MENHYDRO 2010

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NON LINEAR SYSTEMS OF CONSERVATION LAWS

Linear Systems - Hyperbolicity

Let A be a constant matrix in $\mathcal{M}_p$ and $W_0 \in L^\infty(\mathbb{R})^p$, consider the system:

$$\begin{cases} \dfrac{\partial W}{\partial t} + A \dfrac{\partial W}{\partial x} = 0 & x \in \mathbb{R} \\ & t > 0 \\ W(x,0) = W_0(x) \end{cases}$$

We assume that the system is strictly hyperbolic

$\longrightarrow$ $A = R \Lambda R^{-1}$; $\Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \dots \lambda_p)$

and $\lambda_1 < \lambda_2 < \dots < \lambda_p$

$R = [r_1, r_2, \dots r_p]$ is the right eigenvectors matrix

i.e.: $A r_j = \lambda_j r_j$

The rows l_i of R^{-1} are the lef eigenvectors of the system:

i.e.: $l_i A = \lambda_i l_i$ and $l_i r_j = \delta_{ij}$

$$R^{-1} = \begin{bmatrix} - l_1 - \\ \vdots \\ - l_i - \\ \vdots \\ - l_p - \end{bmatrix}$$

SOLUTION OF THE LINEAR SYSTEM

Proposition: The solution of the system (SL) is given by:

$$\boxed{W(x,t) = \sum_{j=1}^{p} \left[\ell_j . W_0(x - \lambda_j t) \right] . r_j}$$

Proof: let's make the change of variable: $V = R^{-1} W$ $\quad (v_j = \ell_j . W)$

$$\Longleftrightarrow \quad W = RV = \sum_{j=1}^{p} v_j \, r_j$$

System (SL) $\Longrightarrow$

$$\begin{cases} \dfrac{\partial V}{\partial t} + \Lambda \dfrac{\partial V}{\partial x} = 0 \quad \left(\dfrac{\partial v_j}{\partial t} + \lambda_j \dfrac{\partial v_j}{\partial x} = 0 \right) \\ V(x,0) = R^{-1} W_0(x) = V_0(x) \end{cases}$$

$$v_j(x,t) = v_j(x - \lambda_j t, 0) = \ell_j . W_0(x - \lambda_j t)$$

and $W(x,t) = RV(x,t) = \sum_{j=1}^{p} v_j(x,t) \, r_j$

SELF SIMILARITY

- <u>proposition</u>:

The solution of the Riemann Problem

$$\frac{\partial W(x,t)}{\partial t} + \frac{\partial}{\partial x} F(W(x,t)) = 0$$

$$W(x,0) = W_0(x) = \begin{cases} W_L & \text{if } x<0 \\ W_R & \text{if } x>0 \end{cases}$$

is self-similar

i.e: $W(x,t) = H\left(\frac{x}{t}\right)$

<u>Proof</u>: for $\alpha > 0$ put $y = \alpha x$ and $\tau = \alpha t$

Let $U(y,\tau) = W(x,t) = W\left(\frac{y}{\alpha}, \frac{\tau}{\alpha}\right)$

Remark $U(y, \tau=0) = W_0\left(\frac{y}{\alpha}\right) = \begin{cases} W_L & \text{if } y<0 \\ W_R & \text{if } y>0 \end{cases}$

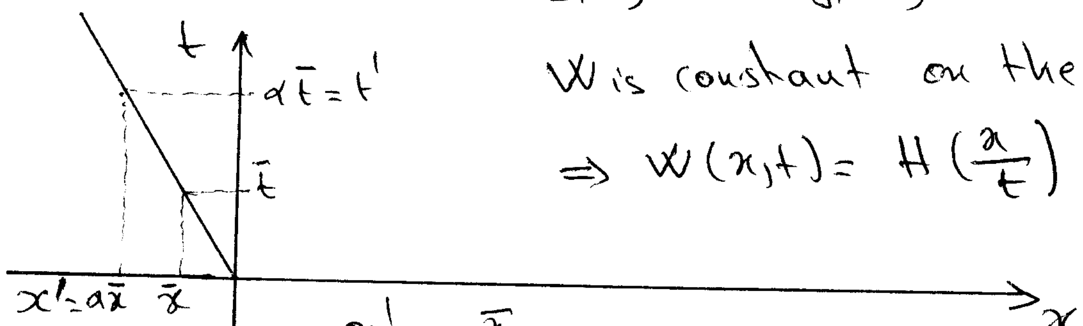
$$\frac{\partial U}{\partial \tau} = \frac{1}{\alpha}\frac{\partial W}{\partial t} \quad \text{and} \quad \frac{\partial F(U)}{\partial y} = \frac{1}{\alpha}\frac{\partial F(W)}{\partial x}$$

