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Lower functions for the support of super-Brownian motion

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Abstract

The aim of this paper is to describe the minimum speed at which a super-Brownian motion starting at the Dirac mass at 0 moves away from its initial point. More precisely, we consider the class of functions $\{\varphi_\kappa(t)=\sqrt{2t(\log(1/t)+\kappa\log\log(1/t))},\ \kappa\in\mathbb{R}\}$ and then determine the values of κ such that the support of super-Brownian motion exits the ball of radius $\varphi_\kappa(t)$ before time t, for every t small enough.

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1. Introduction

The aim of this paper is to give precise information on the minimum speed at which a super-Brownian motion goes away from its starting point. For standard d-dimensional Brownian motion this problem has been solved by Chung (1948). For super-Brownian motion, partial results were obtained by Tribe (1989), and Dawson and Vinogradov (1994). We give here an optimal refinement of their results.

Let $X=(X_t,\ t\geqslant 0)$ denote under $\mathbb{P}_{\delta_0}$ a super-Brownian motion in $\mathbb{R}^d$, starting at δ_0, the Dirac mass at 0. For $t\geqslant 0$, we denote by supp X_t the topological support of the measure X_t, by

$$R_t=\sup\{|y|;\ \ y\in\operatorname{supp}X_t\},$$

the "maximal distance" covered by super-Brownian motion at time t (with the convention $R_t=0$ if $X_t=0$), and by

$$R_t^*=\sup\{R_{t'};\ \ 0\leqslant t'\leqslant t\},$$

the "maximal distance" covered by super-Brownian motion before t. Our main result gives a precise lower bound for R_t^*.

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Theorem 1. *Let $\kappa \in \mathbb{R}$. The following assertions are equivalent.*

(i) $\kappa < d/2$;

(ii) $\mathbb{P}_{\delta_0}$ *almost surely, there exists* $t_0 \in (0, \mathrm{e}^{-1})$ *such that for* $t \in (0, t_0]$,

$$R_t^* \geqslant \sqrt{2t(\log(1/t) + \kappa \log\log(1/t))}.$$

Let us compare Theorem 1 to previous results. Tribe (1989) proved that

$$\lim_{t \to 0} \frac{R_t}{\sqrt{2t \log(1/t)}} = 1, \quad \mathbb{P}_{\delta_0}-\text{a.s.} \tag{1}$$

It easily follows that the lower bound

$$R_t^* \geqslant \sqrt{ct \log(1/t)}, \quad \mathbb{P}_{\delta_0}-\text{a.s. for } t \text{ small}$$

holds if $c < 2$.

Moreover, Dawson and Vinogradov (1994) proved that if $\kappa < d/2 - 1$, the inequality

$$R_t^* > \sqrt{2t(\log(1/t) + \kappa \log\log(1/t))}$$

holds $\mathbb{P}_{\delta_0}$-a.s., for t sufficiently small. Theorem 1 shows that the critical value of κ in the previous inequality is $d/2$ and not $d/2 - 1$.

Remark. Theorem 1 is closely related to Dhersin and Le Gall (1998) where an integral test characterizing the maximal speed of R_t is derived. Moreover, using that if $\eta > 0$ and $u \geqslant 0$ is small then $1 + (\frac{1}{2} - \eta)u \leqslant (1+u)^{1/2} \leqslant 1 + (\frac{1}{2} + \eta)u$, Theorem 1 and Dhersin and Le Gall (1998) (Theorem 1) enable to establish a precise description of the behavior of R_t^* for small t. This result is an optimal refinement of Dawson and Vinogradov (1994), (Theorem 1.7).

Corollary 2. *If* $\varepsilon > 0$, *then* $\mathbb{P}_{\delta_0}$ *almost surely, there exists* $t_0 \in (0, \mathrm{e}^{-1})$ *such that for* $t \in (0, t_0]$,

$$1 + \left(\frac{d}{4} - \varepsilon\right) \frac{\log\log(1/t)}{\log(1/t)} \leqslant \frac{R_t^*}{\sqrt{2t \log(1/t)}} \leqslant 1 + \left(1 + \frac{d}{4} + \varepsilon\right) \frac{\log\log(1/t)}{\log(1/t)}.$$

As usual for these problems, the key step in the proof of Theorem 1 is a precise estimation of the probability of the event $\{R_t \geqslant a\}$. This is the purposal of Lemma 3 in Section 3. In fact, what we need is not an estimate under the probability $\mathbb{P}_{\delta_0}$, but under the (infinite) canonical measure of super-Brownian motion. To this aim, we use the so-called Brownian snake introduced by Le Gall (see e.g. Le Gall, 1993, 1994) as a useful tool to investigate properties of super-Brownian motion. The construction of this path-valued Markov process is briefly recalled in Section 2. We also recall some connections between the Brownian snake and partial differential equations, which are used in our proofs. Finally, the proof of Theorem 1 easily follows from Lemma 3, excursion theory and the Borel–Cantelli Lemma.

