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On the Harish-Chandra Schwartz space of $G(F)\backslash G(\mathbb{A})$

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Abstract

We study the Harish-Chandra Schwartz space of an adelic quotient $G(F)\backslash G(\mathbb{A})$. We state a conjectural spectral decomposition of it in terms of parabolic induction. We verify a cuspidal version of this conjecture under additional hypotheses on the group G, which are known to be satisfied for $G = \mathrm{GL}_n$.

1 Introduction

In the harmonic analysis of reductive groups over a local field, an important companion of the Plancherel formula is the Paley-Wiener Theorem for the Harish-Chandra Schwartz space (cf. [Art75, Ber88, Wal03]). In this paper we will be interested in the Harish-Chandra Schwartz space of an adelic quotient. It was defined in this context by Bernstein [Ber88] (see also §2 below). We will state a conjectural Paley-Wiener Theorem for this space. Unfortunately, in the global setup, we can formulate (let alone prove) a reasonable statement only assuming a conjectural nontrivial analytic condition on the intertwining operators, which is known completely only for the groups GL_n. This condition has both a global aspect, (UT), and a local aspect, (WR). The property (UT) (uniform temperedness) is pertaining to the global normalization factor of the intertwining operator while the property (WR) (weak Ramanujan) is pertaining to the poles of the local normalized intertwining operators. They are explicated in §3. Assuming these properties, we prove a *cuspidal* version of the Paley-Wiener Theorem (Theorem 4.5). The main step is a majorization of cuspidal Eisenstein series which is uniform in the spectral and the group variable (Proposition 5.1).

The property (WR) had been considered earlier (in an equivalent form) in the work of Werner Müller on the spectral side of Arthur's trace formula. Specifically, it was shown in [Mül02] that (WR) implies the absolute convergence of the spectral side of Arthur's trace formula. For GL_n, (WR) directly follows from the results of Luo-Rudnick-Sarnak [LRS99] (see [MS04]). Subsequently, a different approach for the absolute convergence of the spectral side of the trace formula which avoids (WR) was given in [FLM11, FL11]. On the other hand, for the analysis of the Harish-Chandra Schwartz space it seems that the properties (UT) and (WR), at least in some form, are indispensable. We also mention that both (UT) and (WR) were encountered in the analytic study of Jacquet's relative trace formula in [Lap06], where a majorization of cuspidal Eisenstein series (in a slightly weaker form) was also considered.

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We mention that in a different context, an approach for the (conjectural) Paley-Wiener Theorem for the (much smaller) space of rapidly decreasing functions was set forth by Casselman – cf. [Cas04, Cas89]. The analysis of the two problems turns out to be quite different. However, they both have a common goal, namely to study the cohomology (in various contexts) of arithmetic groups (see [Cas84] for such an approach). We will not go into any further detail here but refer the reader to [Fra98, FS98] for a related theme.

The paper is organized as follows. After introducing notation we recall the definition and the basic properties of the Harish-Chandra Schwartz space (§2). We also recall some standard facts from reduction theory. Then, in §3 we discuss the analytic properties which are required for the main result. We prove them for GL_n and conjecture that they hold (even in a stronger form) in general. In §4 we give the conjectural Paley-Wiener statement in this context. We also state our main result, Theorem 4.5, which is the cuspidal version of this conjecture, conditional on the analytic properties discussed above. The technical heart of the proof is the majorization of the Eisenstein series and their derivatives (in both the group and the spectral variables) near the imaginary axis. This is carried out in §5 using the Maass-Selberg relations. Finally, we prove the main result in §6 by a simple induction, together with the standard principle of approximation by the constant term.

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1.1 Notation

Let G be a reductive group over a number field F. We fix throughout a maximal F-split torus T_0. Any F-parabolic subgroup P of G containing T_0 admits a unique Levi decomposition $P = M \ltimes U$ with Levi part M containing T_0. We refer to the M's arising this way simply as the Levi subgroups of G containing T_0 and denote this set by $\mathcal{L}$.

For any algebraic group Y over F we write $\mathfrak{a}_Y^* = X^*(Y) \otimes \mathbb{R}$ where $X^*(Y)$ is the lattice of F-rational characters of Y; let $\mathfrak{a}_Y$ be the dual space of $\mathfrak{a}_Y^*$. We also set

$$Y(\mathbb{A})^1 = \bigcap_{\chi \in X^*(Y)} \operatorname{Ker} |\chi|_{\mathbb{A}^*}$$

where we extend any $\chi \in X^*(Y)$ to a homomorphism $\chi : Y(\mathbb{A}) \to \mathbb{A}^*$ and take the standard norm on the ideles. We have a surjective homomorphism

$$H_Y : Y(\mathbb{A}) \to \mathfrak{a}_Y$$

given by $e^{\langle \chi, H_Y(y) \rangle} = |\chi(y)|$, $\chi \in X^*(Y)$, $y \in Y(\mathbb{A})$. The kernel of H_Y is $Y(\mathbb{A})^1$. Denote by $r(Y)$ the dimension of $\mathfrak{a}_Y$. Finally, we write δ_Y for the modulus function of $Y(\mathbb{A})$.

Fix a minimal F-parabolic subgroup $P_0 = M_0 \ltimes U_0$ containing T_0. Its Levi part M_0 is the centralizer of T_0. We set $\delta_0 = \delta_{P_0}$.

For any $M \in \mathcal{L}$, the set of Levi subgroups containing M will be denoted by $\mathcal{L}(M)$ and the set of parabolic subgroups whose Levi part equals (resp. contains) M will be denoted by $\mathcal{P}(M)$ (resp. $\mathcal{F}(M)$). (For simplicity we write $\mathcal{F} = \mathcal{F}(M_0)$ for the set of semistandard parabolic subgroups.) The parabolic subgroup opposite to P containing M is denoted by $\overline{P}$. We have canonically $\mathfrak{a}_P = \mathfrak{a}_M = \mathfrak{a}_{T_M}$ for any $P \in \mathcal{P}(M)$ where T_M is the split part of the center of M. We write $R(T_M, G)$ for the set of reduced roots of T_M on $\operatorname{Lie} G$. For any $P = M \ltimes U \in \mathcal{P}(M)$ we write $\Sigma_P = R(T_M, U) \subseteq \mathfrak{a}_M^*$ for the set of reduced roots of T_M on $\operatorname{Lie} U$ and Δ_P for the subset of simple roots. Similarly $\Delta_P^\vee \subseteq \mathfrak{a}_M$ will denote the set of simple co-roots.

We set $\mathfrak{a}_0 = \mathfrak{a}_{P_0} = \mathfrak{a}_{M_0} = \mathfrak{a}_{T_0}$ and for any $M \in \mathcal{L}$ we view $\mathfrak{a}_M$ canonically as a subspace of $\mathfrak{a}_0$ with a canonical projection $\mathfrak{a}_0 \to \mathfrak{a}_M$. Similarly for $\mathfrak{a}_M^*$. We denote by Δ_0 the simple roots of T_0 on $\operatorname{Lie}(U_0)$. We endow $\mathfrak{a}_0$ with a W-invariant inner product where $W = W^G = N_{G(F)}(T_0)/M_0$ is the Weyl group of G. We write the corresponding norms on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ by $\|\cdot\|$. We write $\mathfrak{a}_{0,+}$ for the positive Weyl chamber of $\mathfrak{a}_0$, i.e.

$$\mathfrak{a}_{0,+} = \{X \in \mathfrak{a}_0 : \langle \alpha, X \rangle > 0 \text{ for all } \alpha \in \Delta_0\}.$$

Note that $\mathfrak{a}_{0,+}$ is invariant under translation by $\mathfrak{a}_G$.

For any $M, L \in \mathcal{L}$ we write $W(M, L)$ for the set of right W^M-cosets of elements of W such that $wMw^{-1} = L$.

Fix a maximal compact subgroup $\mathbf{K}$ of $G(\mathbb{A})$ in good position with respect to P_0. Let $\mathbf{K}_\infty = \mathbf{K} \cap G(F_\infty)$ where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$.

For any $P = M \ltimes U \in \mathcal{F}$ the homomorphism $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ extends to a left-$U(\mathbb{A})$ right-$\mathbf{K}$-invariant map $H_P : G(\mathbb{A}) \to \mathfrak{a}_M$. In particular, we write $H = H_{P_0}$. Note that if P is standard then H_P is the composition of H with the projection $\mathfrak{a}_0 \to \mathfrak{a}_M$.

Let A_0 be the identity component (in the usual Hausdorff topology) of $T_0(\mathbb{R}) \subseteq T_0(\mathbb{A})$ where we view $\mathbb{R} \hookrightarrow \mathbb{A}$ via $\mathbb{R} \hookrightarrow \mathbb{A}_{\mathbb{Q}} \hookrightarrow \mathbb{A} = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} F$. Then H restricts to an isomorphism $H : A_0 \to \mathfrak{a}_0$. We write $X \mapsto e^X$ for the inverse isomorphism. Similarly for any $M \in \mathcal{L}$ let $A_M = A_0 \cap T_M$, so that $H_M : A_M \to \mathfrak{a}_M$ is an isomorphism.

We will use the following notational convention. Suppose that X and Y are certain quantities (depending on parameters). We write $X \ll Y$ if there exists a constant $c > 0$ such that $|X| \le cY$. If the constant c depends on additional data, say D, we write $X \ll_D Y$.

For instance, for any $g, x \in G(\mathbb{A})$ we have

$$\|H(gx) - H(g)\| \le \sup_{k \in \mathbf{K}} \|H(kx)\|.$$

Therefore, for any compact subset $C \subseteq G(\mathbb{A})$ we have

$$\|H(gx) - H(g)\| \ll_C 1 \text{ for all } g \in G(\mathbb{A}),\ x \in C. \tag{1}$$

Fix $T_1 \in \mathfrak{a}_0$ throughout such that $\langle \alpha, T_1 \rangle$ is sufficiently small for all $\alpha \in \Delta_0$. Let

$$\mathfrak{s} = \{g \in G(\mathbb{A}) : H(g) \in T_1 + \mathfrak{a}_{0,+}\} = \{pe^X k : p \in P_0(\mathbb{A})^1, X \in T_1 + \mathfrak{a}_{0,+}, k \in \mathbf{K}\}$$

and $\mathfrak{s}^1 = \mathfrak{s} \cap G(\mathbb{A})^1$, so that $\mathfrak{s} = A_G \mathfrak{s}^1$. Clearly, $\mathfrak{s}$ is an open left $P_0(F)$-invariant subset of $G(\mathbb{A})$. By reduction theory we have $G(F)\mathfrak{s} = G(\mathbb{A})$ and $G(F)\mathfrak{s}^1 = G(\mathbb{A})^1$.

In particular, for any non-negative measurable function f on $G(F)\backslash G(\mathbb{A})$ we have

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)\,dg \le \int_{P_0(F)\backslash\mathfrak{s}} f(g)\,dg$$
$$= \int_{P_0(F)\backslash P_0(\mathbb{A})^1} \int_{T_1+\mathfrak{a}_{0,+}} \int_{\mathbf{K}} \delta_0(e^X)^{-1} f(pe^X k)\,dp\,dX\,dk. \quad (2)$$

Another basic fact is that there exists a compact subset Ω of $G(\mathbb{A})$ (which can be taken in the form $\Omega_0\mathbf{K}$ where $\Omega_0 \subseteq P_0(\mathbb{A})^1$) such that

$$\mathfrak{s} \subseteq P_0(F)\{e^X : X \in T_1 + \mathfrak{a}_{0,+}\}\Omega. \quad (3)$$

We write $C(X)$ for the space of complex valued continuous functions on a topological space X. If X is a smooth manifold, we denote by $C^m(X)$, $m \in \mathbb{N}$ (resp., C^∞) the subspace of m-times continuously differentiable (resp., smooth) functions in $C(X)$. Similarly we use the notation $C_c(X)$, $C_c^m(X)$, $C_c^\infty(X)$ for the subspaces of compactly supported functions in $C(X)$, $C^m(X)$ and $C^\infty(X)$ respectively.

The space $G(\mathbb{A})$ is not a smooth manifold, but for any compact open subgroup $K \subseteq G(\mathbb{A}_{\mathrm{fin}})$, $G(\mathbb{A})/K$ is a smooth manifold (namely, countably many copies of $G(F_\infty)$). We define $C^\infty(G(\mathbb{A}))$ to be the union over all compact open subgroups $K \subseteq G(\mathbb{A}_{\mathrm{fin}})$ of $C^\infty(G(\mathbb{A})/K)$. Similarly for any closed subgroup H of $G(\mathbb{A})$ let $C^\infty(H\backslash G(\mathbb{A}))$ denote the space of smooth left H-invariant functions on $G(\mathbb{A})$ which are right K-invariant for some compact open subgroup K of $G(\mathbb{A}_{\mathrm{fin}})$. For brevity we will refer to a compact open subgroup K of $G(\mathbb{A}_{\mathrm{fin}})$ simply as a "level" of G.

We denote by $\mathfrak{g}$ the complexified Lie algebra of $G(F_\infty)$. Its universal algebra will be denoted by $\mathcal{U}(\mathfrak{g})$ and the center of the latter by $\mathfrak{z}$.

Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space over $\mathbb{R}$. Denote by $\mathcal{D}(V)$ the graded ring of translation invariant differential operators on V (isomorphic to the symmetric algebra of V). Occasionally, we will also view elements of $\mathcal{D}(V)$ as holomorphic differential operators on $V_\mathbb{C}$ with constant coefficients. Given $D \in \mathcal{D}(V)$ we sometimes write $D_v f(v)$ for $(Df)(v)$ in order to emphasize the variable of differentiation. Given a Fréchet space $\mathcal{F}$ we denote by $\mathcal{S}(V;\mathcal{F})$ the Fréchet space of smooth functions $f : V \to \mathcal{F}$ such that

$$\sup_{v\in V}(1 + \|v\|)^n \mu(D_v f(v)) < \infty$$

for any $n \in \mathbb{N}$, $D \in \mathcal{D}(V)$ and a continuous seminorm μ on $\mathcal{F}$.

For an inductive limit $U = \cup\mathcal{F}_n$ of Fréchet spaces we write $\mathcal{S}(V;U) = \cup\mathcal{S}(V;\mathcal{F}_n)$ with the inductive limit topology.

2 Schwartz space of automorphic forms

In this section we recall the definition of the Schwartz space in this context. For a more flexible and elaborate setup we refer the reader to [Fra98].

Define the function Ξ on $\mathfrak{s}$ by

$$\Xi(pk) = \delta_{P_0}(p)^{\frac{1}{2}},\ p \in P_0(\mathbb{A}), H(p) \in T_1 + \mathfrak{a}_{0,+}, k \in \mathbf{K}.$$

This function plays the role of the Harish-Chandra standard spherical function (which is usually denoted by the same letter) in the local case (cf. [Wal03]).

For any $T \in \mathfrak{a}_0$ define

$$\mathfrak{s}_{>T} = \{g \in G(\mathbb{A}) : H(g) \in T + \mathfrak{a}_{0,+}\}.$$

Thus, $\mathfrak{s} = \mathfrak{s}_{>T_1}$. We will also set

$$\mathfrak{s}_D = \{g \in G(\mathbb{A}) : H(g) \in D\}$$

for any subset $D \subseteq \mathfrak{a}_0$.

The following Lemma is a standard result in reduction theory (see e.g. [LW13, §3.5] or [Fra98, §2.1]). For convenience we provide a proof.

