NUMERICAL MODELING OF TRANSIENT FLOWS INVOLVING EROSION AND DEPOSITION OF SEDIMENT

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- The main concern of the sediment transport (or morphodynamics) is to determine the evolution of bed levels for hydrodynamics systems such as rivers, estuaries, bays and other nearshore regions where water flows interact with the bed geometry.
- Example of applications include among others, beach profile changes due to severe wave climates, seabed response to dredging procedures or imposed structures, and harbour siltation.

One-dimensional sediment transport model [2]

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = \frac{E-D}{1-p}, \tag{1a}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) = -gh\left(\frac{\partial Z}{\partial x} - S_f\right) - \frac{(\rho_s - \rho_w)g}{2\rho}h^2\frac{\partial c}{\partial x} - \frac{(\rho_0 - \rho)(E-D)u}{\rho(1-p)}, \tag{1b}$$

$$\frac{\partial (hc)}{\partial t} + \frac{\partial (huc)}{\partial x} = E - D, \tag{1c}$$

$$\frac{\partial Z}{\partial t} + A\xi\frac{\partial u^3}{\partial x} = -\frac{E-D}{1-p}, \tag{1d}$$

Variables and parameters

h, u, Z and c : water depth, velocity, bed elevation, and sediment concentration.

g : gravitational acceleration

p : bed sediment porosity

E and D : sediment entrainment and deposition fluxes

$\xi = \frac{1}{1-p}$

$\rho = \rho_w(1-c) + \rho_s c$: density of the water-sediment mixture

$\rho_0 = \rho_w p + \rho_s(1-p)$: density of the saturated bed

A : Grass constant for the sediment transport flux

$S_f = \frac{n_b^2 u^2}{h^{4/3}}$: friction slope

n_b : Manning roughness coefficient.

Deposition of sediments

The exchange between the water and the bed is a vital process for morphological evolution.
For deposition of non-cohesive sediment, this study uses the relation :

$$D = w(1 - C_a)^m C_a, \tag{2}$$

where
w is the settling velocity of a single particle in water at rest.
C_a is the near-bed volumetric sediment concentration.
$C_a = \alpha_c c$ where α_c is a coefficient larger than unity. In order so that the near-bed concentration does not exceed $(1 - p)$, the coefficient α_c is computed by $\alpha_c = min(2, \frac{1-p}{c})$.

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Entrainment of sediments

For the entrainment of cohesive material the following relation is used :

$$E = \begin{cases} \varphi(\theta - \theta_c) u h^{-1} d^{-0.2}, & \text{if } \theta \geq \theta_c \\ 0, & \text{else} \end{cases} \quad (3)$$

φ controles the erosion forces, θ_c is the critical value of Shield parameter for the initiation of sediment motion.

The Shields parameter $\theta = \dfrac{u_*^2}{sgd}$, with $u_*^2 = \sqrt{\dfrac{f}{8}}|u|$, is the friction velocity.

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Vector form presentation of the system (1)

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{S}(\mathbf{W}). \tag{4}$$

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hc \\ Z \end{pmatrix}, \qquad \mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huc \\ A\xi u^3 \end{pmatrix},$$

Vector form presentation of the system (1)

$$\mathbf{S} = \begin{pmatrix} \dfrac{E-D}{1-p} \\ -gh(\dfrac{\partial Z}{\partial x} + S_f) - \dfrac{(\rho_s - \rho_w)g}{2\rho} h^2 \dfrac{\partial c}{\partial x} - \dfrac{(\rho_0 - \rho)(E-D)u}{\rho(1-p)} \\ E-D \\ -\dfrac{E-D}{1-p} \end{pmatrix}.$$

The system (1) can be reformulated in an advective form as

$$\frac{\partial \mathcal{W}}{\partial t} + \mathcal{B}(\mathcal{W})\frac{\partial \mathcal{W}}{\partial x} = \mathcal{G}(\mathcal{W}), \qquad (5)$$

$$\mathcal{W} = \begin{pmatrix} h \\ hu \\ hc \\ Z \end{pmatrix}, \qquad \mathcal{G}(\mathcal{W}) = \begin{pmatrix} \dfrac{E-D}{1-p} \\ -ghS_f - \dfrac{\rho_0(E-D)}{\rho(1-p)h}u \\ E-D \\ -\dfrac{E-D}{1-p} \end{pmatrix},$$

$$\mathcal{B}(\mathcal{W}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh - u^2 - \dfrac{(\rho_s - \rho_w)}{2\rho} ghc & 2u & \dfrac{(\rho_s - \rho_w)}{2\rho} gh & gh \\ -uc & c & u & 0 \\ -3A\xi \dfrac{u^3}{h} & 3A\xi \dfrac{u^2}{h} & 0 & 0 \end{pmatrix}$$