2. The Brownian snake and super-Brownian motion

In this section, we briefly recall the basic facts concerning the Brownian snake, and its connection with super-Brownian motion.

The Brownian snake is a Markov process with values in the set of stopped paths. A stopped path is a pair (w,ζ), where $\zeta\geqslant 0$ and $w:\mathbb{R}_+\to\mathbb{R}^d$ is a continuous mapping such that $w(t)=w(\zeta)$ for every $t\geqslant\zeta$. The real ζ is called the lifetime of the path. We always abuse notation, and simply write w for (w,ζ). We also use the notation $\hat{w}=w(\zeta)$ for the tip of the path. We endow the set $\mathscr{W}$ of all stopped paths with the distance

$$d(w,w')=\sup_{t\geqslant 0}|w(t)-w'(t)|+|\zeta-\zeta'|.$$

Let $x\in\mathbb{R}^d$ be a fixed point. We denote by $\mathscr{W}_x$ the set of all stopped paths with initial point $w(0)=x$, and by $\underline{x}$ the trivial path of $\mathscr{W}_x$ with lifetime $\zeta=0$.

The Brownian snake with initial point x is the continuous strong Markov process $W=(W_s,\ s\geqslant 0)$ in $\mathscr{W}_x$ whose law is characterized as follows.

(i) If ζ_s denotes the lifetime of W_s, the process $(\zeta_s,\ s\geqslant 0)$ is a reflecting Brownian motion in $\mathbb{R}_+$.

(ii) Conditionally on $(\zeta_s,\ s\geqslant 0)$, the process W is a time-inhomogeneous Markov process whose transition kernels are characterized by the following properties: If $0\leqslant s<s'$,

∘ $W_{s'}(t)=W_s(t)$ for every $t\leqslant m(s,s'):=\inf_{[s,s']}\zeta_r$;

∘ $(W_{s'}(m(s,s')+t)-W_{s'}(m(s,s'))$, $0\leqslant t\leqslant\zeta_{s'}-m(s,s'))$ is a Brownian motion in $\mathbb{R}^d$, independent of W_s.

We may and will assume that the process $(W_s,\ s\geqslant 0)$ is the canonical process on the space $C(\mathbb{R}_+,\mathscr{W})$ of all continuous functions from $\mathbb{R}_+$ into $\mathscr{W}$.

Heuristically, we can see W_s as a Brownian path in $\mathbb{R}^d$ whose random lifetime ζ_s evolves like reflecting Brownian motion. Furthermore, when ζ_s decreases, the path W_s is "erased"; when ζ_s increases, the path W_s is extended by "adding" independent pieces of d-dimensional Brownian motion at its tip.

As 0 is regular and recurrent for reflecting Brownian motion, $\underline{x}$ is regular and recurrent for the Brownian snake. We denote by $\mathbb{N}_x$ the associated excursion measure, normalized by

$$\mathbb{N}_x[M>1]=\tfrac{1}{2},$$

where $M=\sup_{s\geqslant 0}\zeta_s$. The law of (ζ_s) under $\mathbb{N}_x$ is the usual Itô measure of positive excursions of linear Brownian motion. Using this remark, it is easy to see that $\mathbb{N}_x$ satisfies the following useful scaling property: If $\lambda>0$, we define $W_s^{(\lambda)}\in\mathscr{W}_x$ by

$$\zeta_s^{(\lambda)}=\lambda^{-2}\zeta_{\lambda^4 s},\qquad W_s^{(\lambda)}(t)-x=\lambda^{-1}(W_{\lambda^4 s}(\lambda^2 t)-x),\quad s\geqslant 0,\ t\geqslant 0.$$

Then the law under $\mathbb{N}_x$ of the process $W^{(\lambda)}$ is $\lambda^{-2}\mathbb{N}_x$.