Lemma 2.1 *For any $T_1' \in \mathfrak{a}_0$ we have*

$$\|H(x)\| \ll_{T_1'} 1 + \|H(\gamma x)\| \tag{4}$$

for any $x \in \mathfrak{s}_{>T_1'}$ and $\gamma \in G(F)$, and if moreover $\gamma x \in \mathfrak{s}_{>T_1'}$ then

$$\|H(\gamma x) - H(x)\| \ll_{T_1'} 1. \tag{5}$$

In addition,

$$\left|\{\gamma \in P_0(F)\backslash G(F) : \gamma x \in \mathfrak{s}_{>T_1'}\}\right| \ll_{T_1'} 1 \tag{6}$$

for any $x \in G(\mathbb{A})$. In particular, if $x \in \mathfrak{s}_{>T_1'}$ then

$$\mathfrak{s} \cap G(F)x \subseteq \mathfrak{s}_{H(x)+B} \tag{7}$$

where B is a bounded set of $\mathfrak{a}_0$ depending only on T_1'. If $H(x) \in \mathfrak{a}_{0,+}$ and sufficiently regular then in fact $G(F)x \cap \mathfrak{s} = P_0(F)x$.

Proof Suppose $x \in \mathfrak{s}_{>T_1'}$ and $\gamma \in G(F)$. Let $X = H(x)$ and $Y = H(\gamma x)$. By assumption $X \in T_1' + \mathfrak{a}_{0,+}$. Fix representatives $n_w \in N_G(T_0)$, $w \in W$. Multiplying γ on the left by an element of $P_0(F)$ and using the Bruhat decomposition, we may assume without loss of generality that $\gamma = n_w u$ where $w \in W$ and $u \in U_w^- = U_0 \cap w^{-1}\overline{U_0}w$. Let $x = ptk$ where $p \in P_0(\mathbb{A})^1$, $t \in A_0$ and $k \in \mathbf{K}$. Then $X = H(t)$ and

$$Y = wX + H(n_w u')$$

where $u' = t^{-1}p^{-1}upt \in U_0(\mathbb{A})$. It is well known (e.g., [LW13, Lemme 3.3.2]) that

$$H(n_w u') = \sum_{\beta \in \Phi_w} d_\beta \beta^\vee$$

where

$$\Phi_w = \{\beta \in \Phi_+ : w^{-1}\beta < 0\}$$

and $d_\beta \leq d$ where d is a constant. Thus,

$$w^{-1}H(n_w u') - H(n_w u') = \sum_{\beta \in \Phi_w} d_\beta (w^{-1}\beta^\vee - \beta^\vee).$$

We conclude that $w^{-1}H(n_w u')-H(n_w u')$ is a sum of simple co-roots with coefficients bounded from below and that

$$\|H(n_w u')\| \ll \sum |d_\beta| \ll 1 + \|w^{-1}H(n_w u') - H(n_w u')\|.$$

Since $X \in T_1' + \mathfrak{a}_{0,+}$, we have $wX - X = \sum_{\beta\in\Phi_w} c_\beta \beta^\vee$ with $c_\beta \le c$ where c is a constant depending only on T_1'. From the relation

$$w^{-1}Y - Y = X - wX - H(n_w u') + w^{-1}H(n_w u') = \sum_{\beta\in\Phi_w} (d_\beta w^{-1}\beta^\vee - (c_\beta + d_\beta)\beta^\vee) \tag{8}$$

we conclude that

$$\|wX - X\| \ll \sum |c_\beta| \ll_{T_1'} 1 + \|w^{-1}Y - Y\|$$

and

$$\|H(n_w u')\| \ll 1 + \|w^{-1}H(n_w u') - H(n_w u')\| \ll_{T_1'} 1 + \|w^{-1}Y - Y\|$$

Thus,

$$\|Y - X\| = \|wX - X + H(n_w u')\| \ll_{T_1'} 1 + \|w^{-1}Y - Y\| \le 1 + 2\|Y\|$$

and hence,

$$\|X\| \le \|Y - X\| + \|Y\| \ll_{T_1'} 1 + \|Y\|$$

which is the relation (4).

Moreover, if $\gamma x \in \mathfrak{s}_{>T_1'}$, that is, if $Y \in T_1' + \mathfrak{a}_{0,+}$, then we also have $w^{-1}Y - Y = \sum_{\beta\in\Phi_{w^{-1}}} c'_\beta \beta^\vee$ with $c'_\beta \le c$. (Note that $w^{-1}\Phi_w = -\Phi_{w^{-1}}$.) The relation (8) immediately implies that $|c_\beta|$, $|c'_\beta|$, $|d_\beta|$, $\beta \in \Phi_w$ are bounded in terms of T_1'. Therefore, the same holds for $wX - X$, $H(n_w u')$ and $Y - X = wX - X + H(n_w u')$. This proves (5).

In addition, since $wX - X$ is bounded and $X \in T_1' + \mathfrak{a}_{0,+}$, we infer that X lies in a compact translate of $\mathfrak{a}_L$ where L is the smallest standard Levi subgroup containing n_w. Also, since $H(n_w u')$ is bounded, u' lies in a compact subset of $U_w^-(\mathbb{A})$. Write $t = t_1 t_2$ where $t_1 \in A_L$ and t_2 is in a compact set. Let $p' = t_1^{-1} p t_1 \in P_0(\mathbb{A})^1$ and write $p' = p_1 p_2$ where $p_1 \in P_0(F)$ and p_2 lies in a fixed compact set of $P_0(\mathbb{A})^1$. Since $L \supseteq U_w^-$, t_1 commutes with u and we have $p_2 t_2 u' t_2^{-1} p_2^{-1} = p_1^{-1} u p_1$. Therefore, $p_1^{-1} u p_1$ lies in a fixed compact set C of $U_0(\mathbb{A})$. However, $\left|\{v \in U_0(F) : p_1^{-1} v p_1 \in C\}\right| = |U_0(F) \cap C|$ is finite and bounded. This gives (6).

Finally, the last statement follows from the fact that if $wX - X$ is bounded and $X \in \mathfrak{a}_{0,+}$ is sufficiently regular then $w = 1$.

□

For $g \in G(\mathbb{A})$ let $\sigma(g) = 1 + \min_{x\in\mathfrak{s}\cap G(F)g} \|H(x)\|$. The following properties are clear.

1. σ is left $G(F)$-invariant.

2. If $x, x' \in \mathfrak{s}$ and $x' \in G(F)x$ then $1 + \|H(x')\| \ll 1 + \|H(x)\|$. (Lemma 2.1, or alternatively, [MW95, I.2.2].) Therefore, $1 + \max_{x\in\mathfrak{s}\cap G(F)g} \|H(x)\| \ll \sigma(g)$ for all $g \in G(\mathbb{A})$ and

   $$\sigma(x) \le 1 + \|H(x)\| \ll \sigma(x)$$

   for all $x \in \mathfrak{s}$.

3. $\log \Xi(x) \ll \sigma(x) \ll \max(1, \log \Xi(x))$ for any $x \in \mathfrak{s}^1$.

4. For any compact set $C \subseteq G(\mathbb{A})$ we have

$$\sigma(xy) \ll_C \sigma(x) \text{ for all } x \in G(\mathbb{A}), y \in C.$$

5. For $n \gg 1$ (in fact, for any $n > r/2$ where r is the F-rank of G) we have

$$\Xi\sigma^{-n} \text{ (or equivalently } \Xi(1+\|H\|)^{-n}) \in L^2(P_0(F)\backslash\mathfrak{s}). \tag{9}$$

For any $f \in C_c(G(\mathbb{A}))$ let K_f be the automorphic kernel

$$K_f(x,y) = \sum_{\gamma\in G(F)} f(x^{-1}\gamma y), \quad x, y \in G(F)\backslash G(\mathbb{A}).$$

Similarly, let

$$\tilde{K}_f(x,y) = \int_{A_G} \sum_{\gamma\in G(F)} f(ax^{-1}\gamma y)\, da, \quad x, y \in A_G G(F)\backslash G(\mathbb{A}).$$

The following Lemma is also proved in [LW13, §12.2].

Lemma 2.2 *Let $C \subseteq G(\mathbb{A})$ be compact and $n = 0, 1, 2, \ldots$. Then*

1. $\operatorname{vol}(G(F)\backslash G(F)xC) \ll_C \Xi(x)^{-2}$ *for all $x \in \mathfrak{s}$. (The volume is taken in $G(F)\backslash G(\mathbb{A})$.)*
2. *For any $x \in \mathfrak{s}$ and $y \in G(\mathbb{A})$ we have*

$$\sum_{\gamma\in G(F)} 1_C(x^{-1}\gamma y)\sigma(x)^n\sigma(y)^{-n} \ll_{C,n} 1_{G(F)xC}(y)\Xi(x)^2. \tag{10}$$

Consequently, for any $f \in C_c(G(\mathbb{A}))$ we have

1. $K_f(x,y)\sigma(x)^n\sigma(y)^{-n} \ll_{f,n} \Xi(x)^2$ *for any $x \in \mathfrak{s}, y \in G(\mathbb{A})$.*
2. $\|K_f(x,\cdot)\sigma^{-n}\|_{L^2(G\backslash G(\mathbb{A}))} \ll_{f,n} \sigma(x)^{-n}\Xi(x)$ *for all $x \in \mathfrak{s}$.*

Similarly, we have

$$\|\tilde{K}_f(x,\cdot)\sigma^{-n}\|_{L^2(G\backslash G(\mathbb{A})^1)} \ll_{f,n} \sigma(x)^{-n}\Xi(x), \quad x \in \mathfrak{s}^1.$$

Proof It follows from (1) and (7) that $G(F)xC \cap \mathfrak{s} \subseteq \mathfrak{s}_{H(x)+B}$ where B is a bounded subset of $\mathfrak{a}_0$ depending only on C. The first part follows from (2) since

$$\operatorname{vol}(P_0(F)\backslash\mathfrak{s}_{H(x)+B}) \ll_C \Xi(x)^{-2}.$$

To prove (10), we first observe that the left-hand side of (10) is supported (with respect to y) in $G(F)xC$. Therefore, $\sigma(x)^n\sigma(y)^{-n}$ is bounded in terms of C and n only, and we can ignore it from the estimation. It remains to bound the cardinality of $G(F) \cap xCy^{-1}$. Clearly, if $\gamma_1, \gamma_2 \in G(F) \cap xCy^{-1}$ then $\gamma_1\gamma_2^{-1} \in G(F) \cap xCC^{-1}x^{-1}$. Therefore, by passing to a larger C, it suffices to bound the size of $G(F) \cap xCx^{-1}$. Once again, using (3), upon enlarging C and multiplying x on the left by an element of $P_0(F)$, we may assume that $x = t = e^X \in A_0$

with $X \in T_1 + \mathfrak{a}_{0,+}$. Suppose that $\gamma \in G(F) \cap tCt^{-1}$. Using the Bruhat decomposition, write $\gamma = u_1 a n_w u_2$ where $w \in W$, $a \in M_0(F)$, $u_1 \in U_0(F)$, $u_2 \in U_w^-(F)$, and we recall that $U_w^- = U_0 \cap w^{-1}\overline{U_0}w$. Since $t^{-1}\gamma t \in C$, $H(t^{-1}\gamma t)$ is bounded. On the other hand, we may write $t^{-1}\gamma t = u_1' t^{-1} a n_w t u_2'$ where $u_1' = t^{-1}u_1 t \in U_0(\mathbb{A})$ and $u_2' = t^{-1}u_2 t \in U_w^-(\mathbb{A})$. Therefore

$$H(t^{-1}\gamma t) = wX - X + H(n_w u_2').$$

As in the proof of Lemma 2.1, we have $wX - X = \sum_{\beta \in \Phi_w} c_\beta \beta^\vee$ and $H(n_w u_2') = \sum_{\beta \in \Phi_w} d_\beta \beta^\vee$ with $c_\beta, d_\beta \le c$ (depending only on T_1) and $\Phi_w = \{\beta \in \Phi_+ : w^{-1}\beta < 0\}$. It follows that both $wX - X$ and $H(n_w u_2')$ are bounded. Let L be the smallest standard Levi subgroup containing n_w. The boundedness of $wX - X$ implies that t is in a compact translate of A_L, while the boundedness of $H(n_w u_2')$ implies that u_2' lies in a compact subset of $U_w^-(\mathbb{A})$. Thus, upon enlarging C, we may assume that $t \in A_L$. In this case t commutes with u_2, so that $u_2' = u_2$. Hence, u_2 is confined to a compact set (hence, a finite set) depending only on C. Also, since $t^{-1}\gamma t = u_1' a n_w u_2$, it follows that a is confined to a finite set and u_1' is confined to a compact set D of $U_0(\mathbb{A})$, both depending only on C. Finally, we can bound the number of possible u_1's since $t^{-1}u_1 t = u_1'$ and

$$\left|\{v \in U_0(F) : t^{-1}vt \in D\}\right| = \left|\{Y \in \operatorname{Lie} U_0(F) : \operatorname{Ad}(t)^{-1}Y \in \log(D)\}\right| \ll \delta_0(t).$$

The second part follows.

The next two statements of the Lemma are an immediate consequence. Finally, the last statement is proved by a similar argument.

□

Recall the regular representation R of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$. Thus,

$$R(f)\varphi(x) = \int_{G(\mathbb{A})} f(g)\varphi(xg)\, dg = \int_{G(F)\backslash G(\mathbb{A})} K_f(x,y)\varphi(y)\, dy$$

From the Cauchy-Schwarz inequality we get

Corollary 2.3 *Let $f \in C_c(G(\mathbb{A}))$ and $n = 0, 1, 2, \dots$. Then for any function ϕ on $G(F)\backslash G(\mathbb{A})$ such that $\sigma^n\phi \in L^2(G(F)\backslash G(\mathbb{A}))$ we have*

$$\sup_{x \in \mathfrak{s}} \Xi(x)^{-1}\sigma(x)^n \left|R(f)\phi(x)\right| \ll_f \|\sigma^n\phi\|_{L^2(G(F)\backslash G(\mathbb{A}))}.$$

For future use we will need a variant of this Corollary. Recall Arthur's truncation operator Λ^T (with $T \in \mathfrak{a}_0$) on locally bounded functions on $G(F)\backslash G(\mathbb{A})^1$ [Art80]. We will always implicitly assume that $T \in \mathfrak{a}_{0,+}$ is sufficiently regular.

Corollary 2.4 *Let $f \in C_c(G(\mathbb{A}))$. Then for any locally bounded measurable function ϕ on $G(F)\backslash G(\mathbb{A})$ which is (A_G, χ)-equivariant with respect to a unitary character χ of A_G we have*

$$R(f)\phi(x) \ll_f \Xi(x)\|\Lambda^T\phi\|_{L^2(G(F)\backslash G(\mathbb{A})^1)}$$

for any $x \in \mathfrak{s}^1$ where T is a fixed translate of $H(x)$ (depending on the support of f).

Proof Indeed, let $f^\chi(g) = \int_{A_G} f(ag)\chi(a)\, da$. By [Lap06, Lemma 6.2] and the assumption of T we may write

$$R(f)\phi(x) = \int_{G(\mathbb{A})^1} f^\chi(y)\phi(xy)\, dy = \int_{G(\mathbb{A})^1} f^\chi(y)\Lambda^T\phi(xy)\, dy = \int_{G(\mathbb{A})^1} f^\chi(x^{-1}y)\Lambda^T\phi(y)\, dy.$$

Hence,

$$|R(f)\phi(x)| \le \int_{G\backslash G(\mathbb{A})^1} \tilde{K}_{|f|}(x,\cdot) \left|\Lambda^T\phi(y)\right|\, dy.$$

The corollary follows from the last part of Lemma 2.2 and the Cauchy-Schwarz inequality. □

Recall the following standard result (cf. [Art78, Lemma 4.1]).

