \varepsilon^{-1/3},\, U_1 < 2\big) \ge c_4 \sqrt{\varepsilon}\, \log\Big(\frac{1}{\varepsilon}\Big), \tag{2.10}$$

$$\mathbb{P}\big(\,\beta_\theta < \varepsilon (U_1^*)^2\big) \ge c_5 \sqrt{\varepsilon}\, \log\Big(\frac{1}{\varepsilon}\Big). \tag{2.11}$$

Proof. We only have to treat the case when ε is sufficiently small. Write $a \vee b$ for $\max(a, b)$. By independence and scaling,

$$(\beta_1, U_\theta^*) = (\beta_1, U_\theta \vee \widetilde{U}_\theta) \stackrel{\text{law}}{=} (\beta_1, U_\theta \vee \theta \widetilde{U}_1),$$

where "$\stackrel{\text{law}}{=}$" stands for identity in law. Let $\widehat{U}$ and $\widetilde{U}$ denote two independent copies of the process U, and independent of $\{W(s);\, 0 \le s \le H_1\}$, then by the strong Markov and scaling properties,

$$\begin{aligned}
(\beta_1, U_\theta^*) &\stackrel{\text{law}}{=} \big(\beta_1, (U_1 + \theta - 1) \vee (\,\widehat{U}_{\theta-1}) \vee \theta \widetilde{U}_1\big) \\
&\stackrel{\text{law}}{=} \big(\beta_1, (U_1 + \theta - 1) \vee (\theta - 1)\, \widehat{U}_1 \vee \theta \widetilde{U}_1\big),
\end{aligned}$$

which, in view of the relation $U_1 + \theta - 1 \le \theta U_1$, implies that

$$\begin{aligned}
\mathbb{P}\big(\,\beta_1 < \varepsilon (U_\theta^*)^2\big) &\le \mathbb{P}\Big(\, \frac{\sqrt{\beta_1}}{\theta\sqrt{\varepsilon}} \le U_1 \vee \widehat{U}_1 \vee \widetilde{U}_1\Big) \\
&\le 2\mathbb{P}\Big(\, \frac{\sqrt{\beta_1}}{\theta\sqrt{\varepsilon}} \le U_1 \vee \widetilde{U}_1\Big).
\end{aligned}$$

Observe that the probability expression on the right hand side is

$$\begin{aligned}
&\le \mathbb{P}\Big(U_1 \vee \widetilde{U}_1 > \frac{1}{\theta\sqrt{\varepsilon \log(1/\varepsilon)}}\Big) + \mathbb{P}\Big(\beta_1 < \frac{1}{\log(1/\varepsilon)},\, U_1 > 2\Big) \\
&\quad + \mathbb{P}\Big(\, \frac{\sqrt{\beta_1}}{\theta\sqrt{\varepsilon}} \le 2 + \widetilde{U}_1,\, \widetilde{U}_1 \le \frac{1}{\theta\sqrt{\varepsilon \log(1/\varepsilon)}}\Big).
\end{aligned}$$

The first probability term is easy to estimate. Indeed, since $\mathbb{P}(U_1 > x) = 1/x$ for all $x > 1$, we have,

$$\mathbb{P}\big(U_1 \vee \widetilde{U}_1 > x\big) \le 2\mathbb{P}\big(U_1 > x\big) \le \frac{2}{x}, \qquad x > 1.$$

To estimate the second probability expression, note that $U_1 = 1 - \inf_{0\le s\le \beta_1} W(s)$, which implies, for all $0 < u < 1$,

$$\mathbb{P}\big(\beta_1 < u,\, U_1 > 2\big) \le \mathbb{P}\Big(-\inf_{0\le s\le u} W(s) > 1\Big) \le \exp\Big(-\frac{1}{2u}\Big). \tag{2.12}$$

Finally, thanks to the independence of β_1 and $\widetilde{U}_1$, we have, by conditioning on $\widetilde{U}_1$ and using Lemma 2.3, for all $0 < u < 1$ and $v > 2$ such that $u(2+v) < 1$,

$$\begin{aligned}\mathbb{P}\big(\sqrt{\beta_1} \le u(2+\widetilde{U}_1),\, \widetilde{U}_1 \le v\big) &\le c_2\, \mathbb{E}\Big[\, u(2+U_1)\mathbb{1}_{\{U_1 \le v\}}\Big] \\ &= c_2\, u \int_1^v (2+x)\,\frac{\mathrm{d}x}{x^2} \\ &\le c_6\, u \log v.\end{aligned}$$

Assembling these pieces and we obtain:

$$\mathbb{P}\big(\,\beta_1 < \varepsilon(U_\theta^*)^2\big) \le 4\theta\sqrt{\varepsilon\log(1/\varepsilon)} + 2\sqrt{\varepsilon} + c_7\,\sqrt{\varepsilon}\log\Big(\frac{1}{\varepsilon}\Big),$$

which readily yields (2.9).

To check (2.10), observe that the probability term on the left hand side of (2.10) is greater than (or equal to)

$$\mathbb{P}\big(\beta_1 < \varepsilon\, \widetilde{U}_1^2,\, \widetilde{U}_1 < \varepsilon^{-1/3}\big) - \mathbb{P}\big(\beta_1 < \varepsilon^{1/3},\, U_1 > 2\big),$$

which, by means of Lemma 2.3 and (2.12) (and recalling that $\mathbb{P}(U_1 > x) = 1/x$ for $x > 1$), is

$$\begin{aligned}&\ge c_1\, \mathbb{E}\Big[\,\sqrt{\varepsilon}\, U_1\, \mathbb{1}_{\{U_1 < \varepsilon^{-1/3}\}}\Big] - \exp\Big(-\frac{1}{2\varepsilon^{1/3}}\Big) \\ &\ge c_8\sqrt{\varepsilon}\, \log\Big(\frac{1}{\varepsilon}\Big) - \exp\Big(-\frac{1}{2\varepsilon^{1/3}}\Big),\end{aligned}$$

proving (2.10).

We actually have already proved (2.11), implicitly. Indeed, by independence and scaling,

$$\begin{aligned}\mathbb{P}\big(\,\beta_\theta < \varepsilon(U_1^*)^2\big) &\ge \mathbb{P}\Big(\,\beta_\theta < \varepsilon\widetilde{U}_1^2,\, \widetilde{U}_1 < \frac{\theta}{\sqrt{\varepsilon}}\,\Big) \\ &= \mathbb{P}\Big(\,\beta_1 < \theta^{-2}\,\varepsilon\widetilde{U}_1^2,\, \widetilde{U}_1 < \frac{\theta}{\sqrt{\varepsilon}}\,\Big),\end{aligned}$$

which leads to (2.11) again by conditioning on $\widetilde{U}_1$ and using Lemma 2.3. $\square$

Lemma 2.5. *Fix $\theta > 1$. Let $f > 1$ be a non-decreasing function, and let*

$$\mathcal{J}(f) \overset{\text{def}}{=} \int^{\infty} \frac{\log f(r)}{r\sqrt{f(r)}}\,\mathrm{d}r.$$

(i) *If $\mathcal{J}(f) < \infty$, then*

$$\liminf_{r\to\infty} f(r)\,\frac{\beta_r}{(U^*_{\theta r})^2} = \infty, \qquad \text{a.s.} \tag{2.13}$$

(ii) *If $\mathcal{J}(f) = \infty$, then along the subsequence $r_n \overset{\text{def}}{=} \theta^n$,*

$$\liminf_{n\to\infty} f(r_n)\,\frac{\beta_{\theta r_n}}{(U^*_{r_n})^2} = 0, \qquad \text{a.s.} \tag{2.14}$$

Remark 2.6. Since $U^*_{\theta r} \geq \theta r$, an immediate consequence of (2.13) is that, almost surely for all large r,

$$\beta_r > \frac{r^2}{(\log r)^3}.$$

Proof of Lemma 2.5. Let $f > 1$ be non-decreasing, and $r_n \overset{\text{def}}{=} \theta^n$. It is easily seen that

$$\mathcal{J}(f) < \infty \iff \sum_n \frac{\log f(r_n)}{\sqrt{f(r_n)}} < \infty. \tag{2.15}$$

To prove (2.13), let us assume $\mathcal{J}(f) < \infty$, which implies that f goes to infinity. Fix $\lambda > 0$ and consider the events

$$E_n \overset{\text{def}}{=} \Big\{ f(r_n)\,\beta_{r(n)} < \lambda\,(U^*_{\theta r(n+1)})^2 \Big\},$$

for sufficiently large n, say $n \geq n_0$. (For typesetting reason, we have written $r(n)$ and $r(n+1)$ for r_n and r_{n+1} respectively). It follows from (2.9) that for $n \geq n_0$,

$$\mathbb{P}(E_n) \leq c_3 \sqrt{\frac{\lambda}{f(r_n)}}\,\log \frac{f(r_n)}{\lambda},$$

which, according to (2.15), is summable for n. Applying the Borel–Cantelli lemma and using monotonicity, we obtain:

$$\liminf_{r\to\infty} f(r)\,\frac{\beta_r}{(U^*_{\theta r})^2} \geq \lambda, \qquad \text{a.s.}$$

This yields (2.13) by sending λ to infinity.

