

Abundance of one dimensional non uniformly hyperbolic attractors for surface endomorphisms.

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We are interested in the following family of endomorphisms:

$$f_{a,B} : (x, y) \mapsto (x^2 + a + y, 0) + B(x, y)$$

where a is a real number and B is any fixed C^2 small function.

The motivation goes back to a question of Pesin-Yurchenko [4] in reaction-diffusion PDEs. Here is the answer to the question:

Theorem 1. *For every B of C^2 -norm b sufficiently small, there exists a subset $\Lambda_B \subset [-2, 0)$ with positive Lebesgue measure such that for every $a \in \Lambda_B$, the map $f_{a,B}$ preserves a physical, ergodic, hyperbolic SRB probability μ .*

We recall that μ is *physical* if its basin has positive Lebesgue measure.

The measure μ is *SRB* if its conditional measure with respect to any unstable manifold is absolutely continuous with respect to its Lebesgue measure.

The family of maps $f_{a,B}$ looks familiar.

In the case $B = 0$, the map $f_{a,B}$ preserves the horizontal real line, its restriction is the quadratic map $f_a(x) = x^2 + a$. Restricted to this case the above theorem is Jakobson's one.

In the case $B = b \cdot (0, x)$, the map $f_{a,B}$ is the Hénon map, with Jacobian $-b$. Restricted to this case, the above theorem is Benedicks-Carleson's one for the parameters exclusion [2], and Benedicks-Young's one for the SRB existence [3].

What is new in the statement is that the dynamics might be non-invertible with nasty singularities, and only C^2 (instead of C^3 for all the previous techniques).

Also our proof is different from [2] since it is based on a generalization of the geometric and combinatorial formalism of Yoccoz puzzles [5]. However many analytical and probabilistic ideas are mostly included in the easy parts of [2] and [6]. We will focus our exposition on some of the new aspects of this proof: the geometric operations and the characterization of suitable dynamics.

Before going into the details, we shall remind that the quadratic map f_{-2} is semi-conjugated to the map of the circle $\theta \mapsto 2\theta$ via the map $\theta \mapsto 2 \cos \theta$. This give us some hyperbolicity for free for perturbations of f_{-2} and $f_{-2,0}$.

Also for a close to -2 , the quadratic map f_a has two repulsive fixed points $\mathcal{A} \approx -1$ and $\mathcal{B} \approx 2$. So the map $f_{a,B}$ has also two hyperbolic fixed points $\mathcal{A} \approx (-1, 0)$ and $\mathcal{B} \approx (2, 0)$.

Regular dynamics. To construct an SRB in the one dimensional case, we construct a *Markov structure* $(\alpha)_{\alpha \in \mathcal{Y}}$ of a subinterval S of $[-\mathcal{B}, \mathcal{B}]$.

The set \mathcal{Y} is formed by almost disjoint segments S_α endowed with an integer n_α such that f^{n_α} sends bijectively S_α onto S with some distortion bounds and uniform hyperbolicity (hyperbolic times).

The *Markov piece* $\{S_\alpha\}_\alpha$ cannot cover all S , but has to cover almost it. Even more, we ask for the integrability of the function $\sum n_\alpha \mathbb{1}_{S_\alpha}$. If such a Markov structure exists, then the quadratic map f_a preserves an SRB measure.

For other perspectives, we can assume furthermore that S equals to $[\mathcal{A}, -\mathcal{A}]$. Then, the Markov pieces of \mathcal{Y} are *puzzle pieces* and the quadratic map is *regular*.

We generalize this concept to the two dimensional case, by taking instead of $[\mathcal{A}, -\mathcal{A}]$ a continuous family of *flat and stretched (FS)* curves $\Sigma = (S^t)_{t \in T}$ parametrized by a compact metric space T . By FS we mean that each of them is C^2 -close to the horizontal segment $[-1, 1]$ with end points in a local stable manifold of \mathcal{A} close to an arc of the parabola $x^2 + y + a = -1$.

Each of these curves S is still endowed with a partition by *puzzle pieces* $\mathcal{Y}(S)$: these are segments S_α of S sent by f^{n_α} onto FS curves S^α . As before we suppose some uniform distortion bound and hyperbolicity. Also we ask S^α to belong to Σ and the map $(t, \alpha) \in \prod_{t \in T} \mathcal{Y}(S^t) \mapsto t \cdot \alpha \in T$ to be surjective.

Eventually the function on $X := \prod_t S^t$ equal to $\sum_{\alpha \in \cup_{t \in T} \mathcal{Y}(S^t)} n_\alpha \mathbb{1}_{\{t\} \times S_\alpha^t}$ is supposed to be (more than) measurable and, restricted to each S^t , integrable.

In such a case the map $f = f_{a,B}$ is *regular* and then preserves an, ergodic, hyperbolic, SRB measure μ . Actually, the union of unstable manifolds $\cup_{t \in T} S^t$ has positive μ -measure.

Strongly regular dynamics. The main difficulty is to prove the abundance of regular maps. For this end, we define independently to the parameters selection the strongly regular maps. The definition is combinatorial and geometric. By developing a few classical and easy techniques of [2] and [6] in an independent last part, we show the regularity of strongly regular dynamics. The algebraization of the geometric operations is more original.