Lemma 2.5 *Let G' be a real reductive group. Then for any $m \in \mathbb{N}$ there exist $f_1, f_2 \in C_c^m(G')$ and $X \in \mathcal{U}(\mathfrak{g}')$ such that $f_1 * X + f_2$ is the Dirac distribution at the identity of G'. Consequently, if π is a Banach representation of G' and V is the Fréchet space of smooth vectors then the following conditions are equivalent for a seminorm μ on V.*

1. *μ is continuous.*
2. *There exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G')$ the seminorm $v \mapsto \mu(\pi(f)v)$ is continuous.*
3. *There exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G')$ we have $\mu(\pi(f)v) \ll_f \|v\|$.*

For any irreducible representation π of $G(F_\infty)$ we will write

$$\Lambda_\pi = \sqrt{\lambda_\pi^2 + \lambda_\tau^2} \tag{11}$$

where τ is a minimal $\mathbf{K}_\infty$-type of π and λ_π (resp. λ_τ) is the Casimir eigenvalue of π (resp. τ). (λ_τ does not depend on the choice of τ.) We will also write $\|\tau\|$ for the norm of the highest weight of τ, that is $\lambda_\tau = \|\tau\|^2$.

Corollary 2.6 *The following two conditions are equivalent for $\phi \in C^\infty(G(F)\backslash G(\mathbb{A}))$.*

1. *For any $X \in \mathcal{U}(\mathfrak{g})$ and $n \in \mathbb{N}$ we have $\sigma^n \cdot R(X)\phi \in L^2(G(F)\backslash G(\mathbb{A}))$.*
2. *For any $X \in \mathcal{U}(\mathfrak{g})$ and $n \in \mathbb{N}$ we have*

$$\sup_{x\in\mathfrak{s}} \Xi(x)^{-1}\sigma(x)^n\, |R(X)\phi(x)| < \infty. \tag{12}$$

Denote by $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$ the space of functions satisfying the conditions above. This space is invariant under translation by $G(\mathbb{A})$. Given a level K of G, the two sets of seminorms $\|\sigma^n \cdot R(X)\phi\|_{L^2(G(F)\backslash G(\mathbb{A}))}$, $X \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N}$ and (12) give rise to the same Fréchet space $\mathcal{S}(G(F)\backslash G(\mathbb{A}))^K$ of right-K-invariant functions in $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$.

In one direction this follows from (9). In the other direction this follows from Corollary 2.3 and Lemma 2.5, applied, for any $n \in \mathbb{N}$, to the space of right K-invariant functions φ on $G(F)\backslash G(\mathbb{A})$ such that $\sigma^n \varphi \in L^2(G(F)\backslash G(\mathbb{A}))$.

We call $\mathcal{S}(G(F)\backslash G(\mathbb{A})) = \cup_K \mathcal{S}(G(F)\backslash G(\mathbb{A}))^K$, equipped with the inductive limit topology, the Harish-Chandra Schwartz space of $G(F)\backslash G(\mathbb{A})$.

Similarly, let $P = M \ltimes U$ be a standard parabolic subgroup of G. Define

$$\mathfrak{s}^P = \{g \in G(\mathbb{A}) : \langle \alpha, H(g) - T_1 \rangle > 0 \text{ for all } \alpha \in \Delta_0^M\}.$$

We have

1. $G(\mathbb{A}) = P(F)\mathfrak{s}^P$.
2. There exists C such that $\left|P_0(F)\backslash(\mathfrak{s}^P \cap P(F)x)\right| \le C$ and the diameter of $H(P_0(F)\mathfrak{s}^P \cap P(F)x)$ is bounded by C for any $x \in G(\mathbb{A})$.

Define the function Ξ^P on $\mathfrak{s}^P$ by

$$\Xi^P(pk) = \delta_0(p)^{\frac{1}{2}}, \quad p \in P_0(\mathbb{A}), \ k \in \mathbf{K}, \ \langle \alpha, H(p) - T_1 \rangle > 0 \text{ for all } \alpha \in \Delta_0^M.$$

For $g \in G(\mathbb{A})$ let $\sigma^P(g) = 1 + \min_{x \in \mathfrak{s}^P \cap P(F)g} \|H(x)\|$. Clearly, σ is left $U(\mathbb{A})P(F)$-invariant.

We will define the Harish-Chandra Schwartz space of $P(F)U(\mathbb{A})\backslash G(\mathbb{A})$ (denoted by $\mathcal{S}(P(F)U(\mathbb{A})\backslash G(\mathbb{A}))$) to be the inductive limit over K of the spaces of right-K-invariant functions $\phi \in C^\infty(P(F)U(\mathbb{A})\backslash G(\mathbb{A}))$ such that the seminorms

$$\|(\sigma^P)^n \cdot R(X)\phi\|_{L^2(P(F)U(\mathbb{A})\backslash G(\mathbb{A}))}, \quad X \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N},$$

are finite. Alternatively, we could use the equivalent sequence of seminorms

$$\sup_{x \in \mathfrak{s}^P} \Xi^P(x)^{-1} \sigma^P(x)^n \left|R(X)\phi(x)\right|, \quad X \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N}.$$

It is also easy to see that the map $\phi \mapsto (k \mapsto \delta_P^{-\frac{1}{2}} \phi(\cdot k))$ defines a topological isomorphism between $\mathcal{S}(P(F)U(\mathbb{A})\backslash G(\mathbb{A}))$ and the space of smooth functions $f : \mathbf{K} \to \mathcal{S}(M(F)\backslash M(\mathbb{A}))$ which are right-K-invariant for some level K and such that $f(umk) = f(k)[\cdot m]$ for any $u \in U(\mathbb{A}) \cap \mathbf{K}$, $m \in M(\mathbb{A}) \cap \mathbf{K}$, $k \in \mathbf{K}$.

Note that

$$\sigma\big|_{\mathfrak{s}} \ll \sigma^P\big|_{\mathfrak{s}} \ll \sigma\big|_{\mathfrak{s}} \quad \text{and} \quad \Xi^P\big|_{\mathfrak{s}} = \Xi. \tag{13}$$

3 Growth conditions

3.1 (G, M)-families

Following Arthur ([Art81]), a (G, M)-family is a collection of smooth functions $c_P \in C^\infty(\mathrm{i}\mathfrak{a}_M^*)$, $P \in \mathcal{P}(M)$, satisfying the compatibility relations

$$c_P \equiv c_{P'} \text{ on the hyperplane } \langle \Lambda, \alpha^\vee \rangle = 0$$

whenever P, P' are adjacent along the root α, i.e. when $\Sigma_{\overline{P}} \cap \Sigma_{P'} = \{\alpha\}$. For any such a (G, M)-family one defines the function

$$c_M = \sum_{P \in \mathcal{P}(M)} \frac{c_P}{\theta_P}$$

where

$$\theta_P(\Lambda) = v_M^{-1} \prod_{\alpha \in \Delta_P} \langle \Lambda, \alpha^\vee \rangle$$

and v_M is the co-volume of the lattice spanned by the co-roots in $\mathfrak{a}_M$. The basic result [Art81, Lemma 6.2] is that $c_M \in C^\infty(\mathrm{i}\mathfrak{a}_M^*)$.

Occasionally, we will also consider (G, M)-families of meromorphic functions on $\mathfrak{a}_{M,\mathbb{C}}^*$ which are holomorphic on $\mathrm{i}\mathfrak{a}_M^*$. Then c_M will also be a meromorphic function on $\mathfrak{a}_{M,\mathbb{C}}^*$, holomorphic on $\mathrm{i}\mathfrak{a}_M^*$

If $(c_P)_{P \in \mathcal{P}(M)}$ is a (G, M)-family and $Q = L \ltimes V \in \mathcal{F}(M)$ we may consider the (L, M)-family

$$c_R^Q = c_{R \ltimes V} \quad R \in \mathcal{P}^L(M).$$

Note that $R \ltimes V$ is the unique parabolic subgroup of G contained in Q which intersects L in R. Correspondingly $c_M^Q \in C^\infty(\mathrm{i}\mathfrak{a}_M^*)$ is defined.

If $(c_P)_{P \in \mathcal{P}(M)}$ and $(d_P)_{P \in \mathcal{P}(M)}$ are (G, M)-families, then so is their product $(c_P d_P)_{P \in \mathcal{P}(M)}$. To compute $(cd)_M$ we use a formula of Arthur ([Art88, Proposition 7.1 and Corollary 7.4])[1]: there exist constants α_{Q_1,Q_2} for all pairs $Q_1, Q_2 \in \mathcal{F}(M)$, with α_{Q_1,Q_2} nonzero only if $\mathfrak{a}_M^G = \mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2} = \mathfrak{a}_{L_1}^G \oplus \mathfrak{a}_{L_2}^G$, such that

$$(cd)_M = \sum_{Q_1, Q_2 \in \mathcal{F}(M)} \alpha_{Q_1,Q_2} c_M^{Q_1} d_M^{Q_2}. \tag{14}$$

Note that the constants α_{Q_1,Q_2} are not uniquely determined – they depend on certain auxiliary choices. For our purposes the exact value of α_{Q_1,Q_2} is immaterial.

Consider now a (G, M)-family of a special form as in [Art82, §7] (cf. [Lap06, §4]). Namely, suppose that for any reduced root $\beta \in R(T_M, G)$ we are given $c_\beta \in C^\infty(\mathrm{i}\mathbb{R})$ with $c_\beta(0) = 1$. Let

$$c_P(\Lambda) = \prod_{\beta \in \Sigma_P} c_\beta(\langle \Lambda, \beta^\vee \rangle), \quad P \in \mathcal{P}(M). \tag{15}$$

This is clearly a (G, M)-family. By [Lap06, (4.4)] for any $Q = L \ltimes V$ with $L \in \mathcal{L}(M)$ we have

$$c_M^Q(\Lambda) = \sum_{\mathfrak{B}_1, \mathfrak{B}_2} \alpha_{\mathfrak{B}_1,\mathfrak{B}_2} \prod_{\beta \in \mathfrak{B}_1} \frac{c_\beta(\langle \Lambda, \beta^\vee \rangle) - 1}{\langle \Lambda, \beta^\vee \rangle} \prod_{\beta \in \mathfrak{B}_2} c_\beta(\langle \Lambda, \beta^\vee \rangle), \tag{16}$$

where the sum is over disjoint subsets $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq R(T_M, Q)$ such that $\mathfrak{B}_2 \supseteq R(T_M, V)$ and $\mathfrak{B}_1$ forms a basis for $(\mathfrak{a}_M^L)^*$, and $\alpha_{\mathfrak{B}_1,\mathfrak{B}_2}$ are certain constant (whose values are unimportant for us).

We will use one more elementary fact about (G, M)-families.

[1] This is only stated for the value at 0 but the argument is valid for any Λ.

Lemma 3.1 *Let $\delta > 0$. Suppose that $(c_P(\lambda))_{P\in\mathcal{P}(M)}$ is a (G,M)-family of functions which are holomorphic and bounded on the strip $\|\mathrm{Re}\,\lambda\| < \delta$. Then for any $\epsilon > 0$, c_M is bounded on $\|\mathrm{Re}\,\lambda\| < \delta - \epsilon$ and in this region*

$$c_M(\lambda) \ll_{\delta,\epsilon} \sum_{P\in\mathcal{P}(M)} \sup_{\|\mathrm{Re}\,\lambda\|<\delta} |c_P(\lambda)| .$$

Proof Fix $\lambda_0 \in \mathfrak{a}_M^*$ in general position such that $\|\lambda_0\| = 1$. Let

$$c_0 = \min_{\alpha\in R(T_M,G)} |\langle \lambda_0, \alpha^\vee\rangle|$$

and assume that $c_0 > 0$. Let $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. By the pigeonhole principle there exists $0 \le r \le \epsilon$ such that

$$\Big| |\langle\lambda,\alpha^\vee\rangle| - r\,|\langle\lambda_0,\alpha^\vee\rangle| \Big| > c_0 \frac{\epsilon}{2\,|R(T_M,G)|+1}$$

for all $\alpha \in R(T_M,G)$. For such r we have

$$\theta_P(\lambda + z\lambda_0)^{-1} \ll_{\delta,\epsilon} 1$$

for all $z \in \mathbb{C}$ with $|z| = r$. Hence, if $\|\mathrm{Re}\,\lambda\| < \delta - \epsilon$ then

$$|c_M(\lambda)| \le \sup_{|z|=r} |c_M(\lambda + z\lambda_0)| \ll_{\delta,\epsilon} \sum_{P\in\mathcal{P}(M)} \sup_{\|\mathrm{Re}\,\lambda\|<\delta} |c_P(\lambda)| .$$

□

3.2 Intertwining operators

Let M be a Levi subgroup. Consider the discrete part $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ of $L^2(A_M M(F)\backslash M(\mathbb{A}))$, namely the closure of the sum of the irreducible subrepresentations of $L^2(A_M M(F)\backslash M(\mathbb{A}))$. We write

$$L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})) = \hat{\oplus}_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))} L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))_\pi \qquad (17)$$

where $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))_\pi$ is the π-isotypic component of $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$. For any level K_M of M we write $\Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$ for the subset of $\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ consisting of the π's such that the K_M-fixed part is non-zero.

For any $P \in \mathcal{P}(M)$ we write L^2_P for $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ in the sense of L^2-induction. We can identify L^2_P with the Hilbert space $L^2(A_M M(F)U(\mathbb{A})\backslash G(\mathbb{A}), \delta_P^{\frac12})$ of measurable functions $\varphi : M(F)U(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$ (up to functions which vanish almost everywhere) such that for almost all $g \in G(\mathbb{A})$ the function $m \mapsto \delta_P(m)^{-\frac12}\varphi(mg)$ belongs to $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ and $\|\varphi\|_P^2 := \int_{M(F)A_M U(\mathbb{A})\backslash G(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty$. Corresponding to (17) we write $L^2_P = \hat{\oplus}_\pi L^2_{P,\pi}$. We denote the regular representation of $G(\mathbb{A})$ on L^2_P by I_P, or simply by I if P is clear from the context. For any level K of G we write $(L^2_P)^K$ for the K-fixed part of L^2_P. Moreover, for any $\tau \in \widehat{\mathbf{K}_\infty}$ we denote by $(L^2_{P,\pi})^{\tau,K}$ the $(\mathbf{K}_\infty,\tau)$-isotypic part of $(L^2_{P,\pi})^K$. The space $(L^2_{P,\pi})^{\tau,K}$ is finite-dimensional.

We also have the representations $I_P(\lambda) = I(\lambda)$ on L^2_P, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ given by $(I_P(g,\lambda)\varphi)_\lambda(x) = \varphi_\lambda(xg)$, $x, g \in G(\mathbb{A})$ where $\varphi_\lambda(x) = \varphi(x)e^{\langle\lambda,H(x)\rangle}$.

For any $P = M \ltimes U, Q = L \ltimes V \in \mathcal{F}$ and $w \in W(M, L)$ let $M_{Q|P}(w, \lambda) : L^2_P \to L^2_Q$, $\lambda \in \mathrm{i}\mathfrak{a}^*_M$ be the unitary intertwining operators defined in [Art82, §1]. If $M = L$ and $w = 1$ then we simply write $M_{Q|P}$. In general, if $P' = w^{-1}Qw$ then up to an immaterial constant

$$M_{Q|P}(w, \lambda) \text{ is the composition of left translation by } w^{-1} \text{ with } M_{P'|P}(\lambda) \tag{18}$$

[ibid., (1.4)].

