High Performance meshfree methods for fluid flows Computation

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- Let us consider the conservation law defined by

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{w}) + \frac{\partial}{\partial y}\mathbf{g}(\mathbf{w}) &= 0, \qquad (x,y) \in \mathbb{R}^2, \quad t > 0, \\ \mathbf{w}(x,y,0) &= \mathbf{w}_0(x,y), \quad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{1}$$

  with $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$ is a scalar function, $\mathbf{f}(\mathbf{w})$ and $\mathbf{g}(\mathbf{w})$ are linear or nonlinear functions.
- This kind of PDEs are widely used for numerical simulations of physical, biological and environmental phenomena.
- The goal of this work is to propose and study a robust and stable meshfree method to solve accurately this PDE in complex shaped domains.

- The problem is discretized in a set of N collocation point $x = \{x_1, \ldots, x_N\}$ called centers.
- For each center, the local RBF method is formulated as a local interpolation of the form:

$$f^{[i]}(x) = \sum_{j \in I_{i,m}} \lambda_j(t)\phi(\|x - x_j\|_2), \tag{2}$$

  where $\Lambda = \{\lambda_j\}$ are the expansion coefficients of the RBF method to be determined, ϕ is a radial basis function and $I_{i,m}$ is a local vector that contains the reference node i and indices of collocation points belonging in the local stencil.
- By imposing the interpolation condition in each point of the stencil, we obtain

$$f^{[i]}(x_j) = f(x_j), \quad j \in I_{i,m}. \tag{3}$$

- This produces the linear $m \times m$ system

$$B\Lambda = f(I_{i,m}),$$

to be solved to local expansion coefficients Λ.

- The elements of the interpolation matrix B are

$$b_{kj} = \phi(\|x_k - x_j\|), \quad k, j \in I_{i,m}. \tag{4}$$

- If the local interpolation matrix B is invertible, expansion coefficients Λ exist and are given by

$$\Lambda = B^{-1} f(I_{i,m}). \tag{5}$$

- To calculate an approximation of partial derivatives of flux functions $f(u)$ at the reference point i, the differentiation operator $\mathcal{L}$ is applied as

$$\mathcal{L}f(x_i) = \sum_{j \in I_{i,m}} \lambda_j \mathcal{L}\phi(\|x_i - x_j\|_2), \tag{6}$$

which can be written in scalar product by

$$\mathcal{L}f(x_i) = h \cdot \Lambda,$$

where Λ is a column vector with m elements and h is a line vector with m elements defined by

$$h_j = \mathcal{L}\phi(\|x_i - x_j\|), \quad j \in I_{i,m}.$$

Using equation (5), coefficients Λ can be replaced in the previous equation by

$$\mathcal{L}f(x_i) = h \cdot (B^{-1} f(I_{i,m})) = (h \cdot B^{-1}) f(I_{i,m}) = \mathcal{D} \cdot f(I_{i,m}), \qquad (7)$$

where $\mathcal{D}$ is a vector containing the m differentiation weights ω_j. Thus, partial differentiation of the flux function $f(u)$ are computed by simply multiplying the flux values in the points belonging in the local stencil with the local differentiation weights

$$\mathcal{L}f(x_i) = \sum_{j \in I_{i,m}} \omega_j f(x_j), \quad j \in I_{i,m}. \qquad (8)$$

We obtain the semi-discrete form in the reference point i

$$\frac{\partial u}{\partial t}|_{x_i} + \mathcal{L}_x f(u)_i + \mathcal{L}_y g(u)_i = 0,$$

which can be rewritten as

$$\frac{du}{dt}|_{x_i} = F(u)_i + G(u)_i,$$

with

$$F(u)_i = -\mathcal{L}_x f(u)_i = -\sum_{j \in I_{i,m}} \chi_j f(u_j)$$

and

$$G(u)_i = -\mathcal{L}_y g(u)_i = -\sum_{j \in I_{i,m}} \Upsilon_j g(u_j),$$

where χ and Υ are the local differentiation weights associated to the partial derivatives that respect x and y respectively.

- An explicit first order Euler scheme is obtained by

$$u_i^{n+1} = u_i^n - \Delta t \left(\sum_{j \in I_{i,m}} \chi_j f(u_j) + \sum_{j \in I_{i,m}} \Upsilon_j g(u_j) \right). \tag{9}$$

- To ensure a stable, high-order local meshfree scheme, we use the following techniques:
  - A predictor/corrector scheme based on the characteristics methods at the midpoints;
  - An upwind RBF-MUSCL scheme with stencil adaptation and slope limiters incorporating;
  - An upwind RBF-MUSCL scheme with state reconstruction and slope limiters incorporating.