We now describe one basic connection between the Brownian snake and partial differential equations. These results are due to Dynkin (1992), and the formulation in terms of the Brownian snake follows from Le Gall (1994). We write $\sigma = \inf\{s>0; \zeta_s = 0\}$ for the lifetime of the excursion. Let us introduce $\mathscr{G} = \{(\zeta_s, \hat{W}_s); 0 \leqslant s \leqslant \sigma\}$ the graph of the snake excursion. Let Γ be a domain in $\mathbb{R} \times \mathbb{R}^d$, and for $r \in \mathbb{R}$, $\Gamma^{(r)} = \{(t-r, y); (t,y) \in \Gamma, t \geqslant r\} \subset \mathbb{R}_+ \times \mathbb{R}^d$. Then the function u defined by

$$u(r,x) = \mathbb{N}_x[\mathscr{G} \cap (\Gamma^{(r)})^{\mathrm{c}} \neq \emptyset], \quad (r,x) \in \Gamma$$

solves the parabolic semi-linear differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2} \Delta u = 2u^2 \tag{2}$$

in Γ.

Let us now explain the construction of super-Brownian motion via the Brownian snake. We denote by $L_s^t(\zeta)$ the local time at level t at time s of the Brownian excursion $(\zeta_\cdot)$. For $t>0$, we define the random finite measure $\mathscr{X}_t$ on $\mathbb{R}^d$ as follows: If φ is any nonnegative continuous function on $\mathbb{R}^d$,

$$\langle \mathscr{X}_t, \varphi \rangle = \int_0^\sigma \varphi(\hat{W}_s)\, \mathrm{d}L_s^t(\zeta).$$

Then, the distribution under $\mathbb{N}_x$ of measure valued process $(\mathscr{X}_t,\ t>0)$ is the so-called canonical measure of super-Brownian motion started at the Dirac measure δ_x (see El Karoui and Roelly (1991) for canonical measures of general superprocesses). This means (see Le Gall (1993)) that if $\mathscr{N}(\mathrm{d}\omega)$ is a Poisson point measure on $\mathscr{C}(\mathbb{R}_+, \mathscr{W}_x)$ with intensity $\mathbb{N}_x$, then the measure valued process $(X_t,\ t \geqslant 0)$ defined by $X_0 = \delta_x$ and for $t>0$,

$$X_t = \int \mathscr{N}(\mathrm{d}\omega) \mathscr{X}_t(\omega)$$

is a super-Brownian motion. This connection between super-Brownian motion and the excursion measure of the Brownian snake will be used in Section 4 for the proof of Theorem 1.

3. Hitting probabilities

Our aim in this section is to give bounds on the probability that, at a fixed time t, super-Brownian motion under its canonical measure hits the complement of a large-ball centered at 0. More precisely, we introduce for $t>0$

$$r_t = \sup\{\hat{W}_s;\ \zeta_s = t,\ 0 \leqslant s \leqslant \sigma\}$$

and

$$r_t^* = \sup\{r_{t'};\ 0 \leqslant t' \leqslant t\}.$$

These quantities are analogues of R_t and R_t^* under the canonical measure.

Lemma 3. *There exist two positive constants* α and β *such that, if* $t>0$ *and* $a\geqslant\sqrt{t}$ *then*

$$\frac{\alpha}{t}\left(\frac{a}{\sqrt{t}}\right)^d \exp-\frac{a^2}{2t} \leqslant \mathbb{N}_0[r_t\geqslant a] \leqslant \mathbb{N}_0[r_t^*\geqslant a]\leqslant\frac{\beta}{t}\left(\frac{a}{\sqrt{t}}\right)^d \exp-\frac{a^2}{2t}.$$

Remark. The upper bound of Lemma 3 was previously obtained by Dawson et al. (1989) (Theorem 3.3(b)) in a slightly different form. To make the present work self contained, we will provide a short proof of this upper bound.

We first state without proof a simple lemma about usual Brownian motion. If $x\in\mathbb{R}^d$, (B_t) under P_x is a d-dimensional Brownian motion starting at x.