So $$\frac{\partial U}{\partial \tau} + \frac{\partial}{\partial y} F(U) = 0$$

then $U(y,\tau) = W(y,\tau) = W(\alpha x, \alpha t) = W(x,t)$

W is constant on the rays $\frac{x}{t} = cst$

$\Rightarrow W(x,t) = H\left(\frac{x}{t}\right)$



$$\frac{x'}{t'} = \frac{\bar{x}}{\bar{t}}$$

THE RIEMANN PROBLEM

consider the initial value problem:

$$(RP)\quad \begin{cases} \dfrac{\partial W}{\partial t} + A\,\dfrac{\partial W}{\partial x} = 0 \\[2mm] W_0(x) = \begin{cases} W_L & \text{if } x < 0 \\ W_R & \text{if } x > 0 \end{cases} \end{cases}$$

Remark: $W_R - W_L = \sum_{k=1}^{p} (\beta_k - \alpha_k) r_k$

if $W_L = \sum_{k=1}^{p} \alpha_k r_k$ and $W_R = \sum_{k=1}^{p} \beta_k r_k$

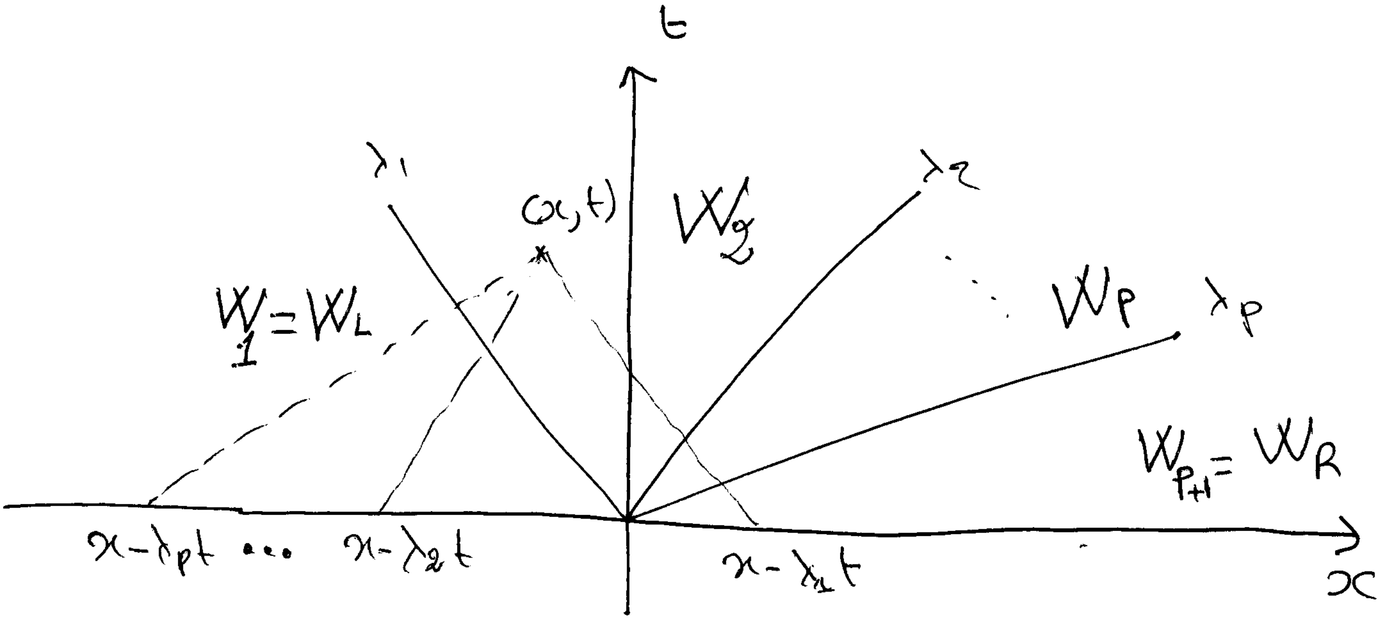
<u>Proposition:</u>

The solution of problem (RP) is made of constant states, separated by characteristic curves $C_k: \dfrac{x}{t} = \lambda_k$ in the frame (x,t).

The solution shows a Jump $[W]_k = (\beta_k - \alpha_k) r_k$ across the k-characteristic C_k.

λ_k is the speed of propagation of the discontinuity $[W]_k$.

Definition: The jump of the family k propagating at constante velocity λ_k, is called: k-wave.



