A FAST FINITE VOLUME SOLVER FOR HYDROSTATIC SHALLOW WATER FLOWS

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Contribution : FVC

We present a new finite volume method for the numerical solution of shallow water models for either flat or non-flat topography. The method is simple, accurate and avoids the solution of Riemann problems during the time integration process.

The performance of the method is also demonstrated by comparing the results and the cpu times obtained using some other FV schemes.

1D Formulation

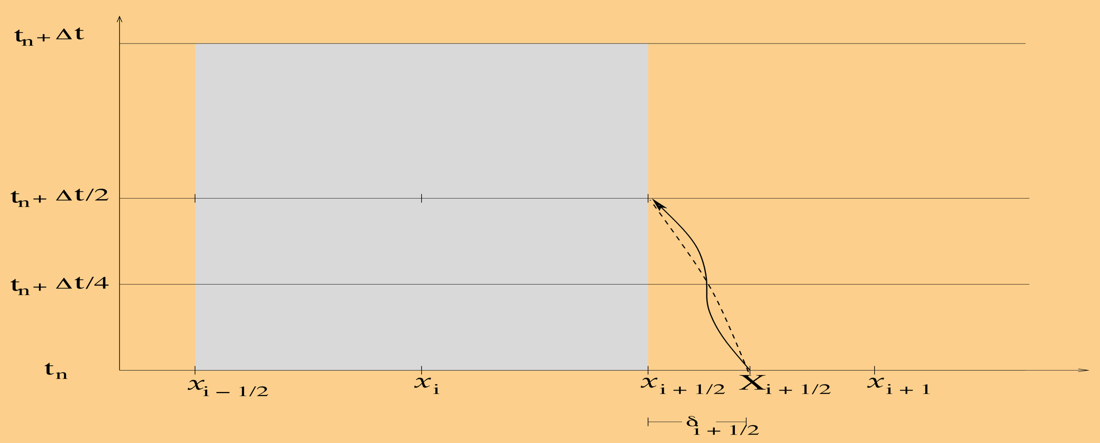
In a vector form, the proposed Finite Volume Characteristics (FVC) method reads

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t \frac{\mathcal{F}^n_{i+1/2} - \mathcal{F}^n_{i-1/2}}{\Delta x} + \Delta t \mathcal{Q}_i^n,$$

where $\mathcal{F}^n_{i\pm1/2} = \mathbf{F}(\mathbf{W}^n_{i\pm1/2})$ are the numerical fluxes at $x = x_{i\pm1/2}$ and time t_n. Here, $\mathbf{W}^n_{i\pm1/2}$ is constructed with *the method of characteristics* applied to *the advective version* of the considered system.

The characteristic curves are solutions of the initial-value problem

$$\begin{aligned} \frac{dX_{i+1/2}(\tau)}{d\tau} &= u_{i+1/2}\Big(\tau, X_{i+1/2}(\tau)\Big), \\ X_{i+1/2}(t_n + \Delta t/2) &= x_{i+1/2}. \end{aligned}$$



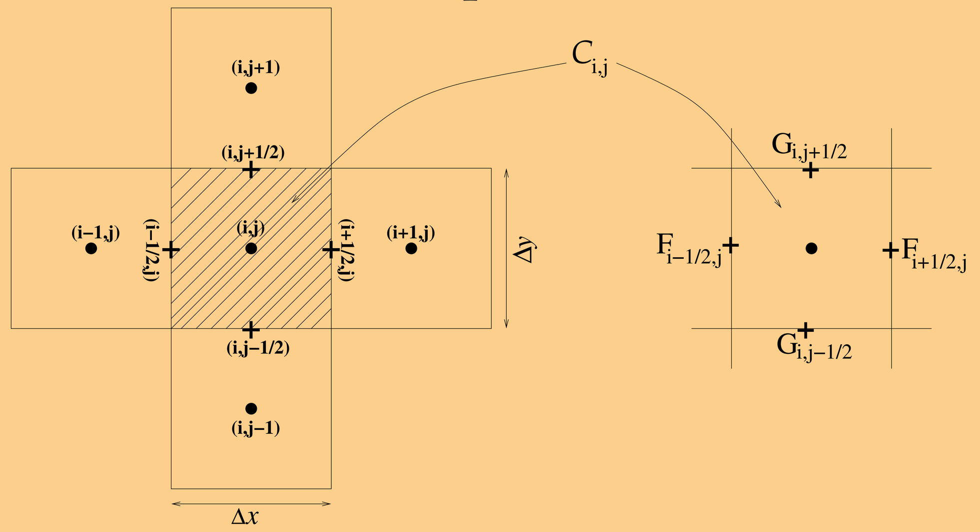
Once the characteristics curves $X_{i+1/2}(t_n)$ are known, a solution at the cell interface $x_{i+1/2}$ is reconstructed.

