A note on the empty balls left by a critical branching Wiener process

Dedicated to Professors Endre Csáki and Pál Révész on the occasion of their 70-th birthday

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Abstract. In this note, we partially confirm some conjectures of P. Révész [10] on the critical branching Wiener process.

1 Introduction

The spatial branching process is one of the simplest models that describe a system of particles combining branching property with spatial motion. We consider here a critical branching Wiener process which is denoted by $(Z_n, n \geq 0)$. At time 0, Z_0 is a Poisson point measure on $\mathbb{R}^d$ whose intensity is the Lebesgue measure: for any measurable $A \subset \mathbb{R}^d$,

$$\mathbb{P}\Big(\#\{ \text{ points of } Z_0 \text{ fall in A } \} = k\Big) = \frac{|A|^k}{k!} e^{-|A|}, \qquad k \geq 0,$$

where $|A|$ denotes the Lebesgue measure of A. Every point of Z_0 is associated with a particle which moves, independently each from other, according to the following rules:

- a particle starts from $x \in \mathbb{R}^d$ and executes a d-dimensional Wiener process during an unit time;
- arriving at the new location at time 1 the particle dies and gives offsprings:

$$\mathbb{P}\Big(\# \text{ offsprings } = 0\Big) = \mathbb{P}\Big(\# \text{ offsprings } = 2\Big) = \frac{1}{2};$$

- each offspring, if exists, starting from where its ancestor died, executes an independent d-dimensional Wiener process and repeats the above steps and so on. All Wiener processes and offspring numbers are assumed to be independent;
- there is no collision between particles.

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Denote by $\lambda(n,x)$ the number of particles living at time n and at position $x \in \mathbb{R}^d$. The process Z_n taking values in positive measures, is defined by

$$Z_n = \sum_x \lambda(n,x)\delta_{\{x\}},$$

the above sum makes sense because there are only countable $x \in \mathbb{R}^d$ such that $\lambda(n,x) > 0$.

The measures-valued process Z_n is called a critical branching Wiener process. The above model and more generally the branching random fields were presented and studied in detail by Révész [9] and his book [7]. Let us also mention some recent references in various settings: Kesten [5] and Révész [8] (critical case), Chen [3], Révész [11] and Révész, Rosen and Shi [12] (supercritical case), and Csáki, Révész and Shi [4] (coalescing random walk).

This note is devoted to the studies of the asymptotic behaviors of Z_n, more precisely the empty balls left by (Z_n). We aim at two conjectures arisen in Révész [10]. Let $\alpha > 0$ and define

$$\begin{aligned}
\mathcal{B}(x,r) &\stackrel{\text{def}}{=} \{y \in \mathbb{R}^d : |y-x| \le r\}, \\
\mathcal{B}(r) &\stackrel{\text{def}}{=} \mathcal{B}(0,r), \\
R(n) &\stackrel{\text{def}}{=} \sup\{r > 0 : \langle Z_n, 1_{\mathcal{B}(r)}\rangle = 0\}, \\
R(\alpha,n) &\stackrel{\text{def}}{=} \sup\{0 < r < n^\alpha : \exists\, x \in \mathcal{B}(n^\alpha - r), \langle Z_n, 1_{\mathcal{B}(x,r)}\rangle = 0\}.
\end{aligned}$$

In other words, $R(n)$ is the radius of the largest ball around the origin which does not contain any particles at time n and $R(\alpha,n)$ is the radius of the largest empty ball contained in $\mathcal{B}(n^\alpha)$ at time n. Let us quote the following results in the two-dimensional case (Révész [9], Theorem 6.3 and Révész [10], Theorem 4):

Theorem A (Révész [9] and [10]) *Let $d = 2, \alpha > \frac{1}{2}$. We have*

$$\limsup_{n\to\infty} \frac{R(n)}{\sqrt{n \log\log n}} = (2\pi)^{-1/2}, \qquad \text{a.s.} \tag{1.1}$$

$$\frac{2\alpha-1}{2\pi} \le \liminf_{n\to\infty} \frac{R(\alpha,n)}{\sqrt{n\log n}} \le \limsup_{n\to\infty} \frac{R(\alpha,n)}{\sqrt{n\log n}} \le \frac{\alpha}{\pi}, \qquad \text{a.s.} \tag{1.2}$$

The above results show a clear image on the almost surely asymptotic behaviors of $R(n)$ and $R(\alpha,n)$, moreover it has been conjectured by Révész [10] that the limit (instead of $\liminf$ and $\limsup$) in (1.2) should exist and equal $\frac{2\alpha-1}{2\pi}$.

