ADAPTIVE NUMERICAL SIMULATION OF SEDIMENT TRANSPORT IN SHALLOW WATER FLOWS

F. Benkhaldoun*, S. Sahmim[x], M. Seaid[+]

* Université Paris 13 , [x]Ecole Polytechnique de Tunisie, Tunisia, [+]School of Engineering, University of Durham

March 24, 2009

Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to $1D$ Shallow Water flow and equilibrium propertie
- Application of *SRNH* scheme to pollutant transport in the strait of Gibraltar
- Application of *SRNH* scheme to a moving bed problem
- Conclusion and future work

Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to $1D$ Shallow Water flow and equilibrium propertie
- Application of *SRNH* scheme to pollutant transport in the strait of Gibraltar
- Application of *SRNH* scheme to a moving bed problem
- Conclusion and future work

Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to $1D$ Shallow Water flow and equilibrium propertie
- Application of *SRNH* scheme to pollutant transport in the strait of Gibraltar
- Application of *SRNH* scheme to a moving bed problem
- Conclusion and future work

Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to $1D$ Shallow Water flow and equilibrium propertie
- Application of *SRNH* scheme to pollutant transport in the strait of Gibraltar
- Application of *SRNH* scheme to a moving bed problem
- Conclusion and future work

Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to $1D$ Shallow Water flow and equilibrium propertie
- Application of *SRNH* scheme to pollutant transport in the strait of Gibraltar
- Application of *SRNH* scheme to a moving bed problem
- Conclusion and future work

Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to $1D$ Shallow Water flow and equilibrium propertie
- Application of *SRNH* scheme to pollutant transport in the strait of Gibraltar
- Application of *SRNH* scheme to a moving bed problem
- Conclusion and future work

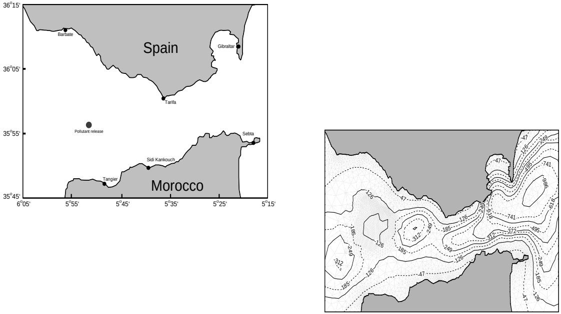


Figure: Definition of the strait of Gibraltar (left) and its bathymetry (right).

Introduction:

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using approximate states instead of approximate fluxes, was developed. The formulation of the scheme is based on the sign of genuine or approximate jacobian of the system considered.

Introduction:

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using approximate states instead of approximate fluxes, was developed. The formulation of the scheme is based on the sign of genuine or approximate jacobian of the system considered.

Introduction:

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using approximate states instead of approximate fluxes, was developed. The formulation of the scheme is based on the sign of genuine or approximate jacobian of the system considered.

Introduction:

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using approximate states instead of approximate fluxes, was developed. The formulation of the scheme is based on the sign of genuine or approximate jacobian of the system considered.

Introduction:

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using approximate states instead of approximate fluxes, was developed. The formulation of the scheme is based on the sign of genuine or approximate jacobian of the system considered.

Introduction:

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using approximate states instead of approximate fluxes, was developed. The formulation of the scheme is based on the sign of genuine or approximate jacobian of the system considered.

SRNH scheme for non linear systems

SRNH scheme for $1D$ non homogeneous systems of balance laws:

$$\left\{ \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = Q(x, W) \text{ in } X = D \times]0, T[\right. \qquad (1)$$

$$\begin{cases} W^n_{i+\frac{1}{2}} & = \frac{1}{2}\left(W^n_{i+1} + W^n_i\right) - \frac{1}{2}\,\mathrm{sgn}\left[\mathcal{B}^n_{i+\frac{1}{2}}\right]\left(W^n_{i+1} - W^n_i\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}^n_{i+\frac{1}{2}}\right|^{-1} Q^n_{i+\frac{1}{2}} \\ W^{n+1}_i & = W^n_i - r\left(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\right) + \Delta t Q^n_i \end{cases} \qquad (2)$$

with : $\mathcal{B}^n_{i+\frac{1}{2}} = \left(\mathcal{R}\Lambda^\star\mathcal{R}^{-1}\right)\left(V\left(W^n_i, W^n_{i+1}\right)\right)$ is an approximation of the jacobian matrix calculated at the average state $V\left(W^n_i, W^n_{i+1}\right)$.

SRNH scheme for non linear systems

SRNH scheme for $1D$ non homogeneous systems of balance laws:

$$\left\{ \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = Q(x, W) \text{ in } X = D\times]0, T[\right. \qquad (1)$$

$$\begin{cases} W^n_{i+\frac{1}{2}} & = \frac{1}{2}\left(W^n_{i+1} + W^n_i\right) - \frac{1}{2}\,\text{sgn}\left[\mathcal{B}^n_{i+\frac{1}{2}}\right]\left(W^n_{i+1} - W^n_i\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}^n_{i+\frac{1}{2}}\right|^{-1} Q^n_{i+\frac{1}{2}} \\ W^{n+1}_i & = W^n_i - r\left(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\right) + \Delta t Q^n_i \end{cases} \qquad (2)$$

with : $\mathcal{B}^n_{i+\frac{1}{2}} = \left(\mathcal{R}\Lambda^\star\mathcal{R}^{-1}\right)\left(V\left(W^n_i, W^n_{i+1}\right)\right)$ is an approximation of the jacobian matrix calculated at the average state $V\left(W^n_i, W^n_{i+1}\right)$.

