A COMPARATIVE STUDY OF SOME TIME INTEGRATION SCHEMES FOR THE FINITE VOLUME SOLUTION OF HETEROGENUOUS DIFFUSION

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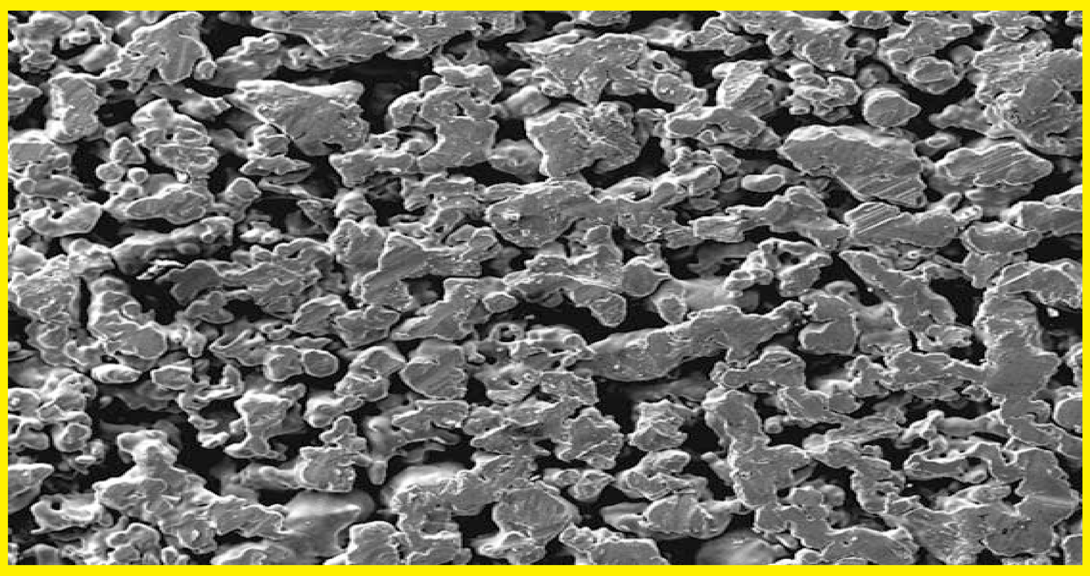
Introduction

We present a numerical comparative study for two time stepping schemes applied to the finite volume discretization of diffusion equations with heterogeneous diffusion coefficients. The cell-centered finite volume method is used to discretize the gradient operator and a mesh adaptation technique is adopted to improve the efficiency of the considered methods.



Objectives

To develop accurate and efficient finite volume method for solvingheterogeneous diffusion problems.

To use adaptive finite volume to simulate the flow transport in porous media.

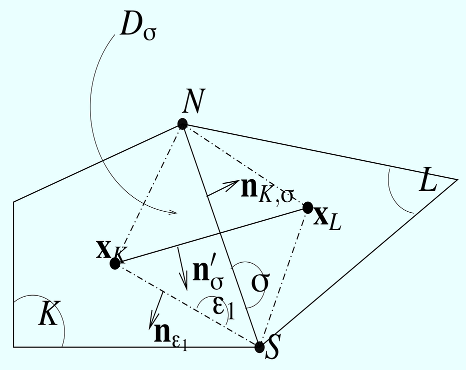
To validate developed methods with numerical solutions obtained using other methods.

Finite Volume Discretization

Our main concern in the present study is on the finite volume discretization of the two-dimensional gradient operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$ resulting from the weak formulation of the diffusion equations. To this end we discretize the spatial domain $\bar{\Omega} = \Omega \cup \partial\Omega$ in conforming triangular elements K_i as $\bar{\Omega} = \cup_{i=1}^{N} K_i$, with N is the total number of elements. Each triangle represents a control volume and the variables are located at the geometric centers of the cells. To discretize the diffusion operators we adapt the so-called cell-centered finite volume method based on a Green-Gauss diamond reconstruction. Hence, a co-volume, D_σ, is first constructed by connecting the barycentres of the elements that share the edge σ and its endpoints as shown in below figure. Then, the discrete gradient operator ∇_σ is evaluated at an inner edge σ as

$$\nabla_\sigma u_h = \frac{1}{2\text{meas}(D_\sigma)}\left((u_L - u_K)\text{meas}(\sigma)\mathbf{n}_{K,\sigma} + (u_S - u_N)\text{meas}(s_\sigma)\mathbf{n}'_\sigma\right), \qquad (1)$$

where u_h is the finite volume discretization of a generic function u, meas(D) denotes the area of the element D, $\mathbf{n}_{K,\sigma}$ denotes the unit outward normal to the surface σ, u_K and u_L are the values of the solution u_h in the elements K and L, respectively. In (1), u_S and u_N are the values of the solution u_h at the co-volume nodes approximated by a linear interpolation from the values on the cells sharing the same vertex S and N, respectively.



