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Maass Cusp Forms with Quadratic Integer Coefficients

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1 Introduction

Let W be the space of weight-zero cuspidal automorphic forms of level N on $GL(2)$ over $\mathbb{Q}$. For each n such that $(n, N) = 1$, denote by $T(n)$ the nth Hecke operator, and by T the algebra over $\mathbb{C}$ which they generate. If $B = \{f_i\}$ is a W-basis of simultaneous eigenvectors of T, then for each $f_i \in B$ let $\lambda_{f_i}(n)$ be the eigenvalue of $T(n)$ at f_i, normalized so as to lie in the interval $[-2, 2]$ under the Ramanujan conjecture. A striking result of Sarnak [9] says that if the L-series,

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}, \tag{1.1}$$

of an eigenform $f \in B$ has integral coefficients, then f must be of Galois type. The proof utilizes the functorial transfers for the second and third symmetric powers on $GL(2)$ established, respectively, by Gelbart and Jacquet in [2] and by Kim and Shahidi in [4]. The condition on the integrality of the coefficients would follow from the stronger supposition that a one-dimensional invariant subspace V of W exists upon which T acts as integral scalars. Stating the theorem in this way, we might try to extend the implication to higher-dimensional T-invariant integral subspaces V. The L-series coefficients of any f in such an n-dimensional subspace V would lie as integers in a number field of degree n over $\mathbb{Q}$, although conversely it is not true that all forms in W having this latter property necessarily belong to V. The more general result would again state that the $f \in V$ are of Galois type. We accomplish this extension for a two-dimensional V using the techniques of [9] and the remaining fourth symmetric power transfer established by Kim in [3].

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Let V be a two-dimensional $\mathbb{T}$-invariant irreducible subspace of W admitting a basis with respect to which the operators in $\mathbb{T}$ can be realized as 2×2 matrices with rational integer entries. If this subspace is irreducible, upon diagonalization these matrices have entries in the integer ring of a quadratic extension K of $\mathbb{Q}$. Thus, if $f \in V$ is a simultaneous eigenvector of $\mathbb{T}$, its L-series has quadratic integer coefficients. As in [9], we seek to prove that the form f is of Galois type. In this paper, we show that in fact this can be done, provided the quadratic field K is not $\mathbb{Q}(\sqrt{5})$. This single exception can be explained by observing that, if f is attached through the Artin conjecture to an even icosahedral Galois representation with trivial central character, its coefficients lie as integers in $\mathbb{Q}(\sqrt{5})$. Without higher symmetric power transfers at our disposal however, it is impossible for our method of proof to fully distinguish the analytic properties of its L-function. Henceforth, the subspace V will be such that its Hecke eigenvalues do not lie in the field $\mathbb{Q}(\sqrt{5})$.

Theorem 1.1. Let $V \subset W$ be as above. Then V comprises forms associated to a Galois representation of either dihedral, tetrahedral, or octahedral type. In particular, V has Laplacian eigenvalue $1/4$ and satisfies the Ramanujan conjecture at every finite place. $\square$

It should be remarked that with the strength of full functoriality, from which follow the conjectures of Ramanujan and Sato-Tate, this result becomes transparent, for a finite number of eigenvalues in an interval cannot be continuously equidistributed. The advantage of theorems of this type is that they give partial progress toward Ramanujan in an altogether different manner than improving local bounds on the Satake parameters. Indeed, a finite number of functorial lifts, along with their cuspidality conditions, suffice for proving that those automorphic forms which satisfy a certain integrality condition on their coefficients must also satisfy the Ramanujan conjecture.

2 Preliminaries

2.1

Let ρ be an irreducible two-dimensional complex Galois representation of a number field F over $\mathbb{Q}$. For each prime p, unramified in F, let $\mathbf{Frob}_p$ denote the Frobenius class at p. Then the partial Artin L-function is defined on a suitable right half-plane as

$$L(s,\rho) = \prod_p \det\bigl(1 - \rho(\mathbf{Frob}_p)p^{-s}\bigr)^{-1} = \prod_p \bigl(1 - \lambda_\rho(p)p^{-s} + \omega_\rho(p)p^{-2s}\bigr)^{-1}, \tag{2.1}$$

the product being taken over all unramified primes. If the numbers $\lambda_\rho(p)$ lie in a totally real or purely imaginary number field, then from the unicity of ω_ρ, it follows that $\omega_\rho^2 = 1$. The image of ρ must therefore lie in

$$\mathrm{GL}_2^{(m)}(\mathbb{C}) := \left\{ g \in \mathrm{GL}_2(\mathbb{C}) \mid (\det g)^m = 1 \right\}, \quad \text{for } m = 1 \text{ or } 2. \tag{2.2}$$

By explicitly constructing all subgroups of $\mathrm{GL}_2^{(m)}(\mathbb{C})$ having quadratic integer traces, we discover in this section which trace fields can actually occur in the setting of our problem.