SRNH scheme for non linear systems

SRNH scheme for $1D$ non homogeneous systems of balance laws:

$$\left\{ \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = Q(x,W) \text{ in } X = D\times]0,T[\right. \qquad (1)$$

$$\begin{cases} W^n_{i+\frac{1}{2}} & = \frac{1}{2}\left(W^n_{i+1} + W^n_i\right) - \frac{1}{2}\operatorname{sgn}\left[\mathcal{B}^n_{i+\frac{1}{2}}\right]\left(W^n_{i+1} - W^n_i\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}^n_{i+\frac{1}{2}}\right|^{-1} Q^n_{i+\frac{1}{2}} \\ W^{n+1}_i & = W^n_i - r\left(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\right) + \Delta t Q^n_i \end{cases} \qquad (2)$$

with : $\mathcal{B}^n_{i+\frac{1}{2}} = \left(\mathcal{R}\Lambda^\star\mathcal{R}^{-1}\right)\left(V\left(W^n_i, W^n_{i+1}\right)\right)$ is an approximation of the jacobian matrix calculated at the average state $V\left(W^n_i, W^n_{i+1}\right)$.

Application to the $1D$ Shallow Water equations with irregular topography

Let us consider the Shallow water equations :

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial F(W)}{\partial x} = Q(x, W), \ (x, t) \in \mathcal{D} \times \mathbb{R}_+^\times, \ \mathcal{D} \subset \mathbb{R} \\ W(x, 0) = W_0(x), \ x \in \mathcal{D} \end{cases} \quad (3)$$

$$W(x, t) = (h(x, t), hu(x, t))^T$$

$$F(W(x, t)) = \left(hu(x, t), hu^2(x, t) + \frac{1}{2} g h^2(x, t) \right)^T$$

$$Q(x, W(x, t)) = \left(0, -gh(x, t) \frac{dz(x)}{dx} \right)^T.$$

Application to the $1D$ Shallow Water equations with irregular topography

Let us consider the Shallow water equations :

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial F(W)}{\partial x} = Q\left(x, W\right), \ (x,t) \in \mathcal{D} \times \mathbb{R}_{+}^{\times}, \ \mathcal{D} \subset \mathbb{R} \\ W(x,0) = W_0(x), \ x \in \mathcal{D} \end{cases} \tag{3}$$

$$W(x,t) = \left(h(x,t), hu(x,t)\right)^{T}$$

$$F(W(x,t)) = \left(hu(x,t), hu^2(x,t) + \frac{1}{2}gh^2(x,t)\right)^{T}$$

$$Q(x, W(x,t)) = \left(0, -gh(x,t)\frac{dz(x)}{dx}\right)^{T}.$$

Application to the $1D$ Shallow Water equations with irregular topography

Let us consider the Shallow water equations :

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial F(W)}{\partial x} = Q\left(x, W\right), \ (x,t) \in \mathcal{D} \times \mathbb{R}_{+}^{\times}, \ \mathcal{D} \subset \mathbb{R} \\ W(x,0) = W_0(x), \ x \in \mathcal{D} \end{cases} \quad (3)$$

$$W(x,t) = \left(h(x,t), hu(x,t)\right)^T$$

$$F(W(x,t)) = \left(hu(x,t), hu^2(x,t) + \frac{1}{2}gh^2(x,t)\right)^T$$

$$Q(x, W(x,t)) = \left(0, -gh(x,t)\frac{dz(x)}{dx}\right)^T.$$

Application to the $1D$ Shallow Water equations with irregular topography

Let us consider the Shallow water equations :

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial F(W)}{\partial x} = Q\left(x, W\right), \ (x,t) \in \mathcal{D} \times \mathbb{R}_{+}^{\times}, \ \mathcal{D} \subset \mathbb{R} \\ W(x,0) = W_0(x), \ x \in \mathcal{D} \end{cases} \tag{3}$$

$$W(x,t) = \left(h(x,t), hu(x,t)\right)^T$$

$$F(W(x,t)) = \left(hu(x,t), hu^2(x,t) + \frac{1}{2}gh^2(x,t)\right)^T$$

$$Q(x, W(x,t)) = \left(0, -gh(x,t)\frac{dz(x)}{dx}\right)^T.$$

The *SRNH* scheme for problem (3) may be written:

$$\begin{cases} W_{i+\frac{1}{2}}^n & = \frac{1}{2}\left(W_{i+1}^n + W_i^n\right) - \frac{1}{2}\,\text{sgn}\left[\mathcal{B}_{i+\frac{1}{2}}^n\right]\left(W_{i+1}^n - W_i^n\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}_{i+\frac{1}{2}}^n\right|^{-1} Q_{i+\frac{1}{2}}^n \\ W_i^{n+1} & = W_i^n - r\left(F(W_{i+\frac{1}{2}}^n) - F(W_{i-\frac{1}{2}}^n)\right) + \Delta t Q_i^n \end{cases} \tag{4}$$

with

$$Q_{i+\frac{1}{2}}^n = -\frac{g}{2\Delta x}\left(h_i^n + h_{i+1}^n\right)\left[\begin{array}{c} 0 \\ z_{i+1} - z_i \end{array}\right],$$

and

$$Q_i^n = \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x, W(x,t))\,dxdt.$$

The *SRNH* scheme for problem (3) may be written:

$$\begin{cases} W_{i+\frac{1}{2}}^n & = \frac{1}{2}\left(W_{i+1}^n + W_i^n\right) - \frac{1}{2}\,\mathrm{sgn}\left[\mathcal{B}_{i+\frac{1}{2}}^n\right]\left(W_{i+1}^n - W_i^n\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}_{i+\frac{1}{2}}^n\right|^{-1} Q_{i+\frac{1}{2}}^n \\ W_i^{n+1} & = W_i^n - r\left(F(W_{i+\frac{1}{2}}^n) - F(W_{i-\frac{1}{2}}^n)\right) + \Delta t Q_i^n \end{cases} \tag{4}$$

with

$$Q_{i+\frac{1}{2}}^n = -\frac{g}{2\Delta x}\left(h_i^n + h_{i+1}^n\right)\begin{bmatrix} 0 \\ z_{i+1} - z_i \end{bmatrix},$$

and

$$Q_i^n = \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x, W(x,t))\,dxdt.$$