For every $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ we denote the restriction of $M_{Q|P}(\lambda)$ to $L^2_{P,\pi}$ by $M_{Q|P}(\pi, \lambda)$. We normalize $M_{Q|P}(\pi, \lambda)$ as in [Art82, §6] (cf. [Art89, Theorem 2.1]). Thus, we write

$$M_{Q|P}(\pi, \lambda) = n_{Q|P}(\pi, \lambda) N_{Q|P}(\pi, \lambda)$$

where $N_{Q|P}$ are the normalized intertwining operators satisfying the properties [Art82, (6.3)–(6.6)] and $n_{Q|P}$ are the normalizing factors. In particular, $N_{Q|P}(\pi, \lambda) = \prod_v N_{Q|P}(\pi_v, \lambda)$ where $N_{Q|P}(\pi_v, \lambda)$, $\lambda \in \mathrm{i}\mathfrak{a}^*_M$ are unitary operators. The normalizing factors are given by

$$n_{Q|P}(\lambda) = \prod_{\beta \in \Sigma_Q \cap \Sigma_{\overline{P}}} n_\beta(\pi, \langle \lambda, \beta^\vee \rangle) = \prod_{\beta \in \Sigma_{\overline{Q}} \cap \Sigma_P} n_\beta(\pi, \langle \lambda, \beta^\vee \rangle)^{-1} \tag{19}$$

where $n_\beta(\pi, z)$ are certain meromorphic functions on $\mathbb{C}$ which are holomorphic on $\mathrm{i}\mathbb{R}$. We have the functional equations

$$n_{-\beta}(\pi, z) n_\beta(\pi, -z) = 1,$$
$$\overline{n_\beta(\pi, z)} = n_{-\beta}(\pi, \bar{z}).$$

Thus,

$$n_\beta(\pi, z)^{-1} = \overline{n_\beta(\pi, -\bar{z})}, \tag{20}$$

so that $|n_\beta(\pi, \mathrm{i}t)| = 1$ for $t \in \mathbb{R}$.

3.3

We will now state the first analytic condition which is crucial for our analysis. Recall the notation (11).

Definition 3.2 We say that G satisfies uniform temperedness (UT) if there exist $k, l > 0$ such that for any maximal parabolic subgroup $P = M \ltimes U$ of G and level K_M of M, there exists a constant $c > 0$ such that for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$ we have

$$n_\alpha(\pi, z) \ll_{K_M} (1 + \Lambda_{\pi_\infty} + |z|)^k \tag{21}$$

in the region $|\mathrm{Re}\, z| < c(1 + \Lambda_{\pi_\infty} + |\mathrm{Im}\, z|)^{-l}$, where $\Sigma_P = \{\alpha\}$.

This growth condition was introduced (in a slightly different form) in [Lap06]. Of course, if we are not interested in optimizing the exponents we could take $k = l$.

If G satisfies (UT) then by Cauchy's formula we get a bound similar to (21) (with a larger k and with c replaced by $c/2$) for any derivative of $n_\alpha(\pi, z)$. By the mean value theorem we also get

$$\frac{n_\alpha(\pi, z) - n_\alpha(\pi, z')}{z - z'} \ll_{K_M} (1 + \Lambda_{\pi_\infty} + |z| + |z'|)^k \tag{22}$$

provided that $|\mathrm{Re}\, z| < c(1 + \Lambda_{\pi_\infty} + |\mathrm{Im}\, z|)^{-l}$ and $|\mathrm{Re}\, z'| < c(1 + \Lambda_{\pi_\infty} + |\mathrm{Im}\, z'|)^{-l}$.

Lemma 3.3 *Suppose that G and its Levi subgroups satisfy (UT) and let $M \in \mathcal{L}$. Then there exists $l > 0$ and*

- *for any $D \in \mathcal{D}(\mathfrak{a}_M^*)$ there exists $k > 0$*
- *for any level K_M of M there exists $c > 0$*

such that for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$ we have

$$D_\lambda n_{P'|P}(\pi, \lambda) \ll_{D,K_M} (1 + \|\lambda\| + \Lambda_{\pi_\infty})^k$$

in the region

$$\mathcal{R}_{\pi,c,l} = \{\lambda \in \mathfrak{a}_{M,\mathbb{C}}^* : \|\mathrm{Re}\,\lambda\| < c(1 + \Lambda_{\pi_\infty} + \|\mathrm{Im}\,\lambda\|)^{-l}\}.$$

Similarly, for any $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ let $\nu_Q(P, \pi, \lambda, \Lambda)$, $Q \in \mathcal{P}(M)$ be the (G, M)-family (in Λ) given by

$$\nu_Q(P, \pi, \lambda, \Lambda) = n_{Q|P}(\pi, \lambda)^{-1} n_{Q|P}(\pi, \lambda + \Lambda).$$

Then there exists $k > 0$ such that for any π and $R \in \mathcal{F}(M)$ we have

$$\nu_M^R(P, \pi, \lambda, \Lambda) \ll_{K_M} (1 + \|\lambda\| + \|\Lambda\| + \Lambda_{\pi_\infty})^k$$

provided that $\lambda, \lambda + \Lambda \in \mathcal{R}_{\pi,c,k}$.

Proof The first part follows from (19), (21) and the Leibniz rule. To prove the second part we write the (G, M)-family $\nu_Q(P, \pi, \lambda, \Lambda)$ in the form (15) with

$$c_\beta(z) = \begin{cases} n_\beta(\pi, \langle\lambda, \beta^\vee\rangle)^{-1} n_\beta(\pi, \langle\lambda, \beta^\vee\rangle + z) & \text{if } \beta \in \Sigma_{\overline{P}}, \\ 1 & \text{otherwise} \end{cases}$$

(cf. [Art82, p. 1323]). The required estimate follows from (16), (20) and (22).

□

It is reasonable to conjecture that any reductive group satisfies (UT). In fact, it is conceivable that we can take *any* $k, l \in \mathbb{R}^{>0}$ in the definition of (UT) (and consequently, in Lemma 3.3). However, at this stage we can only prove (UT) for the general linear group.

3.4

Proposition 3.4 *The group $G = \mathrm{GL}_r$ satisfies (UT).*

We will prove the Proposition below. The proof is based on known analytic properties of the Rankin-Selberg L-function which we now recall. Let π_i be cuspidal representations of $\mathrm{GL}_{n_i}(\mathbb{A})$, $i = 1, 2$, whose central character is trivial on the scalar matrices with (the same) positive real scalar in the archimedean places. (The integers n_1, n_2 as well as the field F will be fixed in the estimates below.) Let $L(s, \pi_1 \times \tilde{\pi}_2)$ and $\epsilon(s, \pi_1 \times \tilde{\pi}_2)$ be the completed Rankin-Selberg L-function and epsilon factors defined in [JPSS83]. We have the functional equation

$$L(s, \pi_1 \times \tilde{\pi}_2) = \epsilon(s, \pi_1 \times \tilde{\pi}_2) L(1 - s, \tilde{\pi}_1 \times \pi_2)$$

where $\epsilon(s, \pi_1 \times \tilde{\pi}_2) = \epsilon_0 q^{\frac{1}{2}-s}$ for some ϵ_0 of modulus 1 and $q \in \mathbb{N}$ (the arithmetic conductor of $\pi_1 \times \tilde{\pi}_2$). The function $[s(1-s)]^{\delta_{\pi_1,\pi_2}} L(s, \pi_1 \times \tilde{\pi}_2)$ is an entire function of order one where $\delta_{\pi_1,\pi_2} = 1$ if $\pi_2 = \pi_1$ and 0 otherwise (see e.g. [RS96]).

We write

$$L(s, \pi_1 \times \tilde{\pi}_2) = L_\infty(s, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty}) L^\infty(s, \pi_1 \times \tilde{\pi}_2)$$

where $L_\infty(s, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})$ is the Archimedean part of the Rankin-Selberg L-function. We write the latter as $\prod_{j=1}^m \Gamma_{\mathbb{R}}(s - \alpha_j)$ for certain complex parameters $\alpha_1, \dots, \alpha_m$ where $m = n_1 n_2 [F : \mathbb{Q}]$. Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ (for the usual Γ function). By the Jacquet-Shalika bounds, $L_\infty(s, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})$ is holomorphic for $\operatorname{Re} s \geq 1$, so that $\operatorname{Re} \alpha_j < 1$ for all j. In fact, by [MS04, Proposition 3.3] (which is based on [LRS99]) we have

$$1 - \operatorname{Re} \alpha_j > \frac{1}{n_1^2 + 1} + \frac{1}{n_2^2 + 1}. \tag{23}$$

Following [IS00] we define the analytic conductor $\mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s) = q \prod_{j=1}^m (1 + |s - \alpha_j|)$ and $\mathfrak{c}(\pi_1 \times \tilde{\pi}_2) = \mathfrak{c}(\pi_1 \times \tilde{\pi}_2, 0)$. Similarly, we define $\mathfrak{c}(\pi)$ for any cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$.

We have

$$\mathfrak{c}(\pi_1 \times \tilde{\pi}_2) \leq \mathfrak{c}(\pi_1)^{n_2} \mathfrak{c}(\pi_2)^{n_1} \tag{24}$$

The p-adic aspect follows from [BH97] and the Archimedean aspect boils down to an easy GL_2 computation (cf. [RS96, Appendix]).

A basic fact which is proved using the Phragmén-Lindelöf principle is polynomial growth on vertical strips for $L^\infty(s, \pi_1 \times \tilde{\pi}_2)$. For our purposes it suffices to know that there exist $k > 0$ and $c > 0$ (independent of π_1, π_2) such that

$$\left(\frac{s-1}{s}\right)^{\delta_{\pi_1,\pi_2}} L^\infty(s, \pi_1 \times \tilde{\pi}_2) \ll \mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s)^k, \tag{25}$$

in the region $\operatorname{Re} s \geq 1 - c\mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s)^{-k}$. As before, this implies bounds of a similar nature for all derivatives. In fact it follows from the results of [Li10] that we can take *any* $k \in \mathbb{R}^{>0}$ but we will not need to use this fact.[2] This is still far off from the "correct" estimates which can be obtained assuming both the generalized Riemann Hypothesis and the Ramanujan Hypothesis, e.g.

$$\left(\frac{s-1}{s}\right)^{\delta_{\pi_1,\pi_2}} L^\infty(s, \pi_1 \times \tilde{\pi}_2) \ll (\log \log \mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s))^m, \quad \operatorname{Re} s = 1.$$

Another property that we will use is the following result due to Brumley ([Bru06], see also Appendix A below) giving coarse lower bounds on Rankin-Selberg L-functions at the edge of the critical strip.

Proposition 3.5 (Brumley) *There exist $k > 0$ and $c > 0$ (independent of π_1, π_2) such that*

$$\left(\frac{s}{s-1}\right)^{\delta_{\pi_1,\pi_2}} L^\infty(s, \pi_1 \times \tilde{\pi}_2)^{-1} \ll (\mathfrak{c}(\pi_1) + \mathfrak{c}(\pi_2) + |s|)^k \tag{26}$$

in the region $\operatorname{Re} s \geq 1 - c\mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s)^{-k}$.

[2] The case $\pi_1 = \pi_2$ is not formally covered by [ibid.] but it follows from the technique.

Remark 3.6 The bound (26) can certainly be improved in many cases. For instance, it is well known that

$$\zeta(s)^{-1} \ll \log(1+|s|), \quad \operatorname{Re} s = 1,$$

and under the Riemann Hypothesis we have

$$\zeta(s)^{-1} \ll \log\log(2+|s|), \quad \operatorname{Re} s = 1,$$

which is (up to determining the constant) best possible. More generally, as in the case of the upper bounds, if we assume both the generalized Ramanujan Hypothesis and the generalized Riemann Hypothesis then we would have the "correct" bound

$$\left(\frac{s}{s-1}\right)^{\delta_{\pi_1,\pi_2}} L^{\infty}(s, \pi_1 \times \tilde{\pi}_2)^{-1} \ll (\log\log \mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s))^m, \quad \operatorname{Re} s = 1.$$

For our purposes the bound (26) suffices.

Note that it follows from Stirling's formula that for any $c > 0$ there exists $k > 0$ such that

$$\frac{\Gamma_{\mathbb{R}}(-s+\overline{\beta})}{\Gamma_{\mathbb{R}}(s+\beta)} \ll_c 1 + |s+\beta|^k \tag{27}$$

uniformly for $\operatorname{Re} s \in [-c/2, c/2]$ and $\operatorname{Re} \beta \geq c$.

Proof (of Proposition 3.4) Assume that $\pi = \pi_1 \otimes \pi_2 \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$. Writing $\Sigma_{\overline{P}} = \{\beta\}$ we have

$$\xi(s) := n_\beta(\pi, -s) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{\epsilon(s, \pi_1 \times \tilde{\pi}_2) L(s+1, \pi_1 \times \tilde{\pi}_2)} = \frac{L(1-s, \tilde{\pi}_1 \times \pi_2)}{L(1+s, \pi_1 \times \tilde{\pi}_2)}.$$

Assume first that π is cuspidal. We write ξ as the product of the following three factors

$$A(s) = \left(\frac{s}{1-s}\right)^{\delta_{\pi_1,\pi_2}} L^{\infty}(1-s, \tilde{\pi}_1 \times \pi_2),$$
$$B(s) = \prod_{j=1}^{m} \frac{\Gamma_{\mathbb{R}}(1-s-\overline{\alpha_j})}{\Gamma_{\mathbb{R}}(1+s-\alpha_j)},$$
$$C(s) = \left(\frac{1-s}{s}\right)^{\delta_{\pi_1,\pi_2}} L^{\infty}(s+1, \pi_1 \times \tilde{\pi}_2)^{-1}.$$

For $A(s)$ we apply (25). For $B(s)$ we apply (27) with $\beta = 1 - \alpha_j$, $j = 1, \dots, m$ and $c = 1/(n_1^2+1) + 1/(n_2^2+1)$ which is applicable by (23). For $C(s)$ we apply (26). To finish the proof in the cuspidal case, it remains to use (24) and to note that $\mathfrak{c}(\pi) \ll_{K_M} \Lambda_{\pi_\infty}$ since the arithmetic conductor is bounded in terms of K_M.

Now, we drop the assumption that π_1 and π_2 are cuspidal. By [MW89] there exist factorizations $n_i = (2d_i+1)m_i$ (d_i half-integers) and cuspidal representations σ_i on $\mathrm{GL}_{m_i}(\mathbb{A})$, $i = 1, 2$ such that π_i is obtained by taking residues of Eisenstein series induced from $\sigma_i \, |\det \cdot|^{d_i} \otimes \cdots \otimes \sigma_i \, |\det \cdot|^{-d_i}$. Assume without loss of generality that $d_1 \leq d_2$. Then

$$\xi(s) = \prod_{j=-d_1}^{d_1} \frac{L(d_2+j+1-s, \tilde{\sigma}_1 \times \sigma_2)}{L(d_2+j+1+s, \sigma_1 \times \tilde{\sigma}_2)}.$$

We can apply the same argument as above for each factor.

□

3.5

We turn to the local aspect of the analytic condition on G. We formulate a property about the local components of an irreducible representation occurring in the discrete spectrum of a Levi subgroup of G.