Klein, in [6], has classified all finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$. We lift them to subgroups of $\mathrm{GL}_2^{(m)}(\mathbb{C})$, requiring additionally that the traces be quadratic integers. For dihedral subgroups of $\mathrm{PGL}_2(\mathbb{C})$, the condition of a quadratic determinate already ensures rational integrality of the traces. These lifts are

$$\begin{aligned} U_2 &= \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \subset \mathrm{GL}_2^{(1)}(\mathbb{C}), \\ V_2 &= \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \subset \mathrm{GL}_2^{(2)}(\mathbb{C}), \end{aligned} \tag{2.3}$$

both of which have image in $\mathrm{PGL}_2(\mathbb{C})$ equal to the Klein four group D_2; and

$$\begin{aligned} U_3 &= \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \pm \begin{bmatrix} -i & i \\ 0 & i \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \right\}, \\ V_3 &= \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right\} \end{aligned} \tag{2.4}$$

in $\mathrm{GL}_2^{(1)}(\mathbb{C})$ and $\mathrm{GL}_2^{(2)}(\mathbb{C})$ respectively, both having an image in $\mathrm{PGL}_2(\mathbb{C})$, the order-6 dihedral group D_3. Now let $\zeta^8 = 1$ be a primitive 8th root of unity. The group

$$U_4 = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} \zeta^r & \zeta^s \\ -\overline{\zeta^s} & \overline{\zeta^r} \end{bmatrix} \;\middle|\; r, s \in \{1, 3, 5, 7\} \right\} \cup U_2 \subset \mathrm{GL}_2^{(1)}(\mathbb{C}) \tag{2.5}$$

has image in PGL_2 isomorphic to A_4, the group of tetrahedral rotations. The trace field for both the dihedral and tetrahedral groups is evidently $\mathbb{Q}$.

For brevity, we give the generating matrices for the remaining groups. Let ζ again denote a primitive 8th root of unity. The matrices

$$\begin{pmatrix} \zeta & \zeta \\ -\overline{\zeta} & \overline{\zeta} \end{pmatrix}, \qquad \begin{pmatrix} \zeta & 0 \\ 0 & \overline{\zeta} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.6}$$

generate an octahedral subgroup of $\mathrm{PGL}_2(\mathbb{C})$. This group lifts to a finite subgroup of $\mathrm{GL}_2^{(1)}(\mathbb{C})$ and $\mathrm{GL}_2^{(2)}(\mathbb{C})$ with traces lying as integers in the field $\mathbb{Q}(\sqrt{2})$. Finally, we let $\epsilon \in \mathbb{C}$ satisfy $\epsilon^5 = 1$ and set $p = (\epsilon^4 - \epsilon)/\sqrt{5}$, $q = (\epsilon^2 - \epsilon^3)/\sqrt{5}$. Then the following matrices generate an icosahedral subgroup of $\mathrm{PGL}_2(\mathbb{C})$:

$$\begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} p & q \\ q & -p \end{pmatrix}. \tag{2.7}$$

One could look at [1, page 73] for the derivation of this fact. This group lifts to a finite subgroup of $\mathrm{GL}_2^{(1)}(\mathbb{C})$ with traces lying as integers in the field $\mathbb{Q}(\sqrt{5})$.

2.2

Retaining the notation of the introduction, we let $\{f, g\}$ be a basis of V of simultaneous eigenvectors for T with eigenvalues $\lambda_f(n)$ and $\lambda_g(n)$ for every n prime to N. Though a priori the field of definition of $\lambda_f(n)$ and $\lambda_g(n)$ may depend on n, it can easily be seen that one quadratic field K contains them all. For, with respect to the integral basis, T embeds in the matrix algebra $M_2(\mathbb{Z})$ as a commutative subalgebra. As such, it is isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}(\sqrt{d})$. In the first case, the eigenforms have rational integer coefficients, and by the work of Sarnak [9], they must be of either dihedral or tetrahedral type. We therefore restrict our attention to the latter case, when $f \neq g$, and denote by $K = \mathbb{Q}(\sqrt{d})$ the smallest field containing the coefficients.