The *SRNH* scheme for problem (3) may be written:

$$\begin{cases} W_{i+\frac{1}{2}}^n & = \frac{1}{2}\left(W_{i+1}^n + W_i^n\right) - \frac{1}{2}\,\mathrm{sgn}\left[\mathcal{B}_{i+\frac{1}{2}}^n\right]\left(W_{i+1}^n - W_i^n\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}_{i+\frac{1}{2}}^n\right|^{-1} Q_{i+\frac{1}{2}}^n \\ W_i^{n+1} & = W_i^n - r\left(F(W_{i+\frac{1}{2}}^n) - F(W_{i-\frac{1}{2}}^n)\right) + \Delta t Q_i^n \end{cases} \tag{4}$$

with

$$Q_{i+\frac{1}{2}}^n = -\frac{g}{2\Delta x}\left(h_i^n + h_{i+1}^n\right)\begin{bmatrix} 0 \\ z_{i+1} - z_i \end{bmatrix},$$

and

$$Q_i^n = \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x, W(x,t))\,dxdt.$$

Definition

$W(x,t)$ is a static stationary solution of the system if $\dfrac{\partial W}{\partial t} = 0$ and $u(x,t) = 0$. In this case, one has $h(x,t) + z(x) =$ canstant.

Definition

A finite volume scheme is said to verify the exact $\mathcal{C}$-property if it preserves the equilibrium state:

$h_i^n + z_i = c \qquad \text{and} \qquad u_i^n = 0 \quad \forall (i,n) \in \mathbb{Z} \times \mathbb{N}.$

Proposition

If the source term, in the second step of the scheme, is discretized as follows : $(Q_i^n)_1 = 0$, *and*

i) $(Q_i^n)_2 = -\frac{g}{4\Delta x}\left(h_{i+\frac{1}{2}}^n + h_{i-\frac{1}{2}}^n\right)(z_{i+1} - z_{i-1})$, *or*

ii) $(Q_i^n)_2 = -\frac{g}{8\Delta x}\left(h_{i+1}^n + 2h_i^n + h_{i-1}^n\right)(z_{i+1} - z_{i-1})$ *then the scheme (4) respects the exact $\mathcal{C}$-property.*

Definition

$W(x,t)$ is a static stationary solution of the system if $\dfrac{\partial W}{\partial t} = 0$ and $u(x,t) = 0$. In this case, one has $h(x,t) + z(x) =$ canstant.

Definition

A finite volume scheme is said to verify the exact $\mathcal{C}$-property if it preserves the equilibrium state:
$h_i^n + z_i = c \qquad \text{and} \qquad u_i^n = 0 \quad \forall (i,n) \in \mathbb{Z} \times \mathbb{N}.$

Proposition

If the source term, in the second step of the scheme, is discretized as follows : $(Q_i^n)_1 = 0$, *and*
i) $(Q_i^n)_2 = -\frac{g}{4\Delta x}\left(h_{i+\frac{1}{2}}^n + h_{i-\frac{1}{2}}^n\right)(z_{i+1} - z_{i-1})$, *or*
ii) $(Q_i^n)_2 = -\frac{g}{8\Delta x}\left(h_{i+1}^n + 2h_i^n + h_{i-1}^n\right)(z_{i+1} - z_{i-1})$ *then the scheme (4) respects the exact* $\mathcal{C}$*-property.*

Definition

$W(x,t)$ is a static stationary solution of the system if $\dfrac{\partial W}{\partial t} = 0$ and $u(x,t) = 0$. In this case, one has $h(x,t) + z(x) =$ canstant.

Definition

A finite volume scheme is said to verify the exact $\mathcal{C}$-property if it preserves the equilibrium state:

$h_i^n + z_i = c \qquad \text{and} \qquad u_i^n = 0 \quad \forall (i,n) \in \mathbb{Z} \times \mathbb{N}.$

Proposition

If the source term, in the second step of the scheme, is discretized as follows : $(Q_i^n)_1 = 0$*, and*

i) $(Q_i^n)_2 = -\frac{g}{4\Delta x}\left(h^n_{i+\frac{1}{2}} + h^n_{i-\frac{1}{2}}\right)(z_{i+1} - z_{i-1})$ *, or*

ii) $(Q_i^n)_2 = -\frac{g}{8\Delta x}\left(h^n_{i+1} + 2h^n_i + h^n_{i-1}\right)(z_{i+1} - z_{i-1})$ *then the scheme (4) respects the exact* $\mathcal{C}$*-property.*

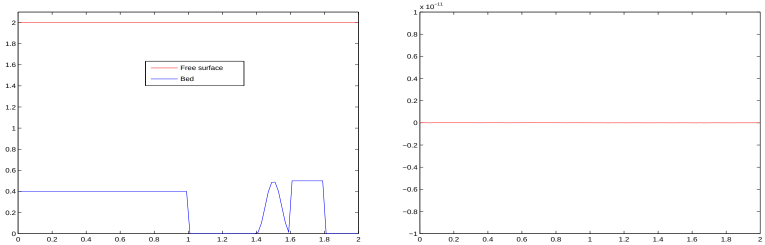
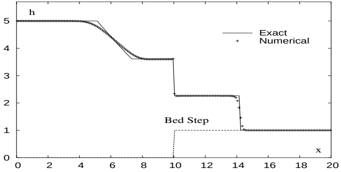


Figure: Bed, free surface and Water momentum $t = 10s$



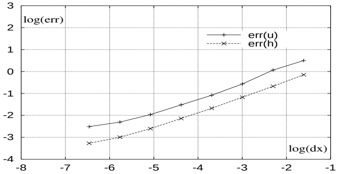


Figure: Water depth at t=0.7s and Error plot (slope $\simeq$ 0.65)

Application of *SRNH* scheme to non homogeneous 2D Shallow Water flows :

The system considered may by written as follows :

$$\begin{cases} h_{,t} + (hu)_{,x} + (hv)_{,y} = 0 \\ (hu)_{,t} + (hu^2)_{,x} + (huv)_{,y} + g\left(\frac{h^2}{2}\right)_{,x} = -gh(Z_f)_{,x} \\ (hv)_{,t} + (huv)_{,x} + (hv^2)_{,y} + g\left(\frac{h^2}{2}\right)_{,y} = -gh(Z_f)_{,y}, \end{cases} \quad (5)$$

where h is the water level, $\mathbf{u} = {}^t(u, v)$ the water velocity and Z_f the bottom height.

