A GENERALIZED RUSANOV METHOD FOR SAINT-VENANT EQUATIONS WITH VARIABLE HORIZONTAL DENSITY

F. Benkhaldoun*, K. Mohamed†, M. Seaïd‡

* LAGA, Université Paris 13
93430 Villetaneuse, France

†Department of Computer Science,
Taibah University Madinah, KSA

‡ School of Engineering, University of Durham
South Road, Durham DH1 3LE, UK

Introduction

Shallow water flows with variable horizontal density occur in many hydraulic phenomena *e.g.*, river discharge in the ocean. We present a class of finite volume methods for the numerical solution of Saint-Venant equations with variable horizontal density. The model is based on coupling the Saint-Venant equations for the hydraulic variables with a suspended sediment transport equation for the concentration variable. To approximate the numerical solution of the considered models we propose a generalized Rusanov method which is well-balanced, conservative, non-oscillatory and suitable for Saint-Venant equations for which Riemann problems are difficult to solve.

Objectives

To develop a robust finite volume method for solving Saint-Venant equations with variable horizontal density. To validate developed methods with numerical solutions obtained using other methods.

The Model

The Saint-Venant equations with variable horizontal density can be formulated in a conservative form as

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{Q}(\mathbf{W}), \qquad (1)$$

where $\mathbf{W} = (\rho h, \rho hu, \rho_s hc)^T$, $Q(\mathbf{W}) = \left(0, -g\rho h\frac{\partial Z}{\partial x}, 0\right)^T$, $F(W) = \left(\rho hu, \rho hu^2 + \frac{1}{2}g\rho h^2, \rho_s huc\right)^T$. To close the system, the density is updated as

$$\rho = \rho_w + (\rho_s - \rho_w)\, c, \qquad (2)$$

where ρ_s is the sediment density and c is the depth-averaged concentration of the suspended sediment. It is easy to verify that the system (1) is hyperbolic.

A Generalized Rusanov Method

To formulate our finite volume method, we integrate the equation (1) with respect to time and space over the domain $[t_n, t_{n+1}] \times [x_{i-1/2}, x_{i+1/2}]$ to obtain

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}(\mathbf{W}_{i+1/2}^n) - \mathbf{F}(\mathbf{W}_{i-1/2}^n)\right) + \Delta t \mathbf{Q}_i^n, \qquad (3)$$

where $\mathbf{W}_i^n$ is the time-space average of the solution $\mathbf{W}$ in the domain $[x_{i-1/2}, x_{i+1/2}]$ at time t_n and $\mathbf{F}(\mathbf{W}_{i\pm 1/2}^n)$ is the numerical flux at $x = x_{i\pm 1/2}$ and time t_n. In general, the construction of numerical fluxes requires a solution of Riemann problems at the interfaces $x_{i\pm 1/2}$. In order to avoid these difficulties and reconstruct an approximation of $\mathbf{W}_{i+1/2}^n$, we integrate the equation (1) over a control domain $[t_n, t_n + \theta_{i+1/2}^n] \times [x_i, x_{i+1}]$ containing the point $(t_n, x_{i+1/2})$, and we have intermediate state given by

$$\mathbf{W}_{i+1/2}^n = \frac{1}{2}(\mathbf{W}_i^n + \mathbf{W}_{i+1}^n) - \frac{\theta_{i+1/2}^n}{\Delta x}\left(F(\mathbf{W}_{i+1}^n) - F(\mathbf{W}_i^n)\right) + \theta_{i+1/2}^n Q_{i+1/2}^n. \qquad (4)$$

In order to complete the implementation of the above finite volume method the parameters $\theta_{i+1/2}^n$ and $Q_{i+1/2}^n$ have to be selected. Based on the stability analysis for conservation laws with source terms, the variable $\theta_{i+1/2}^n$ is selected as

$$\theta_{i+1/2}^n = \alpha_{i+1/2}^n \bar{\theta}_{i+1/2}; \; \bar{\theta}_{i+1/2} = \frac{\Delta x}{2S_{i+1/2}^n}; \; S_{i+1/2}^n = \max_{k=1,2,3}\left(\max\left(\left|\lambda_{k,i}^n\right|, \left|\lambda_{k,i+1}^n\right|\right)\right). \qquad (5)$$

Selection of the parameter $\alpha_{i+1/2}^n$

It is clear that by setting $\alpha_{i+1/2}^n = 1$ the proposed finite volume method reduces to the well-established Rusanov method for linear systems of conservation laws, whereas for $\alpha_{i+1/2}^n = (\Delta t/\Delta x) S_{i+1/2}^n$ one recovers the well-known Lax-Wendroff scheme. Another choice of the slopes $\alpha_{i+1/2}^n$ leading to a first-order scheme is $\alpha_{i+1/2}^n = \tilde{\alpha}_{i+1/2}^n$ with

$$\tilde{\alpha}_{i+1/2}^n = \frac{S_{i+1/2}^n}{s_{i+1/2}^n}, \qquad (6)$$

where

$$s_{i+1/2}^n = \varepsilon + \min_{k=1,2,3}\left(\min\left(\left|\lambda_{k,i}^n\right|, \left|\lambda_{k,i+1}^n\right|\right)\right). \qquad (7)$$