Definition 3.7 We say that G satisfies the weak Ramanujan property (WR) with constant $c > 0$ if for any

- maximal parabolic subgroup $P = MU$ with corresponding fundamental weight ϖ,
- $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$,
- place v of F,
- level K of G,
- $\tau \in \widehat{\mathbf{K}_\infty}$,

the normalized intertwining operators $N_{\overline{P}|P}(\pi_v, s\varpi)$ on $(L^2_{P,\pi})^{\tau,K}$ is holomorphic in the strip $|\text{Re}\, s| < c$.

A closely related property was considered in [Mül02]. (See Lemma 3.11 below.)

For our purposes we may allow c to depend on K, and even inverse polynomially on Λ_{π_∞}. However, this weaker property doesn't seem to be any easier to prove.

For GL_n the property (WR) was established in [MS04] as a consequence of the results of Luo-Rudnick-Sarnak [LRS99] and the properties of the local intertwining operators. More precisely, we have

Theorem 3.8 *[MS04, Proposition 4.2] The group $G = \text{GL}_n$ (and hence any Levi subgroup of G) satisfies (WR) with constant $c = 2/(n^2 + 1)$.*

Remark 3.9 Following the argument of [MS04] it is easy to see that the Ramanujan Hypothesis for GL_n for all n is equivalent to (WR) for all GL_n's with constant $c = 1$. More generally, we expect (WR) to hold for any G with $c = \frac{1}{2}$ (which is best possible for the symplectic group of rank two).

Lemma 3.10 *Let V be a normed vector space and let $z_1, \dots, z_m \notin \mathbf{S^1}$. Suppose that $A : \mathbb{C} \setminus \{z_1, \dots, z_m\} \to V$ is such that $(z - z_1) \dots (z - z_m)A(z)$ is a polynomial on $\mathbb{C}$ of degree $\leq n$ with coefficients in V. Assume that $\|A(z)\| \leq 1$ for all $z \in \mathbf{S^1}$ and that there exist $r < 1 < R$ such that $|z_i| \notin [r, R]$ for all i. Then for any $0 < \epsilon < (R - r)/2$ and $k = 0, 1, 2 \dots$ we have*

$$\sup_{r+\epsilon<|z|<R-\epsilon} \|A^{(k)}(z)\| \ll_{k,m,n,R,\epsilon} 1. \tag{28}$$

Proof By Cauchy's formula it is enough to prove this for $k = 0$. Consider

$$B(z) = \Big(\prod_{|z_i|<1} \frac{z - z_i}{1 - \overline{z_i} z} \Big) A(z).$$

Then $B(z)$ is holomorphic for $|z| \le 1$ and by the maximum modulus principle we have $\|B(z)\| \le 1$ for $|z| \le 1$. We infer that

$$\|A(z)\| \le \left(\frac{2}{\epsilon}\right)^m$$

for $1 \ge |z| \ge r + \epsilon$. Similarly,

$$z^{n-m}\Big(\prod_{|z_i|>1} \frac{z - z_i^{-1}}{1 - \overline{z_i}^{-1} z}\Big) A(1/z)$$

is holomorphic and bounded in absolute value by 1 for $|z| \le 1$ and therefore

$$\|A(z)\| \le \left(\frac{2R}{\epsilon}\right)^m R^{\max(m,n)}$$

for $1 \le |z| \le R - \epsilon$.

□

For Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ we will write $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ for the Banach space of bounded operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ with the standard operator norm. If $\mathfrak{H}_2 = \mathfrak{H}_1$ we simply write $\mathcal{B}(\mathfrak{H}_1)$.

Lemma 3.11 *Suppose that G and its Levi subgroups satisfy (WR) with constant c. Then there exists $c' > 0$ such that for any $M \in \mathcal{L}$ and $D \in \mathcal{D}(\mathfrak{a}_M^*)$ there exists $k > 0$ such that for any level K of G, $P, P' \in \mathcal{P}(M)$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ we have*

$$\|D_\lambda N_{P'|P}(\pi_v, \lambda)\|_{\mathcal{B}((L^2_{P,\pi})^K, (L^2_{P',\pi})^K)} \ll_{D,K} 1$$

for any finite place v and

$$\|D_\lambda N_{P'|P}(\pi_v, \lambda)\|_{\mathcal{B}((L^2_{P,\pi})^\tau, (L^2_{P',\pi})^\tau)} \ll_D (1 + \|\tau\|)^k$$

for any $v|\infty$ and $\tau \in \widehat{\mathbf{K}_v}$. Both estimates are valid for $\|\mathrm{Re}\,\lambda\| < c'$.

Proof By factoring the intertwining operator and passing to a smaller Levi subgroup if necessary, we can reduce to the case where P is maximal and $P' = \overline{P}$. The archimedean case follows from [MS04, Corollary A.3] (which is stated only for $\lambda \in \mathrm{i}\mathfrak{a}_M^*$ but is valid in the larger region) and the assumption on G. In the non-archimedean case we can write $N_{P'|P}(\pi_v, s\varpi)\big|_{(L^2_{P,\pi})^K} = A_v(q_v^{-s})$ where A_v is a rational function whose degree is bounded in terms of K only. By the assumption on G we can apply Lemma 3.10 with $R = q_v^c$ and $r = q_v^{-c}$. We used of course the fact that only finitely many v's need to be considered (depending on K).

□

Let $M \in \mathcal{L}$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$, $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. Define $(\mathfrak{N}_Q(P, \pi, \lambda, \Lambda))_{Q \in \mathcal{P}(M)}$ to be the operator valued (G, M)-family (in Λ) given by

$$\mathfrak{N}_Q(P, \pi, \lambda, \Lambda) = N_{Q|P}(\pi, \lambda)^{-1} N_{Q|P}(\pi, \lambda + \Lambda) = N_{P|Q}(\pi, \lambda) N_{Q|P}(\pi, \lambda + \Lambda). \qquad (29)$$

From Lemma 3.11 (with $D = \mathrm{Id}$) and Lemma 3.1 we infer

Corollary 3.12 *Suppose that G and its Levi subgroups satisfy (WR). Then there exists $c' > 0$ and $k > 0$ such that for any level K of G we have*

$$\|\mathfrak{N}_M^S(P,\pi,\lambda,\Lambda)\|_{\mathcal{B}((L^2_{P,\pi})^{\tau,K})} \ll_K (1+\|\tau\|)^k$$

for any $S \in \mathcal{F}(P)$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\tau \in \widehat{\mathbf{K}_\infty}$ provided that $\|\mathrm{Re}\,\lambda\|, \|\mathrm{Re}\,\Lambda\| < c'$.

3.6

Finally, we combine the properties (UT) and (WR). Namely, we say that G satisfies property (HP) (hereditary property) if G and its Levi subgroups satisfy (UT) and (WR). By the above, the group $G = \mathrm{GL}_n$ satisfies (HP), and we expect that any reductive group satisfies (HP).

From Lemmas 3.3 and 3.11 and (18) we conclude

Corollary 3.13 *Suppose that G satisfies (HP) and let $M \in \mathcal{L}$. Then there exists $l > 0$ and*

- *for any $D \in \mathcal{D}(\mathfrak{a}_M^*)$ there exists $k > 0$,*
- *for any level K there exists $c > 0$,*

such that for any $P' \in \mathcal{P}(M')$, $w \in W(M,M')$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\tau \in \widehat{\mathbf{K}_\infty}$ we have

$$\|D_\lambda M_{P'|P}(w,\lambda)\big|_{(L^2_{P,\pi})^{\tau,K}}\|_{\mathcal{B}((L^2_{P,\pi})^{\tau,K},(L^2_{P',w\pi})^{\tau,K})} \ll_K (1+\|\lambda\|+\Lambda_{\pi_\infty}+\|\tau\|)^k$$

for $\lambda \in \mathcal{R}_{\pi,c,l}$.

Let $(\mathfrak{M}_Q(P,\lambda,\Lambda))_{Q\in\mathcal{P}(M)}$ be the operator valued (G,M)-family

$$\mathfrak{M}_Q(P,\lambda,\Lambda) = M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda+\Lambda)\big|_{(L^2_P)^{\mathbf{K}\text{-fin},\mathfrak{z}\text{-fin}}} \tag{30}$$

where

$$(L^2_P)^{\mathbf{K}\text{-fin},\mathfrak{z}\text{-fin}} = \sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})),\tau\in\widehat{\mathbf{K}_\infty},K} (L^2_{P,\pi})^{\tau,K}$$

is the $\mathfrak{z}$-finite, $\mathbf{K}$-finite part of L^2_P (where K ranges over the compact open subgroups of $G(\mathbb{A}_{\mathrm{fin}})$). We write $\mathfrak{M}_Q(P,\pi,\lambda,\Lambda)$ for the restriction of $\mathfrak{M}_Q(P,\lambda,\Lambda)$ to $L^2_{P,\pi}$.

Corollary 3.14 *Assume that G satisfies (HP). Then there exists $k > 0$ and for any level K there exists $c > 0$ such that for any $M \in \mathcal{L}$, $R \in \mathcal{F}(M)$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\tau \in \widehat{\mathbf{K}_\infty}$ we have*

$$\|\mathfrak{M}_M^R(P,\pi,\lambda,\Lambda)\|_{\mathcal{B}((L^2_{P,\pi})^{\tau,K})} \ll_K (1+\|\lambda\|+\|\Lambda\|+\|\tau\|+\Lambda_{\pi_\infty})^k$$

provided that $\lambda, \lambda+\Lambda \in \mathcal{R}_{\pi,c,k}$.

Proof Using the normalization of the intertwining operators we may write

$$\mathfrak{M}_Q(P,\pi,\lambda,\Lambda) = \nu_Q(P,\pi,\lambda,\Lambda)\mathfrak{N}_Q(P,\pi,\lambda,\Lambda)$$

where the (G,M)-family $\nu_Q(P,\pi,\lambda,\Lambda)$ is given by

$$\nu_Q(P,\pi,\lambda,\Lambda) = n_{Q|P}(\pi,\lambda)^{-1}n_{Q|P}(\pi,\lambda+\Lambda)$$

and $\mathfrak{N}_Q$ was defined in (29). Applying the product formula (14) to

$$\mathfrak{M}_Q^R(P,\pi,\lambda,\Lambda) = \nu_Q^R(P,\pi,\lambda,\Lambda)\mathfrak{N}_Q^R(P,\pi,\lambda,\Lambda),$$

it remains to invoke Lemma 3.3 and Corollary 3.12.

□

4 Conjectural spectral decomposition of the Harish-Chandra Schwartz space

Recall the space L^2_P defined in the previous section. We denote its smooth part by $\mathcal{A}_P$. Thus, $\mathcal{A}_P$ is the space of $\varphi \in C^\infty(M(F)U(\mathbb{A})\backslash G(\mathbb{A}))$ such that for all $X \in \mathcal{U}(\mathfrak{g})$ and $g \in G(\mathbb{A})$ the function $m \mapsto \delta_P(m)^{-\frac{1}{2}} X\varphi(mg)$ belongs to $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$. We endow $\mathcal{A}_P$ with the inductive limit of $\mathcal{A}_P^K = \mathcal{A}_P \cap (L^2_P)^K$ topologized by the seminorms $\|X\varphi\|_P$, $X \in \mathcal{U}(\mathfrak{g})$. We similarly write $\mathcal{A}_{P,\pi}$.

Let $\mathcal{A}_{P,\mathfrak{z}\text{-fin}}$ be the subspace of $\mathcal{A}_P$ consisting of $\mathfrak{z}$-finite functions. It is the algebraic direct sum of $\mathcal{A}_{P,\pi}$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$.

The following results are standard.

Lemma 4.1 *For any $X \in \mathcal{U}(\mathfrak{g})$ there exist a seminorm μ on $\mathcal{A}_P$ and $m \in \mathbb{N}$ such that $\|I(X,\lambda)\varphi\|_P \le (1+\|\lambda\|)^m \mu(\varphi)$ for all $\varphi \in \mathcal{A}_P$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.*

Lemma 4.2 *The topology on $\mathcal{A}_P^K$ is given by the seminorms*

$$\Big(\sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^K, \tau \in \widehat{\mathbf{K}_\infty}} (1 + \Lambda_{\pi_\infty} + \|\tau\|)^k \|p^K_{\pi,\tau}\varphi\|_P^2 \Big)^{\frac{1}{2}}, \quad k \in \mathbb{N},$$

where $p^K_{\pi,\tau} : (L^2_P)^K \to (L^2_{P,\pi})^{\tau,K}$ is the orthogonal projection.

Recall that $\mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_P)$ is the union over K of the Fréchet spaces $\mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}^K_P)$ of smooth functions $\varphi : \mathrm{i}\mathfrak{a}^*_M \to \mathcal{A}^K_P$ such that the seminorms

$$\sup_{\lambda \in \mathrm{i}\mathfrak{a}^*_M} (1+\|\lambda\|)^n \|I(X)(D_\lambda \varphi(\lambda))\|_P$$

are finite for any $X \in \mathcal{U}(\mathfrak{g})$ and $D \in \mathcal{D}(\mathrm{i}\mathfrak{a}^*_M)$. We endow $\mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_P)$ with the inductive limit topology. By Lemma 4.2, we have

Lemma 4.3 *The topology on $\mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}^K_P)$ is given by the seminorms*

$$\sup_{\lambda \in \mathrm{i}\mathfrak{a}^*_M} \Big(\sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^K, \tau \in \widehat{\mathbf{K}_\infty}} (1 + \Lambda_{\pi_\infty} + \|\tau\| + \|\lambda\|)^k \|p^K_{\pi,\tau} D_\lambda \varphi(\lambda)\|_P^2 \Big)^{\frac{1}{2}},$$

$D \in \mathcal{D}(\mathrm{i}\mathfrak{a}^*_M)$, $k \in \mathbb{N}$.

Let $P = M \ltimes U$ and $P' = M' \ltimes U'$ be standard parabolic subgroups and let $w \in W(M, M')$. For any $\varphi \in \mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_P)$ define $\mathcal{I}_w\varphi(w\lambda) = M_{P'|P}(w,\lambda)\varphi(\lambda)$. By Lemma 4.3 and Corollary 3.13 we get

Lemma 4.4 *Suppose that G satisfies the property (HP). Then the map $\varphi \mapsto \mathcal{I}_w\varphi$ defines an isomorphism of topological vector spaces $\mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_P) \xrightarrow{\sim} \mathcal{S}(\mathrm{i}\mathfrak{a}^*_{M'}; \mathcal{A}_{P'})$.*

Denote by $\mathcal{M}$ the set of standard Levi subgroups.

Define

$$(\oplus \mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_P))^W = \{(f_M)_{M \in \mathcal{M}} : f_M \in \mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_P), f_{M'} \equiv \mathcal{I}_w(f_M), \forall w \in W(M,M')\}.$$

Similarly define $C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P)$ to be the space of compactly supported smooth functions $\varphi : \mathrm{i}\mathfrak{a}_M^* \to \mathcal{A}_P$ with image in $\mathcal{A}_P^K$ for some K, and let

$$(\oplus C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P))^W = \{(\varphi_M)_{M\in\mathcal{M}} \in (\oplus \mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P))^W : \varphi_M \in C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P)\}.$$

We also define $L^2(\mathrm{i}\mathfrak{a}_M^*; L_P^2)$ to be the space of measurable functions $\varphi : \mathrm{i}\mathfrak{a}_M^* \to L_P^2$ such that $\int_{\mathrm{i}\mathfrak{a}_M^*} \|\varphi(\lambda)\|_P^2 \, d\lambda < \infty$ (up to functions which are zero almost everywhere) and

$$\begin{aligned}(\oplus L^2(\mathrm{i}\mathfrak{a}_M^*; L_P^2))^W = \{&(\varphi_M)_{M\in\mathcal{M}} : \varphi_M \in L^2(\mathrm{i}\mathfrak{a}_M^*; L_P^2), \\ &\varphi_{M'}(w\lambda) = M_{P'|P}(w,\lambda)\varphi_M(\lambda) \text{ for almost all } \lambda \in \mathrm{i}\mathfrak{a}_M^*, \forall w \in W(M,M')\},\end{aligned}$$

with the inner product

$$\|(\varphi_M)\|^2 = \sum_{M\in\mathcal{M}} n_M \int_{\mathrm{i}\mathfrak{a}_M^*} \|\varphi_M(\lambda)\|_P^2 \, d\lambda$$

where $n_M = \sum_{M'\in\mathcal{M}} |W(M,M')|$.