Application of *SRNH* scheme to non homogeneous 2D Shallow Water flows :

The system considered may by written as follows :

$$\begin{cases} h_{,t} + (hu)_{,x} + (hv)_{,y} = 0 \\ (hu)_{,t} + (hu^2)_{,x} + (huv)_{,y} + g\left(\frac{h^2}{2}\right)_{,x} = -gh(Z_f)_{,x} \\ (hv)_{,t} + (huv)_{,x} + (hv^2)_{,y} + g\left(\frac{h^2}{2}\right)_{,y} = -gh(Z_f)_{,y}, \end{cases} \tag{5}$$

where h is the water level, $\mathbf{u} = {}^t(u, v)$ the water velocity and Z_f the bottom height.

To calculate the predictor phase of *SRNH* scheme, one projects the equations on each interface e_{ij}, and gets the following system (**Abgrall 03)**

$$(U_\eta)_t + (F_\eta)_{,\eta} = Q(x, y, U_\eta) \tag{6}$$

with

$$U_\eta = (h, hu_\eta, hu_\tau)^T, \qquad F_\eta = \left(hu_\eta, hu_\eta^2 + g\frac{h^2}{2}, hu_\eta u_\tau\right)^T,$$

$$\text{et} \qquad Q(x, y, U_\eta) = (0, -gh(Z_f)_{,\eta}, 0)^T,$$

$u_\eta = \mathbf{u} \cdot \eta$, $u_\tau = \mathbf{u} \cdot \tau$, η and τ the normal and the tangential vector to the interface, and $(.)_{,\eta}$ the derivate along the normal vector η.

To calculate the predictor phase of *SRNH* scheme, one projects the equations on each interface e_{ij}, and gets the following system (**Abgrall 03)**

$$(U_\eta)_t + (F_\eta)_{,\eta} = Q(x, y, U_\eta) \tag{6}$$

with

$$U_\eta = (h, hu_\eta, hu_\tau)^T, \qquad F_\eta = \left(hu_\eta, hu_\eta^2 + g\frac{h^2}{2}, hu_\eta u_\tau\right)^T,$$

$$\text{et} \qquad Q(x, y, U_\eta) = (0, -gh(Z_f)_{,\eta}, 0)^T,$$

$u_\eta = \mathbf{u}\cdot\eta$, $u_\tau = \mathbf{u}\cdot\tau$, η and τ the normal and the tangential vector to the interface, and $(.)_{,\eta}$ the derivate along the normal vector η.

To calculate the predictor phase of *SRNH* scheme, one projects the equations on each interface e_{ij}, and gets the following system (**Abgrall 03)**

$$(U_\eta)_t + (F_\eta)_{,\eta} = Q(x, y, U_\eta) \tag{6}$$

with

$$U_\eta = (h, hu_\eta, hu_\tau)^T, \qquad F_\eta = \left(hu_\eta, hu_\eta^2 + g\frac{h^2}{2}, hu_\eta u_\tau\right)^T,$$

$$\text{et} \qquad Q(x, y, U_\eta) = (0, -gh(Z_f)_{,\eta}, 0)^T,$$

$u_\eta = \mathbf{u}\cdot\eta$, $u_\tau = \mathbf{u}\cdot\tau$, η and τ the normal and the tangential vector to the interface, and $(.)_{,\eta}$ the derivate along the normal vector η.

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t \mathbf{W} + \partial_x \Big(\mathbf{F}(\mathbf{W}) - \tilde{\mathbf{F}}(\mathbf{W})\Big) + \partial_y \Big(\mathbf{G}(\mathbf{W}) - \tilde{\mathbf{G}}(\mathbf{W})\Big) = \mathbf{Q}(\mathbf{W}), \quad (7)$$

where $\mathbf{W}$ and $\mathbf{Q}$ are the vectors of conserved variables and source terms, $\mathbf{F}$ and $\mathbf{G}$ are the convection tensor fluxes, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are the diffusion tensor fluxes

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hv \\ hC \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gh(S_{0x}+S_{fx}) \\ -gh(S_{0y}+S_{fy}) \\ hQ \end{pmatrix},$$

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \\ huC \end{pmatrix}, \qquad \mathbf{G}(\mathbf{W}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \\ hvC \end{pmatrix},$$

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t \mathbf{W} + \partial_x \left(\mathbf{F}(\mathbf{W}) - \tilde{\mathbf{F}}(\mathbf{W}) \right) + \partial_y \left(\mathbf{G}(\mathbf{W}) - \tilde{\mathbf{G}}(\mathbf{W}) \right) = \mathbf{Q}(\mathbf{W}), \quad (7)$$

where $\mathbf{W}$ and $\mathbf{Q}$ are the vectors of conserved variables and source terms, $\mathbf{F}$ and $\mathbf{G}$ are the convection tensor fluxes, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are the diffusion tensor fluxes

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hv \\ hC \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gh(S_{0x}+S_{fx}) \\ -gh(S_{0y}+S_{fy}) \\ hQ \end{pmatrix},$$

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \\ huC \end{pmatrix}, \qquad \mathbf{G}(\mathbf{W}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \\ hvC \end{pmatrix},$$

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t \mathbf{W} + \partial_x \left(\mathbf{F}(\mathbf{W}) - \tilde{\mathbf{F}}(\mathbf{W}) \right) + \partial_y \left(\mathbf{G}(\mathbf{W}) - \tilde{\mathbf{G}}(\mathbf{W}) \right) = \mathbf{Q}(\mathbf{W}), \quad (7)$$

where $\mathbf{W}$ and $\mathbf{Q}$ are the vectors of conserved variables and source terms, $\mathbf{F}$ and $\mathbf{G}$ are the convection tensor fluxes, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are the diffusion tensor fluxes