Lemma 4. *There exist two positive constants* α' *and* β' *such that, if* $t>0$ *and* $a\geqslant\sqrt{t}$ *then*

$$\alpha'\left(\frac{a}{\sqrt{t}}\right)^{d-2} \exp-\frac{a^2}{2t} \leqslant P_0[|B_t|\geqslant a] \leqslant P_0\left[\sup_{0\leqslant s\leqslant t}|B_s|\geqslant a\right]\leqslant\beta'\left(\frac{a}{\sqrt{t}}\right)^{d-2} \exp-\frac{a^2}{2t}.$$

Proof of Lemma 3. First of all, using the scaling property under $\mathbb{N}_0$ and a continuity argument it is sufficient to prove the result for $t=1$ and $a\geqslant 2$, which we assume throughout the proof.

We first prove the upper bound. We denote by $\Gamma_a\subset\mathbb{R}_+\times\mathbb{R}^d$ the domain such that

$$\Gamma_a^c=(\mathbb{R}_+\times\mathbb{R}^d)\backslash\Gamma_a=\{(t,y)\in\mathbb{R}_+\times\mathbb{R}^d;\ 0\leqslant t\leqslant 1, |y|\geqslant a\}.$$

Let us recall that if $r\in\mathbb{R}$, then $\Gamma_a^{(r)}=\{(t-r,y);(t,y)\in\Gamma_a, t\geqslant r\}$. Then, we saw in Section 2 that the function

$$u(r,x)=\mathbb{N}_x[\mathscr{G}\cap(\Gamma_a^{(r)})^c\neq\emptyset],\quad (r,x)\in\Gamma_a$$

solves Eq. (2) in Γ_a.

We look for an upper bound on $\mathbb{N}_0[r_1^*\geqslant a]=u(0,0)$. Let us fix $b\in[a/2,a)$ $(\subset[1,\infty))$, introduce Γ_b as above, and denote by $\tau_b=\inf\{t\geqslant 0;\ |B_t|>b\}$, with the usual convention $\inf\emptyset=+\infty$. Let us remark that the process $(t,B_t)_{t\geqslant 0}$ hits the boundary of Γ_b if and only if $\tau_b<1$, and then τ_b denotes the hitting time.

Using Itô's formula, and the fact that u solves Eq. (2), it is easy to prove that the process

$$M_r=u(r\wedge\tau_b,B_{r\wedge\tau_b})\exp-2\int_0^{r\wedge\tau_b}u(s,B_s)\,\mathrm{d}s,\quad r\geqslant 0$$

is a P_0-martingale. Moreover, if we fix $(r,x)\in\Gamma_b$,

$$\begin{aligned}u(r,x) &= \mathbb{N}_x[\mathscr{G}\cap(\Gamma_a^{(r)})^{\mathrm{c}}\neq\emptyset]\\ &\leqslant \mathbb{N}_0[\mathscr{G}\cap(\mathbb{R}_+\times B(0,a-b)^{\mathrm{c}})\neq\emptyset]\\ &\leqslant (a-b)^{-2}\mathbb{N}_0[\mathscr{G}\cap(\mathbb{R}_+\times B(0,1)^{\mathrm{c}})\neq\emptyset]<\infty,\end{aligned} \tag{3}$$

by a scaling argument. Note also that $u(r,x)=0$ if $r\geqslant 1$.

Hence (M_r) is a bounded martingale. By applying the optional stopping theorem, bound (3), and Lemma 4, we get

$$\begin{aligned}u(0,0) &= E_0\left[\mathbf{1}_{\{\tau_b<1\}}u(\tau_b,B_{\tau_b})\exp-2\int_0^{\tau_b}u(s,B_s)\,\mathrm{d}s\right]\\ &\leqslant \mathbb{N}_0[\mathscr{G}\cap(\mathbb{R}_+\times B(0,1)^{\mathrm{c}})\neq\emptyset](a-b)^{-2}P_0\left[\sup_{0\leqslant t\leqslant 1}|B_t|\geqslant b\right]\\ &\leqslant \beta'\mathbb{N}_0[\mathscr{G}\cap(\mathbb{R}_+\times B(0,1)^{\mathrm{c}})\neq\emptyset](a-b)^{-2}b^{d-2}\exp-\frac{b^2}{2}.\end{aligned}$$

The proof of the upper bound is completed by taking $b=a-1/a$.