It remains to show (2.14). Suppose that the integral in (2.15) diverges. In view of the form of the test, we can assume, without loss of generality, that (for all large r),

$$\log r \le f(r) \le (\log r)^3. \tag{2.16}$$

(This is well-known, and can be checked by a deterministic argument, see for example Csáki [6]). Fix $\lambda > 0$, and define

$$\begin{aligned}
E_n &\stackrel{\text{def}}{=} \left\{ f(r_n)\, \beta_{r(n+1)} < \lambda\, \widetilde{U}^2_{r(n)} \right\} \\
G_n &\stackrel{\text{def}}{=} \left\{ \widetilde{U}_{r(n)} < \frac{r_n \sqrt{f(r_n)}}{\log f(r_n)},\ U_{r(n+1)} \le 2r_{n+1} \right\}, \\
D_n &\stackrel{\text{def}}{=} E_n \cap G_n.
\end{aligned}$$

By independence, scaling and (2.10), for all $n \ge n_0$,

$$\begin{aligned}
\mathbb{P}(D_n) &= \mathbb{P}\Big(f(r_n)\, \beta_1 < \frac{\lambda\, r_n^2}{r_{n+1}^2}\, \widetilde{U}_1^2,\ \widetilde{U}_1 < \frac{\sqrt{f(r_n)}}{\log f(r_n)},\ U_1 \le 2\Big) \\
&\ge \mathbb{P}\Big(f(r_n)\, \beta_1 < \lambda\theta^{-2}\, \widetilde{U}_1^2,\ \widetilde{U}_1 < \big(\frac{\lambda\theta^{-2}}{f(r_n)}\big)^{-1/3},\ U_1 \le 2\Big) \\
&\ge c_9\, \frac{\log f(r_n)}{\sqrt{f(r_n)}}. \qquad (2.17)
\end{aligned}$$

This implies $\sum_n \mathbb{P}(D_n) = \infty$. In order to apply the Borel–Cantelli lemma, we have to estimate the second moment $\mathbb{P}(D_i \cap D_j)$, for $j > i$. Recall H from (2.1), and define

$$\widehat{W}(t) \stackrel{\text{def}}{=} W(t + H_{r(i+1)}) - r_{i+1}; \qquad t \ge 0,$$

which is Brownian motion independent of $\mathcal{F}_{H_{r(i+1)}}$. We can define $\widehat{H}$ and $\widehat{\beta}$ for $\widehat{W}$, exactly in the same way as H and β are for W.

There are three possible situations:

$$\begin{aligned}
&\text{Case 1: } \beta_{r(j+1)} < H_{r(i+1)}, \\
&\text{Case 2: } \beta_{r(j+1)} > H_{r(i+1)},\ \ j - i < 3(\log i)/\log\theta, \\
&\text{Case 3: } \beta_{r(j+1)} > H_{r(i+1)},\ \ j - i \ge 3(\log i)/\log\theta.
\end{aligned}$$

Case 1 is equivalent to

$$\inf_{0 \le t \le \widehat{H}_{r(j+1)-r(i+1)}} \widehat{W}(t) > -U_{r(i+1)}.$$

Since $D_i \subset \{U_{r(i+1)} \le 2r_{i+1}\}$, we have

$$\begin{aligned}
\mathbb{P}\Big(D_i,\ D_j,\ \text{Case } 1\Big) &\le \mathbb{P}\Big(D_i, \inf_{0\le t\le \widehat{H}_{r(j+1)-r(i+1)}} \widehat{W}(t) > -2r_{i+1}\Big)\\
&= \mathbb{P}(D_i)\,\mathbb{P}\Big(\inf_{0\le t\le H_{r(j+1)-r(i+1)}} W(t) > -2r_{i+1}\Big)\\
&= \mathbb{P}(D_i)\,\frac{2r_{i+1}}{r_{j+1}+r_{i+1}}\\
&\le 2\theta^{-(j-i)}\,\mathbb{P}(D_i),
\end{aligned}$$

which gives

$$\sum_{n_0\le i<j\le n} \mathbb{P}\Big(D_i,\ D_j,\ \text{Case } 1\Big) \le c_{10}\sum_{i=n_0}^{n} \mathbb{P}(D_i). \tag{2.18}$$

In Cases 2 and 3,

$$\beta_{r(j+1)} = H_{r(i+1)} + \widehat{\beta}_{r(j+1)-r(i+1)}. \tag{2.19}$$

Observing that $D_j \subset \{\beta_{r(j+1)} < \lambda r_j^2/(\log f(r_j))^2\}$, and recalling (2.16),

$$\begin{aligned}
\mathbb{P}\Big(D_i,\ D_j,\ \text{Case } 2\Big) &\le \mathbb{P}\Big(D_i,\ \widehat{\beta}_{r(j+1)-r(i+1)} < \frac{\lambda\, r_j^2}{(\log f(r_j))^2}\Big)\\
&= \mathbb{P}(D_i)\,\mathbb{P}\Big(\beta_{r(j+1)-r(i+1)} < \frac{\lambda\, r_j^2}{(\log f(r_j))^2}\Big)\\
&\le \mathbb{P}(D_i)\,\mathbb{P}\Big(\beta_{r(j+1)-r(i+1)} < \frac{\lambda\, r_j^2}{(\log\log r_j)^2}\Big).
\end{aligned}$$

Since $r(j+1)-r(i+1) \ge (\theta-1)\, r_j$, we have, by means of Lemma 2.3,

$$\mathbb{P}\Big(D_i,\ D_j,\ \text{Case } 2\Big) \le \mathbb{P}(D_i)\frac{c_{11}}{\log\log r_j} \le \frac{c_{12}}{\log j}\,\mathbb{P}(D_i).$$

Therefore,

$$\sum_{n_0\le i<j\le n} \mathbb{P}\Big(D_i,\ D_j,\ \text{Case } 2\Big) \le c_{13}\sum_{i=n_0}^{n} \mathbb{P}(D_i). \tag{2.20}$$

To treat Case 3, we first make a general observation:

$$\widetilde{U}_{r(j)} = \max\big(r_j - r_i + \widetilde{U}_{r(i)},\ Z_{r(j)-r(i)}\big),$$

where $Z_{r(j)-r(i)}$ is distributed as $U_{r(j)-r(i)}$, independent of $\{W(t);\, t \geq 0\}$ and $\widetilde{U}_{r(i)}$. Since by (2.16), $D_i \subset \{\, \widetilde{U}_{r(i)} \leq r_i(\log r_i)^2\}$, and since $r_i(\log r_i)^2 \leq r_j - r_i$ in Case 3, we have, on D_i (and in Case 3),

$$\begin{aligned}\widetilde{U}_{r(j)} &\leq r_j - r_i + r_i(\log r_i)^2 + Z_{r(j)-r(i)}\\ &\leq 2(r_j - r_i) + Z_{r(j)-r(i)}\\ &\leq 3Z_{r(j)-r(i)}.\end{aligned}$$

In light of (2.19), this leads to:

$$\begin{aligned}\mathbb{P}\big(D_i,\ D_j,\ \text{Case 3}\big) &\leq \mathbb{P}\Big(\, D_i,\ \widehat{\beta}_{r(j+1)-r(i+1)} < \frac{9Z^2_{r(j)-r(i)}}{f(r_j)}\Big)\\ &= \mathbb{P}(D_i)\, \mathbb{P}\Big(\, \beta_{r(j+1)-r(i+1)} < \frac{9\,\widetilde{U}^2_{r(j)-r(i)}}{f(r_j)}\Big)\\ &= \mathbb{P}(D_i)\, \mathbb{P}\Big(\beta_1 < \frac{9(r_j - r_i)^2}{(r_{j+1} - r_{i+1})^2 f(r_j)}\, \widetilde{U}_1^2\Big).\end{aligned}$$

In Case 3, $(r_j - r_i)/(r_{j+1} - r_{i+1}) \leq c_{14}$. By (2.9), for $0 < \varepsilon \leq 1/2$,

$$\mathbb{P}\big(\beta_1 < \varepsilon\, \widetilde{U}_1^2\big) \leq \mathbb{P}\big(\beta_1 < \varepsilon\, (U_1^*)^2\big) \leq c_3\, \sqrt{\varepsilon}\log\Big(\frac{1}{\varepsilon}\Big).$$

Therefore,

$$\mathbb{P}\big(D_i,\ D_j,\ \text{Case 3}\big) \leq c_{15}\, \mathbb{P}(D_i)\, \frac{\log f(r_j)}{\sqrt{f(r_j)}},$$

which, in view of (2.17), yields

$$\sum_{n_0 \leq i < j \leq n} \mathbb{P}\big(D_i,\ D_j,\ \text{Case 3}\big) \leq c_{16}\, \Big(\sum_{i=n_0}^{n} \mathbb{P}(D_i)\Big)^2. \tag{2.21}$$

Since $\sum_n \mathbb{P}(D_n) = \infty$, combining (2.18), (2.20) and (2.21) together with Kochen and Stone's Borel–Cantelli lemma ([26]) gives $\mathbb{P}(\limsup_n D_n) > 0$. A fortiori, with positive probability,

$$\liminf_{n\to\infty}\ f(r_n)\, \frac{\beta_{\theta r_n}}{(U^*_{r_n})^2} \leq \lambda.$$

The above clearly is a tail event, which, by means of a 0–1 argument (and by ultimately sending λ to 0^+), completes the proof of (2.14). $\square$

3. Diffusions with random potentials

Let $\{V(t);\, t \geq 0\}$ and $\{V(-t);\, t \geq 0\}$ be adapted and locally bounded processes with $V(0) = 0$, independent of the standard Brownian motion $\{\mathbb{B}(t);\, t \geq 0\}$. Consider the process X defined by $X(0) = 0$ and

$$\mathrm{d}X(t) = \mathrm{d}\mathbb{B}(t) - \frac{1}{2} V'(X(t))\, \mathrm{d}t. \tag{3.1}$$

We call X **diffusion with random potential** V.