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hv \\ hC \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gh(S_{0x}+S_{fx}) \\ -gh(S_{0y}+S_{fy}) \\ hQ \end{pmatrix},$$

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \\ huC \end{pmatrix}, \qquad \mathbf{G}(\mathbf{W}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \\ hvC \end{pmatrix},$$

Pollutant Transport in the Strait of Gibraltar :

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t \mathbf{W} + \partial_x \left(\mathbf{F}(\mathbf{W}) - \tilde{\mathbf{F}}(\mathbf{W})\right) + \partial_y \left(\mathbf{G}(\mathbf{W}) - \tilde{\mathbf{G}}(\mathbf{W})\right) = \mathbf{Q}(\mathbf{W}), \quad (7)$$

where $\mathbf{W}$ and $\mathbf{Q}$ are the vectors of conserved variables and source terms, $\mathbf{F}$ and $\mathbf{G}$ are the convection tensor fluxes, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are the diffusion tensor fluxes

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hv \\ hC \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gh(S_{0x}+S_{fx}) \\ -gh(S_{0y}+S_{fy}) \\ hQ \end{pmatrix},$$

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \\ huC \end{pmatrix}, \qquad \mathbf{G}(\mathbf{W}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \\ hvC \end{pmatrix},$$

$$\tilde{\mathbf{F}}(\mathbf{W}) = \left(0, 0, 0, D_{xx}\partial_x (hC) + D_{xy}\partial_y (hC)\right)^T$$
$$\tilde{\mathbf{G}}(\mathbf{W}) = \left(0, 0, 0, D_{yx}\partial_x (hC) + D_{yy}\partial_y (hC)\right)^T$$

where D_{xx}, D_{xy}, D_{yx} and D_{yy} are entries of the diffusion matrix $\mathbf{D}$ assumed to be nonnegative. $S_{0x} = \partial_x Z$, $S_{0y} = \partial_y Z$, with $Z(x, y)$ denotes the bottom topography, while S_{fx} and S_{fy} are the friction losses along the x- and y-direction, and are defined by $S_{fx} = \eta^2 \frac{u\sqrt{u^2+v^2}}{h^{4/3}}$, $S_{fy} = \eta^2 \frac{v\sqrt{u^2+v^2}}{h^{4/3}}$, where η is the Manning roughness coefficient.

$$\tilde{\mathbf{F}}(\mathbf{W}) = \left(0, 0, 0, D_{xx}\partial_x (hC) + D_{xy}\partial_y (hC)\right)^T$$

$$\tilde{\mathbf{G}}(\mathbf{W}) = \left(0, 0, 0, D_{yx}\partial_x (hC) + D_{yy}\partial_y (hC)\right)^T$$

where D_{xx}, D_{xy}, D_{yx} and D_{yy} are entries of the diffusion matrix $\mathbf{D}$ assumed to be nonnegative. $S_{0x} = \partial_x Z,\ S_{0y} = \partial_y Z,$ with $Z(x, y)$ denotes the bottom topography, while S_{fx} and S_{fy} are the friction losses along the x- and y-direction, and are defined by $S_{fx} = \eta^2 \frac{u\sqrt{u^2+v^2}}{h^{4/3}},\ S_{fy} = \eta^2 \frac{v\sqrt{u^2+v^2}}{h^{4/3}},$ where η is the Manning roughness coefficient.

$$\tilde{\mathbf{F}}(\mathbf{W}) = \left(0, 0, 0, D_{xx}\partial_x (hC) + D_{xy}\partial_y (hC)\right)^T$$

$$\tilde{\mathbf{G}}(\mathbf{W}) = \left(0, 0, 0, D_{yx}\partial_x (hC) + D_{yy}\partial_y (hC)\right)^T$$

where D_{xx}, D_{xy}, D_{yx} and D_{yy} are entries of the diffusion matrix $\mathbf{D}$ assumed to be nonnegative. $S_{0x} = \partial_x Z,\ S_{0y} = \partial_y Z,$ with $Z(x, y)$ denotes the bottom topography, while S_{fx} and S_{fy} are the friction losses along the x- and y-direction, and are defined by $S_{fx} = \eta^2 \frac{u\sqrt{u^2 + v^2}}{h^{4/3}},\ S_{fy} = \eta^2 \frac{v\sqrt{u^2 + v^2}}{h^{4/3}},$ where η is the Manning roughness coefficient.

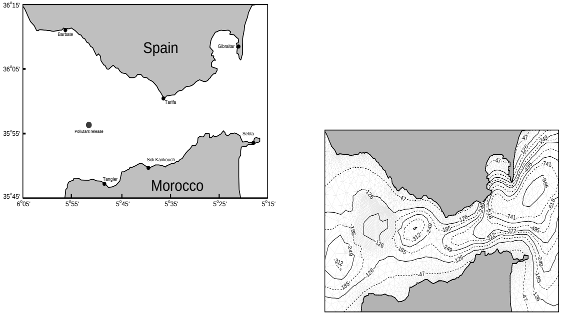


Figure: Definition of the strait of Gibraltar (left) and its bathymetry (right).