The proof of the lower bound is more involved. First of all, using the Cauchy–Schwarz inequality, we get

$$\mathbb{N}_0[r_1\geqslant a]=\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})\neq 0]\geqslant\frac{(\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})])^2}{\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})^2]}. \tag{4}$$

In order to estimate the first and second moments of $\mathscr{X}_1(B(0,a)^{\mathrm{c}})$ we recall that (see e.g. Le Gall and Perkins, 1995, Proposition 2.2)

$$\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})]=P_0[|B_1|\geqslant a], \tag{5}$$

$$\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})^2]=4E_0\left[\int_0^1\mathrm{d}u(P_{B_u}[|B_{1-u}|\geqslant a])^2\right]. \tag{6}$$

Using Lemma 4 and Eq. (5), we get

$$\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})]\geqslant\alpha' a^{d-2}\exp-\frac{a^2}{2}. \tag{7}$$

The proof of the upper bound on the second moment is more technical. In what follows, we denote by $c_1,c_2,\ldots$ positive constants independent of a. Using Eq. (6), we get

$$\mathbb{N}_0[\mathscr{X}_1(B(0,a)^{\mathrm{c}})^2]\leqslant 4(I_1+I_2),$$

where

$$I_1=E_0\left[\int_0^1\mathrm{d}u\,\mathbf{1}_{\{|B_u|\geqslant a\}}\right],$$

$$I_2=E_0\left[\int_0^1\mathrm{d}u\,\mathbf{1}_{\{|B_u|<a\}}(P_{B_u}[\tau_a<1-u])^2\right].$$

By Lemma 4,

$$\begin{aligned} I_1 &\leqslant \beta' \int_0^1 \mathrm{d}u \left(\frac{a}{\sqrt{u}}\right)^{d-2} \exp -\frac{a^2}{2u} \\ &\leqslant \beta' a^2 \int_{a^2}^{\infty} \mathrm{d}v\, v^{d/2-3} \exp -\frac{v}{2} \\ &\leqslant c_1 a^{d-4} \exp -\frac{a^2}{2}. \end{aligned} \tag{8}$$

On the other hand, using the explicit density of $|B_u|$ and Lemma 4,

$$\begin{aligned} I_2 &\leqslant c_2 \int_0^1 \mathrm{d}u \int_0^a \mathrm{d}\rho\, \rho^{d-1} u^{-d/2} \exp\left(-\frac{\rho^2}{2u}\right)(P_0[\tau_{a-\rho} < 1-u])^2 \\ &\leqslant c_3 \int_0^1 \mathrm{d}u \int_0^a \mathrm{d}\rho\, \rho^{d-1} u^{-d/2} \exp\left(-\frac{\rho^2}{2u}\right)\left(\frac{a-\rho}{\sqrt{1-u}} \vee 1\right)^{2(d-2)} \\ &\qquad \times \exp\left(-\frac{(a-\rho)^2}{1-u}\right) \\ &= c_3\left(\int_0^{1/2} \mathrm{d}u \ldots + \int_{1/2}^1 \mathrm{d}u \ldots\right) \\ &= c_3(I_2' + I_2''). \end{aligned} \tag{9}$$

To give an upper bound on I_2', let us fix $\gamma = \frac{3}{4}$. Since the function

$$x \to x^{(d-1)/2} \exp -\frac{(1-\gamma)x}{2}$$

is bounded on $\mathbb{R}_+$, we get

$$\begin{aligned} I_2' &\leqslant c_4 \int_0^{1/2} \mathrm{d}u \int_0^a \mathrm{d}\rho\, u^{-1/2} \exp\left(-\frac{\gamma\rho^2}{2u}\right)\left(\frac{a-\rho}{\sqrt{1-u}} \vee 1\right)^{2(d-2)} \exp\left(-\frac{(a-\rho)^2}{1-u}\right) \\ &\leqslant c_5(1 + a^{2(d-2)}) \int_0^{1/2} \mathrm{d}u \int_0^a \mathrm{d}\rho\, p^{(1)}_{u/\gamma}(0,\rho)\, p^{(1)}_{(1-u)/2}(\rho, a), \end{aligned}$$

where $(p_t^{(1)},\ t \geqslant 0)$ are the transition densities of linear Brownian motion. Hence, by the Chapman–Kolmogorov formula, and using the inequalities

$$\frac{1}{2} \leqslant \frac{u}{\gamma} + \frac{1-u}{2} \leqslant \frac{11}{12} \quad \text{for } 0 \leqslant u \leqslant \frac{1}{2},$$

we finally obtain

$$\begin{aligned} I_2' &\leqslant c_5(1 + a^{2(d-2)}) \int_0^{1/2} \mathrm{d}u\, p^{(1)}_{u/\gamma + (1-u)/2}(0, a) \\ &\leqslant c_6 a^{d-4} \exp -\frac{a^2}{2}, \end{aligned}$$

where the last bound follows from crude estimates.

