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Exact Solutions for Shallow Water Equations

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Exact Solutions for non Linear Conservation Laws

Non Linear Systems Considered

We are intersted by fluid flow problems described by such non linear systems :

$$\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} + \frac{\partial G(W)}{\partial y} + \frac{\partial H(W)}{\partial z} = 0 \qquad (1)$$

Examples :

Euler equations in one space dimension

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0 \\ \dfrac{\partial(\rho u)}{\partial t} + \dfrac{\partial\left(\rho u^2 + P\right)}{\partial x} = 0 \\ \dfrac{\partial E}{\partial t} + \dfrac{\partial\left[u\left(E+P\right)\right]}{\partial x} = 0 \end{cases} \tag{2}$$

with ideal gas equation of state : $p = (\gamma - 1)\left(E - \dfrac{1}{2}\rho u^2\right)$, where ρ is fluid density, u the veocity, E the energy, and p : the pressure.

Shallow Water Flow

We consider water flow in a configuration where the water depth is neglectible when compared to the characteristic length of the domain. [3]).
If the bottom is flat, and the friction neglectible, the problem is described by the following system :

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hu)}{\partial x} = 0 \\ \dfrac{\partial (hu)}{\partial x} + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{1}{2} g h^2 \right) = 0 \end{cases} \quad (3)$$

h being the water depth, u the velocity, and g the gravity constant.

Exact solution for 1D scalar problems

Introduction

Consider the scalar problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0 \quad \text{in } \mathbb{R} \times]0, T[\\ u = u(x,t) \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{4}$$

In the sequel, note $X = \mathbb{R} \times [0, T[$.

Example, Burger's equation : $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{u^2}{2}\right) = 0$

Weak solution and jump condition

Smooth solution

If $u \in C^1(X)$, one has : (4) $\Longrightarrow \dfrac{\partial u}{\partial t} + f'(u)\dfrac{\partial (u)}{\partial x} = 0$

then in the frame (x, t), u is constant on the characteristic curve given by :

$$\begin{cases} \dfrac{dx(t)}{dt} = f'\left[u\left(x(t), t\right)\right] \\ x(t = 0) = x_0 \end{cases} \tag{5}$$

One deduce the solution u :

$$u\left(x(t), t\right) = u\left(x(0), 0\right) = u\left(x_0, 0\right) = u_0(x_0)$$

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Note $f'(u) = a(u)$, the characteristic system considered is then :

$$\begin{cases} \dfrac{dx}{dt} = a\left(u_0(x_0)\right) \\ \\ u(x,t) = u_0(x_0) \end{cases}$$

which gives :

$$\begin{cases} x_0 = x - ta\left(u_0(x_0)\right) \\ \\ u(x,t) = u_0(x_0) \end{cases} \tag{6}$$

Applications :

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Applications :

1. Linear case

$$f(u) = cu : \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \Longrightarrow a(u) = c$$

characteristic curve : $x = x_0 + tc \Leftrightarrow x_0 = x - ct$

Solution : $u(x,t) = u_0(x - ct)$

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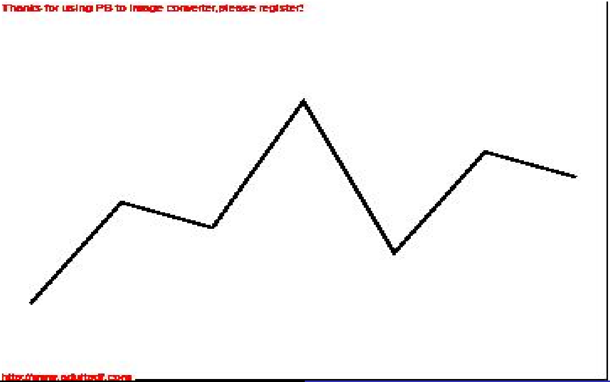