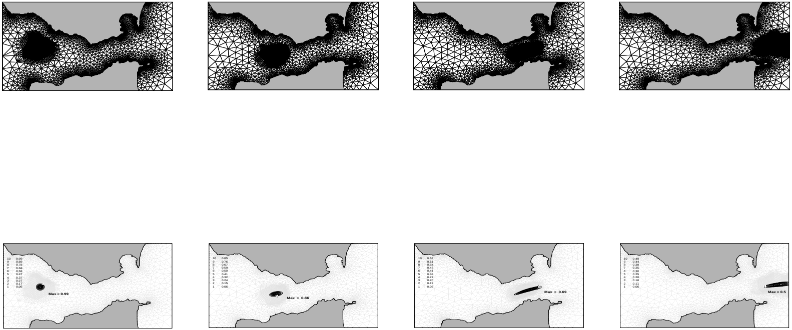
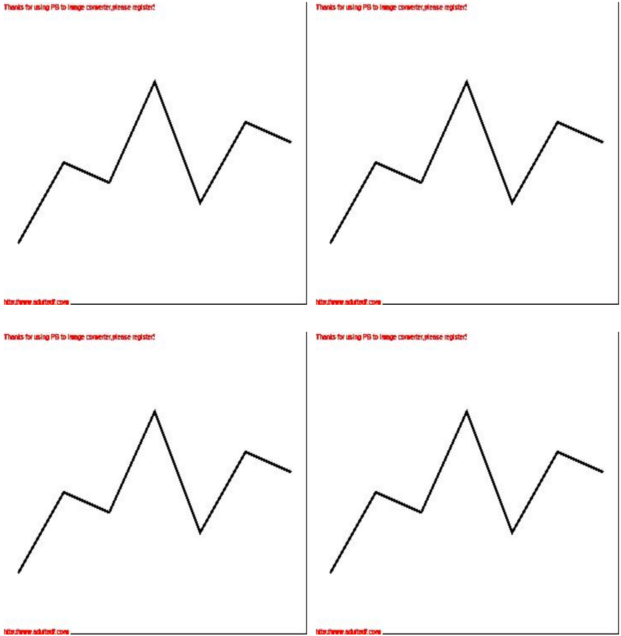


Figure: Adapted meshes (first row) and pollutant concentration (second row) at different simulation times.



Application of *SRNH* scheme to $2D$ sediment transport model :

$$
\begin{aligned}
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0, \\
\partial_t(hu) + \partial_x\left(hu^2 + \frac{1}{2}gh^2\right) + \partial_y\left(huv\right) &= -gh\partial_x Z, \\
\partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{1}{2}gh^2\right) &= -gh\partial_y Z,
\end{aligned} \tag{8}
$$

with the bed-load equation

$$
\partial_t Z + \xi\partial_x q_1 + \xi\partial_x q_2 = 0, \tag{9}
$$

with $\xi = \frac{1}{1-\varepsilon}$ and the sediment transport fluxes are:

$$
q_1 = Au\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad q_2 = Av\left(\sqrt{u^2+v^2}\right)^{m-1}, \tag{10}
$$

Application of *SRNH* scheme to $2D$ sediment transport model :

$$
\begin{aligned}
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0, \\
\partial_t(hu) + \partial_x\left(hu^2 + \frac{1}{2}gh^2\right) + \partial_y\left(huv\right) &= -gh\partial_x Z, \\
\partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{1}{2}gh^2\right) &= -gh\partial_y Z,
\end{aligned} \qquad (8)
$$

with the bed-load equation

$$
\partial_t Z + \xi\partial_x q_1 + \xi\partial_x q_2 = 0, \qquad (9)
$$

with $\xi = \frac{1}{1-\varepsilon}$ and the sediment transport fluxes are:

$$
q_1 = Au\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad q_2 = Av\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad (10)
$$

Application of *SRNH* scheme to $2D$ sediment transport model :

$$
\begin{aligned}
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0, \\
\partial_t(hu) + \partial_x\left(hu^2 + \frac{1}{2}gh^2\right) + \partial_y\left(huv\right) &= -gh\partial_x Z, \\
\partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{1}{2}gh^2\right) &= -gh\partial_y Z,
\end{aligned}
\qquad (8)
$$

with the bed-load equation

$$
\partial_t Z + \xi\partial_x q_1 + \xi\partial_x q_2 = 0, \qquad (9)
$$

with $\xi = \frac{1}{1-\varepsilon}$ and the sediment transport fluxes are:

$$
q_1 = Au\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad q_2 = Av\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad (10)
$$

Application of *SRNH* scheme to $2D$ sediment transport model :

$$\begin{aligned}
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0, \\
\partial_t(hu) + \partial_x\left(hu^2 + \frac{1}{2}gh^2\right) + \partial_y(huv) &= -gh\partial_x Z, \\
\partial_t(hv) + \partial_x(huv) + \partial_y\left(hv^2 + \frac{1}{2}gh^2\right) &= -gh\partial_y Z,
\end{aligned} \qquad (8)$$

with the bed-load equation

$$\partial_t Z + \xi\partial_x q_1 + \xi\partial_x q_2 = 0, \qquad (9)$$

with $\xi = \frac{1}{1-\varepsilon}$ and the sediment transport fluxes are:

$$q_1 = Au\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad q_2 = Av\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad (10)$$

Determination of sign matrix :

To determine the sign matrix, we project the shallow water equations on the local cell outward normal η and tangential $\tau = \eta^{\perp}$ as follows

$$
\begin{aligned}
\partial_t h + \partial_\eta \left(h u_\eta \right) &= 0, \\
\partial_t \left(h u_\eta \right) + \partial_\eta \left(h u_\eta^2 + \frac{1}{2} g h^2 \right) &= -g h \partial_\eta Z, \\
\partial_t \left(h u_\tau \right) + \partial_\eta \left(h u_\eta u_\tau \right) &= 0, \\
\partial_t \left(Z \right) + \partial_\eta \left(A \xi (u_\eta^2 + u_\tau^2) \right) &= 0,
\end{aligned}
\qquad (11)
$$

where $u_\eta = (u, v) \cdot \eta$ and $u_\tau = (u, v) \cdot \tau$ are the normal and tangential velocity, respectively.

Determination of sign matrix :

To determine the sign matrix, we project the shallow water equations on the local cell outward normal η and tangential $\tau = \eta^{\perp}$ as follows

$$\begin{aligned}
\partial_t h + \partial_\eta (h u_\eta) &= 0, \\
\partial_t (h u_\eta) + \partial_\eta \left(h u_\eta^2 + \frac{1}{2} g h^2 \right) &= -g h \partial_\eta Z, \\
\partial_t (h u_\tau) + \partial_\eta (h u_\eta u_\tau) &= 0, \\
\partial_t (Z) + \partial_\eta \left(A \xi (u_\eta^2 + u_\tau^2) \right) &= 0,
\end{aligned} \tag{11}$$

where $u_\eta = (u, v) \cdot \eta$ and $u_\tau = (u, v) \cdot \tau$ are the normal and tangential velocity, respectively.