However, we even do not assume V to be continuous. Therefore, instead of writing the formal derivative of V in (3.1), we really should regard X as a diffusion process whose generator is

$$\frac{1}{2}\, \mathrm{e}^{V(x)} \frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{e}^{-V(x)} \frac{\mathrm{d}}{\mathrm{d}x} \Big).$$

A more convenient way in the study of X is to use diffusion theory to arrive at the following representation (see Brox [3]):

$$X(t) = A^{\leftarrow}(B(T^{\leftarrow}(t))), \qquad t \geq 0. \tag{3.2}$$

Here, B is standard Brownian motion independent of $\{V(x);\, x \in \mathbb{R}\}$,

$$A(x) \overset{\text{def}}{=} \int_0^x \mathrm{e}^{V(y)}\, \mathrm{d}y, \qquad x \in \mathbb{R}, \tag{3.3}$$

$$T(r) \overset{\text{def}}{=} \int_0^r \exp\Big[-2V(A^{\leftarrow}(B(s))) \Big]\, \mathrm{d}s, \qquad r \geq 0, \tag{3.4}$$

and $A^{\leftarrow}$ and $T^{\leftarrow}$ denote the respective inverse functions of A and T. (Of course, we have to assume that almost surely $A(\pm\infty) = \pm\infty$ and $T(\infty) = \infty$, which will be satisfied by the examples of V considered in the paper). We point out that A is the scale function of X.

Let $\{L_B(t,x);\, t \geq 0,\, x \in \mathbb{R}\}$ denote the jointly continuous local time process of B. For any bounded Borel function f, by (3.2),

$$\begin{aligned}
\int_0^t f(X(s))\, \mathrm{d}s &= \int_0^t f(A^{\leftarrow}(B(T^{\leftarrow}(s))))\, \mathrm{d}s \\
&= \int_0^{T^{\leftarrow}(t)} f(A^{\leftarrow}(B(u))) \exp\big(-2V(A^{\leftarrow}(B(u)))\big)\, \mathrm{d}u \\
&= \int_{-\infty}^{\infty} f(A^{\leftarrow}(y)) \exp\big(-2V(A^{\leftarrow}(y))\big)\, L_B\big(T^{\leftarrow}(t), y\big)\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-V(x)}\, L_B\big(T^{\leftarrow}(t), A(x)\big)\, \mathrm{d}x.
\end{aligned}$$

Consequently,

$$L_X(t,x) \stackrel{\text{def}}{=} \mathrm{e}^{-V(x)}\, L_B(T^{\leftarrow}(t), A(x)), \qquad t \geq 0,\ x \in \mathbb{R}, \tag{3.5}$$

is the local time process of X.

The reason for which X interests us is that if the random potential is carefully chosen, then X behaves very much like Sinai's RWRE. Here is a brief description of the choice of this particular random potential (the main idea goes back at least to Schumacher [32]): given $\Xi = \{\xi_j\}_{j\in\mathbb{Z}}$ a random environment satisfying (1.2), and recalling η_j and σ from (1.2)–(1.3) (with $\sigma > 0$), there exists a unique choice of (random) step function $\{V(x);\, x \in \mathbb{R}\}$ with $V(0) = 0$, which is flat on each interval $[n, n+1)$, with jumps $V(n) - V(n-) = \eta_n$ (for $n \in \mathbb{Z}$). More precisely,

$$V(x) \stackrel{\text{def}}{=} \begin{cases} \eta_1 + \cdots + \eta_k, & \text{if } x \in [k, k+1) \text{ for } k \in \mathbb{Z}_+^*, \\ 0, & \text{if } x \in [0,1), \\ -(\eta_0 + \eta_{-1} + \cdots + \eta_{k+1}), & \text{if } x \in [k, k+1) \text{ for } k \in \mathbb{Z}_-^*. \end{cases} \tag{3.6}$$

For this choice of V, we can define a diffusion process X via (3.2). It can be seen that X is recurrent. Define $\mu_0 \stackrel{\text{def}}{=} 0$ and

$$\mu_n \stackrel{\text{def}}{=} \inf\Big\{ t > \mu_{n-1} :\ |X(t) - X(\mu_{n-1})| = 1 \Big\}, \qquad n = 1, 2, \cdots \tag{3.7}$$

It is now possible to compare L_X with local time of Sinai's RWRE. The following is borrowed from [16, (4.12)–(4.13) and Fact 4.3].

Fact 3.1. *Let Ξ satisfy (1.2), with $\sigma > 0$. In a rich probability space, there exists a coupling for RWRE $\{S_n\}_{n\geq 0}$ in random environment Ξ and diffusion process $\{X(t);\, t \geq 0\}$ whose random potential is defined by (3.6), such that with probability one,*

$$\limsup_{n\to\infty} \sup_{k\in\mathbb{Z}} \frac{1}{\sqrt{1+L(n,k)}\, \log n} \sup_{\mu_n \leq s \leq \mu_{n+1}} |L_X(s,k) - 2\xi_k\, L(n,k)| \leq 29, \tag{3.8}$$

where L and L_X are the local times of S_n and X respectively. Moreover,

$$\lim_{n\to\infty} \frac{\mu_n}{n} = 1, \qquad \text{a.s.} \tag{3.9}$$

Our approach essentially goes like this: instead of directly handling L (local time of RWRE), we shall be working on L_X (local time of diffusion with random potential), by exploiting the representation (3.5). Thanks to Fact 3.1, this is sufficient for our needs, at the cost of an extra precision of order $\mathcal{O}(\sqrt{\max_{k\in\mathbb{Z}} L(n,k)}\,\log n)$.

4. Partial sum potential and Brownian movement

This section is devoted to the study of two subjects: (i) partial sum potential V; (ii) Brownian motion B which drives the movement of our diffusion X (see Section 3). For the sake of clarity, they are discussed in distinct subsections.

4.1. Partial sum potential

Let $\Xi = \{\xi_j\}_{j\in\mathbb{Z}}$ be an iid sequence of variables satisfying (1.2), and let $\eta_j = \log((1-\xi_j)/\xi_j)$ (see (1.1)). Therefore, there exists a finite constant $K>0$ such that for all $j\in\mathbb{Z}$,

$$|\eta_j| \le K. \tag{4.1}$$

We assume $\sigma>0$ (see (1.3)), to ensure the randomness of the environment.

Let $\{V(x);\, x\in\mathbb{R}\}$ denote the partial sum potential introduced in (3.6). According to the classical Komlós–Major–Tusnády strong approximation theorem ([27]), possibly in an enlarged probability space, there exists a standard two-sided Brownian motion $\{W(x);\, x\in\mathbb{R}\}$ and finite constants $c_{17}>0$ and $c_{18}>0$ (depending on the distribution of η_0) such that for all $t\ge 1$,

$$\mathbb{P}\Big(\sup_{-t\le x\le t}|V(x)-\sigma W(x)| \ge c_{17}\,\log t\Big) \le \frac{c_{18}}{t^2}. \tag{4.2}$$

We first recall a well-known estimate for the modulus of continuity of W, which is a particular case of Lemma 1.1.1 of Csörgő and Révész [8, p. 24] (taking $\varepsilon=1$, $h=1/r^4$ and $v=\log r$ in their notation).

Fact 4.1. *There exists a universal constant c_{19} such that for any $r>1$,*

$$\mathbb{P}\Big(\sup_{0\le s\le t\le 1,\ t-s\le 1/r^4}|W(t)-W(s)| > \frac{\log r}{r^2}\Big) \le c_{19}\, r^4 \exp\Big(-\frac{(\log r)^2}{3}\Big).$$

Recall H_r from (2.1), and define

$$\beta_r \stackrel{\mathrm{def}}{=} \inf\Big\{t>0:\ W(t) = \inf_{0\le s\le H_r} W(s)\Big\}, \tag{4.3}$$

$$\alpha_r \stackrel{\mathrm{def}}{=} \inf\Big\{t>0:\ W(t) - \inf_{0\le s\le H_r} W(s) < \frac{8c_{17}+4+\sigma}{\sigma}\,\log r\Big\}, \tag{4.4}$$

$$\gamma_r \stackrel{\mathrm{def}}{=} \sup\Big\{t\le H_r:\ W(t) - \inf_{0\le s\le H_r} W(s) < \frac{8c_{17}+4+\sigma}{\sigma}\,\log r\Big\}. \tag{4.5}$$

Observe that α_r, β_r and γ_r are well-defined for all $r \geq r_0$ (so that $r > (8c_{17} + 4 + \sigma)\sigma^{-1}\log r$), and that (4.3) is nothing else but (2.3).

We point out that, despite some ressemblance, our triplet $(\alpha_r, \beta_r, \gamma_r)$ is not a "valley" in the sense of Sinai [35] and Tanaka [37]. The reason for which we are interested in $(\alpha_r, \beta_r, \gamma_r)$ is that the favourite site of the diffusion process X with random potential V, at some suitably chosen random times, lies eventually in $[\alpha_r, \gamma_r]$ (see Lemma 5.1 in Section 5).