Time stepping schemes

For simplicity in the presentation we consider the transient diffusion problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla\cdot(\mathbb{K}(\mathbf{x})\nabla u) &= f(\mathbf{x},t), && (\mathbf{x},t)\in\Omega\times(0,T],\\ u(\mathbf{x},t) &= 0, && (\mathbf{x},t)\in\partial\Omega\times(0,T],\\ u(\mathbf{x},0) &= u_0, && \mathbf{x}\in\Omega, \end{aligned} \qquad (2)$$

Explicit scheme: For the time integration of (2) we use the forward Euler method, the fully discrete version of the diffusion equation (2) reads

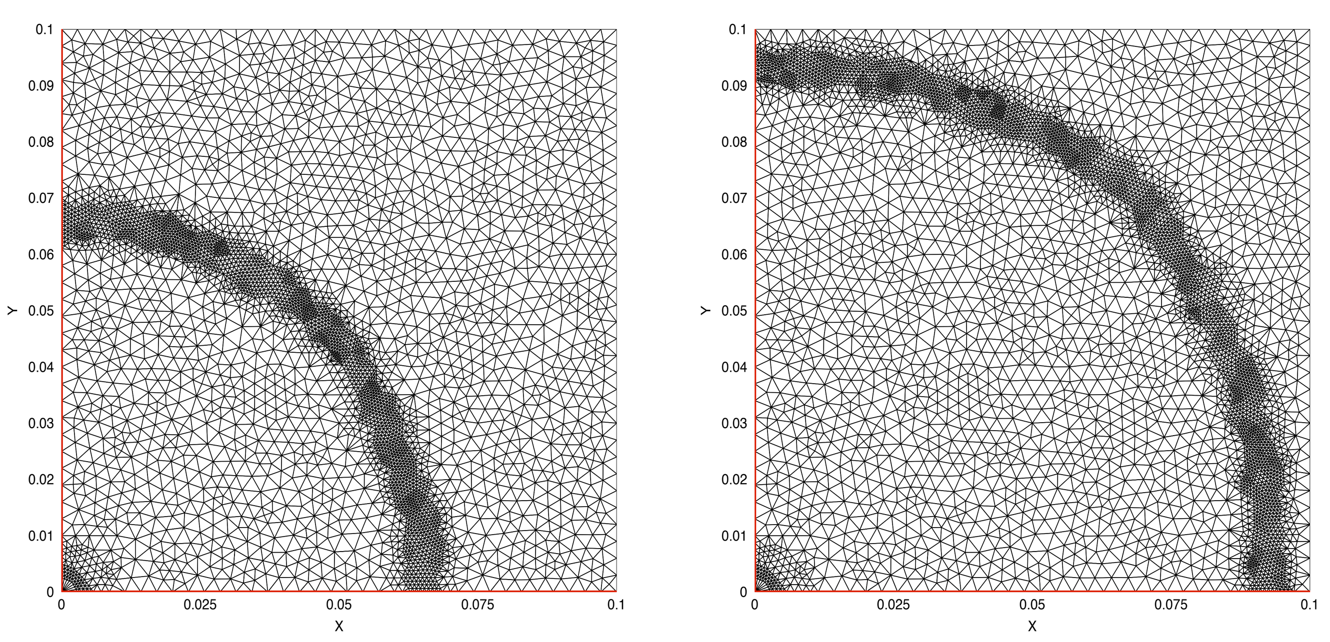
$$\begin{aligned} u_K^0 &= \frac{1}{\text{meas}(K)}\int_K u_0(\mathbf{x})\,d\mathbf{x}, && \forall K\in T,\\ u_K^{n+1} &= u_K^n + \frac{\Delta t}{\text{meas}(K)}\sum_{\sigma\in E_K} F_{K,\sigma}^n\text{meas}(\sigma) + \Delta t f_K^n, && \forall K\in T, \end{aligned} \qquad (3)$$

where E_K is the set of all edges of the control volume K and $F_{K,\sigma}^n$ are the numerical fluxes reconstructed as