Proof of The proposition:

Remark: $v_{0,k}(x) = \ell_k \sum_{j=1}^{p} \gamma_j(x)\, r_j$

with $\gamma_j(x) = \begin{cases} \alpha_j & \text{if } x < 0 \\ \beta_j & \text{if } x > 0 \end{cases}$

But $\ell_k . r_j = \delta_{kj} \Rightarrow v_k(x,t) = \gamma_k(x - \lambda_k t)$

and $W(x,t) = \sum_{k=1}^{p} v_k(x,t)\, r_k$

gives: $\forall (x,t) \,/\, \frac{x}{t} \neq \lambda_k$

$$W(x,t) = \sum_{\frac{x}{t} < \lambda_k} \alpha_k r_k + \sum_{\frac{x}{t} > \lambda_k} \beta_k r_k$$

THE PHASE FRAME

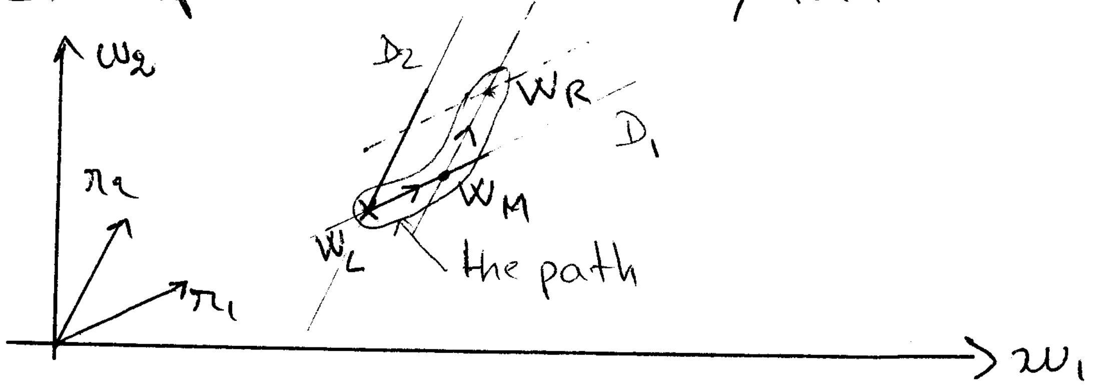
Remark: $$W(x,t) = W_L + \sum_{\frac{x}{t} > \lambda_k} (\beta_k - \alpha_k) r_k$$

$$= W_R + \sum_{\frac{x}{t} < \lambda_k} (\beta_k - \alpha_k) r_k$$

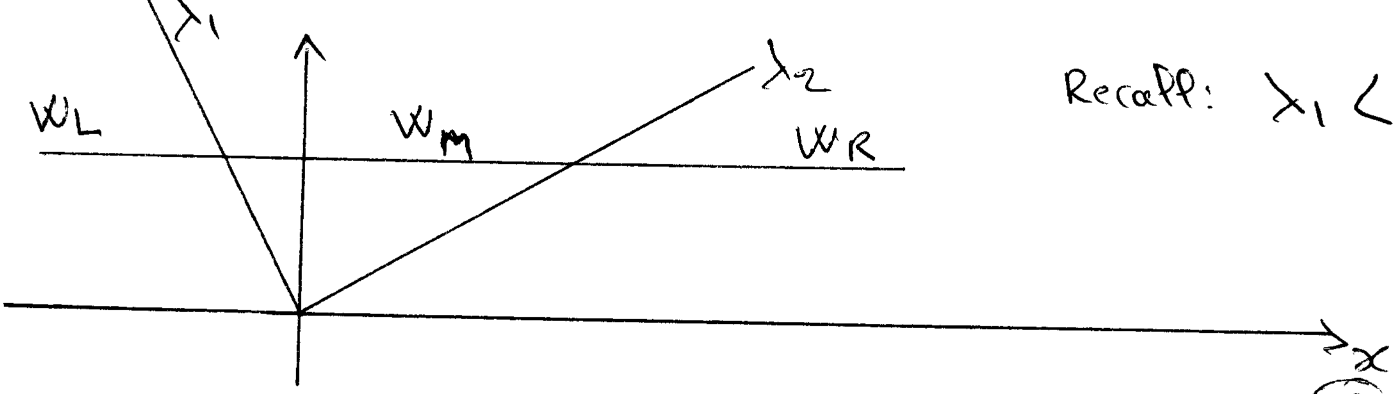
Remark: solving the R.P. consists in a decomposition of the initial discontinuity in several jumps

$$W_R = W_L + \sum_{k=1}^{p} (\beta_k - \alpha_k) r_k$$

Example with a 2×2 system:



Lines D_1 and D_2 give all the states that can be connected to W_L by a 1_wave on a 2_wave.