Lemma 4.2. *Let A be as in (3.3). For all $r \geq r_0$,*

$$\mathbb{P}\big(r^4 A(\gamma_r) > A(H_r)\big) \leq \frac{c_{20}\log r}{r}. \tag{4.6}$$

Proof. Define

$$\begin{aligned}
E_1 &\stackrel{\text{def}}{=} \Big\{ \sup_{0\leq s\leq \gamma_r} W(s) \leq r - \frac{8c_{17}+8+\sigma}{\sigma}\log r \Big\},\\
E_2 &\stackrel{\text{def}}{=} \Big\{ \sup_{0\leq s\leq r^4} |V(s) - \sigma W(s)| < 4c_{17}\log r \Big\},\\
E_3 &\stackrel{\text{def}}{=} \big\{ 1 < H_r < r^4 \big\},
\end{aligned}$$

where c_{17} is the absolute constant in (4.2). By the definitions of A and γ_r, on $E_2 \cap E_3$,

$$\begin{aligned}
A(\gamma_r) &\leq \exp\Big(\sigma \sup_{0\leq s\leq \gamma_r} W(s) + 4c_{17}\log r\Big)\gamma_r\\
&\leq r^4 \exp\Big(\sigma \sup_{0\leq s\leq \gamma_r} W(s) + 4c_{17}\log r\Big).
\end{aligned}$$

Therefore, on $E_1 \cap E_2 \cap E_3$,

$$A(\gamma_r) \leq r^4 \exp\Big(\sigma r - (\sigma + 4c_{17} + 8)\log r\Big).$$

On the other hand, on $E_2 \cap E_3$,

$$\begin{aligned}
A(H_r) &\geq \int_{H_r-1}^{H_r} \mathrm{e}^{V(s)}\,\mathrm{d}s\\
&\geq \exp\Big(\sigma r - 4c_{17}\log r - \sigma \sup_{0\leq s\leq t\leq r^4,\ t-s\leq 1} |W(t) - W(s)|\Big).
\end{aligned}$$

Consequently, by writing $\mathrm{I}_{(4.6)}$ for the probability term on the left hand side of (4.6),

$$\begin{aligned}
\mathrm{I}_{(4.6)} &\leq \mathbb{P}\Big(\sup_{0\leq s\leq t\leq r^4,\ t-s\leq 1} |W(t) - W(s)| > \log r\Big) + \sum_{i=1}^{3}\mathbb{P}(E_i^c)\\
&= \mathbb{P}\Big(\sup_{0\leq s\leq t\leq 1,\ t-s\leq 1/r^4} |W(t) - W(s)| > \frac{\log r}{r^2}\Big) + \sum_{i=1}^{3}\mathbb{P}(E_i^c).
\end{aligned} \tag{4.7}$$

The first probability term on the right hand side is estimated in Fact 4.1. We have to bound $\mathbb{P}(E_i^c)$ for $1 \le i \le 3$. According to (4.2),

$$\mathbb{P}(E_2^c) \le \frac{c_{18}}{r^8}. \tag{4.8}$$

By the usual estimates for Gaussian tails,

$$\mathbb{P}(E_3^c) = \mathbb{P}\big(H_r \le 1\big) + \mathbb{P}\big(H_r \ge r^4\big) \le \exp\Big(-\frac{r^2}{2}\Big) + \frac{1}{r}. \tag{4.9}$$

Finally, to estimate $\mathbb{P}(E_1^c)$, let $G(r) \stackrel{\text{def}}{=} \sup\{t \le H_r : W(t) = 0\}$. Observe that

$$\mathbb{P}(E_1^c) \le \mathbb{P}\big(\gamma_r > G(r)\big) + \mathbb{P}\Big(\sup_{0 \le s \le G(r)} V(s) > r - \frac{8c_{17} + 8 + \sigma}{\sigma} \log r\Big).$$

According to Williams's path decomposition theorem ([38]), $\sup_{0\le s\le G(r)} W(s)$ is uniformly distributed in $(0, r)$. This confirms that the second probability term on the right hand equals $r^{-1}(8c_{17}+8+\sigma)\sigma^{-1}\log r$. Since $\{\gamma_r > G(r)\}$ means $W(t) > -(8c_{17}+4+\sigma)\sigma^{-1}\log r$ for all $0 \le t \le H_r$, it follows that

$$\mathbb{P}(E_1^c) \le \frac{(8c_{17} + 4 + \sigma)\sigma^{-1}\log r}{r + (8c_{17} + 4 + \sigma)\sigma^{-1}\log r} + \frac{8c_{17} + 8 + \sigma}{\sigma}\,\frac{\log r}{r}. \tag{4.10}$$

Assembling (4.7)–(4.10) and using Fact 4.1 yields the lemma. □

4.2. Brownian movement

Let B be standard one-dimensional Brownian motion, whose jointly continuous local time process is denoted by $\{L_B(t,x);\ t \ge 0,\ x \in \mathbb{R}\}$. For $r > 0$, define

$$\varrho(r) \stackrel{\text{def}}{=} \inf\Big\{t > 0 : B(t) > r\Big\}. \tag{4.11}$$

The following is the classical Ray–Knight theorem, see Ray [29], Knight [25] or Revuz and Yor [31, Theorem XI.2.2].

Fact 4.3. *For $r > 0$, $\{L_B(\varrho(r), r - x);\ 0 \le x \le r\}$ is a squared Bessel process of dimension 2, starting from 0.*

Lemma 4.4. *Write*

$$L_B(t, \mathbb{R}_+) \stackrel{\text{def}}{=} \sup_{x \ge 0} L_B(t,x), \qquad t \ge 0. \tag{4.12}$$

For all $v > u > 0$ and $\lambda \geq 2$,

$$\mathbb{P}\Big(\sup_{\varrho(u) \leq t \leq \varrho(v)} \sup_{0 \leq x \leq u/\lambda^2} \frac{L_B(t, \mathbb{R}_+)}{L_B(t,x)} > \lambda \Big) \leq c_{21} \frac{\log \lambda}{\lambda} \Big(1 + \log \frac{v}{u} \Big). \tag{4.13}$$

Proof. Let $N = N(u,v)$ be the integer part of $(\log(v/u))/\log 2$, and let $\mathrm{I}_{(4.13)}$ denote the probability expression on the left hand side of (4.13), then

$$\begin{aligned}
\mathrm{I}_{(4.13)} &\leq \sum_{k=0}^{N} \mathbb{P}\Big(\sup_{\varrho(2^k u) \leq t \leq \varrho(2^{k+1}u)} \sup_{0 \leq x \leq u/\lambda^2} \frac{L_B(t, \mathbb{R}_+)}{L_B(t,x)} > \lambda \Big) \\
&= \sum_{k=0}^{N} \mathbb{P}\Big(\sup_{\varrho(1) \leq t \leq \varrho(2)} \sup_{0 \leq x \leq 2^{-k}/\lambda^2} \frac{L_B(t, \mathbb{R}_+)}{L_B(t,x)} > \lambda \Big) \\
&\leq (N+1)\, \mathbb{P}\Big(\sup_{\varrho(1) \leq t \leq \varrho(2)} \sup_{0 \leq x \leq \lambda^{-2}} \frac{L_B(t, \mathbb{R}_+)}{L_B(t,x)} > \lambda \Big) \\
&\leq (N+1)\, \mathbb{P}\Big(L_B(\varrho(2), \mathbb{R}_+) > 6 \log \lambda \Big) \\
&\qquad + (N+1)\, \mathbb{P}\Big(\inf_{0 \leq x \leq \lambda^{-2}} L_B(\varrho(1), x) < \frac{6 \log \lambda}{\lambda} \Big) \\
&\stackrel{\text{def}}{=} (N+1)\, \mathrm{I}_{(4.14)} + (N+1)\, \mathrm{II}_{(4.14)},
\end{aligned} \tag{4.14}$$

with obvious notation. Let $\{\Re(t);\ 0 \leq t \leq 1\}$ denote a 2-dimensional Bessel process with $\Re(0) = 0$. According to Fact 4.3 and the scaling property,

$$\mathrm{I}_{(4.14)} = \mathbb{P}\Big(\sup_{0 \leq t \leq 1} \Re(t) > \sqrt{3 \log \lambda} \Big) \leq \frac{c_{22}}{\lambda}, \tag{4.15}$$

the last inequality following from the usual Gaussian tail estimate: $\log \mathbb{P}(\sup_{0 \leq t \leq 1} \Re(t) > x) \sim -x^2/2$ (as x goes to infinity).

To estimate $\mathrm{II}_{(4.14)}$, observe that, by Fact 4.3,

$$\begin{aligned}
\mathrm{II}_{(4.14)} &= \mathbb{P}\Big(\inf_{1-\lambda^{-2} \leq t \leq 1} \Re(t) < \frac{\sqrt{6 \log \lambda}}{\sqrt{\lambda}} \Big) \\
&\leq \mathbb{P}\Big(\Re(1) < \frac{\sqrt{\log \lambda}}{\sqrt{\lambda}} \Big) + \mathbb{P}\Big(\sup_{1-\lambda^{-2} \leq t \leq 1} |\Re(1) - \Re(t)| > \frac{\sqrt{\log \lambda}}{\sqrt{\lambda}} \Big).
\end{aligned}$$

The random variable $\Re^2(1)$ being exponential, with mean 2, its density function is bounded above by $1/2$. Therefore for any $y > 0$,

$$\mathbb{P}(\Re(1) < y) \leq \frac{y^2}{2}.$$

On the other hand, since $\Re$ can be realized as the Euclidean modulus of an $\mathbb{R}^2$-valued Brownian motion, say (W_1, W_2), by triangular inequality and time reversal, for any $y > 0$,

$$\mathbb{P}\Big(\sup_{1-\lambda^{-2}\le t\le 1} |\Re(1) - \Re(t)| > y\Big) \le 2\mathbb{P}\Big(\sup_{0\le s\le \lambda^{-2}} |W_1(s)| > \frac{y}{\sqrt{2}}\Big) \le 4\exp\Big(-\frac{y^2\lambda^2}{4}\Big).$$

Consequently,

$$\mathrm{II}_{(4.14)} \le \frac{\log\lambda}{2\lambda} + 4\exp\Big(-\frac{\lambda\log\lambda}{4}\Big).$$

Combining this with (4.15) and (4.14) yields the lemma. □

5. Proof of Theorem 1.2

Let $\Xi = \{\xi_j\}_{j\in\mathbb{Z}}$ be iid random variables satisfying (1.2) (with $\sigma > 0$), and let V be the partial sum process defined by (3.6). We are interested in $\{X(t);\, t \ge 0\}$, the diffusion process with partial sum potential V, driven by the Brownian motion B (see (3.2) for definition).

Let $\{W(x);\, x \in \mathbb{R}\}$ be the Komlós–Major–Tusnády two-sided Brownian motion satisfying (4.2), independent of B. Recall A and T from (3.3)–(3.4). For notational simplification, we write

$$\Theta(r) \stackrel{\text{def}}{=} T(\varrho(A(H_r))), \qquad r > 0, \tag{5.1}$$

where ϱ and H are as in (4.11) and (2.1) respectively. The following lemma confirms that, the favourite site of X, at time $\Theta(r)$, eventually lies in $[\alpha_r, \gamma_r]$, where $(\alpha_r, \beta_r, \gamma_r)$ is defined via (4.3)–(4.5).