We confirm this conjecture:

Theorem 1.1 *Let $d = 2$ and $\alpha > \frac{1}{2}$. We have*

$$\lim_{n\to\infty} \frac{R(\alpha,n)}{\sqrt{n\log n}} = \frac{2\alpha-1}{2\pi}, \qquad \text{a.s.}$$

Concerning on the typical values of $R(n)$, Révész [10] conjectured that

$$\frac{R(n)}{\sqrt{n}} \qquad \text{converges in law when } d = 2, \tag{1.3}$$

$$R(n) \qquad \text{converges in law when } d \geq 3. \tag{1.4}$$

We partially confirm the above conjecture by affirming (1.4):

Theorem 1.2 *Let $d \geq 3$. The radius $R(n)$ converges in law when $n \to \infty$ to some non-degenerated law.*

The above result holds for more general branching mechanisms. Le Gall [6] has shown that the renormalized spatial branching processes converge to the superprocesses. It would be an interesting question to evaluate the limit law, for instance through the superprocess. We also mention Bertoin, Le Gall and Le Jan [2] where they considered general spatial branching processes whose branching times are independent exponential variables. Exploiting the Markov structure of (Z_n), we shall prove Theorem 1.2 in Section 2, whereas the proof of Theorem 1.1 is given in Section 3 by a direct analysis of (Z_n).

2 Convergence in law

We consider a more general branching mechanism in this section. Let $(p_k)_{k\geq 0}$ be a probability measure on $\{0, 1, ...\}$ such that $p_0 > 0$, $\sum_{k=0}^{\infty} k p_k = 1$ and $\sum_n n^2 p_n < \infty$. Defining in the same way as in Introduction the branching Wiener process (Z_n) excepted from the branching rule: a particle dies and gives offsprings according to the probability that $\mathbb{P}(\#\text{ offsprings } = k) = p_k$ for $k \geq 0$. Denote by $\mathcal{M}(\mathbb{R}^d)$ the space of point measures on $\mathbb{R}^d$. Then the process (Z_n) is a Markov process taking values in $\mathcal{M}(\mathbb{R}^d)$. For every $\nu \in \mathcal{M}(\mathbb{R}^d)$, we denote by $\mathbb{P}_\nu$ the law of (Z_n) starting from $Z_0 = \nu$; in particular, (Z_n) under $\mathbb{P}_{Poisson}$ or simply $\mathbb{P}$ means that Z_0 is a Poisson point measure with Lebesgue measure as its intensity. We write $(W, P_x, x \in \mathbb{R}^d)$ to mean that W is a Wiener process in $\mathbb{R}^d$ starting from x under the probability measure P_x.

The following result, whose statement is inspired from Bertoin, Le Gall and Le Jan [2], is an easy consequence of the branching mechanism and the Markov property of (Z_n):

Lemma 2.1 *For a measurable function $g : \mathbb{R}^d \to \mathbb{R}_+$, we define $f(x) = e^{-g(x)}$ and*

$$\Phi_n(f)(x) = \mathbb{E}_{\delta_x} \exp\big(- \langle Z_n, g\rangle\big), \qquad x \in \mathbb{R}^d,\, n \geq 0,$$

where $\langle Z_n, g\rangle$ denotes the integral of g with respect to the point measure Z_n and δ_x the Dirac measure on x. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ and $n \geq m \geq 0$. We have

$$\mathbb{E}_{\mu+\nu} e^{-\langle Z_n, g\rangle} \;=\; \mathbb{E}_\mu e^{-\langle Z_n, g\rangle}\, \mathbb{E}_\nu e^{-\langle Z_n, g\rangle}, \tag{2.1}$$

$$\mathbb{E}_\mu\Big(e^{-\langle Z_n,g\rangle}|\sigma\{Z_k,k\le m\}\Big) = e^{\langle Z_m,\log\Phi_{n-m}(f)\rangle}, \quad (2.2)$$

