# AN EFFICIENT SOLVER FOR DENSITY-DEPENDENT MULTILAYER SHALLOW WATER MODEL


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## Introduction

Geophysical water flows such as lakes and estuarine waters typically exhibit a significant density stratification related to chemical composition. In these water bodies effects related to small density gradients may strongly affect the hydrodyniamics. Density stratification process are therefore often important in environmental flows, and are a key feature in the biogeochemical mechanisms.

## Objectives

The simulation of those flows requires stable, accurate, conservative schemes able to sharply resolve density gradients, to handle efficiently complex topographies and free surfaces deformations, and to capture robustly dry fronts..

The present work is aimed to build a computational model endowed with these properties.

## The model

We consider the one-dimensional multi-layer shallow water equations

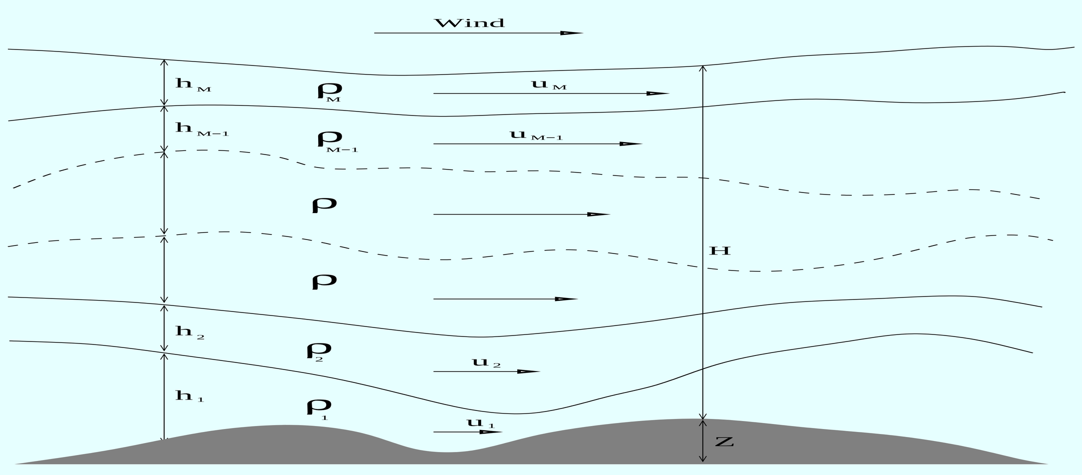
$$\begin{aligned}
\partial_t(\rho_j h_j) + \partial_x(\rho_j h_j u_j) &= 0,\\
\partial_t(\rho_j h_j u_j) + \partial_x\left(\rho_j h_j u_j^2 + \frac{1}{2} g \rho_j h_j^2\right) &= -g\rho_j h_j \left(\partial_x Z + \sum_{k=1}^{j-1} \partial_x h_k\right) - g h_j \sum_{k=j+1}^{M} \partial_x(\rho_k h_k),
\end{aligned}$$

where $j = 1,\ldots,M$ with $M$ is the total number of layers, $\rho_j$ is the water density.
We assume that a sediment transport takes place such that the density depends on space and time variables, *i.e.*, $\rho_j = \rho_j(t,x)$

$$\rho_j = \rho_w + (\rho_{s_j} - \rho_w)\, c_j, \qquad j = 1,\ldots,M,$$

where $\rho_{s_j}$ is the sediment density, and $c_j$ is the depth-averaged concentration.
The equation for mass conservation of species is modeled by

$$\partial_t(\rho_{s_j} h_j c_j) + \partial_x(\rho_{s_j} h_j u_j c_j) = 0, \qquad j = 1,\ldots,M.$$




An equivalent system can be obtained by using the physical variables

$$\begin{aligned}
D_t^{(j)}(\rho_j h_j) + \rho_j h_j \partial_x u_j &= 0,\\
D_t^{(j)} u_j + g\partial_x\left(Z + \frac{1}{2} h_j + \sum_{k=j+1}^{M} h_k\right) &= -\frac{g}{\rho_j}\partial_x\left(\frac{1}{2}\rho_j h_j + \sum_{k=1}^{j-1}(\rho_k h_k)\right),\\
D_t^{(j)}\rho_j &= 0, \qquad j = 1,\ldots,M,
\end{aligned} \tag{1}$$