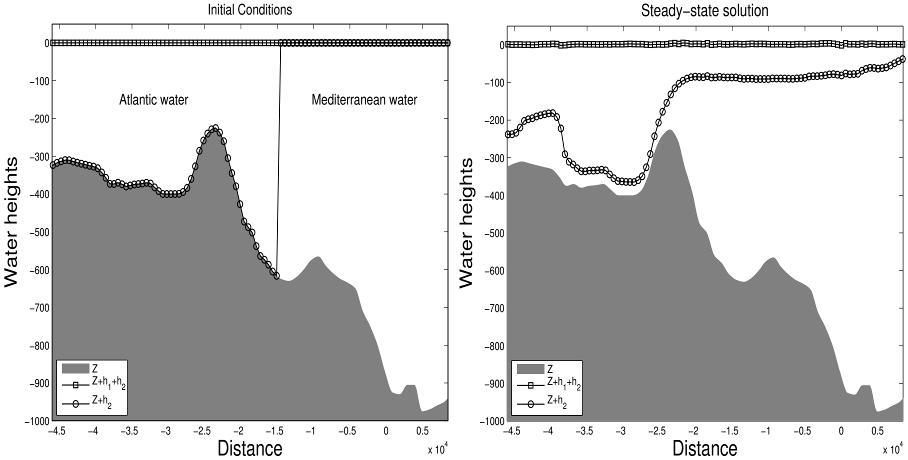

Figure: Lock-exchange between the Mediterranean Sea and the Atlantic ocean, initial solution (left) stationary solution (right).

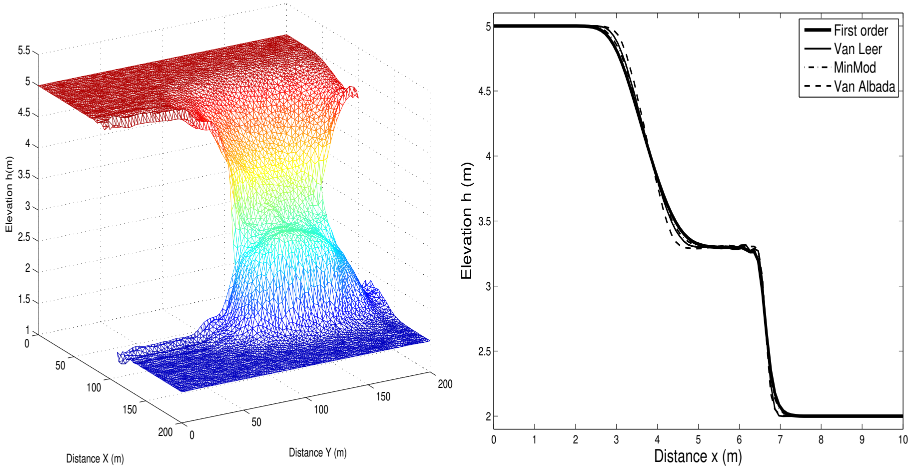

Figure: Dambreak problem: Van Albada solution (left) and profiles comparison of different slope limiters (right).

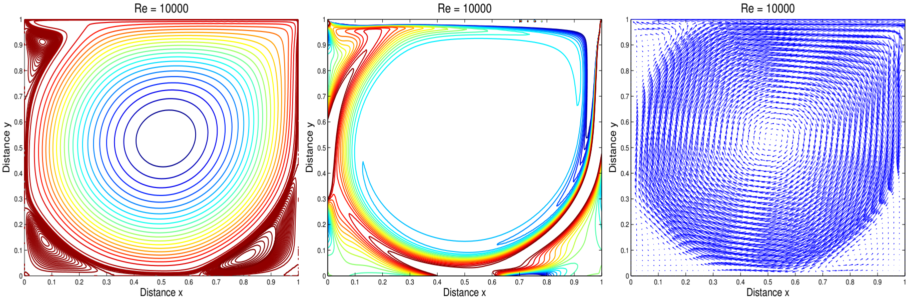

Figure: Lid-Driven Cavity flows for $Re = 10000$.

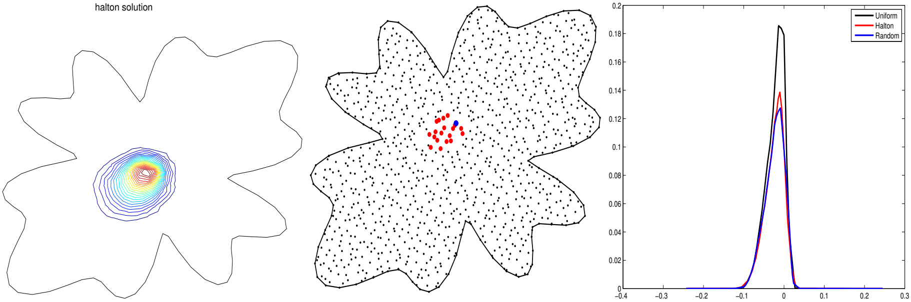

Figure: Burgers problem in a complex domain using different distributions of collocation points.

Thank you for your attention !