The integrality of $T(p)$ implies that $\lambda_g(p) = \lambda_f(p)'$, the Galois conjugate within K. We therefore write f' in place of g. Now, associated to f (resp., f') is a weight-zero irreducible cuspidal automorphic representation π (resp., π') on $\mathrm{GL}_2(\mathbf{A})$, with $\mathbf{A}$ the adele ring of $\mathbb{Q}$. The partial L-function of π on the complex right half-plane $\mathrm{Re}(s) > 1$ is given by

$$L(s, \pi) = \prod_{p \nmid N} \det\left(1 - A_p p^{-s}\right)^{-1} = \prod_{p \nmid N} \left(1 - \lambda_\pi(p) p^{-s} + \omega_\pi(p) p^{-2s}\right)^{-1}, \tag{2.8}$$

where A_p is the matrix of Satake parameters at the prime p, and ω_π is the central character of π. By the same arguments given in Section 2.1 applied to Galois representations, the central character ω_π of π is quadratic.

2.3

We proceed as in [9], demonstrating successively that noncuspidality of each symmetric power lift, together with quadratic integrality of its L-series coefficients, implies the theorem. The arguments in [9] for the second and third symmetric powers when the coefficients were rational integers apply here, making all necessary changes, when the coefficients are quadratic integers. Thus, when $\mathrm{sym}^2\pi$ is noncuspidal, π is monomial, by [2, Theorem 3.3.7]. Using [7], we show that π corresponds to a dihedral representation of the Weil group, which, since it has quadratic integer coefficients, must have a finite order. When $\mathrm{sym}^3\pi$ is noncuspidal, Kim and Shahidi in [5, Proposition 3.3.8] show that π corresponds to a tetrahedral representation ρ of the Weil group. By the integrality assumption on its coefficients, it follows that ρ descends to a Galois representation. Finally, if $\mathrm{sym}^4\pi$ is noncuspidal, π corresponds to an octahedral representation ρ of the Weil group, [5, Proposition 3.3.8]. Since ρ has a finite image in GL_2, it is properly a Galois representation, and again the theorem follows in this case.

2.4

What remains, and what we intend to disprove, is the case when all three symmetric power lifts, $\mathrm{sym}^2\pi$, $\mathrm{sym}^3\pi$, and $\mathrm{sym}^4\pi$, are cuspidal. The cuspidality of these lifts will be the working hypothesis for the remainder of this paper.

The partial L-function $L(s, \mathrm{sym}^k\pi)$ is defined on a suitable right half-plane as

$$L(s, \mathrm{sym}^k\pi) = \prod_{p\nmid N} \det\left(1 - \mathrm{sym}^k(A_p)p^{-s}\right)^{-1}. \tag{2.9}$$

Let χ be the character of the representation sym^k on GL_2. Then we have

$$-\frac{d}{ds}\log L(s, \mathrm{sym}^k\pi) = \sum_{n\geq 1}\sum_p \chi(A_p^n)(\log p)p^{-ns}. \tag{2.10}$$

We will write the above equation as

$$-\frac{d}{ds}\log L(s, \mathrm{sym}^k\pi) = \sum_p \lambda_\pi(p)(\log p)p^{-s} + R^{(k)}(s), \tag{2.11}$$

where

$$R^{(k)}(s) = \sum_{n\geq 2}\sum_p \chi(A_p^n)(\log p)p^{-ns}. \tag{2.12}$$

All the above sums over primes are, and henceforth will be, implicitly taken over those not dividing the level.

Lemma 2.1. On $\mathrm{Re}(s) \geq 1$,

$$-\frac{d}{ds} \log L(s, \mathrm{sym}^k \pi) = \sum_p \lambda_{\mathrm{sym}^k \pi}(p)(\log p)p^{-s} + O(1) \quad (1 \leq k \leq 8). \tag{2.13}$$

□

Proof. We first demonstrate that the inner sum in $R^{(k)}(s)$ converges. Accordingly, set

$$W_n^{(k)}(X) = \sum_{p \leq X} \chi(A_p^n)(\log p)p^{-ns}. \tag{2.14}$$

Taking the absolute values, we obtain

$$|W_n^{(k)}(X)| < \sum_{p \leq X} |\chi(A_p^n)|(\log p)p^{-n\sigma}, \tag{2.15}$$

where $\sigma = \mathrm{Re}(s)$. Now, the coefficient $\chi(A_p^n)$ can be written as an integral polynomial in both $\lambda_\pi(p)$ and $\omega_\pi(p)$, with $\lambda_\pi(p)$ appearing in the leading term to the power nk. Thus

$$|\chi(A_p^n)| \ll |\lambda_\pi(p)|^{nk}, \tag{2.16}$$

from which we get

$$|W_n^{(k)}(X)| \ll_\epsilon \sum_{p \leq X} |\lambda_\pi(p)|^{nk} p^{-n\sigma+\epsilon}. \tag{2.17}$$