$$\Phi_{n+k}(f) = \Phi_n(\Phi_k(f)), \quad n,k\ge 0, \quad (2.3)$$

$$\mathbb{E}_{Poisson}e^{-\langle Z_n,g\rangle} = \exp\Big(-\int_{\mathbb{R}^d}dx(1-\Phi_n(f)(x))\Big), \quad (2.4)$$

$$\Phi_1(f)(x) = \int_{\mathbb{R}^d}dy\, q(x,y)\,\Pi(f(y)), \quad (2.5)$$

where $q_n(x,y)\stackrel{\text{def}}{=}P_x(W_n\in dy)/dy=(2\pi n)^{-d/2}e^{-|x-y|^2/(2n)}$ *denotes the density of Brownian semigroup,* $q(x,y)=q_1(x,y)$ *and* Π *denotes the generating function:* $\Pi(v)\stackrel{\text{def}}{=}\sum_{n\ge 0}p_n v^n, 0\le v\le 1$*.*

Proof: The equalities (2.1) and (2.2) follow from the branching property and from the Markov property of Z respectively. Taking the expectation on both two sides of (2.2) we obtain (2.3). By means of (2.1)

$$\mathbb{E}_{Poisson}e^{-\langle Z_n,g\rangle}=\mathbb{E}_{Poisson}e^{\langle Z_0,\log\Phi_n(f)\rangle}$$

which shows (2.4) according to the exponential formula for a Poisson point measure. Finally (2.5) follows from the definition of Z_1:

$$\Phi_1(f)(x) = \mathbb{E}_{\delta_x}e^{-\langle Z_1,g\rangle}=E_x\sum_{n=0}^\infty p_n\,e^{-ng(W_1)}=E_x\Big(\Pi(f(W_1))\Big),$$

yielding (2.5). □

By virtue of (2.4), we obtain that

$$\mathbb{P}_{Poisson}\Big(R_n\ge r\Big)=\exp\Big(-\int_{\mathbb{R}^d}dx\mathbb{P}_{\delta_x}(R_n<r)\Big), \qquad r>0,\, n\ge 0. \quad (2.6)$$

We now give the proof of Theorem 1.2:

Proof of Theorem 1.2: Observe that $\{R_n\ge r\}=\{\langle Z_n,1_{\mathcal{B}(r)}\rangle=0\}$, where $\mathcal{B}(r)$ denotes the ball centered at the origin and of radius r. It follows that

$$\begin{aligned}\mathbb{P}_{\delta_x}(R_n\ge r) &= \lim_{a\to\infty}\mathbb{E}_{\delta_x}e^{-a\langle Z_n,1_{\mathcal{B}(r)}\rangle}\\ &= \lim_{a\to\infty}\Phi_n(e^{-a1_{\mathcal{B}(r)}})(x)\\ &= \Phi_n(1-\ell_0^{(r)})(x),\end{aligned}$$

where $\ell_0^{(r)}(x)\stackrel{\text{def}}{=}1_{(|x|<r)}$. Define for $n\ge 1$,

$$\ell_n^{(r)}(x)=\mathbb{P}_{\delta_x}(R_n<r)=1-\Phi_n(1-\ell_0^{(r)})(x).$$

Based on (2.5),

$$\begin{aligned}\ell_1^{(r)}(x) &= 1 - \Phi_1(1-\ell_0^{(r)})(x)\\ &= 1 - \int dy\, q(x,y)\Pi(1_{(|y|\ge r)})\\ &= (1-p_0)P_x(|W_1|<r).\end{aligned}$$

Using (2.3), we obtain that

$$\begin{aligned}\ell_n^{(r)}(x) &= 1-\Phi_1(\Phi_{n-1}(1-\ell_0^{(r)}))(x)\\ &= 1-\Phi_1(1-\ell_{n-1}^{(r)})(x)\\ &= \int dy\, q(x,y)\Big(1-\Pi(1-\ell_{n-1}^{(r)})(y))\Big)\\ &\stackrel{\text{def}}{=} \int dy\, q(x,y)\,\widehat{\Pi}(\ell_{n-1}^{(r)}(y)),\end{aligned} \tag{2.7}$$