The eigenvalues corresponding to the matrix $\mathcal{B}(\mathcal{W})$ are :

$$\begin{aligned}
\lambda_1 &= u \\
\lambda_2 &= 2\sqrt{-Q}\,cos(\frac{1}{3}\theta) + \frac{2}{3}u \\
\lambda_3 &= 2\sqrt{-Q}\,cos\left(\frac{1}{3}(\theta + 2\pi)\right) + \frac{2}{3}u \\
\lambda_4 &= 2\sqrt{-Q}\,cos\left(\frac{1}{3}(\theta + 4\pi)\right) + \frac{2}{3}u
\end{aligned}$$

and the corresponding eigenvectors are :

$$\begin{aligned}
e_1 &= \left(1, u, c - \frac{2\rho}{(\rho_s - \rho_w)}, 0\right) \\
e_k &= \left(1, \lambda_k, c, \frac{(\lambda_k - u)^2 - gh}{gh}\right) \qquad k = 2, 3, 4
\end{aligned}$$

SRNH : a splitting predictor corrector scheme

Predictor stage :

$$\begin{aligned} W_i^* &= W_i^n + \Delta t\, \mathcal{G}(W_i^n), \\ W_{i+\frac{1}{2}}^n &= \frac{W_i^* + W_{i+1}^*}{2} - \frac{1}{2}\mathrm{sgn}\left[\mathcal{B}\Big(\nu(W_i^*, W_{i+1}^*)\Big)\right](W_{i+1}^* - W_i^*), \end{aligned}$$

Corrector stage :

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\Big(F(W_{i+\frac{1}{2}}^n) - F(W_{i-\frac{1}{2}}^n)\Big) + \Delta t S_i^n \qquad (6)$$

where

$$\mathcal{B}\Big(\nu(V_i^*, V_{i+1}^*)\Big) = \mathcal{R}_{i+\frac{1}{2}}^* \Lambda_{i+\frac{1}{2}}^* \mathcal{R}^{*\,-1}_{i+\frac{1}{2}},$$

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Bed erosion after dam break

We consider the test example of dam-break flow over movable beds studied in [1].
Here, the experiment is carried out in a rectangular channel 1000 m long.

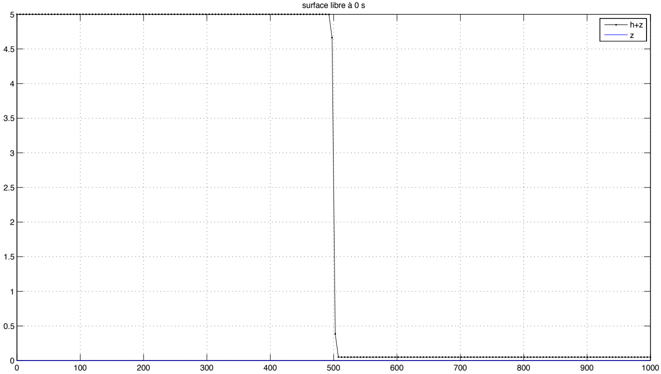


Figure: Initial conditions for the dam break.

Upstream initial water depth $h_l = 5\ m$
Sediment porosity $p = 0.4$
Exner constant $A = 0$
$\Delta x = 10\ m$
Results displayed at time $t = 20\ s$.

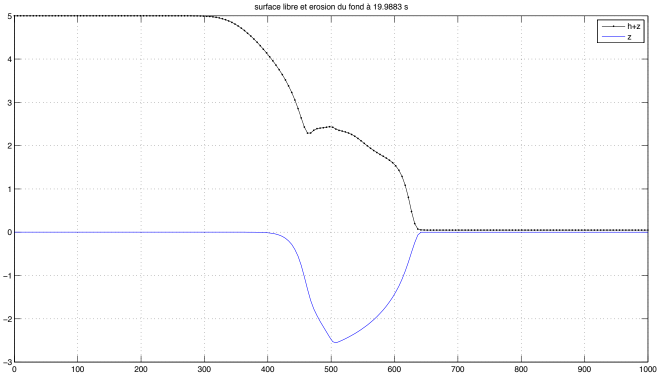


Figure: Computed results at t=20 s.