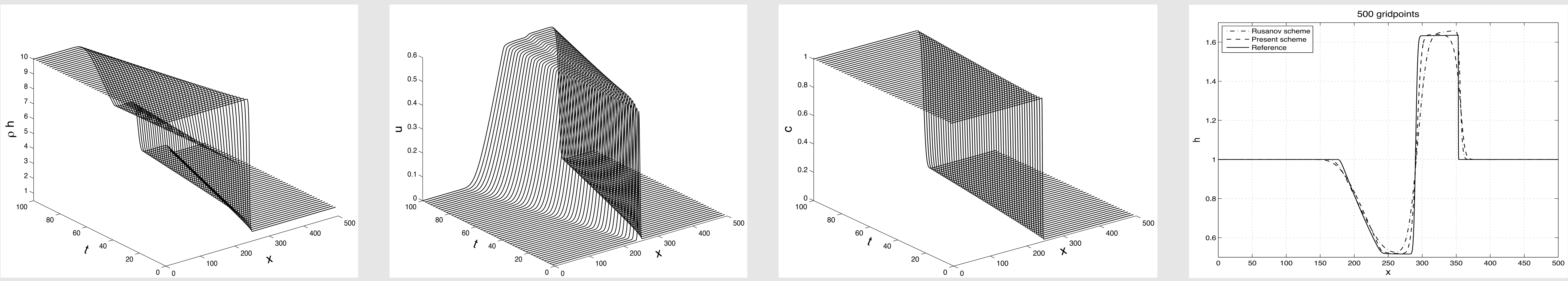
In the current study we incorporate limiters in its reconstruction as

$$\alpha_{i+1/2}^n = \tilde{\alpha}_{i+1/2}^n + \sigma_{i+1/2}^n \Phi\left(r_{i+1/2}\right), \qquad (8)$$

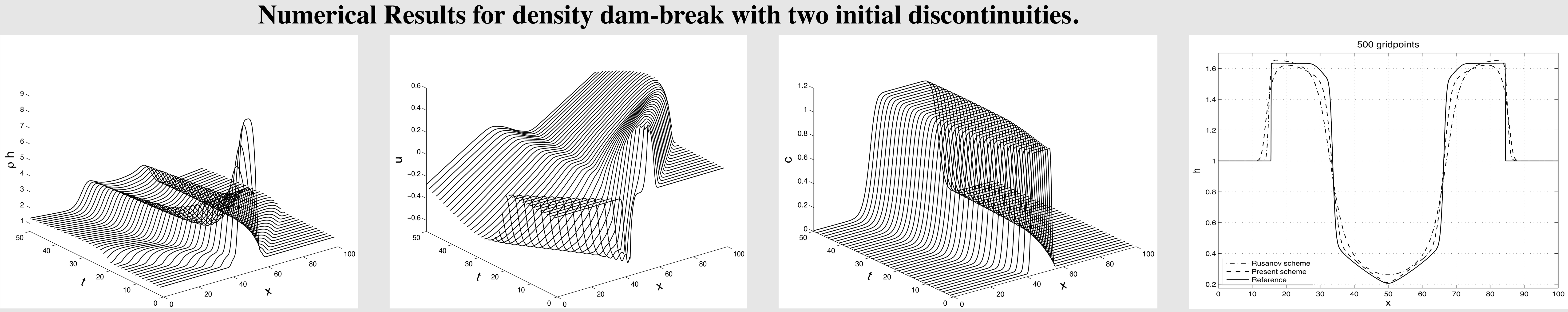
where $\tilde{\alpha}_{i+1/2}^n$ is given by (6) and $\Phi_{i+1/2} = \Phi\left(r_{i+1/2}\right)$ is an appropriate limiter which is defined by using a flux limiter function Φ acting on a quantity that measures the ratio $r_{i+1/2}$ of the upwind change to the local change,. In the present study,

$$\sigma_{i+1/2}^n = \frac{\Delta t}{\Delta x} S_{i+1/2}^n - \tilde{\alpha}_{i+1/2}^n.$$

Numerical Results for density dam-break with single initial discontinuity.



Numerical Results for density dam-break with two initial discontinuities.



Conclusions

We have solved of Saint-Venant equations with variable horizontal density,by using finite volume method which is accurate,well-balanced, conservative,non-oscillatory.

Further Work

To apply the finite volume methods for two layers density variable.