where $\widehat{\Pi}(v)\stackrel{\text{def}}{=}1-\Pi(1-v)$, for $0\le v\le 1$. It is easily checked that $\widehat{\Pi}(v)\le v$, $\widehat{\Pi}(0)=0$, $\widehat{\Pi}(1)=1-p_0<1$ and $\widehat{\Pi}'(0)=1$. Define

$$a_n(r)=\int \ell_n^{(r)}(x)dx, \qquad n\ge 0,\ r>0.$$

Then

$$a_n(r)=\int dx\int dy q(x,y)\widehat{\Pi}(\ell_{n-1}^{(r)}(y))=\int dy\widehat{\Pi}(\ell_{n-1}^{(r)}(y))\le\int dy\ell_{n-1}^{(r)}(y)=a_{n-1}(r),$$

where the second equality follows from the symmetry of semigroup. Hence the limit

$$a_\infty(r)\stackrel{\text{def}}{=}\lim\downarrow a_n(r)\quad\text{exists},\qquad \forall\, r>0,$$

and $a_\infty(r)$ is a nondecreasing function on r. We shall prove that when $d\ge 3$, for every $r>0$,

$$a_\infty(r)>0,\quad a_\infty(0)=0,\quad a_\infty(\infty)=\infty. \tag{2.8}$$

This and (2.6) imply that for every $r>0$,

$$\lim_{n\to\infty}\mathbb{P}_{Poisson}\Big(R_n\ge r\Big)=\lim_{n\to\infty}e^{-a_n(r)}=e^{-a_\infty(r)},$$

yielding Theorem 1.2.

It remains to prove (2.8). Firstly, since $\widehat{\Pi}(v)\le v$, we deduce from (2.7) that

$$\ell_n^{(r)}(x)\le\int dy q(x,y)\ell_{n-1}^{(r)}(y)\stackrel{\text{def}}{=}q*\ell_{n-1}^{(r)}(x)\le\ldots\le q_n*\ell_0^{(r)}(x)=P_x(|W_n|<r),$$

by iteration on the convolution operator $*$. Hence for some constants $c(d), c'(d) > 0$,

$$a_n(r) \leq a_{n-1}(r) \leq ... \leq a_0(r) = c(d)\, r^d, \tag{2.9}$$

$$\ell_n^{(r)}(x) \leq \frac{c' r^d}{n^{d/2}}. \tag{2.10}$$

Elementary calculations show that there exists a convex decreasing function $\kappa(\cdot)$ such that $\widehat{\Pi}(v) = v\,\kappa(v)$ and $\kappa(0) = 1$, $\kappa(1) = 1 - p_0 < 1$. It follows from the convexity of $\kappa(\cdot)$ that

$$a_n(r) = \int dx \ell_{n-1}^{(r)}(x) \kappa(\ell_{n-1}^{(r)}(x)) \geq a_{n-1}(r)\, \kappa\Big(\int dx \frac{(\ell_{n-1}^{(r)}(x))^2}{a_{n-1}(r)} \Big).$$

Note that $\int dx \frac{(\ell_i^{(r)}(x))^2}{a_i(r)} \leq \sup_x \ell_i^{(r)}(x) \leq c' r^d i^{-d/2}$ by (2.10). The function κ being decreasing, it turns out that

$$a_n(r) \geq a_0(r) \prod_{i=1}^{n-1} \kappa\Big(c' r^d i^{-d/2} \Big).$$

The infinite product $\prod_{i=1}^\infty \kappa\Big(c' r^d i^{-d/2} \Big) > 0$ since for large i, $\kappa\Big(c' r^d i^{-d/2} \Big) \sim 1 - c'' r^d i^{-d/2}$ and $d \geq 3$. This proves Theorem 1.2. $\square$

Finally, we mention that by computing the second moment of $\langle Z_n, g \rangle$, we may obtain a lower bound of $a_\infty(r)$ when $r \to \infty$: when $d \geq 3$, there exists some constant $c = c(d) > 0$ such that $a_\infty(r) \geq c\, r^{d-2}$. See also Révész [10].