Determination of sign matrix :

To determine the sign matrix, we project the shallow water equations on the local cell outward normal η and tangential $\tau = \eta^{\perp}$ as follows

$$\begin{aligned} \partial_t h + \partial_\eta (hu_\eta) &= 0, \\ \partial_t (hu_\eta) + \partial_\eta \left(hu_\eta^2 + \frac{1}{2} g h^2 \right) &= -gh\partial_\eta Z, \\ \partial_t (hu_\tau) + \partial_\eta (hu_\eta u_\tau) &= 0, \\ \partial_t (Z) + \partial_\eta \left(A\xi (u_\eta^2 + u_\tau^2) \right) &= 0, \end{aligned} \tag{11}$$

where $u_\eta = (u, v) \cdot \eta$ and $u_\tau = (u, v) \cdot \tau$ are the normal and tangential velocity, respectively.

the system writes in a non conservative form:

$$\frac{\partial U}{\partial t} + \mathcal{A}(U)\frac{\partial U}{\partial \eta} = 0,$$

where

$$\mathcal{A}(U) = \begin{pmatrix} u_\eta & h & 0 & 0 \\ g & u_\eta & 0 & g \\ 0 & 0 & u_\eta & 0 \\ 0 & A\xi(3u_\eta^2 + u_\tau^2) & 2A\xi u_\eta u_\tau & 0 \end{pmatrix},$$

If we put $\lambda_4 = u_\eta$, with $e_4 = [-1, 0, \frac{1}{2A\xi u_\tau}, 1]^T$, then the 3 first eigenvalues have the same expression as ine the 1D case, with here $d = A\xi(3u_\eta^2 + u_t a u^2)$. Their associated eigenvectors are:
$e_k = \left(1, \frac{\lambda_k - u_\eta}{h}, 0, \frac{(\lambda_k - u_\eta)^2 - c^2}{c^2}\right)^T$.

the system writes in a non conservative form:

$$\frac{\partial U}{\partial t} + \mathcal{A}(U)\frac{\partial U}{\partial \eta} = 0,$$

where

$$\mathcal{A}(U) = \begin{pmatrix} u_\eta & h & 0 & 0 \\ g & u_\eta & 0 & g \\ 0 & 0 & u_\eta & 0 \\ 0 & A\xi(3u_\eta^2 + u_\tau^2) & 2A\xi u_\eta u_\tau & 0 \end{pmatrix},$$

If we put $\lambda_4 = u_\eta$, with $e_4 = [-1, 0, \frac{1}{2A\xi u_\tau}, 1]^T$, then the 3 first eigenvalues have the same expression as ine the 1D case, with here $d = A\xi(3u_\eta^2 + u_t a u^2)$. Their associated eigenvectors are:
$e_k = \left(1, \frac{\lambda_k - u_\eta}{h}, 0, \frac{(\lambda_k - u_\eta)^2 - c^2}{c^2}\right)^T.$

the system writes in a non conservative form:

$$\frac{\partial U}{\partial t} + \mathcal{A}(U)\frac{\partial U}{\partial \eta} = 0,$$

where

$$\mathcal{A}(U) = \begin{pmatrix} u_\eta & h & 0 & 0 \\ g & u_\eta & 0 & g \\ 0 & 0 & u_\eta & 0 \\ 0 & A\xi(3u_\eta^2 + u_\tau^2) & 2A\xi u_\eta u_\tau & 0 \end{pmatrix},$$

If we put $\lambda_4 = u_\eta$, with $e_4 = [-1, 0, \frac{1}{2A\xi u_\tau}, 1]^T$, then the 3 first eigenvalues have the same expression as ine the 1D case, with here $d = A\xi(3u_\eta^2 + u_t a u^2)$. Their associated eigenvectors are:

$$e_k = \left(1, \frac{\lambda_k - u_\eta}{h}, 0, \frac{(\lambda_k - u_\eta)^2 - c^2}{c^2}\right)^T.$$

Numerical results

We use a simplified test example of the evolution of an initially hump-shaped bed in a squared channel. The channel is of length 1000 m and the initial bed is defined as

$$Z(0,x,y) = \begin{cases} \sin^2\left(\frac{(x-500)\,\pi}{200}\right)\sin^2\left(\frac{(x-400)\,\pi}{200}\right), & \text{if} \quad (x,y) \in \Omega, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\Omega = [500, 700] \times [400, 600]$ and the initial water level and initial velocity field are

$$h(0,x,x) = 10 - Z(0,x,y), \quad u(0,x,y) = \frac{Q}{h(0,x,y)}, \quad v(0,x,y) = 0, \tag{12}$$

where $m = 3$, the porosity $\epsilon = 0.4$ and the discharge $Q = 10\ m^2/s$.

Numerical results

We use a simplified test example of the evolution of an initially hump-shaped bed in a squared channel. The channel is of length 1000 m and the initial bed is defined as

$$Z(0,x,y) = \begin{cases} \sin^2\left(\frac{(x-500)\,\pi}{200}\right)\sin^2\left(\frac{(x-400)\,\pi}{200}\right), & \text{if} \quad (x,y)\in\Omega, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\Omega = [500,700]\times[400,600]$ and the initial water level and initial velocity field are

$$h(0,x,x) = 10 - Z(0,x,y), \quad u(0,x,y) = \frac{Q}{h(0,x,y)}, \quad v(0,x,y) = 0, \tag{12}$$

where $m = 3$, the porosity $\epsilon = 0.4$ and the discharge $Q = 10\ m^2/s$.