In Figure 2 we present the simulated results obtained for the bed-load and the water surface.

The erosion magnitude and wavefront location are well predicted by the numerical model.

As expected, a hydraulic jump is formed near the initial dam place and propagates upstream.

The effect of erosion due to the dam break is well reproduced.

Moving sand bank experiment

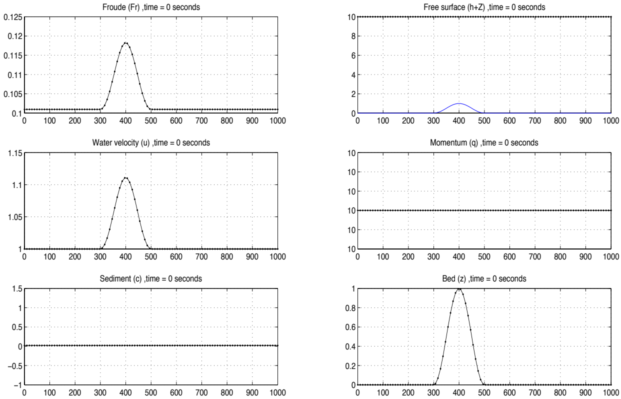


Figure: Grass model for sediments transport, n=200 points, t=0 seconds

Moving sand bank experiment

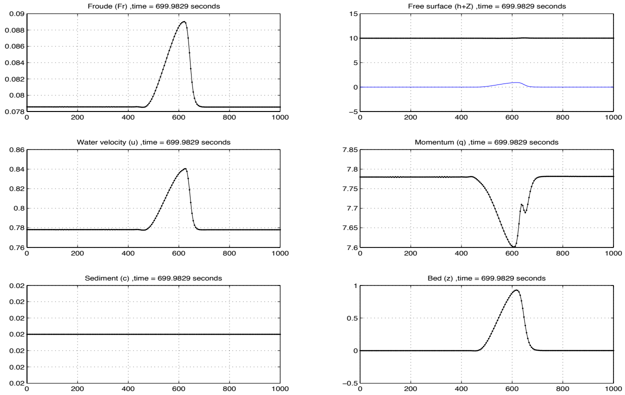
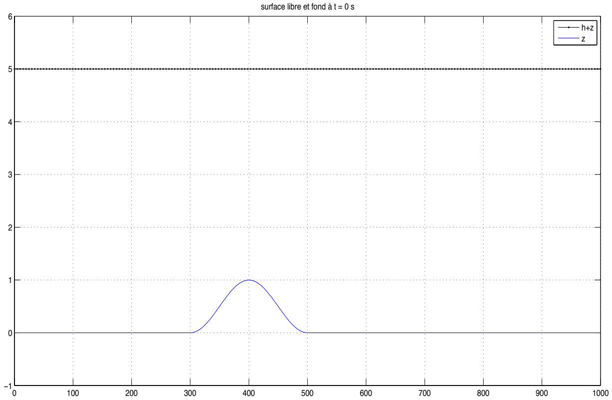


Figure: Sediments transport, n=200 points, A=1, E=D=0, t=700 s

Moving sand bank experiment



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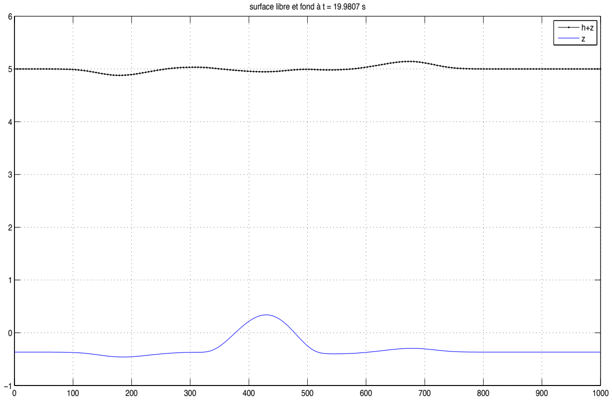