A summation by parts affords

$$\sum_{p \leq X} |\lambda_\pi(p)|^{nk} p^{-n\sigma+\epsilon} \ll_\epsilon \sum_{p \leq X} S_{nk}(p) p^{-n\sigma-1+\epsilon}, \tag{2.18}$$

where we have denoted

$$S_{nk}(X) = \sum_{m \leq X} |\lambda_\pi(m)|^{nk}. \tag{2.19}$$

From the general theory of Rankin-Selberg integrals applied to the function $L(s, \mathrm{sym}^4 \pi \times \mathrm{sym}^4 \pi)$, both factors being cuspidal by assumption, we deduce a bound for the eighth moment

$$\sum_{n \leq X} |\lambda_\pi(n)|^8 \leq CX, \tag{2.20}$$

for some positive constant C. Combining this with the bound on the Hecke eigenvalues $|\lambda_\pi(p)| \le 2p^{1/9}$, obtained by Kim and Shahidi in [4], we have

$$S_{nk}(X) \le \max_{m\le X} \left|\lambda_\pi(m)\right|^{nk-8} \sum_{m\le X} \left|\lambda_\pi(m)\right|^{8} \ll_\epsilon X^{(nk+1)/9+\epsilon}. \tag{2.21}$$

From this it follows that, as long as $k \le 8$, the sum in (2.18) converges on the region $\sigma \ge 1$. With these restrictions on k and s, the inner sum of $R^{(k)}(s)$ converges, and we write, switching the order of summation,

$$R^{(k)}(s) = \sum_p p^{-8/9} \sum_{n\ge 2} \left(p^{-1/9}\right)^n = \sum_p \left(p^{10/9} - 1\right)^{-1}. \tag{2.22}$$

This sum converges and the lemma is proved. ■

Directly from our cuspidality assumption, we know that for integers $1 \le k \le 4$, the function $L(s, \mathrm{sym}^k\pi)$ is invertible at $s = 1$. In fact, we can establish, through Rankin-Selberg factorization (cf. [4]) that $L(s, \mathrm{sym}^k\pi)$ is invertible at $s = 1$ for the extended range of $1 \le k \le 8$. This, along with Lemma 2.1, implies the following identity:

$$\lim_{M\to\infty} \frac{1}{M} \sum_{p\le M} \lambda_{\mathrm{sym}^k\pi}(p) \log p = 0 \quad (1 \le k \le 8). \tag{2.23}$$

2.5

We would like to obtain a result for Rankin-Selberg products of automorphic forms similar to that of Lemma 2.1. Throughout this section, we will denote by Π any one of the following tensor product representations on $GL_{(m+1)(n+1)}$:

$$\mathrm{sym}^m\pi \times \mathrm{sym}^n\pi, \qquad \mathrm{sym}^m\pi \times \mathrm{sym}^n\pi', \qquad \mathrm{sym}^m\pi' \times \mathrm{sym}^n\pi', \tag{2.24}$$

when $1 \le m, n \le 4$. If A_p and B_p, for $p \nmid N$, are the respective diagonal matrices of Satake parameters at the prime p for the factors in Π, then the partial L-function $L(s, \Pi)$ is defined on $\mathrm{Re}(s) > 1$ by

$$L(s, \Pi) = \prod_{p\nmid N} \det\left(1 - \Pi\left(A_p \otimes B_p\right)p^{-s}\right)^{-1}. \tag{2.25}$$

If we let χ denote the character of the representation $\mathrm{sym}^m \otimes \mathrm{sym}^n$, then upon taking the logarithmic derivative we get

$$-\frac{d}{ds}\log L(s, \Pi) = \sum_{j=1}^{\infty} \sum_p \chi\Big(\left(A_p \otimes B_p\right)^j\Big)(\log p)p^{-js}. \tag{2.26}$$

We write the above equation as

$$-\frac{d}{ds}\log L(s,\Pi) = \sum_{p} \lambda_{\Pi}(p)(\log p)p^{-s} + R^{(m,n)}(s), \tag{2.27}$$

where

$$R^{(m,n)}(s) = \sum_{j\geq 2}\sum_{p} \chi\Big(\big(A_p \otimes B_p\big)^j\Big)(\log p)p^{-js}. \tag{2.28}$$

The proof of the following lemma mimics exactly that of Lemma 2.1.