Lemma 5.1. *Let L_X denote the local time of X as in (3.5). For all $s > r \ge r_0$,*

$$\mathbb{P}\Big(\sup_{\Theta(r)\le t\le\Theta(s)}\ \max_{0\le k\le\alpha_r} \frac{L_X(t,k)}{L_X(t,[\beta_r])} > \frac{1}{r^2}\Big) \le c_{23}\,\frac{s\log r}{r^2}, \tag{5.2}$$

$$\mathbb{P}\Big(\max_{k\ge\gamma_r} \frac{L_X(\Theta(r),k)}{L_X(\Theta(r),[\beta_r])} > \frac{1}{r^2}\Big) \le c_{24}\,\frac{\log r}{r}, \tag{5.3}$$

where $[\beta_r]$ stands for the integer part of β_r.

Proof. Let

$$E_2 \stackrel{\text{def}}{=} \Big\{\sup_{0\le s\le r^4} |V(s) - \sigma W(s)| < 4c_{17}\log r\Big\},$$
$$E_3 \stackrel{\text{def}}{=} \big\{1 < H_r < r^4\big\},$$
$$E_4 \stackrel{\text{def}}{=} \Big\{\sup_{0\le s\le t\le r^4,\ t-s\le 1} |W(t) - W(s)| \le \log r\Big\},$$

where c_{17} is the constant in (4.2). By (3.5), on $E_2 \cap E_3 \cap E_4$, for all $\Theta(r) \le t \le \Theta(s)$ and $0 \le k \le \alpha_r$,

$$
\begin{aligned}
\frac{L_X(t,k)}{L_X(t,[\beta_r])} &= \mathrm{e}^{-(V(k)-V([\beta_r]))}\, \frac{L_B(\overleftarrow{T}(t), A(k))}{L_B(\overleftarrow{T}(t), A([\beta_r]))} \\
&\le \mathrm{e}^{8c_{17}\log r}\, \mathrm{e}^{-\sigma(W(k)-W(\beta_r))+\sigma\log r}\, \frac{L_B(\overleftarrow{T}(t), A(k))}{L_B(\overleftarrow{T}(t), A([\beta_r]))} \\
&\le \mathrm{e}^{8c_{17}\log r}\, \mathrm{e}^{-(8c_{17}+4)\log r}\, \frac{L_B(\overleftarrow{T}(t), A(k))}{L_B(\overleftarrow{T}(t), A([\beta_r]))} \qquad (5.4)\\
&\le \frac{1}{r^4}\, \frac{L_B(\overleftarrow{T}(t), \mathbb{R}_+)}{L_B(\overleftarrow{T}(t), A([\beta_r]))},
\end{aligned}
$$

where $L_B(\,\cdot\,, \mathbb{R}_+)$ is defined in (4.12). We have used the definition of α_r (see (4.4)) in (5.4).

Let $\mathrm{I}_{(5.2)}$ denote the probability term on the left hand side of (5.2). Since $t \mapsto \overleftarrow{T}(t)$ is continuous, with $\overleftarrow{T}(\Theta(v)) = \varrho(A(H_v))$, we arrive at (noting that $[\beta_r] \le \gamma_r$):

$$
\begin{aligned}
\mathrm{I}_{(5.2)} &\le \sum_{i=2}^{4}(E_i^c) + \mathbb{P}\Big(\sup_{\varrho(A(H_r))\le u\le \varrho(A(H_s))} \frac{L_B(u,\mathbb{R}_+)}{L_B(u, A([\beta_r]))} > r^2\Big) \\
&\le \sum_{i=2}^{4}(E_i^c) + \mathbb{P}\Big(r^4\, A(\gamma_r) > A(H_r)\Big) \\
&\qquad + \mathbb{P}\Big(\sup_{\varrho(A(H_r))\le u\le \varrho(A(H_s))}\ \sup_{0\le x\le A(H_r)/r^4} \frac{L_B(u,\mathbb{R}_+)}{L_B(u,x)} > r^2\Big).
\end{aligned}
$$

We can apply Lemmas 4.2 and 4.4 respectively to the last two probability expressions on the right hand side, to see that

$$
\mathrm{I}_{(5.2)} \le \sum_{i=2}^{4}(E_i^c) + c_{20}\,\frac{\log r}{r} + c_{21}\frac{2\log r}{r^2}\Big(1 + \mathbb{E}\log\frac{A(H_s)}{A(H_r)}\Big).
$$

Since $\mathbb{E}\log A(H_r) \ge 0$ for large r, and since $A(H_s) \le \mathrm{e}^s H_s$, we have

$$
\mathbb{E}\log\frac{A(H_s)}{A(H_r)} \le s + \mathbb{E}\log H_s \le c_{25}\, s,
$$

the last inequality following from the scaling property. Consequently,

$$
\mathrm{I}_{(5.2)} \le \sum_{i=2}^{4}(E_i^c) + c_{26}\,\frac{s\log r}{r^2}.
$$

This, jointly considered with (4.8), (4.9) and Fact 4.1, yields (5.2).

The proof of (5.3) is along the same lines, using the fact that $W(t) - W(\beta_r) \geq (8c_{17} + 4 + \sigma)\sigma^{-1} \log r$ for all $\gamma_r \leq t \leq H_r$ (as for all $0 \leq t \leq \alpha_r$). $\square$

Now let us look at the supremum of X. By (3.2) and the occupation time formula, for any $t > 0$ and $v > 0$,

$$
\begin{aligned}
\Big\{ \sup_{0\leq s\leq t} X(s) > v \Big\} &= \Big\{ \int_0^{\varrho(A(v))} \exp\Big(-2V(A^{\leftarrow}(B(s)))\Big)\, \mathrm{d}s < t \Big\} \\
&= \Big\{ \int_{-\infty}^{A(v)} \mathrm{e}^{-2V(A^{\leftarrow}(y))} L_B(\varrho(A(v)), y)\, \mathrm{d}y < t \Big\} \\
&= \Big\{ \int_{-\infty}^{v} \mathrm{e}^{-V(z)} L_B(\varrho(A(v)), A(z))\, \mathrm{d}z < t \Big\},
\end{aligned}
$$

using a change of variable $y = A(z)$. Writing

$$
\begin{aligned}
\mathrm{I}_{(5.5)}(v) &\overset{\text{def}}{=} \int_0^{v} \mathrm{e}^{-V(s)} L_B(\varrho(A(v)), A(s))\, \mathrm{d}s, \\
\mathrm{II}_{(5.5)}(v) &\overset{\text{def}}{=} \int_0^{\infty} \mathrm{e}^{-V(-s)} L_B(\varrho(A(v)), A(-s))\, \mathrm{d}s,
\end{aligned}
\tag{5.5}
$$

we have,

$$
\Big\{ \sup_{0\leq s\leq t} X(s) > v \Big\} = \Big\{ \mathrm{I}_{(5.5)}(v) + \mathrm{II}_{(5.5)}(v) < t \Big\}. \tag{5.6}
$$

For brevity, we write

$$
\overline{W}(t) \overset{\text{def}}{=} \sup_{0\leq s\leq t} W(s), \tag{5.7}
$$

$$
W^{\#}(t) \overset{\text{def}}{=} \sup_{0\leq u\leq v\leq t} \big(W(v) - W(u)\big), \qquad t \geq 0. \tag{5.8}
$$

Define, for $r > 0$,

$$
H_r^- \overset{\text{def}}{=} \inf\Big\{ t > 0 : \, W(-t) > r \Big\},
$$

which is the first hitting time at (r, ∞) by $\{W(-t);\, t \geq 0\}$. Let

$$
U^-(r) \overset{\text{def}}{=} \Big| \inf_{0\leq s\leq H_r^-} W(-s) \Big| + r. \tag{5.9}
$$

The following estimate can be found in [17, Lemmas 4.1 and 4.2].

Fact 5.2. *Under (1.2) with $\sigma > 0$, for all sufficiently large v, we can find a measurable event $E(v)$, with $\mathbb{P}(E(v)) \geq 1 - \exp\big(-(\log v)^{3/2}\big)$, such that on $E(v) \cap \{\overline{W}(v) \geq 2(\log v)^4\}$,*

$$
\begin{aligned}
\sigma W^{\#}(v) - (\log v)^4 \leq \log \mathrm{I}_{(5.5)}(v) \leq \sigma W^{\#}(v) + (\log v)^4, \\
\sigma U^-\big(\overline{W}(v) - (\log v)^4\big) \leq \log \mathrm{II}_{(5.5)}(v) \leq \sigma U^-\big(\overline{W}(v) + (\log v)^4\big).
\end{aligned}
$$

Lemma 5.3. *Assume* (1.2) *and* $\sigma > 0$. *Almost surely for all large* v,

$$\sigma W^{\#}(v) - (\log v)^5 \le \log \mathrm{I}_{(5.5)}(v) \le \sigma W^{\#}(v) + (\log v)^5, \tag{5.10}$$

$$\sigma U^{-}\big(\overline{W}(v) - (\log v)^5\big) \le \log \mathrm{II}_{(5.5)}(v) \le \sigma U^{-}\big(\overline{W}(v) + (\log v)^5\big). \tag{5.11}$$

Proof. Apply Fact 5.2 and the Borel–Cantelli lemma (noting that $\overline{W}(v) \ge 2(\log v)^4$ is almost surely realized for all large v), to see that for all large integer n,

$$\sigma W^{\#}(n) - (\log n)^4 \le \log \mathrm{I}_{(5.5)}(n) \le \sigma W^{\#}(n) + (\log n)^4.$$