Numerical results

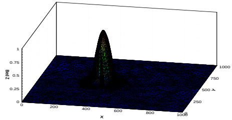
We use a simplified test example of the evolution of an initially hump-shaped bed in a squared channel. The channel is of length 1000 m and the initial bed is defined as

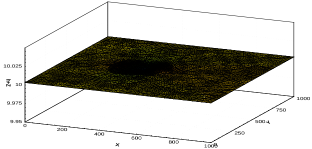
$$Z(0,x,y) = \begin{cases} \sin^2\left(\frac{(x-500)\,\pi}{200}\right)\sin^2\left(\frac{(x-400)\,\pi}{200}\right), & \text{if} \quad (x,y) \in \Omega, \\ 0, & \text{elsewhere}, \end{cases}$$

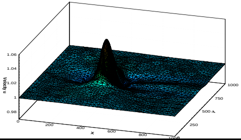
where $\Omega = [500, 700] \times [400, 600]$ and the initial water level and initial velocity field are

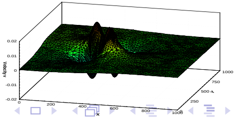
$$h(0,x,x) = 10 - Z(0,x,y), \quad u(0,x,y) = \frac{Q}{h(0,x,y)}, \quad v(0,x,y) = 0, \tag{12}$$

where $m = 3$, the porosity $\epsilon = 0.4$ and the discharge $Q = 10\ m^2/s$.


1
0.75
0.5
0.25
0
Bed Z
0
200
400
600
800
1000
x
250
500
750
1000
y


10.025
10
9.975
9.95
H+Z
0
200
400
600
800
1000
x
250
500
750
1000
y


1.06
1.04
1.02
1
0.98
Velocity u
0
200
400
600
800
x
250
500
750
1000
y


0.02
0.01
0
-0.01
-0.02
Velocity v
0
200
400
600
800
1000
x
250
500
750
1000
y

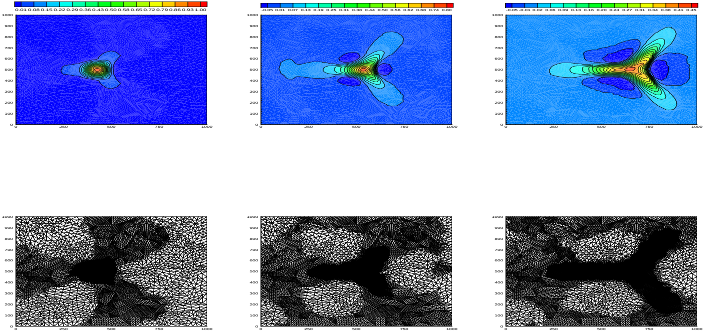
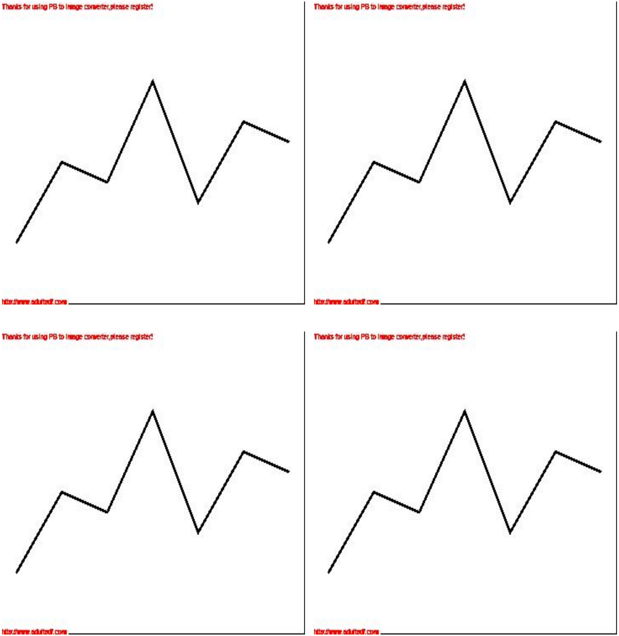


Figure: Bed and mesh evolution for the coupled approach using $A = 1$ at three different times. From left to right $t = 50$, 300 and 600 s.


Thanks for using PB to image converter,please register!
Thanks for using PB to image converter,please register!
Thanks for using PB to image converter,please register!
Thanks for using PB to image converter,please register!

Conclusions and future

- Presentation of a new finite volume scheme designed for non homogeneous systems
- The approximate intermediate state is upwind instead of the numerical flux
- Both homogeneous and non homogeneous part of the system are upwind
- Equilibrium for steady states is respected
- New applications were considered (problems of pollutant transport, Shallow water on a moving bed)
- More complex problems (realistic moving bed problems) are under study

Conclusions and future

- Presentation of a new finite volume scheme designed for non homogeneous systems
- The approximate intermediate state is upwind instead of the numerical flux
- Both homogeneous and non homogeneous part of the system are upwind
- Equilibrium for steady states is respected
- New applications were considered (problems of pollutant transport, Shallow water on a moving bed)
- More complex problems (realistic moving bed problems) are under study

Conclusions and future

- Presentation of a new finite volume scheme designed for non homogeneous systems
- The approximate intermediate state is upwind instead of the numerical flux
- Both homogeneous and non homogeneous part of the system are upwind
- Equilibrium for steady states is respected
- New applications were considered (problems of pollutant transport, Shallow water on a moving bed)
- More complex problems (realistic moving bed problems) are under study

Conclusions and future

- Presentation of a new finite volume scheme designed for non homogeneous systems
- The approximate intermediate state is upwind instead of the numerical flux
- Both homogeneous and non homogeneous part of the system are upwind
- Equilibrium for steady states is respected
- New applications were considered (problems of pollutant transport, Shallow water on a moving bed)
- More complex problems (realistic moving bed problems) are under study

Conclusions and future

- Presentation of a new finite volume scheme designed for non homogeneous systems
- The approximate intermediate state is upwind instead of the numerical flux
- Both homogeneous and non homogeneous part of the system are upwind
- Equilibrium for steady states is respected
- New applications were considered (problems of pollutant transport, Shallow water on a moving bed)
- More complex problems (realistic moving bed problems) are under study

Conclusions and future

- Presentation of a new finite volume scheme designed for non homogeneous systems
- The approximate intermediate state is upwind instead of the numerical flux
- Both homogeneous and non homogeneous part of the system are upwind
- Equilibrium for steady states is respected
- New applications were considered (problems of pollutant transport, Shallow water on a moving bed)
- More complex problems (realistic moving bed problems) are under study