3 Proof of Theorem 1.1: $d = 2$

In view of the lower bound of (1.2), it remains to show the upper bound: For $d = 2$ and $\alpha > \frac{1}{2}$, we have

$$\limsup_{n\to\infty} \frac{R(\alpha, n)}{\sqrt{n \log n}} \leq \frac{2\alpha - 1}{2\pi}, \qquad \text{a.s..} \tag{3.1}$$

Fix an arbitrary constant $a > \frac{2\alpha-1}{2\pi}$. Let n be large and define a sequence of integers $t_n = [e^{n/\log n}]$. Plainly, $\frac{t_{n+1}}{t_n} - 1 \sim \frac{1}{\log n}$ as $n \to \infty$. Define $r_n = \sqrt{an \log n}$. We are going to estimate

$$I_n \stackrel{\text{def}}{=} \mathbb{P}\Big(\exists\, k \in [t_n, t_{n+1}], R(\alpha, k) \geq r_k \Big),$$

where $\mathbb{P}$ should to be understood that Z_0 is a Poisson point measure with Lebesgue measure as its intensity. If we can prove that

$$\sum_n I_n < \infty, \tag{3.2}$$

then a.s. for all large k, $R(\alpha,k)<r_k$ which implies (3.1) since a is arbitrary.

To estimate I_n, we consider a covering of the disc $\mathcal{B}(t_{n+1}^\alpha)$ centered at the origin and of radius t_{n+1}^α by N discs $\{\mathcal{B}(x_j,\sqrt{t_n}),1\le j\le N\}$. We can choose a covering such that N is of order $\frac{t_{n+1}^{2\alpha}}{t_n}$, hence $N\le c t_{n+1}^{2\alpha-1}$ for some constant $c>0$. Now suppose that for some $k\in[t_n,t_{n+1}]$, $R(\alpha,k)\ge r_k$, which means the existence of an empty disc $\mathcal{B}(x,r_k)\subset\mathcal{B}(k^\alpha)$. Since $x\in\mathcal{B}(x_j,\sqrt{t_n})$ for some $1\le j\le N$, the disc $\mathcal{B}(x_j,s_n)$ centered at x_j is empty, where $s_n\overset{\text{def}}{=}r_{t_n}-\sqrt{t_n}\le r_k-\sqrt{t_n}$ and $s_n\sim r_{t_n}=\sqrt{at_n\log t_n}$. It turns out that

$$\begin{aligned}
I_n &\le \mathbb{P}\Big(\exists\, j\in[1,N],\exists\, k\in[t_n,t_{n+1}]:\,\langle Z_k,1_{\mathcal{B}(x_j,s_n)}\rangle=0\Big)\\
&\le \sum_{j=1}^N \mathbb{P}\Big(\exists\, k\in[t_n,t_{n+1}]:\,\langle Z_k,1_{\mathcal{B}(x_j,s_n)}\rangle=0\Big)\\
&= N\,\mathbb{P}\Big(\exists\, k\in[t_n,t_{n+1}]:\,\langle Z_k,1_{\mathcal{B}(s_n)}\rangle=0\Big)
\end{aligned}$$

where the above equality follows from the translation invariance of Poisson point measure. By using the upper bound of N and the exponential formula for Poisson point measure, we obtain that

$$\begin{aligned}
I_n &\le c\,t_{n+1}^{2\alpha-1}\exp\Big(-\int_{\mathbb{R}^2}dx\,\mathbb{P}_{\delta_x}(\forall\, k\in[t_n,t_{n+1}]:\,\langle Z_k,1_{\mathcal{B}(s_n)}\rangle\ge 1)\Big)\\
&= c\,t_{n+1}^{2\alpha-1}\exp\Big(-\int_{\mathbb{R}^2}dx\,\mathbb{P}_{\delta_0}(\forall\, k\in[t_n,t_{n+1}],\,\langle Z_k,1_{\mathcal{B}(x,s_n)}\rangle\ge 1)\Big),
\end{aligned}\tag{3.3}$$

by using the translation from x to 0. Under $\mathbb{P}_{\delta_0}$, the process $(\langle Z_k,1\rangle,k\ge0)$ is a critical Galton-Watson process. It is well-known ([1]) that

$$\mathbb{P}_{\delta_0}\Big(\langle Z_k,1\rangle>0\Big)\sim\frac{2}{k},\qquad k\to\infty.\tag{3.4}$$

Let us consider the tree associated with the Galton-Watson process $(\langle Z_k,1\rangle,k\ge0)$. Its leaves are ordered lexicographically and $\langle Z_k,1\rangle$ is the total number of leaves of height k. Each leaf of height k corresponds to a point of Z_k.