Let $v \in [n, n+1)$. By monotonicity,

$$\log \mathrm{I}_{(5.5)}(v) \ge \log \mathrm{I}_{(5.5)}(n) \ge \sigma W^{\#}(n) - (\log n)^4.$$

It is well-known (see Csörgő and Révész [8, p. 31]) that

$$\limsup_{n\to\infty} (2\log n)^{-1/2} \max_{0\le k\le n} \sup_{0\le s\le 1} |W(k+s) - W(k)| = 1, \qquad \text{a.s.}$$

Therefore, for large $v \in [n, n+1)$, $W^{\#}(v) - W^{\#}(n) \le (3\log n)^{1/2}$, which implies

$$\log \mathrm{I}_{(5.5)}(v) \ge \sigma W^{\#}(v) - \sigma(3\log n)^{1/2} - (\log n)^4 \ge \sigma W^{\#}(v) - (\log v)^5.$$

This yields the lower bound in (5.10). The rest of the lemma can be proved exactly in the same way. $\square$

Proof of Theorem 1.2. Recall H_r and U_r from (2.1) and (2.7) respectively. By definition, $W^{\#}(H_r) = U_r$. Since $U_b - U_a \ge b - a$ for all $b > a > 0$, by the first part of Lemma 5.3, almost surely for all large r,

$$\sigma U_{r/2} \le \log \mathrm{I}_{(5.5)}(H_r) \le \sigma U_{2r}. \tag{5.12}$$

On the other hand, $\overline{W}(H_r) = r$, which, in view of (5.11), yields that for large r,

$$\sigma U^{-}_{r/2} \le \log \mathrm{II}_{(5.5)}(H_r) \le \sigma U^{-}_{2r}. \tag{5.13}$$

By the occupation time formula and (5.6),

$$\mathrm{I}_{(5.5)}(v) + \mathrm{II}_{(5.5)}(v) = T(\varrho(A(v))), \tag{5.14}$$

which, in view of (5.1), implies

$$\mathrm{I}_{(5.5)}(H_r) + \mathrm{II}_{(5.5)}(H_r) = \Theta(r).$$

Using $\max(a,b) \le a+b \le 2\max(a,b)$ (for positive a and b), and in light of (5.12)–(5.13), we arrive at: almost surely for all large r,

$$\sigma \max\big(U_{r/2},\, U^-_{r/2}\big) \le \log\Theta(r) \le \sigma \max\big(U_{3r},\, U^-_{3r}\big). \tag{5.15}$$

Another observation is that by Lemma 2.1, for all large r,

$$|\alpha_r - \beta_r| \le \frac{r^2}{(\log r)^4}, \qquad |\gamma_r - \beta_r| \le \frac{r^2}{(\log r)^4}. \tag{5.16}$$

Let $u_j \stackrel{\text{def}}{=} j^2$ for all $j \ge 1$. Applying Lemma 5.1 to $r = u_j$ and $s = u_{j+1}$, and by virtue of the Borel–Cantelli lemma, we have, almost surely for all large j,

$$\max_{0\le k\le \alpha(u_j)} L_X(t,k) \le u_j^{-2} L_X(t,\mathbb{Z}_+), \qquad t \in [\Theta(u_j), \Theta(u_{j+1})], \tag{5.17}$$

$$\max_{k\ge \gamma(u_j)} L_X(\Theta(u_j),k) \le u_j^{-2} L_X(\Theta(u_j),\mathbb{Z}_+), \tag{5.18}$$

where $L_X(t,\mathbb{Z}_+) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{Z}_+} L_X(t,x)$. Let $\{\mu_n\}_{n\ge 0}$ be the sequence defined in (3.7). For large n, there exists a unique $j = j(n,\omega)$ such that $\mu_n \in [\Theta(u_j), \Theta(u_{j+1}))$. By (3.8),

$$\begin{aligned}\max_{0\le k\le \alpha(u_j)} 2\xi_k L(n,k) &\le 30\sqrt{1+L(n,\mathbb{Z}_+)}\,\log n + \max_{0\le k\le \alpha(u_j)} L_X(\mu_n,k)\\ &\le 30\sqrt{1+L(n,\mathbb{Z}_+)}\,\log n + u_j^{-2} L_X(\mu_n,\mathbb{Z}_+),\end{aligned}$$

the second inequality following from (5.17), with the notation $L(n,\mathbb{Z}_+) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{Z}_+} L(n,x)$. Applying (3.8) once more to see that the above is smaller than (noting that $\xi_k \le 1$)

$$60\sqrt{1+L(n,\mathbb{Z}_+)}\,\log n + 2u_j^{-2} L(n,\mathbb{Z}_+).$$

It is known (see Révész [30, p. 292]) that

$$\lim_{n\to\infty} \frac{(\log\log n)^2}{\log n} \log L(n,\mathbb{Z}_+) = \infty, \qquad \text{a.s.} \tag{5.19}$$

By (1.2), ξ_k is bounded below by a positive constant, this yields

$$\max_{0\le k\le \alpha(u_j)} L(n,k) < L(n,\mathbb{Z}_+), \tag{5.20}$$

i.e. for $\mu_n \in [\Theta(u_j), \Theta(u_{j+1}))$,

$$F(n) > \alpha(u_j).$$

Therefore, by (3.9),

$$\frac{F(n)}{(\log n)^2} \geq \frac{1}{2} \frac{\alpha(u_j)}{(\log \mu_n)^2} \geq \frac{1}{2} \frac{\alpha(u_j)}{(\log \Theta(u_{j+1}))^2}.$$

According to (5.16) and (5.15), this yields

$$\begin{aligned}\frac{F(n)}{(\log n)^2} &\geq \frac{1}{2} \frac{\beta(u_j) - u_j^2/(\log u_j)^4}{\sigma^2 \big(\max(U_{3u_{j+1}}, U^-_{3u_{j+1}})\big)^2} \\ &\geq \frac{1}{2\sigma^2} \frac{\beta(u_j) - u_j^2/(\log u_j)^4}{\big(\max(U_{4u_j}, U^-_{4u_j})\big)^2} \\ &\geq \frac{1}{3\sigma^2} \frac{\beta(u_j)}{\big(\max(U_{4u_j}, U^-_{4u_j})\big)^2}, \qquad (5.21)\end{aligned}$$

the last inequality following from Remark 2.6.

Let $a_n > 1$ be a non-decreasing sequence such that $\sum_n (\log a_n)/(n\sqrt{a_n}\,\log n) < \infty$. Let $\varepsilon > 0$, and define the function

$$g(r) \stackrel{\text{def}}{=} \varepsilon\, a_{[\exp(\sigma r/3)]}, \qquad r \geq 1.$$

Then $\int^\infty (\log g(r))/(r\sqrt{g(r)}\,)\,\mathrm{d}r < \infty$. By Lemma 2.5, for all large r,

$$\frac{\beta(r)}{\big(\max(U_{4r}, U^-_{4r})\big)^2} \geq \frac{3\sigma^2}{g(r)},$$

which, in view of (5.21), yields that, for all large n with $\mu_n \in [\Theta(u_j), \Theta(u_{j+1}))$,

$$\frac{F(n)}{(\log n)^2} \geq \frac{1}{g(u_j)}.$$

Since $\log \mu_n \geq \log \Theta(u_j)$ which, according to (5.15), is greater than $\sigma u_j/2$, it follows from (3.9) that $\log n \geq \sigma u_j/3$. Therefore,

$$\frac{F(n)}{(\log n)^2} \geq \frac{1}{g(3\sigma^{-1}\log n)} = \frac{1}{\varepsilon\, a_n}.$$

This yields the convergent part of Theorem 1.2.

To prove the divergent part of the theorem, consider a non-decreasing sequence $a_n > 1$ such that $\sum_n (\log a_n)/(n\sqrt{a_n}\,\log n) = \infty$. Let $\varepsilon > 0$, and define the function

$$h(r) \stackrel{\text{def}}{=} \frac{3}{\varepsilon}\, a_{[\exp(32r^2)]}, \qquad r \geq 1.$$

Then $\int^\infty (\log h(r))/(r\sqrt{h(r)})\,dr = \infty$. Applying Lemma 2.5 to $\theta = 4$ yields that, there are infinitely many m satisfying

$$\frac{\beta(2^{m+2})}{\left(\max(U_{2^m},\, U^-_{2^m})\right)^2} \leq \frac{1}{h(2^m)}. \tag{5.22}$$

We are now working only with these m satisfying (5.22). Let $j = j(m)$ be such that $u_j \in [2^{m+1}, 2^{m+2})$. There exists a random index $n = n(m)$ such that $\Theta(u_j) \in [\mu_n, \mu_{n+1})$. As in the proof of (5.20), using (5.18) instead of (5.17), we can see that

$$\max_{k \geq \gamma(u_j)} L(n,k) < L(n, \mathbb{Z}_+).$$

Therefore, by (3.9),

$$\frac{F(n)}{(\log n)^2} \leq \frac{\gamma(u_j)}{(\log n)^2} \leq 2\frac{\gamma(u_j)}{(\log \mu_{n+1})^2} \leq 2\frac{\gamma(u_j)}{(\log \Theta(u_j))^2}.$$

Applying (5.16), (5.15) and Remark 2.6 yields

$$\frac{F(n)}{(\log n)^2} \leq 3\frac{\beta(u_j)}{\sigma^2\left(\max(U_{u_j/2}, U^-_{u_j/2})\right)^2},$$

which, according to (5.22), yields

$$\frac{F(n)}{(\log n)^2} \leq \frac{3}{h(u_j/4)}. \tag{5.23}$$

By (2.7) and the laws of the iterated logarithm (see Révész [30, p. 53]), almost surely for all large r,

$$U(r) \leq r + (3H_r \log\log H_r)^{1/2} \leq r(\log r)^2, \tag{5.24}$$

which, in light of (5.15), implies $\log\Theta(r) \leq r^2$. Since $\Theta(u_j) \geq \mu_n$, and $\mu_n/n \to 1$ (see (3.9)), this yields $\log n \leq 2u_j^2$. Going back to (5.23), we have,

$$\frac{F(n)}{(\log n)^2} \leq \frac{3}{h(2^{-1/2}(\log n)^{1/2}/4)} = \frac{\varepsilon}{a_n},$$

completing the proof of the divergent part of Theorem 1.2. □

6. Proof of Theorem 1.1

Throughout the section, $\{S_n\}_{n\ge 0}$ denotes a simple RWRE, whose associated random environment $\Xi = \{\xi_j\}_{j\in\mathbb{Z}}$ satisfies (1.2), see Section 1. Let σ be as in (1.3), with $\sigma > 0$. We first recall the following law of the iterated logarithm (see [17, Theorem 1.3]):

Fact 6.1. *Under* (1.2), *if* $\sigma > 0$,

$$\limsup_{n\to\infty} \frac{\max_{0\le k\le n} S_k}{(\log n)^2 \log\log\log n} = \frac{8}{\pi^2\sigma^2}, \qquad \text{a.s.}$$

Let $L(n,x)$ be the local time process of RWRE, with $F(n)$ the favourite site up to time n (see (1.5)). In view of the trivial relation $F(n) \le \max_{0\le k\le n} S_k$, the upper bound in Theorem 1.1 immediately follows from Fact 6.1.