We write $C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathfrak{z}\text{-fin}}) = \oplus_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))} C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\pi})$.

Recall the Eisenstein series $E(\varphi,\lambda)$. The Eisenstein transform

$$\mathcal{E} : C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathfrak{z}\text{-fin}}) \to C^\infty(G(F)\backslash G(\mathbb{A}))$$

is given by

$$\mathcal{E}(\varphi) = \int_{\mathrm{i}\mathfrak{a}_M^*} E(\varphi(\lambda),\lambda) \, d\lambda.$$

By Langlands, $\mathcal{E}$ extends to a continuous linear map of Hilbert spaces

$$\tilde{\mathcal{E}} : L^2(\mathrm{i}\mathfrak{a}_M^*; L_P^2) \to L^2(G(F)\backslash G(\mathbb{A}))$$

which induces an isomorphism of Hilbert spaces

$$\tilde{\mathcal{E}}^W : (\oplus L^2(\mathrm{i}\mathfrak{a}_M^*; L_P^2))^W \xrightarrow{\sim} L^2(G(F)\backslash G(\mathbb{A})).$$

The inverse of $\tilde{\mathcal{E}}$ is given by $\phi \mapsto (\varphi_M)_M$ where

$$(\varphi_M(\lambda),\psi)_P = \frac{1}{n_M}(\phi, E(g,\psi,\lambda))_{G(F)\backslash G(\mathbb{A})}, \quad \lambda \in \mathrm{i}\mathfrak{a}_M^*, \ \psi \in \mathcal{A}_P.$$

Following Harish-Chandra in the local case (cf. [Art75, Wal92, Wal03]) it is natural to make the following conjecture.

Conjecture 4.1 *The map $\mathcal{E}$ uniquely extends to a continuous linear map*

$$\mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P) \to \mathcal{S}(G(F)\backslash G(\mathbb{A}))$$

which induces an isomorphism of topological vector spaces

$$(\oplus \mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P))^W \xrightarrow{\sim} \mathcal{S}(G(F)\backslash G(\mathbb{A})).$$

Note that a special case of the conjecture is that $\mathcal{A}_G \subseteq \mathcal{S}(G(F)\backslash G(\mathbb{A}))$ if $A_G = 1$. This assertion already seems nontrivial. (The local analogue is the finiteness of discrete series representations with a given K-type.) On the other hand, it is well known that the cuspidal part of $\mathcal{A}_G$ is contained in $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$ (if $A_G = 1$). Our modest goal in this paper is to prove a cuspidal version of Conjecture 4.1 for groups satisfying (HP). More precisely, let $\mathcal{A}_{P,\mathrm{cusp}}$ be the cuspidal part of $\mathcal{A}_P$ and denote by $\mathcal{E}_{\mathrm{cusp}}$ the restriction of $\mathcal{E}$ to $C_c^\infty(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp},\mathfrak{z}\text{-fin}})$. Similarly for $\tilde{\mathcal{E}}_{\mathrm{cusp}}$. Let $L^2_c(G(F)\backslash G(\mathbb{A}))$ be the image of $\oplus L^2(\mathrm{i}\mathfrak{a}_M^*; L^2_{P,\mathrm{cusp}})$ under $\tilde{\mathcal{E}}_{\mathrm{cusp}}$. It is a closed subspace of $L^2(G(F)\backslash G(\mathbb{A}))$. Let $\mathcal{S}_c(G(F)\backslash G(\mathbb{A})) = L^2_c(G(F)\backslash G(\mathbb{A})) \cap \mathcal{S}(G(F)\backslash G(\mathbb{A}))$, a closed subspace of $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$. Then we have

Theorem 4.5 *Assume that G satisfies (HP). Then the map $\mathcal{E}_{\mathrm{cusp}}$ uniquely extends to a continuous linear map*

$$\mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}}) \to \mathcal{S}_c(G(F)\backslash G(\mathbb{A}))$$

which gives rise to an isomorphism of topological vector spaces

$$(\oplus \mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}}))^W \xrightarrow{\sim} \mathcal{S}_c(G(F)\backslash G(\mathbb{A})).$$

5 Majorization of cuspidal Eisenstein series

The key step in the proof of Theorem 4.5, which is of independent interest, will be the following majorization of Eisenstein series in the spirit of [Lap06, Proposition 6.1].

Throughout this section we assume that G satisfies (HP).

Proposition 5.1 *For any $X \in \mathcal{U}(\mathfrak{g})$ there exists $k > 0$ such that*

$$\varphi \mapsto \sup_{\lambda \in \mathrm{i}\mathfrak{a}_M^*} (1 + \|\lambda\|)^{-k} \sup_{x \in \mathfrak{s}^1} \Xi(x)^{-1} \sigma(x)^{(r(G)-r(P))/2} |XE(x, \varphi, \lambda)|$$

is a continuous seminorm on $\mathcal{A}_{P,\mathrm{cusp}}$. Moreover, for any $N > 1$ there exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G(F_\infty))$ and level K of G we have

$$[f * E(\varphi, \lambda)](x) = E(x, I(f, \lambda)\varphi, \lambda) \ll_{f,N,K} \Xi(x)\sigma(x)^{(r(P)-r(G))/2}(1 + \|\lambda\|)^{-N} \|\varphi\|_P \quad (31)$$

for any $\varphi \in \mathcal{A}^K_{P,\mathrm{cusp}}$, $\lambda \in \mathrm{i}\mathfrak{a}_M^*$, $x \in \mathfrak{s}^1$.

Proof Note that by Lemma 2.5 and Lemma 4.1 the first part of the proposition follows from the second part. In fact, the first part is equivalent to the following statement: there exists $k > 0$ such that for any level K of G there exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G(F_\infty))$ we have

$$E(x, I(f, \lambda)\varphi, \lambda) \ll_{f,K} \Xi(x)\sigma(x)^{(r(P)-r(G))/2}(1 + \|\lambda\|)^k \|\varphi\|_P$$

for any $\varphi \in \mathcal{A}^K_{P,\mathrm{cusp}}$, $\lambda \in \mathrm{i}\mathfrak{a}_M^*$, $x \in \mathfrak{s}^1$.

To prove the bound (31) we may assume that $f = f_1 * f_2$ for sufficiently smooth $f_1, f_2 \in C_c(G(F_\infty))$, since any $f \in C_c^m(G(F_\infty))$ is a linear combination of these. Corollary 2.4 applied to $\phi = f_2 * E(\varphi, \lambda)$ and $f = f_1$ will reduce the Proposition to the following Lemma.

Lemma 5.2 *For any $N > 1$ there exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G(F_\infty))$ and level K of G we have*

$$\|\Lambda^T E(I(f,\lambda)\varphi,\lambda)\|_{L^2(G(F)\backslash G(\mathbb{A})^1)} \ll_{f,N,K} \|T\|^{(r(P)-r(G))/2}(1+\|\lambda\|)^{-N}\|\varphi\|_P$$

for any $\varphi \in \mathcal{A}^K_{P,\mathrm{cusp}}$, $\lambda \in \mathrm{i}\mathfrak{a}_M^*$, and $T \in \mathfrak{a}_{0,+}$ sufficiently regular.

Proof We recall the Maass-Selberg relations worked out in [Lan76] and [Art80]. We will follow the discussion in [Art82]. As in [ibid., p. 1310] consider the (G,M)-families (in $\Lambda \in \mathrm{i}\mathfrak{a}_M^*$)

$$c_Q(T,\Lambda) = e^{\langle \Lambda, Y_Q(T)\rangle},$$
$$\mathfrak{M}_Q^T(P,\lambda,\Lambda) = c_Q(T,\Lambda)\mathfrak{M}_Q(P,\lambda,\Lambda),$$

where the Y_Q's are certain affine functions which we don't need to know explicitly and $\mathfrak{M}_Q$ was defined in (30). Then

$$\|\Lambda^T E(\varphi,\lambda)\|^2_{L^2(G(F)\backslash G(\mathbb{A})^1)} = \sum_{s\in W(M,M)} (\mathfrak{M}_M^T(P,\lambda,s\lambda-\lambda)M_{P|P}(s,\lambda)\varphi,\varphi)_P. \qquad (32)$$

We use the product formula (14). It is clear from the formula [ibid., (3.1)] that for any $Q_1 \in \mathcal{F}(M)$ we have $c_M^{Q_1}(T,\Lambda) \ll \|T\|^{r(P)-r(Q_1)}$. By Lemma 4.1 it remains to show that any $N > 1$, $Q_2 \in \mathcal{F}(M)$ and $s \in W(M,M)$ there exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G(F_\infty))$ and level K we have

$$\sup_{\lambda\in\mathrm{i}\mathfrak{a}_M^*}(1+\|\lambda\|)^N\|\mathfrak{M}_M^{Q_2}(P,\lambda,s\lambda-\lambda)M_{P|P}(s,\lambda)I(f,\lambda)\varphi\|_P \ll_{f,N,K} \|\varphi\|_P$$

for all $\varphi \in \mathcal{A}_P^K$. Since $M_{P|P}(s,\lambda)$ is unitary and $M_{P|P}(s,\lambda)I(f,\lambda) = I(f,s\lambda)M_{P|P}(s,\lambda)$, it suffices to prove that

$$\sup_{\lambda\in\mathrm{i}\mathfrak{a}_M^*}(1+\|\lambda\|)^N\|\mathfrak{M}_M^{Q_2}(P,\lambda,s\lambda-\lambda)I_P(f,s\lambda)\|_{\mathcal{B}((L_P^2)^K)} < \infty,$$

or equivalently, that

$$\sup_{\lambda\in\mathrm{i}\mathfrak{a}_M^*}(1+\|\lambda\|)^N \sup_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))}\|\mathfrak{M}_M^{Q_2}(P,\pi,\lambda,s\lambda-\lambda)I_P(f,\pi,s\lambda)\|_{\mathcal{B}((L^2_{P,\pi})^K)} < \infty.$$

Expand $f = \sum_{\tau\in\widehat{\mathbf{K}_\infty}} f_\tau$ according to the left action of $\mathbf{K}_\infty$. Then

$$\begin{aligned}&\|\mathfrak{M}_M^{Q_2}(P,\pi,\lambda,s\lambda-\lambda)I_P(f,\pi,s\lambda)\|_{\mathcal{B}((L^2_{P,\pi})^K)}\\ &\qquad\le \sum_{\tau\in\widehat{\mathbf{K}_\infty}}\|\mathfrak{M}_M^{Q_2}(P,\pi,\lambda,s\lambda-\lambda)\|_{\mathcal{B}((L^2_{P,\pi})^{\tau,K})}\|I_P(f_\tau,\pi,s\lambda)\|_{\mathcal{B}((L^2_{P,\pi})^K)}.\end{aligned}$$

To prove the lemma it therefore remains to apply Corollary 3.14 together with the fact that for any $N > 1$ there exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G(\mathbb{A}))$ we have

$$\|I_P(\pi,f_\tau,s\lambda)\| \ll_{f,K,N} (1+\Lambda_{\pi_\infty}+\|\tau\|+\|\lambda\|)^{-N}.$$

□

Remark 5.3 It is likely that with a finer analysis the factor $\sigma(x)^{(r(P)-r(G))/2}$ can be eliminated from the statement of Proposition 5.1. However, we will not address this issue here.

Remark 5.4 Consider the standard Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, namely, the analytic continuation in $s \in \mathbb{C}$ of

$$E(z,s) = \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}, \quad z = x + \mathrm{i}y, y > 0.$$

Fix $c > 0$. The argument of [Sar04] together with the bounds $\zeta(1+s), \zeta(1+s)^{-1} \ll \log(1+|s|)$, $s \in \mathrm{i}\mathbb{R}$ show that for any $\epsilon > 0$,

$$E(z,s) \ll_{c,\epsilon} y^{\frac{1}{2}} + |s|^{\frac{1}{2}+\epsilon} y^{-\frac{1}{2}}$$

uniformly for $y > c$ and $s \in \mathrm{i}\mathbb{R}$. This bound is essentially sharp for $y > |s|^{\frac{1}{2}}$ but at least if y is confined to a compact set, the argument of [IS95] gives a better exponent in $|s|$. It is an extremely difficult problem to determine precisely the growth of $E(z,s)$. For instance, the estimate $E(\mathrm{i},s) \ll_\epsilon (1+|s|)^\epsilon$, $s \in \mathrm{i}\mathbb{R}$ is equivalent to the Lindelöf hypothesis for the Dedekind zeta function $\zeta_{\mathbb{Q}(\mathrm{i})}$.

Next, we will give a consequence of Proposition 5.1. For $\kappa \geq 0$ and any level K of G, denote by $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A})^1)^K$ the Fréchet space of right-K-invariant functions $\phi \in C^\infty(G(F)\backslash G(\mathbb{A})^1)$ such that

$$\sup_{x \in \mathfrak{s}^1} \Xi(x)^{-1} \sigma(x)^{-\kappa} |R(X)\phi(x)| < \infty$$

for all $X \in \mathcal{U}(\mathfrak{g}^1)$. We also denote by $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A}))^K$ the Fréchet space of right-K-invariant functions $\phi \in C^\infty(G(F)\backslash G(\mathbb{A}))$ such that

$$\sup_{x \in \mathfrak{s}^1, a \in A_G} \Xi(x)^{-1} \sigma(a)^n \sigma(x)^{-\kappa} |R(X)\phi(ax)| < \infty$$

for all $X \in \mathcal{U}(\mathfrak{g})$ and $n \in \mathbb{N}$. (Note that we require ϕ to be a Schwartz function in A_G.) We write $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A})^1)$ for the union over K of $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A}))^K$ with the inductive limit topology. Similarly for $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A}))$. Note that we can identify $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A}))$ with $\mathcal{S}(A_G; A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A})^1))$ by $f \mapsto (a \mapsto f(a\cdot))$.

Corollary 5.5 *Let κ be half the F-rank of G. Then the map $\varphi \mapsto \mathcal{E}(\varphi)$ extends to a continuous linear map from $\mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_{P,\mathrm{cusp}})$ to $A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A}))$.*

Proof It easily follows from Proposition 5.1 that the map

$$\varphi \in \mathcal{S}(\mathrm{i}\mathfrak{a}^*_M; \mathcal{A}_{P,\mathrm{cusp}}) \mapsto \int_{\mathrm{i}(\mathfrak{a}^G_M)^*} E(\varphi(\lambda + \cdot), \lambda)\, d\lambda \in \mathcal{S}(\mathrm{i}\mathfrak{a}^*_G; A^\kappa_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A})^1))$$

is continuous. Composing this map with the Fourier transform on $\mathrm{i}\mathfrak{a}^*_G$ we get $\mathcal{E}_{\mathrm{cusp}}$.