A basic operation is the \star -product. Let $\alpha = \{S_\alpha, n_\alpha\}$ be a puzzle pieces of an FS curve S and let $\beta = \{S_\beta, n_\beta\}$ be a puzzle pieces of the FS curve $S^\alpha = f^{n_\alpha}(S_\alpha)$. Then the pair $\alpha \star \beta = \{f_{|S_\alpha}^{-n_\alpha}(S_\beta), n_\alpha + n_\beta\}$ is a puzzle piece of S .

Before defining the other operation, let us recall that from the Chebychev map f_{-2} , for a close to -2 and then b small enough, a lot of puzzle pieces $s_-^M, \dots, s_-^2, s_+^2, \dots, s_+^M$ match for any FS curves. Here M is the smallest positive integer s.t. $f_a^{M+1}(0)$ belongs to $[\mathcal{A}, -\mathcal{A}]$. The puzzle pieces $(s_\pm^k)_k$ are all disjoint and their union covers all S but a small segment S_\square close to 0. The puzzle pieces $(s_\pm^k)_k$ are called *simple* and satisfy $n_{s_\pm^k} = k$.

To state the strong regularity, we must understand how to construct new puzzle pieces on S_\square . The segment S_\square is sent by f^{M+1} to a curves S^\square which starts from a very small local stable manifold of \mathcal{A} , comes around $(f_a^{M+1}(0), 0)$ and comes back to the same local stable manifold. Let us suppose that S^\square is tangent to a local stable manifold W_c^s of a point $c \in S$. Suppose c is the unique point of an intersection $\cap_{i=0}^\infty S_{c_i}$ of puzzle pieces $c_i = \alpha_1 \star \alpha_2 \star \dots \star \alpha_i$, where $(\alpha_k)_{k=1}^i$ are mostly simple for every $i \geq 1$. Then W_c^s is C^1 -close to a local stable manifold of $f_{a,0}$: an arc of parabola with end points as far from c as $\theta := -(\log b)^{-1}$. By mostly simple we mean for every i :

$$(1) \quad \sum_{\{k \leq i+1, \alpha_k \text{ not simple}\}} n_{\alpha_k} \leq e^{-\sqrt{M}} \sum_{k \leq i} n_{\alpha_k} = e^{-\sqrt{M}} n_{c_i}$$

One can prove that the end points of S_{c_i} have also a long stable manifold close to an arc of a parabola. Let Y_{c_i} be the box bounded by these two stable manifolds and the two horizontal lines $\{y = \pm\theta\}$.

From elementary topology the set $Y_{c_i} \setminus Y_{c_{i+1}}$ intersects the curve S^\square at (0 or) two segments that are pulled back via $f^{M+1}|_{S^\square}$ to segments S_{Δ_-} and S_{Δ_+} . Let $\Delta_\pm := \{S_{\Delta_\pm}, n_\Delta := M + 1 + n_{c_i}\}$.

We note that f^{n_Δ} sends each segment S_{Δ_\pm} to a curve with an end point in a small local stable manifold $W^s(z)$ of an end point z of $S_{\alpha_{i+1}}^{c_i}$. If the image by $f^{n_{c_i}}$ of the tangency point C between S^\square and $W^s(c)$ is not too close to $W^s(z)$, then one shows that the curves $f^{n_\Delta}(S_{\Delta_\pm})$ can be extended to C^1 -FS curves by following the curvature of S^{c_i} after $W^s(z)$. Let S^{Δ_\pm} be these extensions. We note that the images of S_{Δ_\pm} by f^{n_Δ} are strictly included in S^{Δ_\pm} . Therefore Δ_\pm are not Puzzle pieces of S . However the pairs Δ_\pm satisfies hyperbolic times inequalities. Thus if β is a puzzle piece of S^{Δ_\pm} included in $f^{n_\Delta}(S_{\Delta_\pm})$ then $\Delta \star \beta$ defines a *topological* puzzle piece of S included in S^\square : the segment $S_{\Delta \star \beta}$ is sent to a FS curve $S^{\Delta \star \beta}$ with some hyperbolicity. For instance β can be equal to a simple piece s or even of the form $\Delta' \star s$ to make a (topological) puzzle piece $\Delta \star \Delta' \star s$.

Let us state the combinatorial condition implying that the tangency point C is not close to $W^s(z)$, with s_-^2 the simple piece next to \mathcal{A} :

$$(2) \quad \alpha_i = \dots = \alpha_{i+k} = s_-^2 \Rightarrow Mi \geq 20k$$

A *strongly regular* map is endowed with a set Σ of FS curves S equipped with a set $\mathcal{Y}(S)$ of almost disjoint topological puzzle pieces α s.t. S^α belongs to Σ . Also \mathcal{Y} consists of the pieces of the form $\Delta_1 \star \dots \star \Delta_n \star s$, where s is simple and Δ_n is given by the above algorithm from a requested tangency between $(S^{\Delta_1 \dots \Delta_{n-1}})^\square$ and W_c^s , with c a sequence of puzzle pieces in \mathcal{Y} satisfying (1 – 2). Then the topological puzzle pieces of \mathcal{Y} satisfy automatically the distortion and hyperbolic bounds and so are real puzzle pieces.

We can truncate the definition by asking for geometrical conditions with respect to Y_{c_k} instead of W_c^s . Then regarding only the Δ -constructions made from c_j , $j \leq k$. This makes the concept of *k-strong regularity* which is useful to show the abundance of strongly regular maps.

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