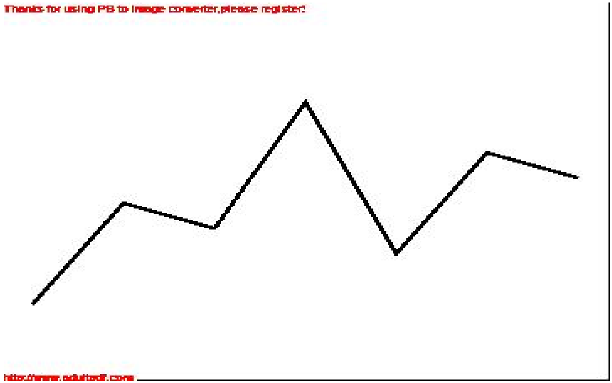
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2. Burger's equation

Consider the non linear equation :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = 0,$$

here $f(u) = \frac{u^2}{2}$, then $a(u) = f'(u) = u$

The characteristic curve is given by : $x = x_0 + tu_0(x_0)$

Consider the different initial conditions :

case 1

$$u_0(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

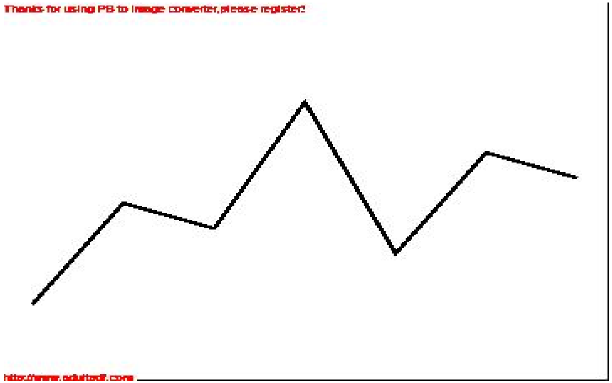
case $x_0 < 0$, $x = x_0$, $u_1(x,t) = u_0(x)$

case $x_0 \geq 0$, $x = x_0 + tx_0$, $u_1(x,t) = u_0(x_0) = u_0\left(\frac{x}{1+t}\right) = \frac{x}{1+t}$

case 2

$$u_0(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0 \end{cases}$$

case $x_0 < 0$, $x = x_0 + t$, $u_2(x,t) = u_0(x-t) = u_0(x_0) = 1$
case $x_0 \geq 0$, $x = x_0$, $u_2(x,t) = u_0(x) = 0$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Discontinuous solution and jump condition :

Theorem

The 3 following assertions are equivalent :

i) u is a weak solution of problem (4) , i.e :

$$\int_0^\infty \int_{\mathbb{R}} \left(u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x} \right) dxdt + \int_{\mathbb{R}} u_0(x) \varphi(x,0) dx = 0,$$

$\forall \varphi \in D\left(\mathbb{R} \times [0, +\infty[\right)$

ii) $\forall R = [x_1, x_2] \times [t_1, t_2] \subset \Omega = \mathbb{R} \times [0, T]$,

$$\int_{\partial R} [u.n_t + f(u).n_x] \, d\sigma = 0$$

Theorem

iii) If u is C^1, *u is classical solution of* $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} f(u) = 0$, *and on a shoc curve* $\Gamma(u_l, u_r)$, *the solution is governed by the jump condition :* $[f(u)] = s\,[u]$.
One defines the jump $[u] = u_r - u_l$, *and the curve* $\Gamma(u_l, u_r)$, *which equation is :* $\dfrac{dx}{dt} = s$, *separates the left and right states* u_l *and* u_r

The jump condition is called the Rankime-Hugoniot condition in gas dynamics.