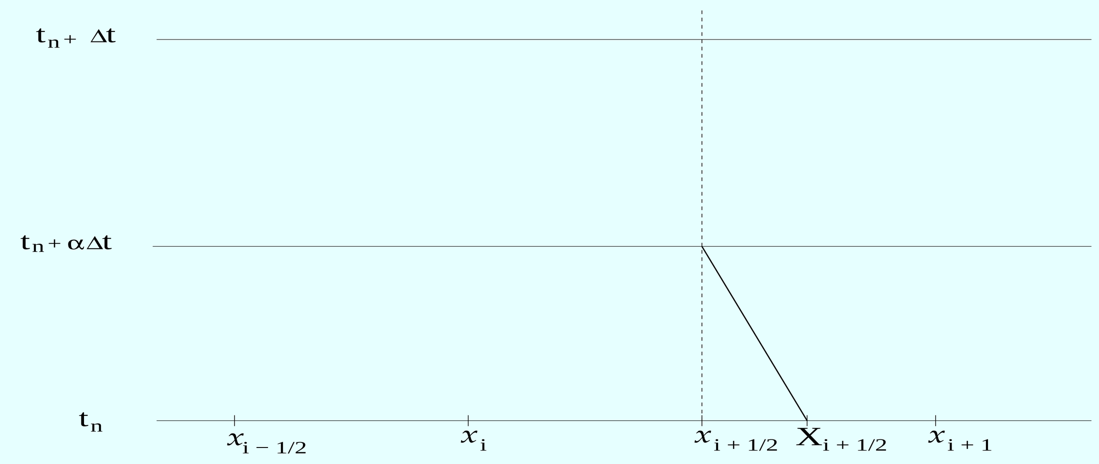
where $D_t^{(j)}\omega = \partial_t\omega + u_j\partial_x\omega, \qquad j = 1,\ldots,M.$

## A new finite volume scheme (FVC)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t \frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{\Delta x} + \Delta t\, \mathcal{Q}_i^n, \tag{2}$$

where $\mathcal{F}_{i\pm1/2}^n = \mathbf{F}(\mathbf{W}_{i\pm1/2}^n)$ are the numerical fluxes at $x = x_{i\pm1/2}$ and time $t_n$.
$\mathbf{W}_{i\pm1/2}^n$ is constructed with *the method of characteristics* applied to *the advective version* of the considered system.
the characteristic curves associated with the equation (1) are solutions of the initial-value problems

$$\begin{aligned}
\frac{dX_{j,i+1/2}(\tau)}{d\tau} &= u_{j,i+1/2}\Big(\tau, X_{j,i+1/2}(\tau)\Big), \quad \tau \in [t_n, t_n + \Delta t/2],\\
X_{j,i+1/2}(t_n + \Delta t/2) &= x_{i+1/2}, \qquad j = 1,\ldots,M.
\end{aligned} \tag{3}$$



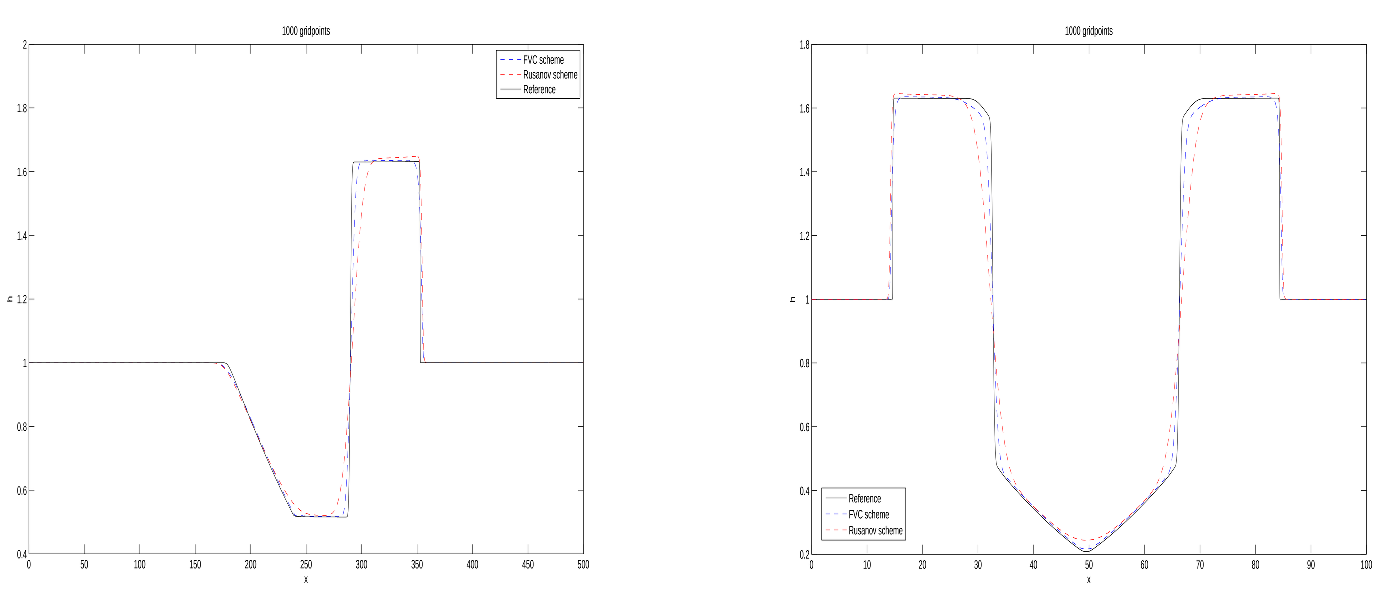

Once the characteristics curves $X_{j,i+1/2}(t_n)$ are known, a solution at the cell interface $x_{i+1/2}$ is reconstructed as

$$\mathbf{U}_{j,i+1/2}^n = \mathbf{U}_j\left(t_n + \Delta t/2, x_{i+1/2}\right) = \tilde{\mathbf{U}}_j\left(t_n, X_{j,i+1/2}(t_n)\right), \tag{4}$$