Lemma 2.2. On $\mathrm{Re}(s) \geq 1$, if Π is a representation in (2.24) with $1 \leq m, n \leq 4$, then

$$-\frac{d}{ds}\log L(s,\Pi) = \sum_{p} \lambda_{\Pi}(p)(\log p)p^{-s} + O(1). \tag{2.29}$$

□

When the two factors in the Rankin-Selberg product are contragredient, the L-function has a simple pole at $s = 1$, and is invertible there otherwise. We conclude from Lemma 2.2 that for any representation Π in (2.24), with $1 \leq m, n \leq 4$,

$$\lim_{M\to\infty} \frac{1}{M}\sum_{p\leq M} \lambda_{\Pi}(p)\log p = \begin{cases} 1, & \text{if the factors in } \Pi \text{ are contragredient,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.30}$$

2.6

To make use of these quantities, we calculate some linearity relations. Denote by $P_n(x)$ the polynomial sending the trace of a matrix in $GL_2^{(2)}(\mathbb{C})$, with determinant ω_π, to the trace of its nth symmetric power. In particular,

$$\begin{aligned} &\lambda_{\mathrm{sym}^n\pi}(p) = P_n\big(\lambda_\pi(p)\big), \\ &\lambda_{\mathrm{sym}^m\pi\times\mathrm{sym}^n\pi'}(p) = P_m\big(\lambda_\pi(p)\big)P_n\big(\lambda_{\pi'}(p)\big). \end{aligned} \tag{2.31}$$

The coefficients of each P_n lie in $\mathbb{Z}[\omega_\pi]$, where ω_π is a symbol satisfying $\omega_\pi^2 = 1$. The following relations hold:

$$\begin{aligned} &P_1(X) = X, \qquad P_2(X) = X^2 - \omega_\pi, \qquad P_3(X) = X^3 - 2\omega_\pi X, \\ &P_4(X) = X^4 - 3\omega_\pi X^2 + 1, \qquad P_5(X) = X^5 - 4\omega_\pi X^3 + 3X, \\ &P_6(X) = X^6 - 5\omega_\pi X^4 + 6X^2 - \omega_\pi, \qquad P_7(X) = X^7 - 6\omega_\pi X^5 + 10X^3 - 4\omega_\pi X, \\ &P_8(X) = X^8 - 7\omega_\pi X^6 + 15X^4 - 10\omega_\pi X^2 + 1. \end{aligned} \tag{2.32}$$

From identity (2.23), we have

$$\lim_{M\to\infty} \frac{1}{M} \sum_{p\le M} P_m\big(\lambda_\pi(p)\big) \log p = 0 \quad (1 \le m \le 8), \tag{2.33}$$

and from identity (2.30), assuming $1 \le m, n \le 4$,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{p\le M} P_m\big(\lambda_\pi(p)\big) P_n\big(\lambda_{\pi'}(p)\big) \log p = \begin{cases} 1, & \text{if } \pi' \simeq \tilde{\pi}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.34}$$

The obvious equalities hold for Rankin-Selberg products $\mathrm{sym}^m\pi \times \mathrm{sym}^n\pi$ and $\mathrm{sym}^m\pi' \times \mathrm{sym}^n\pi'$.

We now have the tools with which to eliminate, case by case, the potential trace fields of the automorphic π afforded by the hypothesis of the theorem.

3 K-imaginary quadratic

3.1

We first treat the case when K is an imaginary quadratic extension of $\mathbb{Q}$, say $K = \mathbb{Q}(\sqrt{-d})$, for some square-free positive integer d. The complex conjugate representation $\overline{\pi} = \otimes_p \overline{\pi}_p$ of π is an irreducible cuspidal automorphic representation of GL_2 over $\mathbb{Q}$, which is given locally by the complex conjugate of the Satake parameters for π_p. As the central character ω_π is unitary, we have $\tilde{\pi} = \overline{\pi}$. On GL_2, there is an isomorphism $\tilde{\pi} \simeq \pi \otimes \omega_\pi^{-1}$, and in the present context, $\omega_\pi^{-1} = \omega_\pi$. For those primes p at which $\omega_\pi(p) = 1$, the Satake parameters, and thus the Hecke eigenvalues, are real, being equal to their complex conjugate. Likewise, for p at which $\omega_\pi(p) = -1$, the Hecke eigenvalues are purely imaginary, being the negative of their complex conjugate. In the complex plane, the coefficients of π lie in $\mathbb{R} \cup i\mathbb{R}$.

We are fortunate in this setting to have a convenient description of the representation π'. The field K is imaginary; the Galois action is therefore complex conjugation, and we have $\pi' \simeq \tilde{\pi}$. We also observe that from the condition that π is not equal to π', the representation π is not self-dual, and the central character is therefore nontrivial.

Throughout the remainder of this section, we will write $\tilde{\pi}$ in place of π'. The following relationships will be used in later calculations:

(i) π and $\mathrm{sym}^3\pi$ are not self-dual;
(ii) $\mathrm{sym}^2\pi$ and $\mathrm{sym}^4\pi$ are self-dual;
(iii) ω_π is nontrivial.