To show the lower bound, we again make use of the diffusion model. From the random environment $\Xi = \{\xi_j\}_{j\in\mathbb{Z}}$, we can define a partial sum process $\{V(x);\, x\in\mathbb{R}\}$ via (3.6), and a diffusion process $\{X(t);\, t\ge 0\}$ with potential V, driven by a Brownian motion B which is independent of Ξ, see (3.2).

Let $\{W(x);\, x\in\mathbb{R}\}$ be the Komlós–Major–Tusnády Brownian motion satisfying (4.2), independent of B. Let (A,T) be as in (3.3)–(3.4). Recall that ϱ is the first hitting time process associated with B, and that L_X is the local time of X, see (4.11) and (3.5) respectively. Fix $\varepsilon > 0$, and define, for large v,

$$E_5(v) \stackrel{\text{def}}{=} \Big\{ \sup_{0\le x\le (1-\varepsilon)v} L_X(T(\varrho(A(v))),x) < \frac{1}{v}\max_{x\in\mathbb{Z}_+} L_X(T(\varrho(A(v))),x) \Big\} \tag{6.1}$$

$$E_6(v) \stackrel{\text{def}}{=} \Big\{ \log T(\varrho(A(v))) \le \frac{(1+4\varepsilon)\sigma\pi}{\sqrt{8}}\,\frac{\sqrt{v}}{\sqrt{\log\log v}} \Big\} \tag{6.2}$$

Assume for the moment that we could show

$$\mathbb{P}\Big(E_5(v)\cap E_6(v), \text{ i.o.}\Big) = 1. \tag{6.3}$$

Let $\omega\in E_5(v)$. There exists a unique index $n = n(v,\omega)$ such that $\mu_n \le T(\varrho(A(v))) < \mu_{n+1}$, where $\{\mu_k\}_{k\ge 0}$ is the sequence defined in (3.7).

By (3.8) and (6.1) (writing again $L(n,\mathbb{Z}_+) \stackrel{\text{def}}{=} \max_{x\in\mathbb{Z}_+} L(n,x)$),

$$\begin{aligned}\max_{0\le k\le (1-\varepsilon)v} 2\xi_k L(n,k) &\le 30\sqrt{1+L(n,\mathbb{Z}_+)}\,\log n + \max_{0\le k\le (1-\varepsilon)v} L_X(\mu_n,k)\\ &\le 30\sqrt{1+L(n,\mathbb{Z}_+)}\,\log n + v^{-1}L_X(\mu_{n+1},\mathbb{Z}_+).\end{aligned}$$

Using (3.8) once more gives

$$\max_{0\le k\le (1-\varepsilon)v} 2\xi_k L(n,k) \le 60\sqrt{1+L(n,\mathbb{Z}_+)}\,\log n + 2v^{-1}L(n,\mathbb{Z}_+).$$

In view of (5.19) and of the boundedness of ξ_k guaranteed by (1.2), we would have

$$\max_{0\le k\le (1-\varepsilon)v} L(n,k) < L(n,\mathbb{Z}_+),$$

i.e. we would have $F(n)\ge (1-\varepsilon)v$. (These lines really are rewritings of the proof leading to (5.20)).

By (3.9), on $E_6(v)$,

$$\log n \le (1+\varepsilon)\log \mu_n \le (1+\varepsilon)\log T(\varrho(A(v))) \le \frac{(1+6\varepsilon)\sigma\pi\sqrt{v}}{\sqrt{8\log\log v}}.$$

Therefore, if $\omega\in E_5(v)\cap E_6(v)$, we would have, for the random index $n=n(v,\omega)$,

$$F(n)\ge \frac{1-\varepsilon}{(1+6\varepsilon)^2}\,\frac{8}{\sigma^2\pi^2}\,(\log n)^2\log\log\log n,$$

i.e. by assuming (6.3), we would obtain the the lower bound in Theorem 1.1.

The rest of this section is devoted to the proof of (6.3). For brevity, write

$$W^\#(u,v) \overset{\text{def}}{=} \sup_{0\le x\le u;\ x\le y\le v} (W(y)-W(x)), \qquad 0\le u\le v.$$

Recall the definitions of $\overline{W}(t)$ and $W^\#(t)$ from (5.7) and (5.8) respectively (thus $W^\#(u) = W^\#(u,u)$). The proof of (6.3) is based on the following two lemmas.

Lemma 6.2. *Assume* (1.2) *with* $\sigma>0$. *Fix* $\varepsilon\in(0,1)$. *Almost surely for all sufficiently large* v,

$$\left|\log \sup_{0\le x\le (1-\varepsilon)v} L_X(T(\varrho(A(v))),x) - \sigma W^\#((1-\varepsilon)v,v)\right| \le (\log v)^5, \tag{6.4}$$

$$\left|\log \max_{x\in\mathbb{Z}_+} L_X(T(\varrho(A(v))),x) - \sigma W^\#(v)\right| \le (\log v)^5. \tag{6.5}$$

Proof of Lemma 6.2. Write $\mathrm{I}_{(6.4)}$ and $\mathrm{I}_{(6.5)}$ respectively for the expressions on the left hand side of (6.4) and (6.5). According to [17, Remark 6.2], for any $\varepsilon\in(0,1)$, there exists $c_{27}>0$ such that for all large v,

$$\mathbb{P}\big(\mathrm{I}_{(6.4)} \ge (\log v)^4\big) \le c_{27}\exp\big(-(\log v)^2\big),$$
$$\mathbb{P}\big(\mathrm{I}_{(6.5)} \ge (\log v)^4\big) \le c_{27}\exp\big(-(\log v)^2\big).$$

The lemma now follows from the Borel–Cantelli lemma and the monotonicity, using the same argument as in the proof of Lemma 5.3. $\square$

Lemma 6.3. *Fix $0 < \varepsilon < 1/30$. There exists a constant $c_{28} > 0$, depending on ε, such that for all $t > 0$ and $0 < x < \sqrt{t}$,*

$$\mathbb{P}\Big(W^{\#}((1-\varepsilon)t, t) < (1-\varepsilon)x;\ (1-\frac{\varepsilon}{3})x < W^{\#}(t) < x;\ \overline{W}(t) < \frac{x}{5} \Big) \geq c_{28} \exp\Big(- \frac{(1+5\varepsilon)\pi^2 t}{8x^2} \Big).$$

Proof of Lemma 6.3. By scaling, it suffices to treat the case $t = 1$. We clearly only have to deal with small x. Let

$$\widehat{W}(t) \stackrel{\text{def}}{=} W(t+1-\varepsilon) - W(1-\varepsilon), \qquad t \geq 0,$$

which is again a Brownian motion, independent of $\{W(s);\ 0 \leq s \leq 1-\varepsilon\}$. We can define $\widehat{W}^{\#}$ and $\overline{\widehat{W}}$ for $\widehat{W}$ exactly in the same way as $W^{\#}$ and $\overline{W}$ for W. Observe that

$$\begin{aligned} W^{\#}(1-\varepsilon, 1) &= \max\big(W^{\#}(1-\varepsilon),\ \overline{\widehat{W}}(\varepsilon) + W(1-\varepsilon) - \inf_{0 \leq s \leq 1-\varepsilon} W(s)\big), \\ \overline{W}(1) &= \max\big(\overline{W}(1-\varepsilon),\ \overline{\widehat{W}}(\varepsilon) + W(1-\varepsilon)\big). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{P}\Big(W^{\#}(1-\varepsilon, 1) < (1-\varepsilon)x;\ (1-\frac{\varepsilon}{3})x < W^{\#}(1) < x;\ \overline{W}(1) < \frac{x}{5} \Big) \\ &\geq \mathbb{P}\Big(W^{\#}(1-\varepsilon) < (1-2\varepsilon)x;\ \overline{W}(1-\varepsilon) \leq \frac{x}{6}; \\ &\qquad (1-\frac{\varepsilon}{3})x < \widehat{W}^{\#}(\varepsilon) < x;\ \overline{\widehat{W}}(\varepsilon) < \varepsilon x \Big) \\ &= \mathbb{P}\Big(W^{\#}(1-\varepsilon) < (1-2\varepsilon)x;\ \overline{W}(1-\varepsilon) \leq \frac{x}{6} \Big) \times \\ &\qquad \times \mathbb{P}\Big((1-\frac{\varepsilon}{3})x < W^{\#}(\varepsilon) < x;\ \overline{W}(\varepsilon) < \varepsilon x \Big). \end{aligned} \tag{6.6}$$