□

For the second part of Theorem 4.5 we will need to extend Proposition 5.1 to the derivatives of $E(\varphi, \lambda)$ in λ. To that end we first extend Proposition 5.1 for $\mathfrak{z}$-finite φ to a larger domain of λ's.

Proposition 5.6 *There exists $l > 0$ and*

- *for any level K of G there exists $c > 0$,*
- *for any $N > 1$ there exists $m \in \mathbb{N}$,*

such that for any $f \in C_c^m(G(F_\infty))$ and $\pi \in \Pi_{\mathrm{cusp}}(M(\mathbb{A}))^{K_M}$ we have

$$f * E(x, \varphi, \lambda) \ll_{f,N,K} \Xi(x) e^{\|\mathrm{Re}\,\lambda\| \sigma(x)} \sigma(x)^{(r(P)-r(G))/2} (1 + \Lambda_{\pi_\infty} + \|\lambda\|)^{-N} \|\varphi\|_P$$

for any $\varphi \in \mathcal{A}^K_{P,\mathrm{cusp},\pi}$, $x \in \mathfrak{s}^1$ and $\lambda \in \mathcal{R}_{\pi,c,l}$.

Proof We proceed as in Proposition 5.1 and Lemma 5.2 and use the notation of the latter. The difference is that now we have to use the Maass-Selberg relations for general $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Thus, instead of (32) we have

$$\|\Lambda^T E(\varphi, \lambda)\|^2_{L^2(G(F)\backslash G(\mathbb{A})^1)} = \sum_{s \in W(M,M)} (\mathfrak{M}^T_M(P, -\overline{\lambda}, s\lambda + \overline{\lambda}) M_{P|P}(s, \lambda)\varphi, \varphi)_P$$

(cf. the proof of [Lap06, Proposition 6.1]). Since $c_M^{Q_1}(T, \Lambda)$ is the Fourier transform of a certain polytope (cf. [Art82, §3]) we get

$$c_M^{Q_1}(T, \Lambda) \ll e^{\|\mathrm{Re}\,\Lambda\| \|T\|} \|T\|^{r(P)-r(G)}.$$

The rest of the proof follows that of Lemma 5.2 except that we have to bound $\mathfrak{M}^T_M(P, \pi, -\overline{\lambda}, \Lambda)$ and $M_{P|P}(s, \lambda)$ off the unitary axis. To that end we apply Corollaries 3.13 (with $D = \mathrm{Id}$) and 3.14.

□

Corollary 5.7 *There exists $l > 0$ and*

- *for any level K of G there exists $c' > 0$,*
- *for any $D \in \mathcal{D}(\mathfrak{a}^*_M)$ there exists $k > 0$,*
- *for any $N > 1$ there exists $m \in \mathbb{N}$,*

such that for any $f \in C_c^m(G(F_\infty))$ and $\pi \in \Pi_{\mathrm{cusp}}(M(\mathbb{A}))^{K_M}$ we have

$$D_\lambda(f * E(x, \varphi, \lambda)) \ll_{f,D,N,K} \Xi(x) e^{\|\mathrm{Re}\,\lambda\| \sigma(x)} \sigma(x)^k (1 + \Lambda_{\pi_\infty} + \|\lambda\|)^{-N} \|\varphi\|_P$$

for any $\varphi \in \mathcal{A}^K_{P,\mathrm{cusp},\pi}$, $x \in \mathfrak{s}^1$ and $\lambda \in \mathcal{R}_{\pi,c',l}$.

This follows from Proposition 5.6 by taking $c' = c/2$ and using Cauchy's formula for a circle of radius $\min(\sigma(x)^{-1}, c'(1 + \Lambda_{\pi_\infty} + \|\mathrm{Im}\,\lambda\|)^{-l})$.

Corollary 5.8 *For any $D \in \mathcal{D}(\mathrm{i}\mathfrak{a}_M^*)$ there exists $k > 0$ and for any $N > 1$ there exists $m \in \mathbb{N}$ such that for any $f \in C_c^m(G(F_\infty))$ and level K of G we have*

$$D_\lambda(f * E(x, \varphi, \lambda)) \ll_{f,D,N,K} \Xi(x)\sigma(x)^k(1 + \|\lambda\|)^{-N}\|\varphi\|_P$$

for any $\varphi \in \mathcal{A}^K_{P,\mathrm{cusp}}$, $x \in \mathfrak{s}^1$ and $\lambda \in \mathrm{i}\mathfrak{a}_M^*$.

This follows from Corollary 5.7 together with the fact that there exists $N > 1$ such that for any K we have

$$\sum_{\pi \in \Pi_{\mathrm{cusp}}(M(\mathbb{A}))^{K_M}} m_{\mathrm{cusp}}(\pi)(1 + \Lambda_{\pi_\infty})^{-N} < \infty$$

where $m_{\mathrm{cusp}}(\pi) = \dim \mathrm{Hom}(\pi, L^2_{\mathrm{cusp}}(A_M M(F)\backslash M(\mathbb{A})))$ (e.g., [Don82]).

6 Proof of main theorem

We will apply induction and approximation by the constant term. For convenience we introduce an auxiliary space of functions on $P_0(F)\backslash\mathfrak{s}$, analogous to $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$. Namely, for any level K of G let $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})^K$ be the Fréchet space of smooth right-K-invariant functions $\phi : P_0(F)\backslash\mathfrak{s} \to \mathbb{C}$ such that for all $n \in \mathbb{N}$ and $X \in \mathcal{U}(\mathfrak{g})$ we have

$$\sup_{x \in \mathfrak{s}} \Xi(x)^{-1}\sigma(x)^n |X\phi(x)| < \infty. \tag{33}$$

We write $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})$ for the union over K of $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})^K$ with the inductive limit topology.

Note that we can identify $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$ with a closed subspace of $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})$.

For any measurable locally bounded function φ on $U_0(F)\backslash G(\mathbb{A})$ and a standard parabolic subgroup $P = M \ltimes U$ let φ_P be the constant term

$$\varphi_P(g) = \int_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\, du, \quad g \in G(\mathbb{A}).$$

The map $\phi \mapsto \mathfrak{d}\phi = \sum_{P \supseteq P_0}(-1)^{r(P)-r(G)}\phi_P$ defines a linear operator on the space of measurable locally bounded functions on $P_0(F)\backslash\mathfrak{s}$.

The following version of Langlands's Lemma is proved exactly as in [MW95, Lemma I.2.10 and Corollary I.2.11].

Proposition 6.1 *Fix $\lambda_0 \in \mathfrak{a}_0^*$. Then $\mathfrak{d}$ defines a continuous map from the space of functions on $P_0(F)\backslash\mathfrak{s}$ defined by the seminorms*

$$\sup_{x \in \mathfrak{s}^1, a \in A_G} \Xi(x)^{-1}\sigma(a)^n e^{-\langle\lambda_0, H(x)\rangle} |X\phi(ax)|, \quad X \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N}$$

to the space of functions on $P_0(F)\backslash\mathfrak{s}$ defined by the seminorms

$$\sup_{x \in \mathfrak{s}^1, a \in A_G} \Xi(x)^{-1}\sigma(a)^n e^{\langle\lambda, H(x)\rangle} |X\phi(ax)|, \quad X \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N}, \lambda \in \mathfrak{a}_0^*.$$

Remark 6.2 The factor $\Xi(x)^{-1}$ can be eliminated from the statement of the proposition without changing its contents. We include it in order to conform to the previous notation.

For our purposes we will only need the following much weaker consequence.

Corollary 6.3 *Fix $\kappa \geq 0$. Then $\mathfrak{d}$ defines a continuous map from $A^{\kappa}_{\mathrm{mg}}(G(F)\backslash G(\mathbb{A}))$ to $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})$.*

More generally, for any standard parabolic subgroup $Q = LV$, we can define $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s}^Q)$ in a similar way (with $\mathfrak{s}$, Ξ and σ replaced by $\mathfrak{s}^Q$, σ^Q and Ξ^Q respectively in (33)), and we can identify $\mathcal{S}(L(F)V(\mathbb{A})\backslash G(\mathbb{A}))$ with a closed subspace of $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s}^Q)$.

Note that by (13), the restriction map $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s}^Q) \to A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})$ is continuous. Therefore,

$$\text{the restriction map } \mathcal{S}(L(F)V(\mathbb{A})\backslash G(\mathbb{A})) \to A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s}) \text{ is continuous} \tag{34}$$

(not necessarily an embedding).

Proof (of Theorem 4.5) The proof will be by induction on the semisimple rank of G. The case where G is anisotropic modulo its center is trivial. For the induction step, we can assume by induction hypothesis that the theorem holds for any proper Levi subgroup of G. The constant term of the cuspidal Eisenstein transform is given by

$$\mathcal{E}(\varphi)_Q = \sum_{M'} \sum_{w \in W(M,M')} \mathcal{E}^Q(\mathcal{I}_w \varphi),$$

where M' ranges over the standard Levi subgroups of L and $\mathcal{E}^Q$ is the relative Eisenstein transform. By the induction hypothesis and Lemma 4.4, we conclude that for any proper Q the map $\mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}}) \to \mathcal{S}(L(F)V(\mathbb{A})\backslash G(\mathbb{A}))$ given by $\varphi \mapsto \mathcal{E}(\varphi)_Q$ is continuous. Thus, by (34), for all $Q \subsetneq G$

$$\text{the map } \varphi \mapsto \mathcal{E}(\varphi)_Q\big|_{P_0(F)\backslash\mathfrak{s}} \text{ from } \mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}}) \text{ to } A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s}) \text{ is continuous.} \tag{35}$$

On the other hand, it follows from Corollaries 5.5 and 6.3 that for κ equals half the F-rank of G, the map $\varphi \mapsto \mathfrak{d}(\mathcal{E}(\varphi))$ defines a continuous linear map from $\mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}})$ to $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})$. From the definition of the map $\mathfrak{d}$ and (35) we conclude that the map $\varphi \mapsto \mathcal{E}(\varphi)\big|_{P_0(F)\backslash\mathfrak{s}}$ from $\mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}})$ to $A_{\mathrm{sch}}(P_0(F)\backslash\mathfrak{s})$ is continuous. Equivalently, the map $\varphi \mapsto \mathcal{E}(\varphi)$ to $\mathcal{S}(G(F)\backslash G(\mathbb{A}))$ is continuous.

To show the second part of Theorem 4.5 we will construct the inverse

$$\iota : \mathcal{S}_c(G(F)\backslash G(\mathbb{A})) \to (\oplus \mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_P))^W$$

to $\mathcal{E}_{\mathrm{cusp}}$ (which will be automatically continuous by the open mapping theorem). Let $\phi \in \mathcal{S}_c(G(F)\backslash G(\mathbb{A}))$. We first claim that for any $\lambda \in \mathrm{i}\mathfrak{a}_M^*$ the linear form $\varphi \mapsto (\phi, E(\varphi, \lambda))_{G(F)\backslash G(\mathbb{A})}$ extends to $L^2_{P,\mathrm{cusp}}$. Indeed, we may assume that ϕ is of the form $R(f_0)\phi_0$ for some $f_0 \in C_c^m(G(F_\infty))$ (with $m \in \mathbb{N}$ arbitrary) and then the claim follows from Proposition 5.1 and the relation

$$(\phi, E(\varphi, \lambda))_{G(F)\backslash G(\mathbb{A})} = (\phi_0, E(I(f_0^*, \lambda)\varphi, \lambda))_{G(F)\backslash G(\mathbb{A})}$$

where $f_0^*(g) = \overline{f_0(g^{-1})}$. Let $\iota_P(\phi)(\lambda) \in L^2_{P,\mathrm{cusp}}$ be such that

$$(\iota_P(\phi)(\lambda), \varphi)_P = \frac{1}{n_M}(\phi, E(\varphi, \lambda))_{G(F)\backslash G(\mathbb{A})}.$$

By a similar reasoning, using Proposition 5.1 once again, we have $\iota_P(\phi)(\lambda) \in \mathcal{A}_{P,\mathrm{cusp}}$, i.e. $\iota_P(\phi)(\lambda)$ is a smooth vector in $L^2_{P,\mathrm{cusp}}$. By the functional equation of the Eisenstein series we have $\iota_{P'}(\phi)(w\lambda) = M_{P'|P}(w,\lambda)(\iota_P(\phi)(\lambda))$ for all $w \in W(M,M')$ and $\lambda \in \mathrm{i}\mathfrak{a}_M^*$. It remains therefore to show that $\iota_P(\phi) \in \mathcal{S}(\mathrm{i}\mathfrak{a}_M^*; \mathcal{A}_{P,\mathrm{cusp}})$. This follows easily from Corollary 5.8.

□

It would be desirable to extend Theorem 4.5, as well as Proposition 5.1, to the non-cuspidal case. We will not discuss this problem here but we mention that a first step in this direction, namely the Maass-Selberg relations for general Eisenstein series, was carried out in [LO12].

A Lower bounds on Rankin-Selberg L-functions, by Farrell Brumley[3]

The purpose of this appendix is to update and improve the statement of [Bru06, Theorem 5] on the lower bounds of Rankin-Selberg L-functions at the edge of the critical strip. We use this opportunity to correct some errors that appeared in [ibid.]. (See footnotes below.)

Fix a number field F and integers n_1, n_2. Write $d = n_1 + n_2$. Let π_1 and π_2 be automorphic cuspidal representations of $\mathrm{GL}_{n_1}(\mathbb{A}_F)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_F)$, respectively. We assume that their central characters are unitary and normalized so that viewed as Hecke characters on $F^*\backslash\mathbb{A}_F^*$, they are trivial on $\mathbb{R}_{>0}$ embedded diagonally.

In the following all constants depend implicitly on F, n_1, n_2. As usual we write $X \ll_\varepsilon Y$ to mean that $|X| \le cY$ for some constant c depending on ε (as well as on F, n_1, n_2).

Theorem A.1 *For every $\varepsilon > 0$ there exists a constant $c_1 > 0$ such that for any two distinct π_1 and π_2 as above and all*

$$\operatorname{Re} s \ge 1 - c_1 \mathfrak{c}(\Pi \times \tilde{\Pi})^{-(\frac{1}{2} - \frac{1}{2d} + \varepsilon)}, \tag{36}$$

we have

$$L^\infty(s, \pi_1 \times \tilde{\pi}_2)^{-1} \ll_\varepsilon \mathfrak{c}(\Pi \times \tilde{\Pi})^{\frac{1}{2} - \frac{1}{2d} + \varepsilon}, \tag{37}$$

where, setting $t = \operatorname{Im} s$,

$$\mathfrak{c}(\Pi \times \tilde{\Pi}) = \mathfrak{c}(\pi_1 \times \tilde{\pi}_1)\mathfrak{c}(\pi_2 \times \tilde{\pi}_2)\mathfrak{c}(\pi_1 \times \tilde{\pi}_2, 1 + \mathrm{i}t)^2. \tag{38}$$

Similarly, for every $\varepsilon > 0$ and n there exists a constant $c_2 > 0$ such that for any cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ we have

$$\frac{s}{s-1} L^\infty(s, \pi \times \tilde{\pi})^{-1} \ll_\varepsilon \mathfrak{c}(\Pi \times \tilde{\Pi})^{\frac{7}{8} - \frac{5}{8n} + \varepsilon}, \tag{39}$$

for $\operatorname{Re} s \ge 1 - c_2 \mathfrak{c}(\Pi \times \tilde{\Pi})^{-(\frac{7}{8} - \frac{5}{8n} + \varepsilon)}$ where $\mathfrak{c}(\Pi \times \tilde{\Pi})$ is as before (with $\pi_2 = \pi_1$).