Example : Consider the Burger's equation :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = 0 \tag{7}$$

and the initial condition : $u_0(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$

First possibiliy : a weak solution with the shoc $\Gamma(0,1)$.
The jump condition gives :

$$[f(u)] = s\,[u] \Longrightarrow \left[\frac{u_r^2}{2} - \frac{u_l^2}{2}\right] = s\,[u_r - u_l] \Longrightarrow s = \tfrac{1}{2} \Longrightarrow$$

$$u(x,t) = \begin{cases} 0 & \text{si} \quad \dfrac{x}{t} < \dfrac{1}{2} \\ 1 & \text{si} \quad \dfrac{x}{t} \geq \dfrac{1}{2} \end{cases}$$

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Second possibility : a continuous weak solution.

$$u(x,t) = \begin{cases} 0 & \text{si} \quad \frac{x}{t} < 0 \\ \frac{x}{t} & \text{si} \quad 0 \leq \frac{x}{t} < 1 \\ 1 & \text{si} \quad \frac{x}{t} \geq 1 \end{cases}$$

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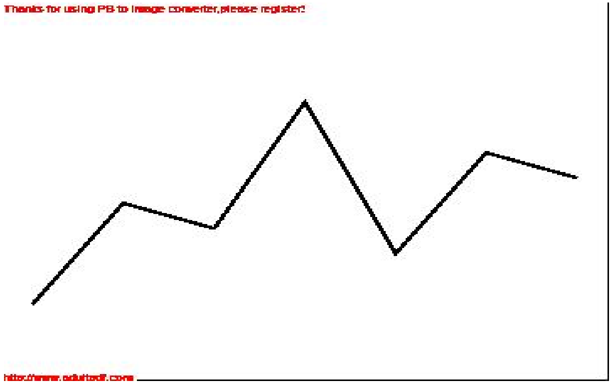
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Physical validation of the solution : the entropy condition

The entropy solution

Definition

A smooth convex function U, is said to be an entropy of the problem, if there exists an entropy flux F such that : $U'(u)f'(u) = F'(u)$.

Definition

a weak solution u of (4) is said entropy solution if

$$\forall \varphi \in D\left(\mathbb{R}\times]0,T]\right) : \int_0^T \int_{\mathbb{R}} \left(U(u)\frac{\partial \varphi}{\partial t} + F(u)\frac{\partial \varphi}{\partial x} \right) dxdt \geq 0, \text{ where}$$

U is an entropy of the problem, and F its entropy flux.

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U is an entropy of the problem, and F its entropy flux.

Remark 1 : An entropy solution respects the entropy condition with the convex (though non derivable) function, $U(u) = \mid u - k \mid$, and the associated entropy flux : $F(u) = sgn(u - k)(f(u) - f(k))$, where $k \in \mathbb{R}$

Remark 2 : Reciprocally, since every convex function belongs to the convex hull of all affine functions, and functions of the form $x \longrightarrow \mid x - k \mid$, a weak solution which respects the entropy condition with the convex function $U(u) = \mid u - k \mid$, is an entropy solution.

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About Entropy

Lemme : There exists a function U which is transported in regions where u is C^1. i.e. $\frac{\partial}{\partial t}U(u) + \frac{\partial}{\partial x}F(u) = 0$

proof : If u is C^1 : $U'(u)\left(\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x}\right) = 0$, if there exists F such that $U'(u)f'(u) = F'(u)$, then $\frac{\partial}{\partial t}U(u) + \frac{\partial}{\partial x}F(u) = 0$

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Consider the regularized problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = u_0(x) \end{cases} \tag{8}$$

Proposition

There exists a unique smooth solution u^ε of the problem (8)

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A piecewise C^1 function u, is an entropy weak solution of (4) if and only if :
i) u is a classical solution in (x, t) regions where u is C^1
ii) On an shoc curve Γ, u satisfies $[F(u)] \leq s\,[U(u)]\,,\ \forall\ (U, F)$ a couple of entropy and antropy flux.

Corollaire

1) If f is strictly convex, then a shoc is entropic if and only if : $f'(u_r) < s < f'(u_l)$

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Application : the first weak solution in example (7) is non entropic, and hence non admissible.

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