Figure: Sediments transport, n=200 points, A=1, $E \neq 0$, $D \neq 0$, t=700 s

Flux limitation

- The main idea behind the construction of flux limiter schemes is to limit the numerical solution's diffusion.
- a high order flux (Lax-Wendroff)
- a low order flux (SRNH)

$$\phi_{i+\frac{1}{2}} = \phi^{O2}_{i+\frac{1}{2}} - (1-\ell_i)(\phi^{O2}_{i+\frac{1}{2}} - \phi^{O1}_{i+\frac{1}{2}})$$

Flux limitation

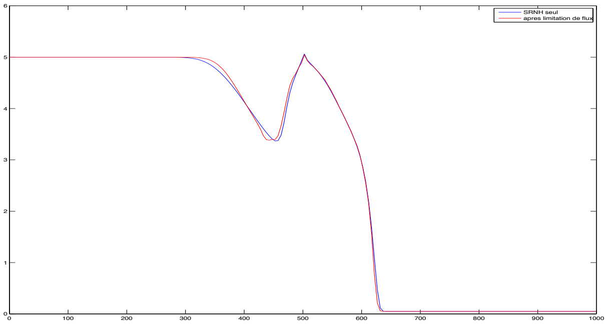


Figure: Comparison between the first and second order of the height *h* for a dam break

Flux limitation

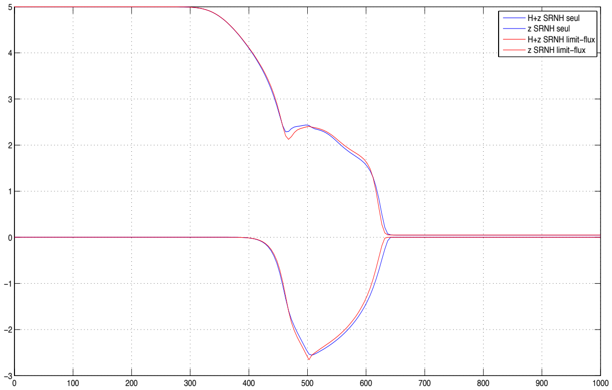


Figure: Comparison of $h + z$ and z, n=200 points, t=20 s

Flux limitation (example treated in [1])

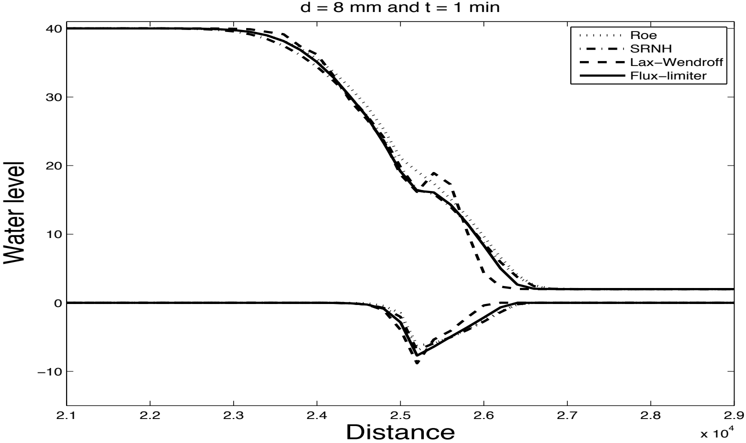


Figure: d=8 mm, t=1 mn

Flux limitation (example treated in [1])

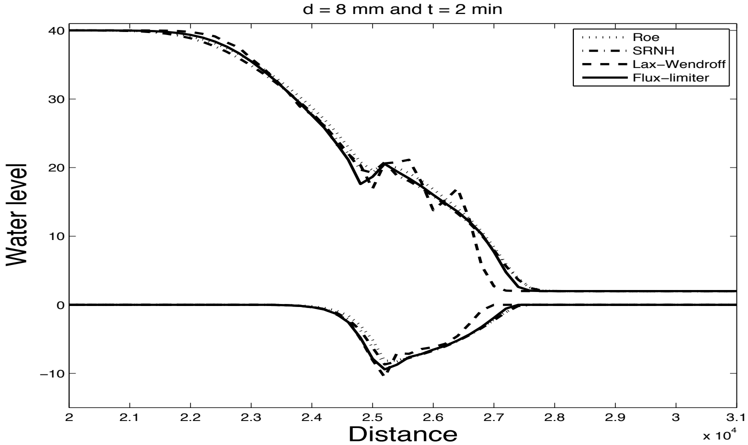


Figure: d=8 mm, t=2 mn

Flux limitation (example treated in [1])