Recall: $\lambda_1 < \lambda_2$

NON LINEAR RIEMANN PROBLEM

consider the problem:

$$(NLRP)\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial}{\partial x} F(W) = 0 \\ W(x,0) = W_0(x) = \begin{cases} W_L & \text{if } x<0 \\ W_R & \text{if } x>0 \end{cases} \end{cases}$$

$F'(W) = A(W) \equiv$ strictly $\mathbb{R}$-diagonalisable

$\lambda_1(W) < \lambda_2(W) < \dots < \lambda_p(W) \quad \forall W$

Hugoniot locus

goal: construct a weak solution made of m discontinuities propagating at speeds: $s_1 < s_2 < \dots s_m$

consider the discontinuity $(\hat{W}, \tilde{W})$, speed s

The jump (Rankine-Hugoniot) condition:

$$F(\tilde{W}) - F(\hat{W}) = s(\tilde{W} - \hat{W})$$

m equations of $m+1$ unknowns $(\tilde{W} + s)$

$\longrightarrow$ a one-parameter family solution

Like in linear case, one writes:

$$\tilde{W}_{\ell} = \hat{W} + u \vec{r}_{\ell}$$

$$\begin{cases} \tilde{W}_{\ell}(u, \hat{W}) = \hat{W} + u \vec{r}_{\ell} \\ s_{\ell}(u, \hat{W}) \end{cases}$$

$\tilde{W}$ connected to $\hat{W}$ by a ℓ-wave.

<u>Remark</u>: $\tilde{W}(0, \hat{W}) = \hat{W}$

<u>Proposition</u>: The curve (Hugoniot Locus) $\tilde{W}_{\ell}(u, \hat{W})$ is tangent to the eigenvector $\vec{r}_{\ell}(\hat{W})$ at $\tilde{W} = \hat{W}$

<u>Proof</u>: R.H.C $\rightarrow F(\tilde{W}_{\ell}) - F(\hat{W}) = s_{\ell}(\tilde{W}_{\ell} - \hat{W})$

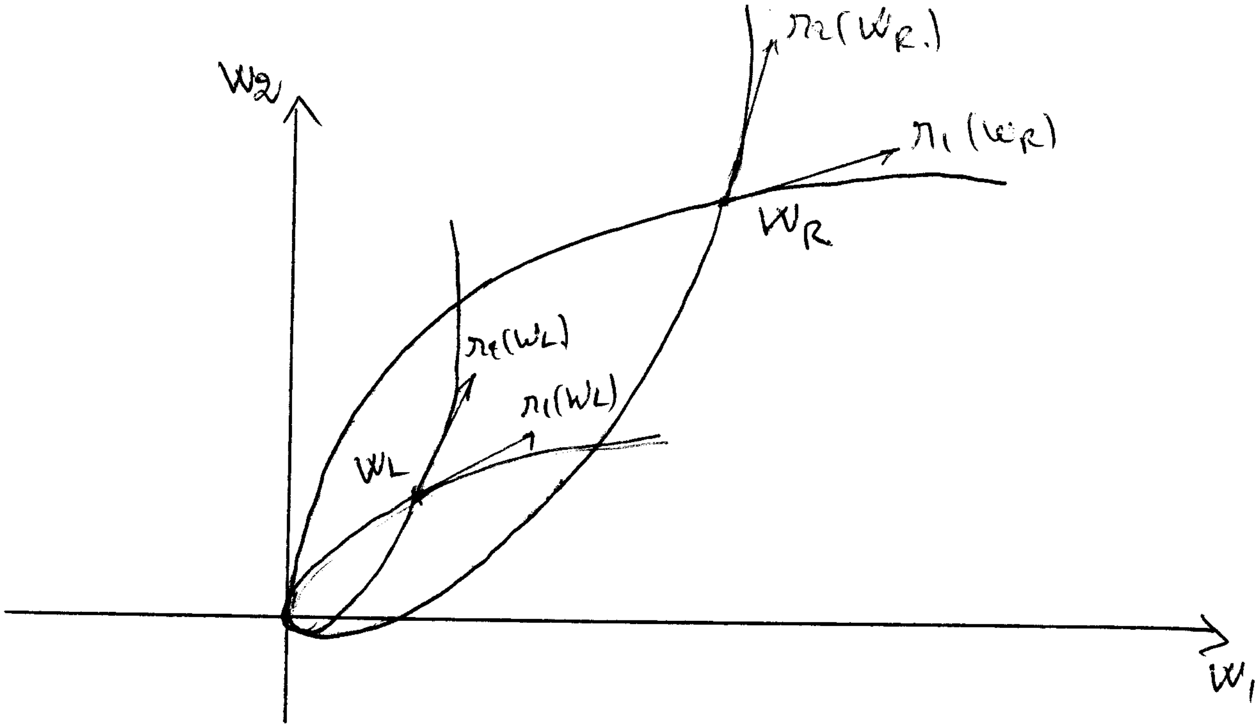
$$\frac{d}{du} \rightarrow A(\tilde{W}_{\ell}) \frac{d\tilde{W}_{\ell}}{du} = s_{\ell} \frac{d\tilde{W}_{\ell}}{du} + (\tilde{W}_{\ell} - \hat{W}) \frac{ds_{\ell}}{du}$$