To get an upper bound on I_2'', we make the change of variables $b=(a-\rho)/\sqrt{1-u}$. Then,

$$I_2'' \leqslant 2^{d/2}a^{d-1}\int_{1/2}^{1} \mathrm{d}u\,\sqrt{1-u}\int_0^{a/\sqrt{1-u}} \mathrm{d}b\,(b\vee 1)^{2(d-2)} \times \exp\left(-b^2-\frac{(a-b\sqrt{1-u})^2}{2u}\right).$$

Since we have

$$b^2+\frac{(a-b\sqrt{1-u})^2}{2u}=\frac{1+u}{2u}\left(b-a\,\frac{\sqrt{1-u}}{1+u}\right)^2+\frac{a^2}{2}+\frac{a^2(1-u)}{2(1+u)},$$

we get for $u\in[\frac{1}{2},1]$,

$$\int_0^{a/\sqrt{1-u}} \mathrm{d}b\,(b\vee 1)^{2(d-2)}\exp\left(-b^2-\frac{(a-b\sqrt{1-u})^2}{2u}\right) \leqslant \mathrm{e}^{-a^2/2}\,\mathrm{e}^{-a^2(1-u)/4}\int_0^{\infty} \mathrm{d}b\,(b\vee 1)^{2(d-2)}\,\mathrm{e}^{-1/2(b-a\sqrt{1-u}/(1+u))^2}.$$

If $d=1$,

$$\int_0^{\infty} \mathrm{d}b\,(b\vee 1)^{2(d-2)}\,\mathrm{e}^{-1/2(b-a\sqrt{1-u}/(1+u))^2} \leqslant \int_{-\infty}^{\infty} \mathrm{d}\alpha\,\mathrm{e}^{-\alpha^2/2}.$$

Otherwise,

$$\begin{aligned}
&\int_0^{\infty} \mathrm{d}b\,(b\vee 1)^{2(d-2)}\,\mathrm{e}^{-1/2(b-a\sqrt{1-u}/(1+u))^2}\\
&\quad\leqslant \int_0^{\infty} \mathrm{d}b\,(1+b^{2(d-2)})\,\mathrm{e}^{-1/2(b-a\sqrt{1-u}/(1+u))^2}\\
&\quad\leqslant 2^{2(d-2)}\int_{-\infty}^{\infty} \mathrm{d}\alpha\left(1+|\alpha|^{2(d-2)}+\left(a\,\frac{\sqrt{1-u}}{1+u}\right)^{2(d-2)}\right)\mathrm{e}^{-\alpha^2/2}\\
&\quad\leqslant c_7(1+(a\sqrt{1-u})^{2(d-2)}).
\end{aligned}$$

Hence,

$$\begin{aligned}
I_2'' &\leqslant c_8 a^{d-1}\,\mathrm{e}^{-a^2/2}\int_{1/2}^{1} \mathrm{d}u\,\sqrt{1-u}(1+(a\sqrt{1-u})^{2(d-2)})\mathrm{e}^{-a^2(1-u)/4}\\
&\leqslant 2c_8 a^{d-4}\mathrm{e}^{-a^2/2}\int_0^{a^2/2} \mathrm{d}v\,\sqrt{v}\,(1+v^{d-2})\mathrm{e}^{-v/4}\\
&\leqslant c_9 a^{d-4}\mathrm{e}^{-a^2/2}.
\end{aligned}$$

Therefore, there exists a positive constant c_{10} such that

$$I_2 \leqslant c_{10}a^{d-4}\mathrm{e}^{-a^2/2}.$$

The upper bounds on I_1 and I_2 complete the proof of Lemma 3. □

4. Proof of Theorem 1

For $\kappa\in\mathbb{R}$, we introduce the function $\varphi_\kappa(t)=\sqrt{2t(\log(1/t)+\kappa\log\log(1/t))}$ defined for $t>0$ small enough.