Remark A.2 Note that this makes the following improvements to the original statement of Theorem 5 of [Bru06].

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1. We have explicated the exponent. With Xiannan Li's recent convexity bound [Li10] on Rankin-Selberg L-functions, this seemed like a good time to put into print an explicit power.

2. We have included the case where $\pi_1 = \pi_2$, which was not considered in [Bru06].

3. We have extended the range of s from the 1-line, as it was originally stated, to the wider region that extends slightly within the critical strip.

We would like to thank Erez Lapid for bringing these improvements to our attention and for suggesting the necessary modifications to the proof to handle the points (2) and (3).

Remark A.3 Besides the narrow (or coarse) zero free regions and lower bounds at the edge of the critical strip for Rankin-Selberg L-functions, we also established in [Bru06, Theorem 7] an effective multiplicity one result for cusp forms on GL_n. The more recent papers [Wan08, LW09] explicate the exponents of this other result, using a more efficient argument that doesn't pass by zero-free regions.

Proof (of Theorem A.1) It suffices to prove (37) for $\mathrm{Re}\, s = 1$. This bound can then be extended to all $\mathrm{Re}\, s \geq 1$ in the following way. On $\mathrm{Re}\, s \geq 2$ one bounds $|L^\infty(s, \pi_1 \times \tilde{\pi}_2)|^{-1}$ by an absolute constant by inserting the Jacquet-Shalika bounds. One then interpolates between the two bounds within the strip $1 \leq \mathrm{Re}\, s \leq 2$ by the Phragmén-Lindelöf principle. Moreover the same bounds can be extended to the small region within the critical strip (36) by applying the convexity bound of Li [Li10] on the derivative of Rankin-Selberg L-functions.

To proceed, we first assume that $\pi_1 \neq \pi_2$. For $t \in \mathbb{R}$ put[4]

$$\Pi = (\pi_1 \otimes |\det|^{\mathrm{i}t/2}) \boxplus (\pi_2 \otimes |\det|^{-\mathrm{i}t/2}). \tag{40}$$

The associated Rankin-Selberg L-function $L(s, \Pi \times \tilde{\Pi})$ factorizes as

$$L(s, \pi_1 \times \tilde{\pi}_1) L(s, \pi_2 \times \tilde{\pi}_2) L(s + \mathrm{i}t, \pi_1 \times \tilde{\pi}_2) L(s - \mathrm{i}t, \tilde{\pi}_1 \times \pi_2). \tag{41}$$

On the right half-plane $\mathrm{Re}\, s > 1$, $L(s, \Pi \times \tilde{\Pi})$ is an absolutely convergent Euler product of degree $d^2[F : \mathbb{Q}]$. Let

$$D(s) = L^\infty(s, \Pi \times \tilde{\Pi})$$

be the finite part L-function. When expanded into a Dirichlet series

$$D(s) = \sum_{\mathfrak{n}} b(\mathfrak{n})\, \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^{-s},$$

the coefficients $b(\mathfrak{n})$ are non-negative. Moreover $D(s)$ has a double pole at $s = 1$ and nowhere else.[5]

We write $D(s)$ in its Laurent series expansion about $s = 1$ as

$$D(s) = \sum_{j=-2}^{\infty} r_j (s-1)^j. \tag{42}$$

[4] In [Bru06] there is a missing minus sign in the exponent of the second isobaric factor of Π.

[5] In [Bru06] we make a big deal of the possibility that $D(s)$ might have a pole at $s = 0$. This in fact never happens because for any cuspidal representation π, $L(s, \pi_v \times \tilde{\pi}_v)$ has a pole at $s = 0$ for all places v, in particular the archimedean ones. (This holds for any generic representation, regardless of its temperedness.)

Theorem 3 of [Bru06] bounds the polar part from below by

$$\mathfrak{c}(\Pi \times \tilde{\Pi})^{-\frac{1}{2}+\frac{1}{2d}-\varepsilon} \ll_\varepsilon |r_{-1}| + |r_{-2}|. \tag{43}$$

Let us recall the proof of (43). Fix a non-negative $\psi \in C_c^\infty(0,\infty)$ such that $\psi\big|_{[1,2]} \equiv 1$. For a parameter $Y \geq 1$ we form the sum

$$S(Y) = \sum_{\mathfrak{n}} b(\mathfrak{n})\psi(\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})/Y).$$

We have

$$S(Y) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Re}\, s=2} D(s)\hat{\psi}(s) Y^s ds.$$

Shifting contours far to the left to $\mathrm{Re}\, s = \sigma$ we pick up no other poles and hence

$$S(Y) = \operatorname*{Res}_{s=1} \hat{\psi}(s) D(s) Y^s + \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Re}\, s=\sigma} D(s)\hat{\psi}(s) Y^s ds. \tag{44}$$

The second term above is

$$O_{\varepsilon,\sigma}(\mathfrak{c}(\Pi \times \tilde{\Pi})^{\frac{1-\sigma}{2}+\varepsilon} Y^\sigma), \tag{45}$$

by Li's convexity bound [Li10]. On the other hand, by [Bru06, Lemma 1]

$$S(Y) \geq \#\{\mathfrak{n} : Y \leq \mathrm{N}_{F/\mathbb{Q}}\, \mathfrak{n}^d \leq 2Y\} \gg Y^{1/d}. \tag{46}$$

(This is an improvement and simplification of [Bru06, Lemma 2]. Thanks to Erez Lapid for pointing this out.) Picking $Y = \mathfrak{c}(\Pi \times \tilde{\Pi})^{B(\sigma)+\varepsilon}$ where

$$B(\sigma) = \frac{1}{2} \cdot \frac{\sigma - 1}{\sigma - 1/d},$$

the second term in (44) is smaller than the lower bound (46). For this value of Y we thus obtain

$$Y^{1/d} \ll \operatorname*{Res}_{s=1} \hat{\psi}(s) D(s) Y^s.$$

Calculating this residue, we find

$$\frac{1}{Y^{1-1/d}} \ll (|r_{-1}| + |r_{-2}|) \log Y.$$

Inserting the value of Y and noting that $B(\sigma)$ tends to $1/2$ (from below) as σ tends to $-\infty$ we obtain (43).

Although we used it in the above reasoning, the bound (43) does not require Li's convexity result as an input. Indeed, if we use only the preconvex bound, the error term in (45) is $O_{\varepsilon,\sigma}(\mathfrak{c}(\Pi \times \tilde{\Pi})^{\frac{c-\sigma}{2}+\varepsilon} Y^\sigma)$ for some potentially large (fixed) constant $c \geq 1$. The exponent $B(\sigma)$ would then have a $\sigma - c$ in the numerator rather than a $\sigma - 1$. Nevertheless the limit of $B(\sigma)$ as $\sigma \to -\infty$ would still be $1/2$ as before.[6] Li's convexity result will be used more crucially in the next paragraph (to optimize exponents).

[6] We point out that in displays (3) and (6) of [Bru06] the absolute values on the parameters $\mu_\pi(v,i)$ and $\mu_{\pi\times\pi'}(v,i,j)$ should not appear. This misprint does not affect the reasoning leading to display (9) of [ibid.].

To complete the argument, the idea is that we can factor out the L-value we're interested in from the polar part $r_{-1}+r_{-2}$. We first let

$$L^\infty(s,\pi_1\times\tilde{\pi}_1)=\sum_{j=-1}^{\infty}A_j(s-1)^j,\quad L^\infty(s,\pi_2\times\tilde{\pi}_2)=\sum_{j=-1}^{\infty}B_j(s-1)^j$$

be the Laurent series expansions about $s=1$. Then

$$r_{-2}=A_{-1}B_{-1}\left|L^\infty(1+\mathrm{i}t,\pi_1\times\tilde{\pi}_2)\right|^2$$

and

$$\begin{aligned}r_{-1}&=(A_{-1}B_0+A_0B_{-1})\left|L^\infty(1+\mathrm{i}t,\pi_1\times\tilde{\pi}_2)\right|^2\\&\quad+2A_{-1}B_{-1}\operatorname{Re}(L^{\infty\prime}(1+\mathrm{i}t,\pi_1\times\tilde{\pi}_2)\overline{L^\infty(1+\mathrm{i}t,\pi_1\times\tilde{\pi}_2)}).\end{aligned}$$

From Li's convexity bound we have

$$A_j\ll_{\varepsilon,j}\mathfrak{c}(\pi_1\times\tilde{\pi}_1)^\varepsilon,\qquad B_j\ll_{\varepsilon,j}\mathfrak{c}(\pi_2\times\tilde{\pi}_2)^\varepsilon,$$

and

$$L^{\infty(k)}(s,\pi_1\times\tilde{\pi}_2)\ll_{\varepsilon,k}\mathfrak{c}(\pi_1\times\tilde{\pi}_2,s)^\varepsilon,$$

along $\operatorname{Re}s=1$. This gives

$$r_{-2},r_{-1}\ll_\varepsilon\mathfrak{c}(\Pi\times\tilde{\Pi})^\varepsilon\left|L^\infty(1+\mathrm{i}t,\pi_1\times\tilde{\pi}_2)\right|.\tag{47}$$

Putting (43) and (47) together yields (37) on the 1-line, as desired.

Now we tackle the more delicate case where $\pi_1=\pi_2(=\pi)$ and $n_1=n_2(=n)$. Here the proof breaks up into two parts according to whether s is close to 1 or not.

If $s=1$ then we already know by [Bru06, Theorem 3] applied to $L^\infty(s,\pi\times\tilde{\pi})$ (as explained above) that

$$A_{-1}=\operatorname*{Res}_{s=1}L^\infty(s,\pi\times\tilde{\pi})$$

is bounded below by

$$\mathfrak{c}(\pi\times\tilde{\pi})^{-\frac{1}{2}+\frac{1}{2n}-\varepsilon}.$$

As before, Li's upper bounds on the derivatives of $L^\infty(s,\pi\times\tilde{\pi})$ allow us to extend this to

$$((s-1)L^\infty(s,\pi\times\tilde{\pi}))^{-1}\ll_\varepsilon\mathfrak{c}(\pi\times\tilde{\pi},s)^{\frac{1}{2}-\frac{1}{2n}+\varepsilon},$$

for all $s=1+it$ with $|t|\le\mathfrak{c}(\pi\times\tilde{\pi})^{-\frac{1}{2}+\frac{1}{2n}-2\varepsilon}$.

It remains to treat the range $|t|\ge\mathfrak{c}(\pi\times\tilde{\pi})^{-\frac{1}{2}+\frac{1}{2n}-2\varepsilon}$. Define Π as in (40) so that

$$L(s,\Pi\times\tilde{\Pi})=L(s,\pi\times\tilde{\pi})^2L(s+\mathrm{i}t,\pi\times\tilde{\pi})L(s-\mathrm{i}t,\pi\times\tilde{\pi}).$$

Again write $D(s)=L^\infty(s,\Pi\times\tilde{\Pi})$ for the finite part L-function. This time $D(s)$ has a double pole at $s=1$ and two additional poles, both simple, at $s=1\pm it$. We continue to use the notation r_j, introduced in (42), for the Laurent series coefficients of $D(s)$ at $s=1$. Let $r_{-1}^{\pm}$ be the residues of $D(s)$ at $s=1\pm it$. The proof of (43) given above gives

$$\mathfrak{c}(\Pi\times\tilde{\Pi})^{-\frac{1}{2}+\frac{1}{4n}-\varepsilon}\ll_\varepsilon|r_{-1}|+|r_{-2}|+\left|r_{-1}^{+}\right|+\left|r_{-1}^{-}\right|.\tag{48}$$

Indeed, the contour shift picks out all of these poles. We calculate

$$r_{-1}^{\pm} = A_{-1} L^{\infty}(1 \pm \mathrm{i}t, \pi \times \tilde{\pi})^2 L^{\infty}(1 \pm 2\mathrm{i}t, \pi \times \tilde{\pi}).$$

If $|t| \geq 1$ then we can apply Li's convexity result simultaneously to $L^{\infty}(1 + \mathrm{i}t, \pi \times \tilde{\pi})$ and $L^{\infty}(1 + 2\mathrm{i}t, \pi \times \tilde{\pi})$ to obtain (47) and

$$r_{-1}^{\pm} \ll_{\varepsilon} \mathfrak{c}(\Pi \times \tilde{\Pi})^{\varepsilon} \left| L^{\infty}(1 \pm \mathrm{i}t, \pi \times \tilde{\pi}) \right|. \tag{49}$$

On the other hand, for $|t| \leq 1$ we claim that

$$r_{-1}, r_{-2}, r_{-1}^{\pm} \ll_{\varepsilon} \left| t L^{\infty}(1 + \mathrm{i}t, \pi \times \tilde{\pi}) \right| |t|^{-3} \, \mathfrak{c}(\pi \times \tilde{\pi}, s)^{\varepsilon}.$$

This follows from the convexity bounds

$$(s-1) L^{\infty}(s, \pi \times \tilde{\pi}) \ll_{\varepsilon} \mathfrak{c}(\pi \times \tilde{\pi}, s)^{\varepsilon}$$

(applied to $s = 1 \pm \mathrm{i}t$ and $s = 1 \pm 2\mathrm{i}t$) and

$$(s-1)^2 (L^{\infty})'(s, \pi \times \tilde{\pi}) \ll_{\varepsilon} \mathfrak{c}(\pi \times \tilde{\pi}, s)^{\varepsilon}$$

which are valid for $\mathrm{Re}\, s = 1$, $s - 1 = O(1)$. We conclude that for $\mathfrak{c}(\pi \times \tilde{\pi})^{-\frac{1}{2} + \frac{1}{2n} - 2\varepsilon} \leq |t| \leq 1$ we have

$$r_{-1}, r_{-2}, r_{-1}^{\pm} \ll_{\varepsilon} \left| t L^{\infty}(1 + \mathrm{i}t, \pi \times \tilde{\pi}) \right| \mathfrak{c}(\Pi \times \tilde{\Pi})^{\frac{3}{4}(\frac{1}{2} - \frac{1}{2n}) + \varepsilon},$$

where we used that $\mathfrak{c}(\Pi \times \tilde{\Pi})$ and $\mathfrak{c}(\pi \times \tilde{\pi})^4$ are within a constant apart. Combining the three regimes we get the required lower bound from (48).

□

Remark A.4 Note that except in the range $\mathfrak{c}(\pi \times \tilde{\pi})^{-\frac{1}{2} + \frac{1}{2n} - 2\varepsilon} \leq |t| \leq 1$ we can improve the bound (39). It would be nice to improve the bound in the above regime as well.

Remark A.5 As in the case of the upper bounds, it would be more natural to replace $\mathfrak{c}(\Pi \times \tilde{\Pi})$ by $\mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s)$ in the formulation of Theorem A.1. This is possible since as explained to us by Guy Henniart (work in progress), one has

$$\mathfrak{c}(\pi_1 \times \tilde{\pi}_1) \mathfrak{c}(\pi_2 \times \tilde{\pi}_2) \leq \mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s)^2.$$

This is a purely local question. In the supercuspidal case we have in fact the inequality

$$\max(\mathfrak{c}(\pi_1 \times \tilde{\pi}_1), \mathfrak{c}(\pi_2 \times \tilde{\pi}_2)) \leq \mathfrak{c}(\pi_1 \times \tilde{\pi}_2, s)$$

for the local conductors – see [BH03, Theorem C]. We take this opportunity to thank Guy Henniart for providing us this information.

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