3.2

Let S be the subspace of $\mathbb{R}[x]$ generated by monomials whose degrees are even and no greater than 8. Elements in S can be considered as functions from $\mathbb{R} \cup i\mathbb{R}$ to $\mathbb{R}$. Define a linear form I on S by

$$I(T) = \lim_{M \to \infty} \frac{1}{M} \sum_{p \leq M} T\big(\lambda_\pi(p)\big) \log p, \tag{3.1}$$

for $T \in S$. We calculate I on the standard basis of S by applying linearity to the relations between powers of the trace of a matrix and the traces of its symmetric powers. The polynomials P_n defined in Section 2.6, the nontriviality ω_π, and identity (2.33), provide the following values for I:

$$I(1) = 1, \qquad I\big(x^2\big) = 0, \qquad I\big(x^4\big) = 2, \qquad I\big(x^6\big) = 0, \qquad I\big(x^8\big) = 14. \tag{3.2}$$

If $K = \mathbb{Q}(i)$, consider the polynomial $T(x) = -(x^2-1)(x^2-4)(x^2+1)(x^2+4)$. As a real-valued function, T is negative at $x = 0$, zero at all other K-integral points in $[-2,2] \cup [-2i, 2i]$, and negative elsewhere in $\mathbb{R} \cup i\mathbb{R}$. The mean trace $I(T)$ accordingly should be nonpositive. Using the identities of (3.2), however, we compute that $I(T(x)) = I(-x^8 + 17x^4 - 16) = 4$. This contradiction eliminates the possibility that $K = \mathbb{Q}(i)$.

If $K = \mathbb{Q}(\sqrt{-d})$, for $d \geq 3$, then set $T(x) = -x^2(x^2-1)(x^2-4)(x^2+3)$. The polynomial T vanishes at all integer points in $[-2,2]$, and is negative outside of this interval. On the K-integral points in $[-2i, 2i]$, the value of T is zero when $d = 3$, and negative when $d \geq 5$. Furthermore, T is negative outside of this interval in $i\mathbb{R}$. The mean trace $I(T)$ should therefore be nonpositive. Once again, however, we obtain a contradiction by computing with the identities of (3.2) the value $I(T(x)) = I(-x^8 + 2x^6 + 11x^4 - 12x^2) = 8$.

3.3

Now, when $K = \mathbb{Q}(\sqrt{-2})$, the above setting for the linear form I does not produce a contradiction. The critical distance d on the imaginary axis beyond which a polynomial $T(x) = -x^2(x^2-1)(x^2-4)(x^2+d)$ must have a root, in order to have a positive mean, is $d = 11/5$—a distance small enough to eliminate all fields $\mathbb{Q}(\sqrt{-d})$, $d \geq 3$, but too great to eliminate $\mathbb{Q}(\sqrt{-2})$. Similarly, the critical distance d within which a polynomial $T(x) = -(x^2-1)(x^2-4)(x^2+d)(x^2+4d)$ must have a root, in order to have a positive mean, is approximately 1.452—a distance small enough only for $\mathbb{Q}(i)$. For $\mathbb{Q}(\sqrt{-2})$ then, it is necessary to make use of the Galois structure afforded by the hypothesis of the theorem. In a two-variable setting that registers $\tilde{\pi}$, as well as π, all quadratic imaginary fields can be eliminated at once, as we will see.

Let $K = \mathbb{Q}(\sqrt{-d})$ for $d \geq 1$. For each prime $p \nmid N$, let $\mathrm{Tr}(p) = \lambda_\pi(p) + \lambda_{\pi'}(p)$ and $\mathrm{Nm}(p) = \lambda_\pi(p)\lambda_{\pi'}(p)$ be the trace and norm in $\mathbb{Z}$ of the coefficients, as elements in K. Then, for $T(x, y)$ in $S = \mathbb{R}[x, y]$, set

$$I(T) = \lim_{M\to\infty} \frac{1}{M} \sum_{p\leq M} T\big(\mathrm{Tr}(p), \mathrm{Nm}(p)\big) \log p, \tag{3.3}$$

for $T \in S$. The linear form I is evaluated at points (x, y) in the standard $\mathbb{Z}^2$ lattice in $\mathbb{R}^2$ such that $x = \mathrm{Tr}(p)$ and $y = \mathrm{Nm}(p)$. Dropping the dependence on p, if $\lambda_\pi = a + b\sqrt{-d}$, then when $x = \mathrm{Tr} = 2a$, we have $y = \mathrm{Nm} = a^2 + db^2 \geq a^2 = (1/4)x^2$. Only those lattice points on or above the parabola $y = (1/4)x^2$ can possibly be represented as trace $\times$ norm of the coefficients. Set

$$T(x, y) = \left(y - \frac{1}{4}x^2\right)(y - 1)^3. \tag{3.4}$$

The function T vanishes on the parabola $y = (1/4)x^2$. Within it, T is zero on the line $y = 1$, and strictly positive above it—a region which for every quadratic imaginary field includes all points $(\mathrm{Tr}(p), \mathrm{Nm}(p))$. Accordingly, T should always have a nonnegative mean. We compute however, using the values

$$\begin{array}{llll} I(y) = 1, & I(y^2) = 2, & I(x^2) = 2, & I(x^2y) = 4, \\ I(y^3) = 4, & I(x^2y^2) = 8, & I(y^4) = 20, & I(x^2y^3) = 48, \end{array} \tag{3.5}$$

that $I(T(x, y)) = -3/2 < 0$, which gives the desired contradiction.