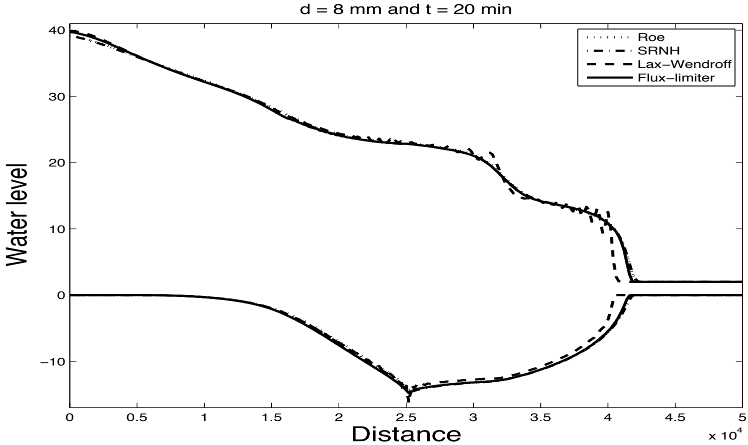
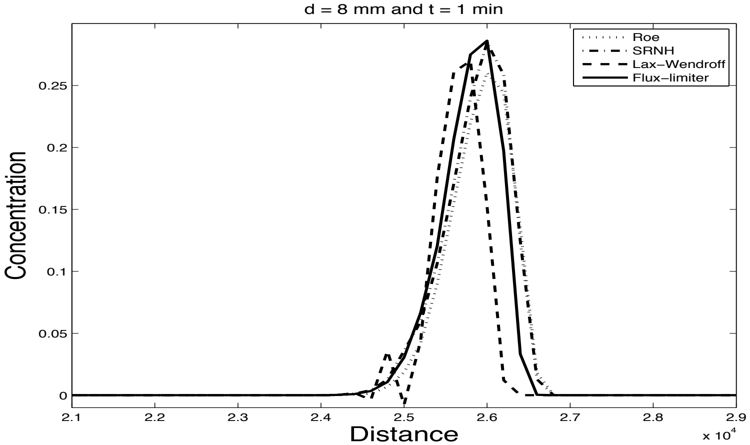


Figure: d=8 mm, t=20 mn

Flux limitation (example treated in [1])



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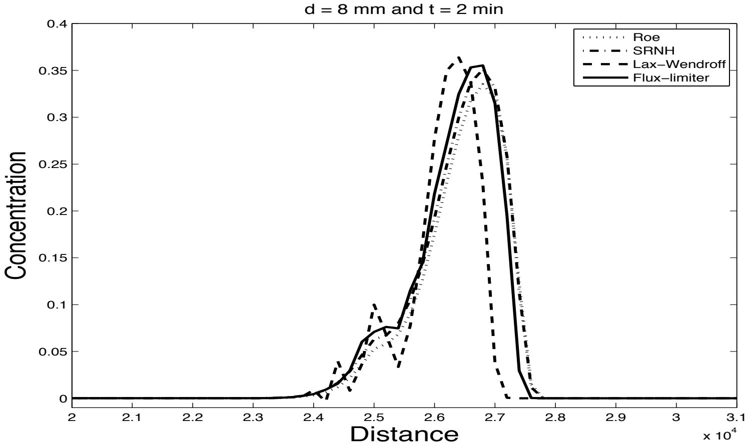