We first fix $\kappa=d/2$. We assume that t is small enough so that $\varphi_{d/2}(t)\geqslant\sqrt{t}$. Since $\mathscr{X}_t$ is distributed under $\mathbb{N}_0$ according to the canonical measure of super-Brownian motion started at δ_0, we get by Lemma 3

$$\begin{aligned}\mathbb{P}_{\delta_0}[R_t^*<\varphi_{d/2}(t)] &= \exp-\mathbb{N}_0[r_t^*\geqslant\varphi_{d/2}(t)]\\ &\geqslant \exp-\frac{\beta}{t}\left(\frac{\varphi_{d/2}(t)}{\sqrt{t}}\right)^d\exp-\frac{\varphi_{d/2}(t)^2}{2t}.\end{aligned}$$

Let us introduce an arbitrary decreasing sequence (t_n) of positive reals which tends to 0. A straightforward calculation shows that

$$t\to\frac{\beta}{t}\left(\frac{\varphi_{d/2}(t)}{\sqrt{t}}\right)^d\exp-\frac{\varphi_{d/2}(t)^2}{2t}$$

is bounded above over $(0,\mathrm{e}^{-1})$. Hence,

$$\mathbb{P}_{\delta_0}\left[\limsup_{n\to\infty}\{R_{t_n}^*<\varphi_{d/2}(t_n)\}\right]\geqslant\limsup_{n\to\infty}\mathbb{P}_{\delta_0}[R_{t_n}^*<\varphi_{d/2}(t_n)]>0.$$

Since the event $\limsup_{n\to\infty}\{R_{t_n}^*<\varphi_{d/2}(t_n)\}$ belongs to the asymptotic σ-field of the Markov process X, Blumenthal's 0–1 law gives

$$\mathbb{P}_{\delta_0}\left[\limsup_{n\to\infty}\{R_{t_n}^*<\varphi_{d/2}(t_n)\}\right]=1.$$

This implies that if $\kappa\geqslant d/2$, (ii) does not hold.

Conversely, let us assume that $\kappa<d/2$, and introduce the sequence $t_n=\exp-\sqrt{n}$. We claim that

$$\sum_{n=0}^{\infty}\mathbb{P}_{\delta_0}[R_{t_{n+1}}^*<\varphi_\kappa(t_n)]<\infty. \tag{10}$$

To prove this claim, we first note that for $n\geqslant N$ large enough, $\varphi_\kappa(t_n)\geqslant\sqrt{t_{n+1}}$ and $(\varphi_\kappa(t_n)^2/2t_n)(t_n/t_{n+1}-1)\leqslant 1$. Then using Lemma 3, we get that there exists a positive constant C such that

$$\begin{aligned}\mathbb{P}_{\delta_0}[R_{t_{n+1}}^*<\varphi_\kappa(t_n)] &= \exp-\mathbb{N}_0[r_{t_{n+1}}^*\geqslant\varphi_\kappa(t_n)]\\ &\leqslant \exp\left[-\frac{\alpha}{t_{n+1}}\left(\frac{\varphi_\kappa(t_n)}{\sqrt{t_{n+1}}}\right)^d\exp-\frac{\varphi_\kappa(t_n)^2}{2t_{n+1}}\right]\\ &\leqslant \exp\left[-\frac{\alpha}{t_n}\left(\frac{\varphi_\kappa(t_n)}{\sqrt{t_n}}\right)^d\left(\exp-\frac{\varphi_\kappa(t_n)^2}{2t_n}\right)\right.\\ &\qquad\left.\times\left(\exp-\frac{\varphi_\kappa(t_n)^2}{2t_n}\left(\frac{t_n}{t_{n+1}}-1\right)\right)\right]\end{aligned}$$

$$\leqslant \exp\left[-\mathrm{e}^{-1}\,\frac{\alpha}{t_n}\left[2\left(\log\frac{1}{t_n}+\kappa\log\log\frac{1}{t_n}\right)\right]^{d/2}\left[t_n\left(\log\frac{1}{t_n}\right)^{-\kappa}\right]\right]$$
$$\leqslant \exp(-Cn^{(d-2\kappa)/4}),$$

which completes the proof of (10).

Hence, using the Borel–Cantelli Lemma, we get that, $\mathbb{P}_{\delta_0}$-a.s., there exists an integer n_0 such that if $n \geqslant n_0$, $R^*_{t_{n+1}} \geqslant \varphi_\kappa(t_n)$. Then, by monotonicity of R^*, we get that if $0 < t \leqslant t_{n_0}$, and n satisfies $t_{n+1} \leqslant t \leqslant t_n$,

$$R^*_t \geqslant R^*_{t_{n+1}} \geqslant \varphi_\kappa(t_n) \geqslant \varphi_\kappa(t),$$

which proves that (ii) holds. □

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