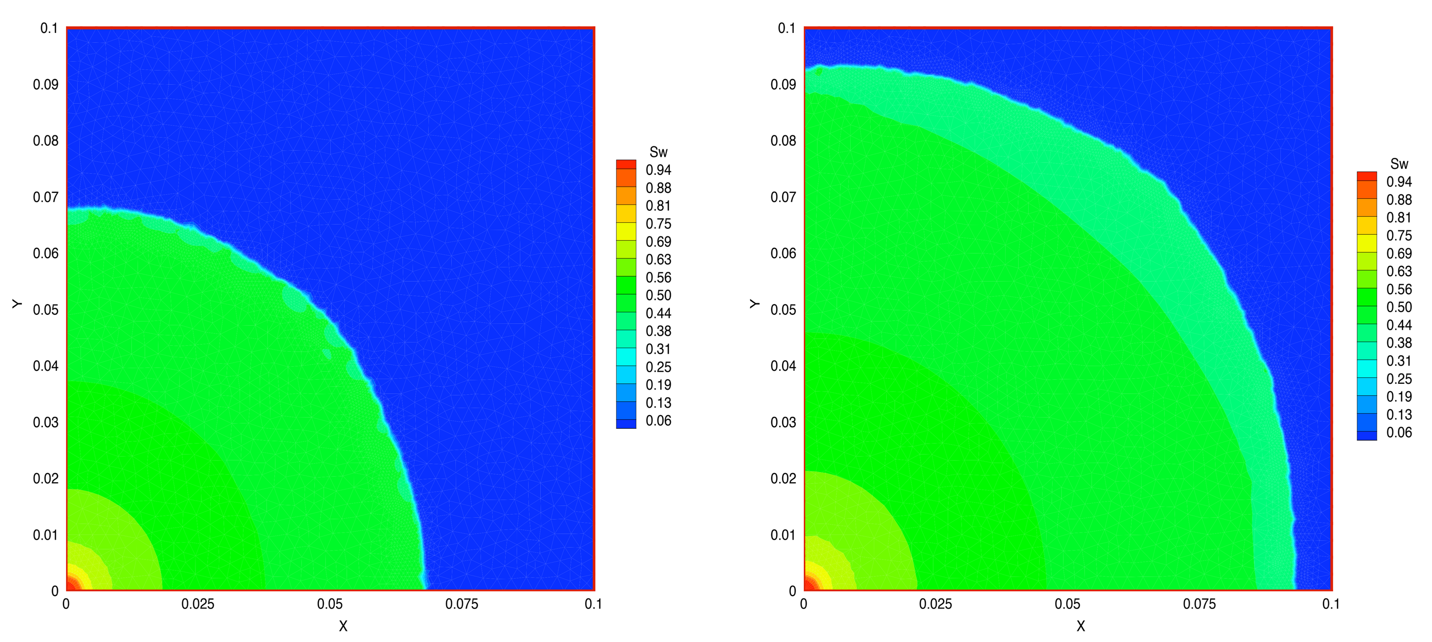
$$F_{K,\sigma}^n = \mathbb{K}_\sigma\nabla_\sigma u_h^n\cdot\mathbf{n}_{K,\sigma}.$$

Implicit scheme: An implicit time stepping scheme for (2) is formulated using the backward Euler scheme as

$$\begin{aligned} u_K^0 &= \frac{1}{\text{meas}(K)}\int_K u_0(\mathbf{x})\,d\mathbf{x}, && \forall K\in T,\\ u_K^{n+1} &= u_K^n + \frac{\Delta t}{\text{meas}(K)}\sum_{\sigma\in E_K} F_{K,\sigma}^{n+1}\text{meas}(\sigma) + +\Delta t f_K^{n+1}, && \forall K\in T. \end{aligned} \qquad (4)$$



Adaptive meshes at time $t = 0.022$ (left) and $t = 0.048$ (right).



Saturation contours at time $t = 0.022$ (left) and $t = 0.048$ (right).

Comparison between explicit and implicit schemes on adaptive mesh solving the problem (2) with $\mathbb{K} = (1+\alpha x)^2\begin{pmatrix}1 & 10^{-2}\\ 10^{-2} & 10^{-6}\end{pmatrix}$. The reaction term f is explicitly calculated such that the exact solution is $U(x,y,t) = \sin(\pi x)\sin(\pi y)\left(1-e^{-\lambda t}\right)$.

	Explicit scheme	Implicit scheme			
	CFL = 1	CFL = 5	CFL = 10	CFL = 50	CFL = 100
min Δt	8.91E-06	4.46E-05	8.91E-05	4.46E-04	8.91E-04
Relative error	1.93E-003	1.92E-03	1.89E-03	1.88E-03	1.87E-03
# time steps	12912881	2582537	1291245	258210	129080
CPU time	88084.50	39642.57	25688.67	11299.60	11378.34
GMRES iter	−−	2	3	9	19
# elements	17256	17249	17248	16480	15507
# nodes	8681	8676	8676	8292	7806

Conclusions

We have investigated a class of time stepping schemes for solving the transient diffusion problems using the finite volume method. The finite volume method uses the cell-centered for the spatial disctretization of the diffusion operator. The method is formulated for unstructured grids and an adaptive procedure is implemented. We have considered both the forward and backward Euler schemes. A comparison with other finite volume methods demonstrates the feasibility of the present algorithms to solve diffusion problems with heterogeneous diffusion coefficients.

Further Work

To apply the finite volume methods for complex flow transport in porous media.