On $\{\langle Z_{t_{n+1}},1\rangle>0\}$, we denote by $(\zeta_i,0\le i\le t_{n+1})$ the chain of leaves such that for any $0\le i<t_{n+1}$, ζ_i is the root of ζ_{i+1}, $\zeta_0=0$ is the origin and $\zeta_{t_{n+1}}$ is the first leaf of height t_{n+1} in the lexicographical order. For every $0\le i\le t_{n+1}$, we denote by $W(i)$ the point of the Z_i associated with ζ_i. Then W forms a standard two-dimensional Wiener process starting from 0, independent of $(\langle Z_k,1\rangle,k\ge0)$. The law of W is denoted by P_0.

Fix a small $\epsilon>0$ such that $2a\pi(1-4\epsilon)^4>2\alpha-1$, we have

$$\begin{aligned}
&\mathbb{P}_{\delta_0}\Big(\forall\, k\in[t_n,t_{n+1}],\,\langle Z_k,1_{\mathcal{B}(x,s_n)}\rangle\ge1\,\Big|\,\langle Z_{t_{n+1}},1\rangle>0\Big)\\
\ge\;& P_0\Big(\forall\, k\in[t_n,t_{n+1}],W_k\in\mathcal{B}(x,s_n)\Big)\\
\ge\;& P_0\Big(|x-W(t_{n+1})|\le(1-\epsilon)s_n,\sup_{t_n\le t\le t_{n+1}}|W(t)-W(t_{n+1})|\le\epsilon s_n\Big).
\end{aligned}$$

Combining this with (3.3), we deduce from (3.4) that

$$
\begin{aligned}
& I_n \\
\le\ & c\, t_{n+1}^{2\alpha-1} \exp\left[-\frac{2-\epsilon}{t_{n+1}} \int_{\mathbb{R}^2} dx\, \mathbb{P}_{\delta_0}\Big(\forall\, k \in [t_n, t_{n+1}],\, \langle Z_k, 1_{\mathcal{B}(x,s_n)}\rangle \ge 1 \Big| \langle Z_{t_{n+1}}, 1\rangle > 0\Big)\right] \\
\le\ & c\, t_{n+1}^{2\alpha-1} \exp\left[-\frac{2-\epsilon}{t_{n+1}} \int_{\mathbb{R}^2} dx\, P_0\Big(|x - W(t_{n+1})| \le (1-\epsilon)s_n,\, \sup_{t_n \le t \le t_{n+1}} |W(t) - W(t_{n+1})| \le \epsilon s_n\Big)\right] \\
=\ & c\, t_{n+1}^{2\alpha-1} \exp\left[-\frac{2-\epsilon}{t_{n+1}}\, \pi\, (1-\epsilon)^2\, s_n^2\, P_0\Big(\sup_{t_n \le t \le t_{n+1}} |W(t) - W(t_{n+1})| \le \epsilon s_n\Big)\right],
\end{aligned}
$$

by Fubini's theorem. Using the standard estimate for the Gaussian tail, we get

$$
\begin{aligned}
P_0\Big(\sup_{t_n \le t \le t_{n+1}} |W(t) - W(t_{n+1})| \le \epsilon s_n\Big) &= P_0\Big(\sup_{0 \le t \le t_{n+1}-t_n} |W(t)| \le \epsilon s_n\Big) \\
&= 1 - P_0\Big(\sup_{0 \le t \le t_{n+1}-t_n} |W(t)| > \epsilon s_n\Big) \\
&= 1 - O(e^{-\epsilon^2 s_n^2/(2(t_{n+1}-t_n))}) \\
&\ge 1 - \epsilon,
\end{aligned}
$$

for all large n. It follows that

$$
I_n \le c\, t_{n+1}^{2\alpha-1}\, e^{-(2-\epsilon)(1-\epsilon)^3 \pi s_n^2/t_{n+1}} \le c\, t_{n+1}^{2\alpha-1}\, e^{-2a\pi(1-4\epsilon)^4 \log t_n} \le 2c\, t_n^{-2a\pi(1-4\epsilon)^4+2\alpha-1},
$$

yielding (3.2) since $-2a\pi(1-4\epsilon)^4 + 2\alpha - 1 < 0$. The proof of Theorem 1.1 is completed. □

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