Figure: d=8 mm, t=2 mn

comparison between some limiters

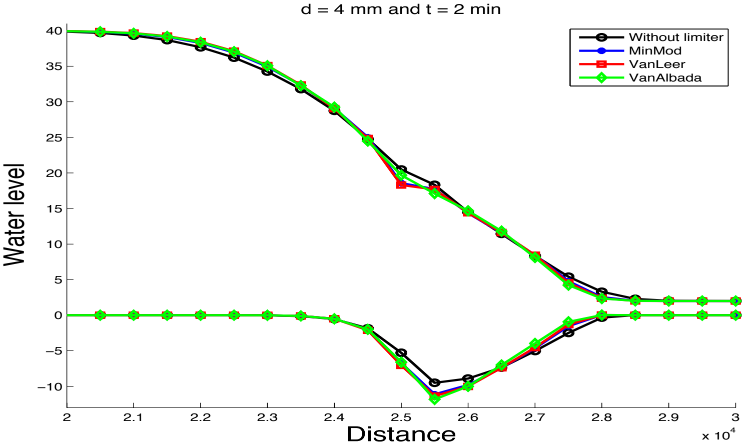


Figure: bed and water level

comparison between some limiters

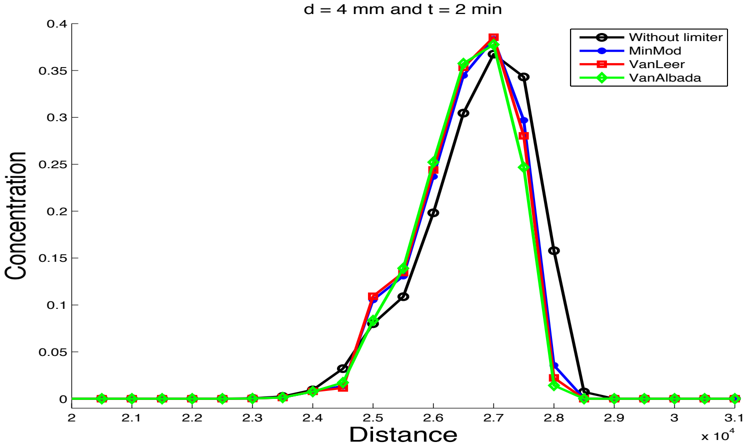
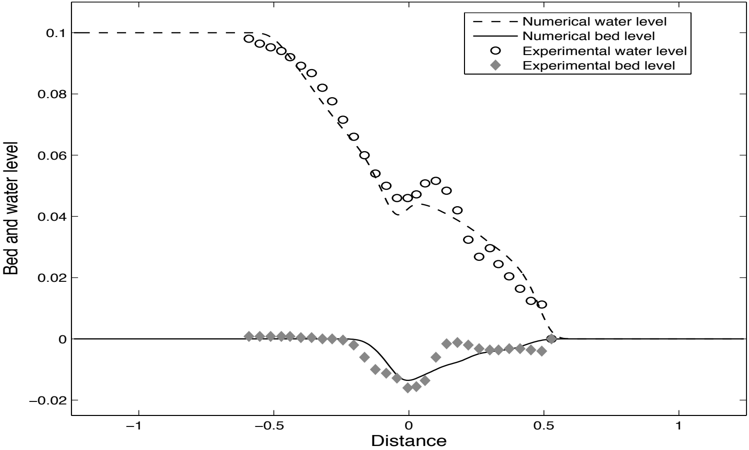


Figure: concentration

comparison with experimental data



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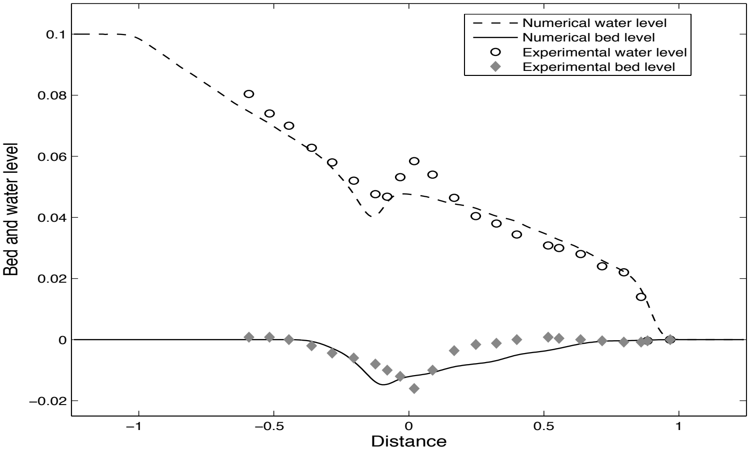


- However when flows with large friction coefficients, with significant water depth or with important wind effects are considered, the horizontal velocity can hardly be approximated by a vertically constant velocity.
- To drop this limitation a multilayer Saint-Venant model has been introduced in [4], where the different layers have their own velocities.
- The interest of the multilayer approach lies in the possibility to obtain a detailed description of the velocity in the flow while keeping a relative simplicity and a great robustness in the numerical procedure.