$$u = 0 \rightarrow A(\hat{W}) \frac{d}{du} \tilde{W}_{\ell}(0, \hat{W}) = s_{\ell}(0, \hat{W}) \frac{d}{du} \tilde{W}_{\ell}(0, \hat{W})$$

Then $\frac{d\tilde{W}_{\ell}(0, \hat{W})}{du}$ is an eigenvector of $A(\hat{W})$ associated to an eigenvalue $s_{\ell}(0, \hat{W})$

but $\lambda_1 < \lambda_2 < \dots \lambda_p$ and $s_1 < s_2 < \dots < s_m$

$$\rightarrow s_{\ell}(0, \hat{W}) = \lambda_{\ell} \quad \text{and} \quad m = p.$$



w_2
$r_2(w_R)$
$r_1(w_R)$
W_R
$r_2(W_L)$
$r_1(W_L)$
W_L
w_1

SHALLOW WATER PROBLEM

EXACT SOLUTION FOR DAM BREAK

consider the system:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0 \\[2ex] \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{g}{2}h^2\right) = 0 \end{cases}$$

Note $q = hu$, $W = \begin{pmatrix} h \\ q \end{pmatrix}$, $F(W) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{g}{2}h^2 \end{pmatrix}$

One gets: $\dfrac{\partial W}{\partial t} + \dfrac{\partial}{\partial x} F(W) = 0$

then $F'(W) = A(W) = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}$

with $c = \sqrt{gh}$

So $A(W) = R(W)\,\Lambda(W)\,R^{-1}(W)$

$\lambda_1(W) = u - c \qquad \lambda_2(W) = u + c$

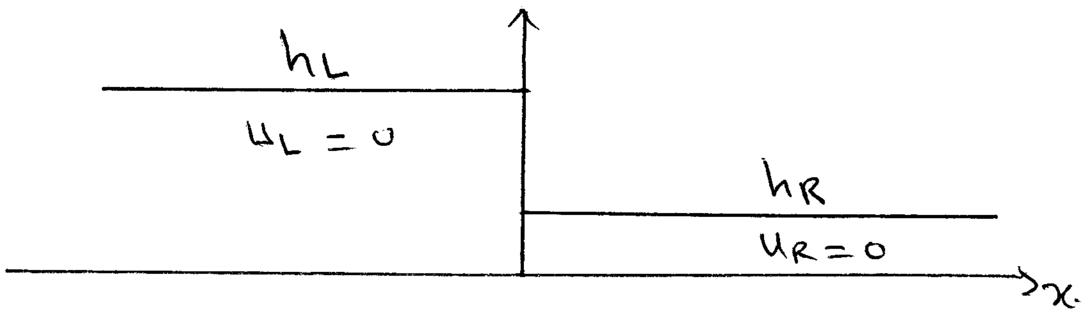
$r_1(W) = \begin{pmatrix} 1 \\ \lambda_1(W) \end{pmatrix} \qquad r_2(W) = \begin{pmatrix} 1 \\ \lambda_2(W) \end{pmatrix}$

Rankine - Hugoniot $\Longrightarrow$

$$[F(w)] = s[w]$$

$$\begin{cases} \tilde{q} - \hat{q} = s(\tilde{h} - \hat{h}) \\ \left(\dfrac{\tilde{q}^2}{\tilde{h}} + \dfrac{g}{2}\tilde{h}^2\right) - \left(\dfrac{\hat{q}^2}{\hat{h}} + \dfrac{g}{2}\hat{h}^2\right) = s(\tilde{q} - \hat{q}) \end{cases}$$