Some models treated with FVC

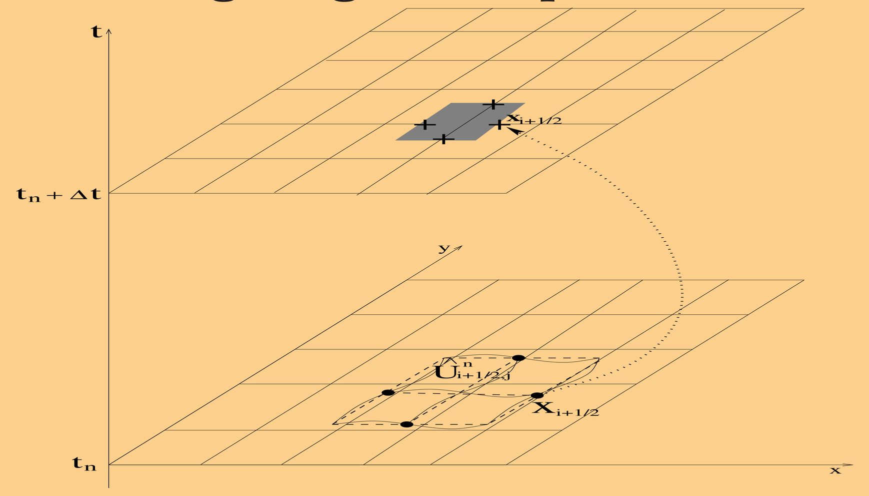
1. **Classical shallow water model**
   [1] F. Benkhaldoun and M. Seaid. *A simple finite volume method for the shallow water equations*. J. Comp. Applied Math. 234, 58-72 (**2010**)
2. **Density-driven shallow water flows**
   [2] F. Benkhaldoun, S. Sari, M. Seaid, *A simple multi-layer finite volume solver for density-driven shallow water flows*, Mathematics and Computers in Simulation, Volume 99, (**2014**), Pages 170-189.

2D Formulation

The Eulerian step



The Lagrangian step



Some models treated with FVC

3. **Multi-layered model**
   [3] E. Audusse, F. Benkhaldoun, S. Sari, M. Seaid, P. Tassi, *A fast finite volume solver for multi-layered shallow water flows with mass exchange*, Journal of Computational Physics, Volume 272, (**2014**), Pages 23-45.
4. **rotational shallow water model**
   [4] F. Benkhaldoun, S. Sari, M. Seaid, *Projection finite volume method for shallow water flows*, Mathematics and Computers in Simulation, Volume 118, (**2015**), Pages 87-101.
5. **Two-dimensional Conservation Laws**
   [5] F. Benkhaldoun, S. Sari, M. Seaid, *A Family of Finite Volume Eulerian-Lagrangian Schemes for Two-dimensional Conservation Laws*, Journal of Computational and Applied Mathematics, Volume 285, (**2015**), Pages 181-202.

Comparison of some test examples

1 Classical dam-break problem [1]

Gridpoints	Roe method	SRNH method	FVC method
500	8.746	13.193	1.008
1000	34.780	52.655	2.707
2000	134.152	210.620	15.756
4000	534.124	834.055	61.096
8000	2178.701	3378.303	249.209

CPU times for classical dam-break on wet bed at $t = 50$ s using $h_r/h_l = 0.005$.

2 Density dam-break problem [2]

Gridpoints	a single initial discontinuity				two initial discontinuities			
	FVC	Rusanov	ROE	SRNH	FVC	Rusanov	ROE	SRNH
100	0.21	0.06	0.78	0.81	0.61	0.17	2.67	2.7
200	0.4	0.19	3.1	3.17	1.33	0.6	10.36	10.6
400	0.94	0.67	12.18	12.58	3.36	2.22	41.37	42.46
800	2.58	2.66	48.83	50.34	9.64	8.73	164.86	172.34
1600	8.48	10.25	193.5	206.58	31.29	34.46	656.72	705.67

CPU times for density dam-break problem at $t = 200$ s using different gridpoints.

3 Multi-layered dam-break problem [3]

Gridpoints	10-layer model		20-layer model	
	Kinetic scheme	fvc scheme	Kinetic scheme	fvc scheme
200	3.2	3.0	5.4	4.8
400	12.5	9.7	20.8	15.5
800	49.1	34.9	81.9	55.4

CPU times for 10-layer and 20-layer models on different meshes for a dam-break at $t = 14$.

4 Circular dam-break problem [4]

Gridpoints	4 s			8 s			20 s		
	FVC	ROE	SRNH	FVC	ROE	SRNH	FVC	ROE	SRNH
50×50	3.74	6.03	6.16	10.66	18	18.2	17.64	29.74	30.02
100×100	23.52	44.63	46.27	70.61	137.22	139.81	115.17	230.29	232.54

CPU times on different meshes for circular dam-break on a flat bottom at different times.

Conclusion

The new method has several advantages. First, it can compute the numerical flux corresponding to the real state of water flow without relying on Riemann problem solvers. Second, reasonable accuracy can be obtained easily and no special treatment is needed to maintain a numerical balance, because it is performed automatically in the integrated numerical flux function. Finally, the proposed approach does not require either nonlinear solution or special front tracking techniques. Furthermore, it has strong applicability to various conservative laws.