4 K-real quadratic

4.1

Next, we consider the case when K is a real quadratic extension of $\mathbb{Q}$. Let S be the real vector space generated by polynomials in two variables x and y of the form

$$x^i, \quad i \leq 8; \qquad y^j, \quad j \leq 8; \qquad x^i y^j, \quad 1 \leq i, j \leq 4. \tag{4.1}$$

Define a linear form I on S by

$$I(T) = \lim_{M\to\infty} \frac{1}{M} \sum_{p \le M} T\big(\lambda_\pi(p), \lambda_{\pi'}(p)\big) \log p, \tag{4.2}$$

for $T \in S$.

We assume for the moment that $\mathrm{sym}^2\pi \simeq \mathrm{sym}^2\pi'$, and then evaluate I at any polynomial $T \in S$ using identity (2.34). This assumption will be in effect until Section 4.3. For brevity, in the list that follows, we give only the nonzero values of I for those monomials in S needed in the analysis:

$$\begin{aligned} &I(1) = 1, \qquad I\big(x^2\big) = I\big(y^2\big) = 1, \qquad I\big(x^2y^2\big) = 2, \\ &I\big(x^4\big) = I\big(y^4\big) = 2, \qquad I\big(x^6\big) = 5, \qquad I\big(x^8\big) = 14. \end{aligned} \tag{4.3}$$

From $\mathrm{sym}^2\pi \simeq \mathrm{sym}^2\pi'$, we deduce that $\lambda_\pi(p) = \pm\lambda_{\pi'}(p)$ for every p. As one eigenvalue is the Galois conjugate of the other, either $\lambda_\pi(p) \in \mathbb{Z}$ or $\sqrt{d}\lambda_\pi(p) \in \mathbb{Z}$. Thus in the x, y-plane, all coefficient pairs $(\lambda_\pi(p), \lambda_{\pi'}(p))$ lie on the line $y = x$, or $y = -x$. Set

$$T_\epsilon(x, y) = (y - x)^2\big(x^2 + y^2 - 4 - \epsilon\big), \tag{4.4}$$

for $\epsilon \ge 0$. When $d \ge 3$, the polynomial T_ϵ is nonnegative on all possible coefficient pairs as long as $\epsilon \le 2$. Indeed, when $K = \mathbb{Q}(\sqrt{3})$, the coefficient pairs off the line $y = x$ lie at distance at least $\sqrt{6}$ from the origin. (This is the closest any off-diagonal coefficient pair can be to the origin, for in the case that $K = \mathbb{Q}(\sqrt{5})$, any K-integer that is plus or minus its Galois conjugate has trivial denominator.) Thus, having as a factor in T_ϵ, the equation for a circle of radius at most $\sqrt{6}$ ensures nonnegativity for the value of T_ϵ at all possible coefficient pairs. Accordingly, T_ϵ should have zero mean-trace. We compute, however, using the identities of (4.3), that for $\epsilon > 0$, $I(T_\epsilon) < -2\epsilon$. Thus for $d \ge 3$, we have obtained a contradiction.

4.2

Now, when $d = 2$, the above arguments do not produce a contradiction, for then $\epsilon = 0$ gives $I(T_\epsilon) = 0$. This is well so, for as the explicit matrix realization of Section 2.1 showed, $\mathbb{Q}(\sqrt{2})$ is precisely the case which occurs for automorphic π coming from octahedral Galois representations. In this case, we construct the subspace V of the theorem, along with its integral basis, as follows. Let ρ be an even octahedral representation of the absolute Galois group of $\mathbb{Q}$. The image of ρ in $GL_2^{(2)}(\mathbb{C})$ is given in Section 2.1. By composing ρ with the nontrivial Galois automorphism of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, which acts entrywise on the image of ρ, we obtain the Galois conjugate octahedral representation ρ'. The results of Langlands [8] and Tunnell [10] produce corresponding cuspidal automorphic representations (π, H_π) and $(\pi', H_{\pi'})$ of GL_2 over $\mathbb{Q}$ preserving L-functions. Moreover, π is weight-zero since ρ is even. If $f \in H_\pi$ and $f' \in H_{\pi'}$ are eigenfunctions of the Hecke subalgebra T, then we take $V = \mathbb{C}\{f, f'\}$. A basis of V with respect to which T acts by integral matrices is $\{(f + f')/2, (f - f')/2\sqrt{2}\}$.