particular case : $\hat{u} \equiv 0 \longrightarrow \hat{q} = 0$

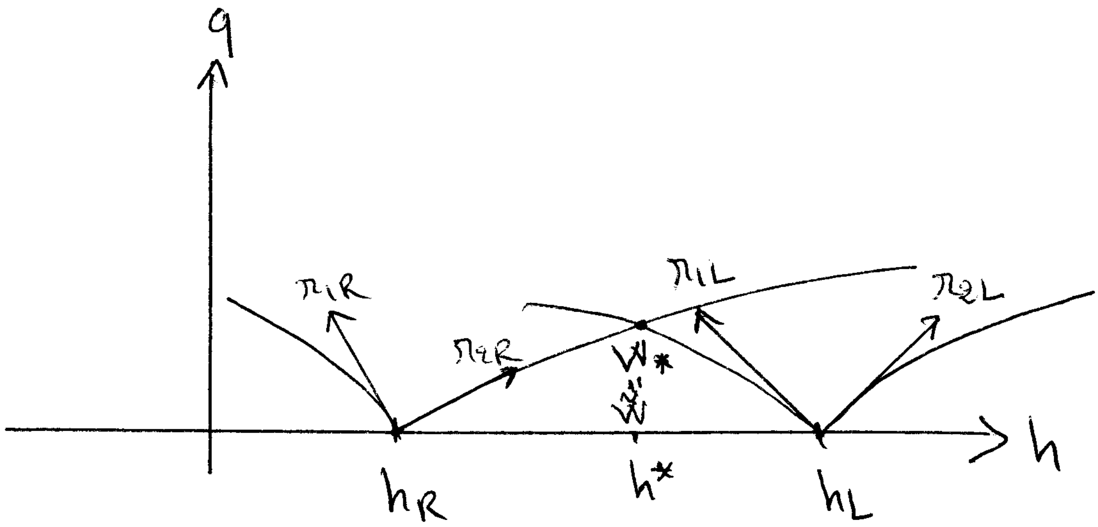


$$\tilde{q} = s(\tilde{h} - \hat{h}) \longrightarrow s = \frac{\tilde{q}}{\tilde{h} - \hat{h}}$$

$$\frac{\tilde{q}^2}{\tilde{h}} + \frac{g}{2}(\tilde{h}^2 - \hat{h}^2) = \frac{\tilde{q}^2}{\tilde{h} - \hat{h}}$$

$$\longrightarrow \tilde{q} = \pm(\hat{h} - \tilde{h})\,\tilde{h}\sqrt{\frac{g}{2}\left(\frac{1}{\hat{h}} + \frac{1}{\tilde{h}}\right)}$$

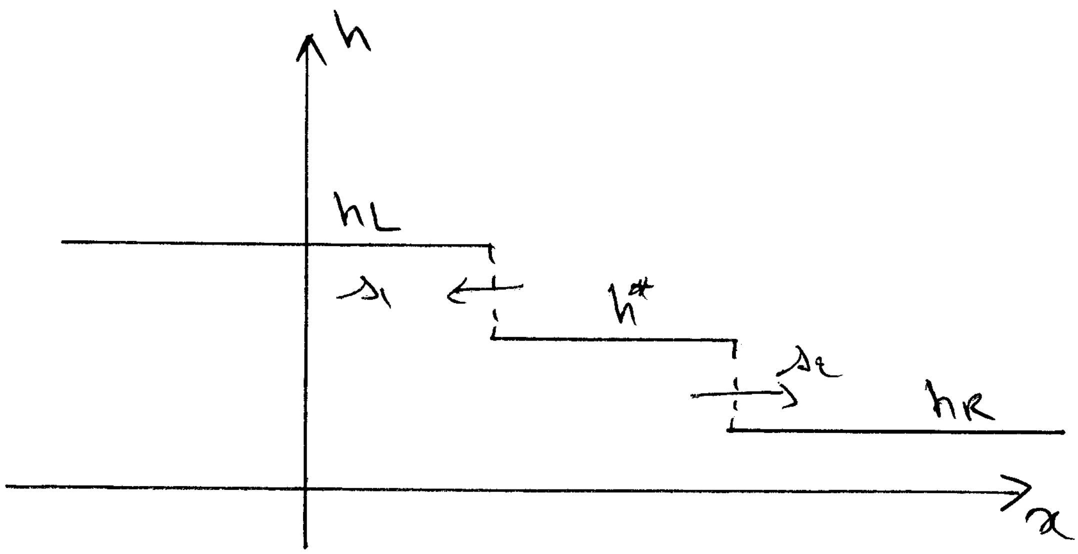
$$\boxed{\tilde{u} = \pm(\hat{h} - \tilde{h})\sqrt{\frac{g}{2}\left(\frac{1}{\hat{h}} + \frac{1}{\tilde{h}}\right)}}$$



Then:

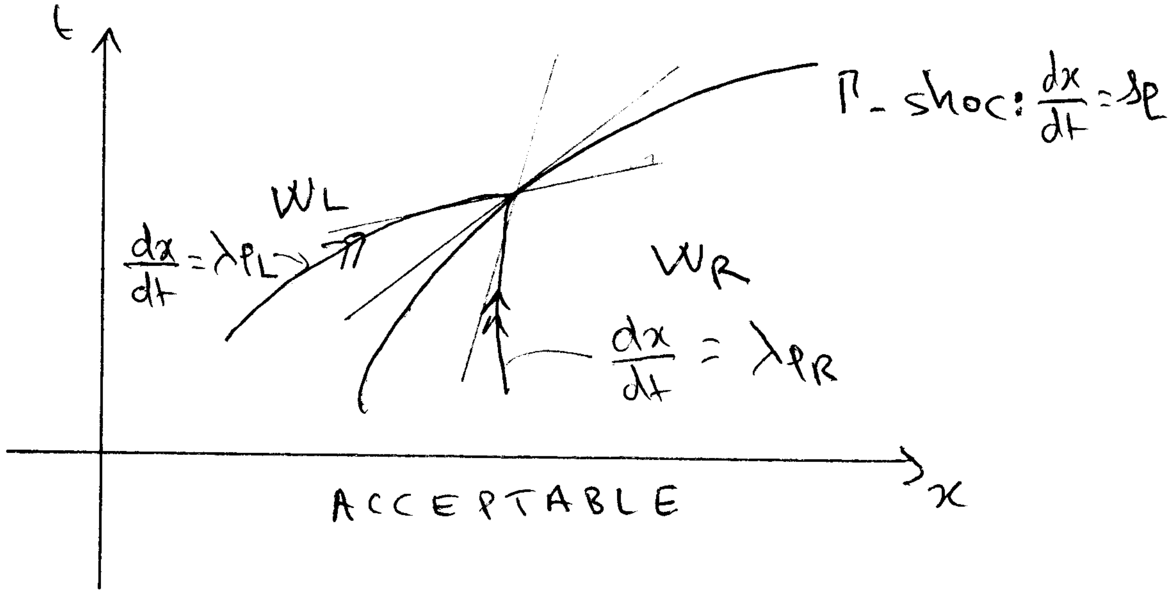
$$\begin{cases} u^* = (h_L - h^*)\sqrt{\dfrac{g}{2}\left(\dfrac{1}{h_L} + \dfrac{1}{h^*}\right)} \\ s_1 = \dfrac{-u^* h^*}{h_L - h^*} \end{cases}$$

$$\begin{cases} u^* = (h^* - h_R)\sqrt{\dfrac{g}{2}\left(\dfrac{1}{h_R} + \dfrac{1}{h^*}\right)} \\ s_2 = \dfrac{u^* h^*}{h^* - h_R} \end{cases}$$

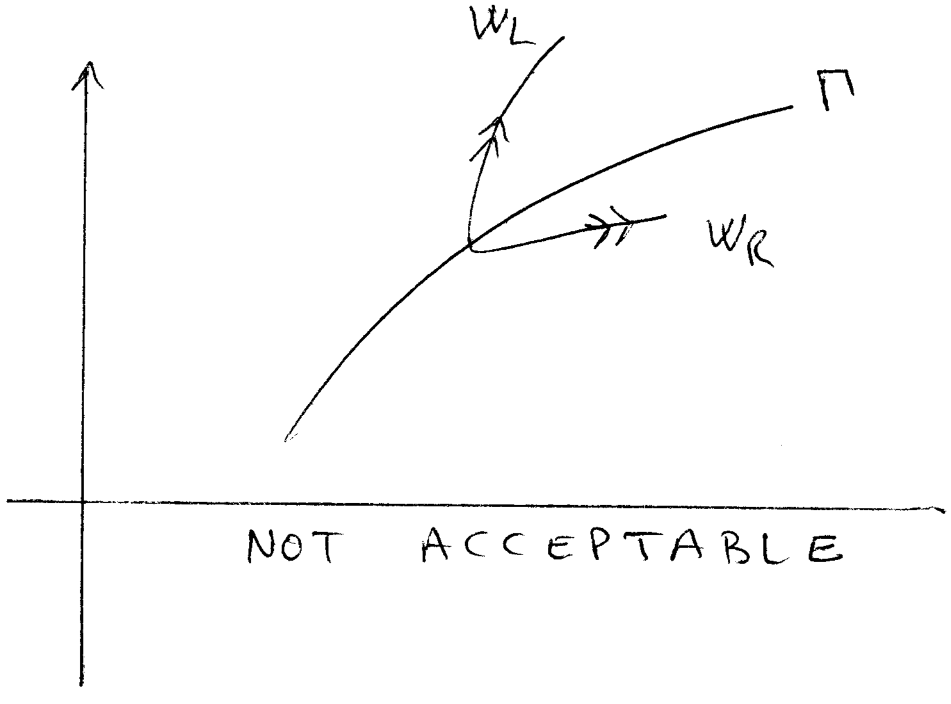


Proposition: (Lax Entropy condition)