from hydrostatic Euler system to multilayer shallow water system [4]

We depart from the free surface hydrostatic Euler system

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (7)$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \quad (8)$$

$$\frac{\partial p}{\partial z} = -g \quad (9)$$

from hydrostatic Euler system to multilayer shallow water system [4]

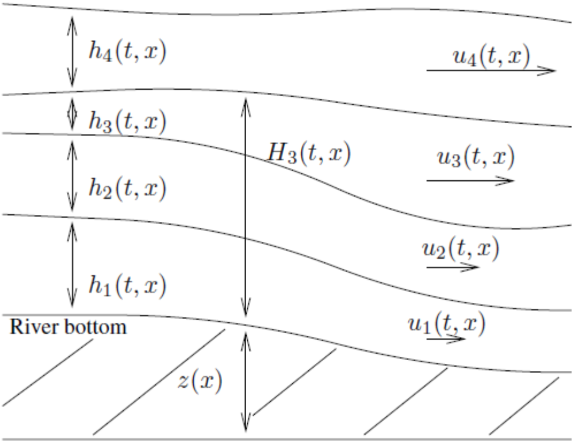
We consider that the flow domain is divided in the vertical direction into N layers of thickness h_α with $N+1$ interfaces $z_{\alpha+1/2(x,t)}$, $\alpha = 0, ..., N$ so that

$$H = \sum_{\alpha=1}^{N} h_\alpha,$$

and

$$z_{\alpha+1/2(x,t)} = z_{b(x)} + \sum_{j=1}^{\alpha} h_j(x,t)$$

Add of a wind source



from hydrostatic Euler system to multilayer shallow water system [4]

From a simple calculus based on the Leibniz rule and using the incompressibility condition (7) integrated over the interval $[z_{\alpha-1/2}(x,t), z_{\alpha+1/2}(x,t)]$ we deduce the mass equation

$$\frac{\partial h_\alpha}{\partial t} + \frac{\partial h_\alpha u_\alpha}{\partial x} = G_{\alpha+1/2}(x,t) - G_{\alpha-1/2}(x,t) \tag{10}$$

where $G_{\alpha+1/2}(x,t) = \frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2}\frac{\partial z_{\alpha+1/2}}{\partial x} - w(x, z_{\alpha+1/2}, t)$,
$u_{\alpha+1/2} = u(x, z_{\alpha+1/2}, t)$,
with the kinematic boundary condition

$$G_{1/2} = 0, \qquad G_{N+1/2} = 0 \tag{11}$$

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from hydrostatic Euler system to multilayer shallow water system [4]

By adding up the equations (10) for $j \leq \alpha$ and using the boundary condition (11)

$$G_{\alpha+1/2}(x,t) = \frac{\partial}{\partial t} \sum_{j=1}^{\alpha} h_j + \frac{\partial}{\partial x} \sum_{j=1}^{\alpha} h_j u_j, \qquad \alpha = 1, \ldots, N$$

from hydrostatic Euler system to multilayer shallow water system [4]

the sum of the equation (10) for all the layers and the boundary condition (11) lead to a global continuty equation for the total water height H

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} \sum_{\alpha=1}^{N} h_\alpha u_\alpha = 0, \tag{12}$$

from hydrostatic Euler system to multilayer shallow water system [4]

the same integration over the interval $[z_{\alpha-1/2}(x,t), z_{\alpha+1/2}(x,t)]$ leads to the momentum equation

$$\frac{\partial h_\alpha u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha < u^2 >_\alpha + \frac{1}{l_\alpha}\frac{g h_\alpha^2}{2} \right) = -g h_\alpha \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}, \qquad (13)$$

where

$$< u^2 >_\alpha (x,t) = \frac{1}{h_\alpha} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u^2(x,z,t) dz$$

from hydrostatic Euler system to multilayer shallow water system [4]

the velocities $u_{\alpha+1/2}$, $\alpha = 1, ..., N-1$ are obtained using an upwinding

$$u_{\alpha+1/2} = \begin{cases} u_\alpha & \text{if } G_{\alpha+1/2} \geq 0 \\ u_{\alpha+1} & \text{if } G_{\alpha+1/2} < 0 \end{cases} \quad (14)$$