The joint law of $(W^{\#}, \overline{W})$ at fixed time has been studied in [17]. In particular, for fixed $0 < a < 1$ and $s > 0$,

$$\mathbb{P}\Big(W^{\#}(s) < y,\ \overline{W}(s) < ay \Big) \sim \frac{4}{\pi} \sin(\frac{\pi a}{2}) \exp\Big(- \frac{\pi^2 s}{8y^2} \Big), \qquad y \to 0.$$

Applying this to both probability expressions on the right hand side of (6.6) (and noting that $(1-\varepsilon)(1-2\varepsilon)^{-2} < 1 + 4\varepsilon$) yields the lemma. $\square$

Proof of (6.3). Fix a small $\varepsilon > 0$. Let U^- be the process introduced in (5.9), and define

$$E_7(v) \stackrel{\text{def}}{=} \Big\{ W^\#((1-\varepsilon)v, v) < W^\#(v) - (\log v)^6;\ W^\#(v) < \frac{(1+3\varepsilon)\pi\sqrt{v}}{\sqrt{8\log\log v}};\\ U^-\big(\,\overline{W}(v) + (\log v)^5\big) < W^\#(v)\ \Big\}.$$

It is an immediate consequence of (5.14), Lemmas 5.3 and 6.2 that almost surely for all large v,

$$E_7(v) \subset E_5(v) \cap E_6(v). \tag{6.7}$$

Let

$$\begin{aligned} v_j &\stackrel{\text{def}}{=} \exp(j^{1+\varepsilon}),\\ x_j &\stackrel{\text{def}}{=} \frac{(1+3\varepsilon)\pi\sqrt{v_j}}{\sqrt{8\log\log v_j}},\\ G(j) &\stackrel{\text{def}}{=} \Big\{ W^\#((1-\varepsilon)v_j, v_j) < (1-\varepsilon)x_j;\ (1-\frac{\varepsilon}{3})x_j \le W^\#(v_j) < x_j;\\ &\qquad \overline{W}(v_j) < \frac{x_j}{4};\ U^-\big(\frac{x_j}{3}\big) < \frac{x_j}{2} \Big\}. \end{aligned}$$

Since $G(j) \subset E_7(v_j)$, and in view of (6.7), the proof of (6.3) is reduced to showing the following:

$$\mathbb{P}\big(G(j),\ \text{i.o.}\big) = 1. \tag{6.8}$$

To this end, write $\mathcal{F}_j \stackrel{\text{def}}{=} \sigma\{W(s), 0 \le s \le v_j;\ W(-t), 0 \le t \le x_j/3\}$. Observe that $G(j)$ is $\mathcal{F}_j$–measurable. By (5.9),

$$U^-\big(\frac{x_j}{3}\big) = \max\Big(U^-\big(\frac{x_{j-1}}{3}\big) + \frac{x_j - x_{j-1}}{3},\ Z_j\Big), \tag{6.9}$$

where Z_j is a variable having the same distribution as $U^-((x_j - x_{j-1})/3)$, and is independent of $\mathcal{F}_{j-1}$ and $\{W(t);\ t \ge 0\}$. Let

$$\widetilde{W}(t) \stackrel{\text{def}}{=} W(t + v_{j-1}) - W(v_{j-1}), \qquad t \ge 0,$$

which is again a Brownian motion. We can define $\overline{\widetilde{W}}$ and $\widetilde{W}^\#$ in the obvious way. Write $\Delta_j \stackrel{\text{def}}{=} v_j - v_{j-1}$. Since $v_{j-1} < (1-\varepsilon)v_j$, we have

$$\begin{aligned} W^\#((1-\varepsilon)v_j, v_j) &\le \max\big(W^\#(v_{j-1}) + \overline{\widetilde{W}}(\Delta_j),\ \widetilde{W}^\#((1-\varepsilon)v_j - v_{j-1}, \Delta_j)\big)\\ &\le \max\big(W^\#(v_{j-1}) + \overline{\widetilde{W}}(\Delta_j),\ \widetilde{W}^\#((1-\varepsilon)\Delta_j, \Delta_j)\big), \end{aligned}$$

and also

$$\overline{W}(v_j) \le \overline{W}(v_{j-1}) + \overline{\widetilde{W}}(\Delta_j).$$

In light of (6.9), this gives

$$G_1(j) \cap G_2(j) \subset G(j),$$

where

$$G_1(j) \stackrel{\text{def}}{=} \Big\{ W^{\#}(v_{j-1}) < \frac{\varepsilon\, x_j}{5};\ U^{-}(\frac{x_{j-1}}{3}) \le \frac{x_j}{6} \Big\},$$

$$G_2(j) \stackrel{\text{def}}{=} \Big\{ \widetilde{W}^{\#}((1-\varepsilon)\Delta_j, \Delta_j) < (1-\varepsilon)x_j;\ \overline{\widetilde{W}}(\Delta_j) < \frac{x_j}{5};$$
$$(1-\frac{\varepsilon}{3})x_j < \widetilde{W}^{\#}(\Delta_j) < x_j;\ Z_j < \frac{x_j}{2} \Big\}.$$

Observe that $G_1(j)$ is $\mathcal{F}_{j-1}$–measurable, and that both $\widetilde{W}$ and Z_j is independent of $\mathcal{F}_{j-1}$. Accordingly,

$$\begin{aligned} \mathbb{P}\big(G(j)\,|\,\mathcal{F}_{j-1}\big) &\ge \mathbb{1}_{G_1(j)}\, \mathbb{P}(G_2(j)) \\ &= \mathbb{1}_{G_1(j)}\, \mathbb{P}\Big(\ Z_j < \frac{x_j}{2}\Big) \ \times \mathrm{I}_{(6.10)}, \end{aligned} \tag{6.10}$$

where

$$\mathrm{I}_{(6.10)} \stackrel{\text{def}}{=} \mathbb{P}\Big(\ W^{\#}((1-\varepsilon)\Delta_j, \Delta_j) < (1-\varepsilon)x_j;\ \overline{W}(\Delta_j) < \frac{x_j}{5};$$
$$(1-\frac{\varepsilon}{3})x_j < W^{\#}(\Delta_j) < x_j\Big).$$

Applying Lemma 6.3 to $t = \Delta_j$ and $x = x_j$ gives

$$\begin{aligned} \mathrm{I}_{(6.10)} &\ge c_{28}\ \exp\Big(-\frac{1+5\varepsilon}{(1+3\varepsilon)^2}\ \frac{v_j - v_{j-1}}{v_j}\ \log\log v_j\Big) \\ &\ge c_{28}\ \exp\Big(-\frac{\log\log v_j}{1+\varepsilon}\Big) \\ &= \frac{c_{28}}{j}. \end{aligned}$$

On the other hand, $\mathbb{P}(U^{-}(a) < b) = (b-a)/b$ for all $0 < a < b$, and Z_j is distributed as $U^{-}((x_j - x_{j-1})/3)$. Therefore, (6.10) leads to:

$$\mathbb{P}\big(G(j)\,|\,\mathcal{F}_{j-1}\big) \ge c_{29}\, j^{-1}\, \mathbb{1}_{G_1(j)}. \tag{6.11}$$

By the usual law of the iterated logarithm, almost surely for all large j,

$$W^{\#}(v_{j-1}) \le 2\sqrt{3v_{j-1}\log\log v_{j-1}} < \frac{\varepsilon\, x_j}{5},$$

whereas by (5.24) (noting that U^- has the same law as U),

$$U^-(\frac{x_{j-1}}{3}) \le \frac{x_{j-1}}{3} \left(\log x_{j-1}\right)^2 < \frac{x_j}{6}.$$

Consequently, $\mathbb{1}_{G_1(j)} = 1$ almost surely for all large j. Going back to (6.11), we have

$$\sum_j \mathbb{P}\big(G(j) \,|\, \mathcal{F}_{j-1}\big) = \infty, \qquad \text{a.s.,}$$

which, according to Lévy's Borel–Cantelli lemma (see [34, p. 518]), implies (6.8). This completes the proof of (6.3). $\square$

7. Favourite sites of diffusions with random potentials

Our proofs of Theorems 1.1 and 1.2 clearly work directly for favourite sites of diffusions with random potentials. Let $\{X(t);\, t \ge 0\}$ be a diffusion process with random potential V, as in (3.1)–(3.4). Its local time L_X is defined in (3.5). As for RWRE, we can define

$$\mathbb{F}_X(t) \stackrel{\text{def}}{=} \Big\{x \in \mathbb{R}_+ \,:\, L_X(t,x) = \sup_{y\in\mathbb{R}_+} L_X(t,y)\Big\},$$

and the favourite site

$$F_X(t) = \max_{x\in\mathbb{F}_X(t)} x.$$

The arguments in Sections 5 and 6 yield the following counterparts of Theorems 1.1 and 1.2 for X:

Theorem 7.1. *If* (4.2) *is satisfied with some constant* $\sigma > 0$, *then*

$$\limsup_{t\to\infty} \frac{F_X(t)}{(\log t)^2 \log\log\log t} = \frac{8}{\pi^2\sigma^2}, \qquad \text{a.s.}$$

Theorem 7.2. *Under* (4.2), *for any non-decreasing function* $f > 1$,

$$\liminf_{t\to\infty} \frac{f(t)}{(\log t)^2} F_X(t) = \begin{cases} 0 \\ \infty \end{cases}, \quad \text{a.s.} \quad \Longleftrightarrow \quad \int^\infty \frac{\log f(t)}{t\sqrt{f(t)}\,\log t}\,\mathrm{d}t \begin{cases} = \infty \\ < \infty \end{cases}.$$

Remark 7.3. In the particular case when X is Brox's diffusion process with Brownian potential (i.e. when V is Brownian motion), (4.2) trivially holds with $\sigma = 1$, and hence Theorems 7.1 and 7.2 apply.

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