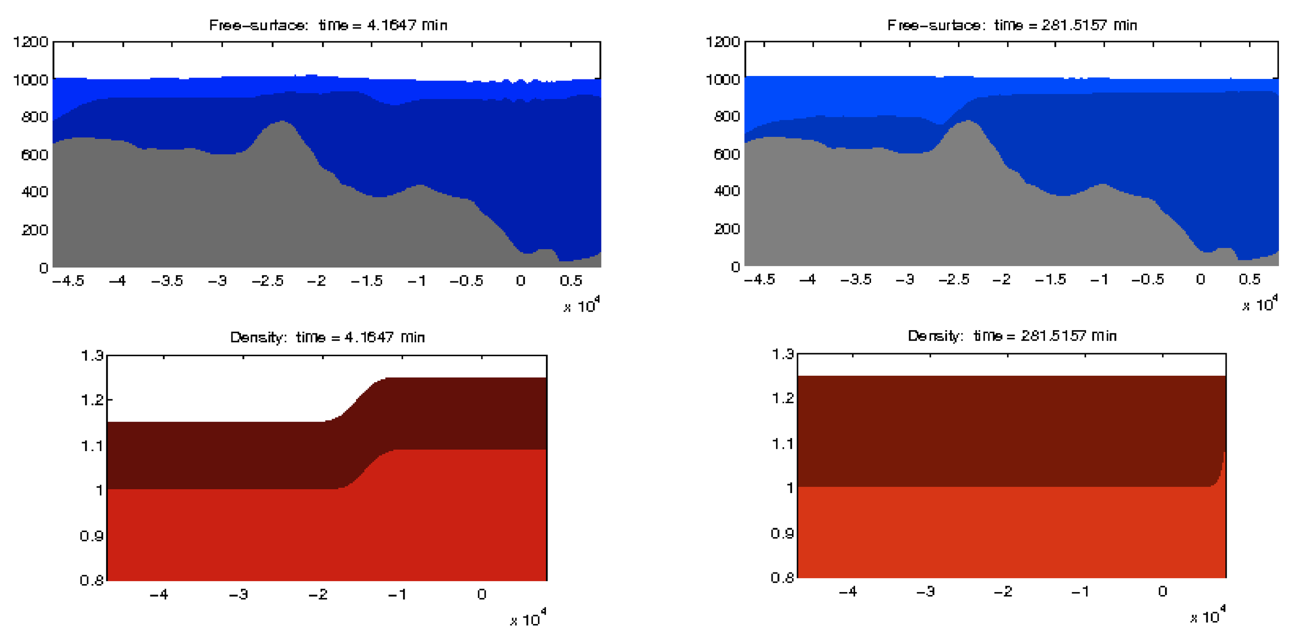
where $\tilde{\mathbf{U}}_j\left(t_n, X_{j,i+1/2}(t_n)\right)$ is the solution at the characteristic foot computed by interpolation from the gridpoints of the control volume where the departure point resides *i.e.*

$$\tilde{\mathbf{U}}_j\left(t_n, X_{j,i+1/2}(t_n)\right) = \mathcal{P}\Big(\mathbf{U}_j\left(t_n, X_{j,i+1/2}(t_n)\right)\Big), \tag{5}$$

where $\mathcal{P}$ represents the interpolating polynomial.




Mnonlayer density dam-break problem with a single initial discontinuity (left) and two initial discontinuities (right) on a flat bottom




Two-layer density dam-break problem with two initial discontinuities on the strait of Gibraltar. (left) : initial state, (right) : steady state.

Comparison between computational times for monolayer density dam-break problem with a single initial discontinuity (example1) and two initial discontinuities (example 2) on a flat bottom at $t = 200\ s$ using different gridpoints.

| | example 1 | | | | example 2 | | | |
|---|---|---|---|---|---|---|---|---|
| Gridpoints | FVC | Runsanov | Roe | SRNH | FVC | Runsanov | Roe | SRNH |
| 100 | 0,21 | 0,06 | 0,78 | 0,81 | 0,61 | 0,17 | 2,67 | 2,7 |
| 200 | 0,4 | 0,19 | 3,1 | 3,17 | 1,33 | 0,6 | 10,36 | 10,6 |
| 400 | 0.94 | 0,67 | 12,18 | 12,58 | 3,36 | 2,22 | 41,37 | 42,46 |
| 800 | 2,58 | 2,66 | 48,83 | 50,34 | 9,64 | 8,73 | 164,86 | 172,34 |
| 1600 | 8,48 | 10,25 | 193,5 | 206,58 | 31,29 | 34,46 | 656,72 | 705,67 |

## Conclusions

- A new model for shallow water flows with variable densities.
- A new finite volume scheme based on the method of characteristics which can be used for solving non hyperbolic systems of conservation laws.
- Combined finite volume characteristics methods perform well.
- Aplication of the method to resolve complicated systems.
- Extension to systems of conservation laws in two space dimensions.
- Application of the method to unstructured grids.