Multilayer shallow water equations

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\sum_{\alpha=1}^{N} h_\alpha u_\alpha = 0, \tag{15}$$

$$\frac{\partial h_\alpha u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + \frac{1}{l_\alpha}\frac{g h_\alpha^2}{2}\right) = -g h_\alpha \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}, \tag{16}$$

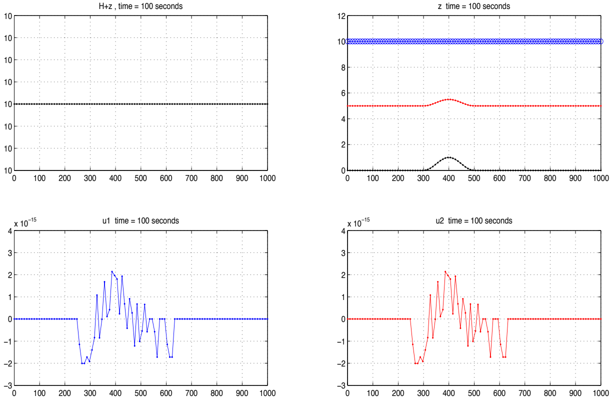
Finite volume scheme [3]

- a new finite volume method for numerical solution of shallow water equations for either flat or non flat topography,
- it consists on a predictor stage and a corrector stage,
- the predictor stage uses the method of characteristcs to reconstruct the numerical fluxes,
- the corrector stage recovers the conservation equations

The two layer shallow water equations

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial h_1 u_1}{\partial x} + \frac{\partial h_2 u_2}{\partial x} &= 0, \\
\frac{\partial h_1 u_1}{\partial t} + \frac{\partial}{\partial x}\left(h_1 u_1^2 + \frac{g}{2} H h_1\right) &= -g h_1 \frac{\partial z_b}{\partial x} + u_{3/2}\left(l \frac{\partial H}{\partial t} + l \frac{H u_1}{\partial x}\right), \\
\frac{\partial h_2 u_2}{\partial t} + \frac{\partial}{\partial x}\left(h_2 u_2^2 + \frac{g}{2} H h_2\right) &= -g h_2 \frac{\partial z_b}{\partial x} - u_{3/2}\left(l \frac{\partial H}{\partial t} + l \frac{H u_1}{\partial x}\right),
\end{aligned}
$$

The two layer shallow water equations



Add of a wind source

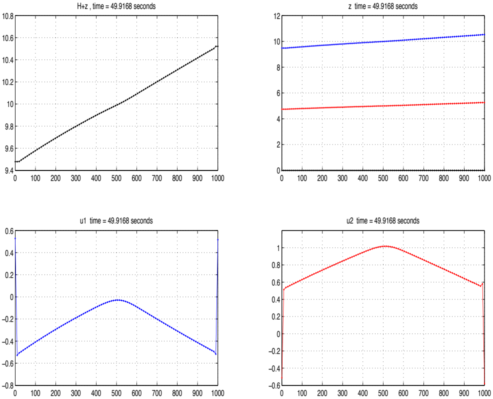
Let's imagine now that a wind source is blowing the free water surface from the left handside

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial h_1 u_1}{\partial x} + \frac{\partial h_2 u_2}{\partial x} &= 0, \\
\frac{\partial h_1 u_1}{\partial t} + \frac{\partial}{\partial x}\left(h_1 u_1^2 + \frac{g}{2} H h_1\right) &= -g h_1 \frac{\partial z_b}{\partial x} + u_{3/2}\left(l \frac{\partial H}{\partial t} + l \frac{H u_1}{\partial x}\right), \\
\frac{\partial h_2 u_2}{\partial t} + \frac{\partial}{\partial x}\left(h_2 u_2^2 + \frac{g}{2} H h_2\right) &= -g h_2 \frac{\partial z_b}{\partial x} - u_{3/2}\left(l \frac{\partial H}{\partial t} + l \frac{H u_1}{\partial x}\right) \\
&\quad + F_{wind},
\end{aligned}
$$

where

$F_{wind} = c_w w|w|,\ c_w = \rho_a(0.75 + 0.067|w|)10^{-3},\ w = 8,\ \rho_a = 1.2$

Add of a wind source



Perspectives

- explore more complicated topographies for the problem sediment transport,
- continue to study the case of two-dimensional sediment transport and mesh adaption,
- continue to study the case of multilayer shallow water equations for the sediment transoprt

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