The symmetric power L-functions of an octahedral π are invertible at $s = 1$ for all powers up to the seventh, but have a pole there at the eighth. In order to rule out the existence of a nonoctahedral π, the polynomial producing the contradiction must be able to distinguish between the two by picking up the behavior of $L(s, \mathrm{sym}^8\pi)$ at $s = 1$. The degree of the polynomial should accordingly be at least 8. We set

$$T(x, y) = x^2\left(x^2 - 1\right)\left(x^2 - 2\right)\left(x^2 - 4\right). \tag{4.5}$$

The seven vertical lines in the vanishing locus of T coincide with the coefficient pairs which satisfy the Ramanujan conjecture. The assumption that $\mathrm{sym}^2\pi \simeq \mathrm{sym}^2\pi'$ ensures that all other coefficient pairs lie to the left or right of these vertical lines. In those two regions, the polynomial T is positive. With these considerations, it should follow that the mean trace of T is nonnegative. We compute, however, using the identities of (4.3), that $I(T) = -1$. This contradiction eliminates the possibility that the π of our hypothesis can be non-Galois. Finally, we check that if π corresponded to an octahedral Galois representation, the value $I(x^8)$ would be 15, $I(T)$ would be 0, and no contradiction then occurs.

4.3

We now go through similar arguments under the complementary assumption that $\mathrm{sym}^2\pi \neq \mathrm{sym}^2\pi'$. This changes the value of $I(x^2y^2)$ listed in (4.3) from 2 to 1. We take $K = \mathbb{Q}(\sqrt{d})$ where $d \neq 5$. If the Ramanujan conjecture holds, the only points $(\lambda_\pi(p), \lambda_{\pi'}(p))$ which contribute to the value of I are those for which both coordinates are bounded in absolute value by 2. Those that are also K-integral lie either on the diagonal $y = x$, as with $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(2, 2)$, and $(-2, -2)$; or off the diagonal and at distance from the origin no less than 2, as with $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$, when $K = \mathbb{Q}(\sqrt{2})$. Set

$$T(x, y) = \left(x^2 + y^2 - 4\right)(y - x)^2. \tag{4.6}$$

The polynomial T is nonnegative in the region bounded away from the circle centered at the origin of radius 2, and also on all Ramanujan points enumerated above. The mean $I(T)$ should therefore be nonnegative. However, using the identities of (4.3), we compute that $I(T) = -2$, a contradiction.

5 The exceptional field $\mathbb{Q}(\sqrt{5})$

Up to this point, we have proven the theorem when the coefficient field is any quadratic extension of $\mathbb{Q}$ except $\mathbb{Q}(\sqrt{5})$. This case must remain outstanding for reasons we now set forth.

If ρ is an even icosahedral representation of the absolute Galois group of $\mathbb{Q}$, with trivial central character, then its range in $GL_2(\mathbb{C})$ is described in Section 2.1 and we can form the Galois conjugate ρ' by composing ρ with the nontrivial element in $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. To ρ and ρ' are associated, through a conjectural Artin correspondence, cuspidal automorphic representations (π, H_π) and $(\pi', H_{\pi'})$ both of which are irreducible constituents of W if $f \in H_\pi$ and $f' \in H_{\pi'}$ are eigenfunctions of the Hecke algebra T, the subspace $V = \mathbb{C}\{f, f'\}$ admits the basis $\{(f + f')/2, (f - f')/2\sqrt{5}\}$ with respect to which the Hecke operators act as integral matrices.

The symmetric power L-functions of f, being equal to those of ρ, are invertible at $s = 1$ for all powers up to the eleventh, and have a pole there at the twelfth. This property serves to test whether a given automorphic form is Galois, for any Maass cusp form whose coefficients are integers in $\mathbb{Q}(\sqrt{5})$ could not possibly correspond to an icosahedral Galois representation if its symmetric power L-functions were invertible at $s = 1$ for all powers up to and including the twelfth. This latter property would presumably hold, via Rankin-Selberg factorization, if there existed symmetric power functorial lifts up through the sixth, and if under every one of these lifts the given form was actually cuspidal. As we have at present only functorial lifts up to the fourth power, and from this, knowledge of the form's symmetric power L-functions only up to the eighth, those Maass forms which, conjecturally at least, are of icosahedral Galois provenance are effectively indistinguishable from those which are not. This explains why in the theorem no claims are made about forms with quadratic integer coefficients inside $\mathbb{Q}(\sqrt{5})$.

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