A shoc in a ℓ-wave family is admissible (entropy satisfying) if:
$$\lambda_\ell(W_L) > s_\ell > \lambda_\ell(W_R)$$
$$\lambda_{\ell L}^{-1} < s_\ell^{-1} < \lambda_{\ell R}^{-1}$$



ACCEPTABLE



NOT ACCEPTABLE

Hence : the first shoc (W_L, W^*) is note acceptable

Indeed :

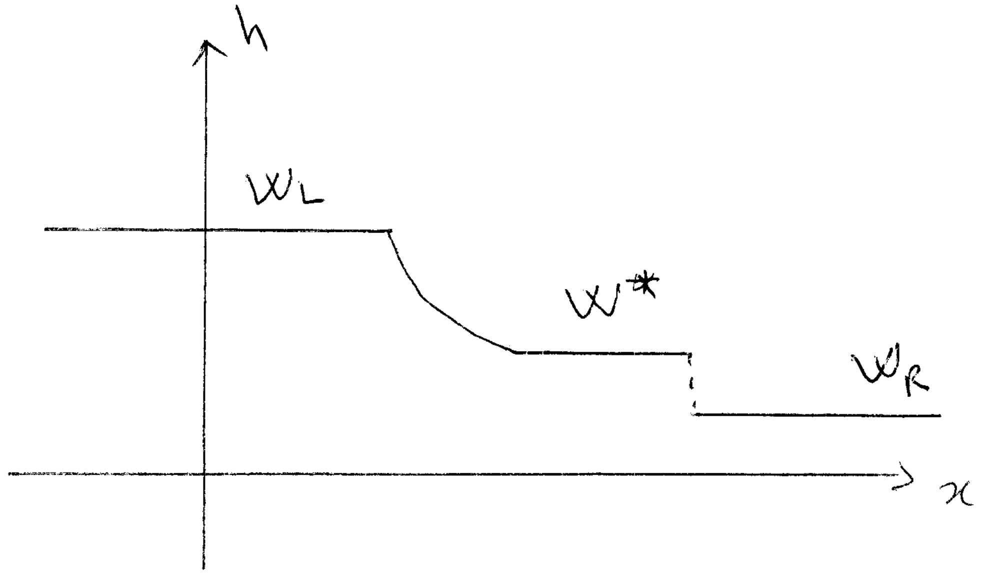
$$\lambda_1(W_L) > s_1 > \lambda_1(W^*) \Rightarrow$$

$$0 - \sqrt{gh_L} \overset{①}{>} \frac{-u^* h^*}{h_L - h^*} \overset{②}{>} u^* - \sqrt{gh^*}$$

$$① \Rightarrow \sqrt{gh_L} < h^* \sqrt{\frac{g}{2}\left(\frac{1}{h_L} + \frac{1}{h^*}\right)}$$

$$\Rightarrow \sqrt{gh_L} < \sqrt{gh^* \tfrac{1}{2}(1 + h^*/h_L)}$$

Impossible since $h^* < h_L$



RAREFACTION WAVE

<u>Proposition</u> : Let $W(x,t) = H\left(\frac{x}{t}\right)$ the solution of The R.P in a region where it is regular, then H is solution of the system:

$$H'(\xi) = \left[\nabla \lambda_p[\cancel{W}H(\xi)] \cdot \vec{r_p}[H(\xi)]\right]^{-1} \vec{r_p}[H(\xi)]$$

$$\xi_1 < \xi < \xi_2$$

$$H(\xi_1) = W_L$$

<u>Proof</u> : $\frac{\partial W}{\partial t} + A(W)\frac{\partial W}{\partial x} = 0 \Rightarrow$

$$-\frac{x}{t^2} H'\left(\frac{x}{t}\right) + A\left(H\left(\frac{x}{t}\right)\right)\left(\frac{1}{t}\right) H'\left(\frac{x}{t}\right) = 0$$

$\Rightarrow A[H(\xi)]\, H'(\xi) = \xi\, H'(\xi) \;/\; \xi = \frac{x}{t}$

$\Rightarrow H'(\xi)$ colinear to an eigenvector

$$\Rightarrow \begin{cases} H'(\xi) = \alpha(\xi)\, r_p[H(\xi)] & (1) \\ \xi = \lambda_p[H(\xi)] & (2) \end{cases}$$

$$\frac{d}{d\xi} (2) \Rightarrow 1 = \nabla \lambda_p[H(\xi)]\, H'(\xi)$$

$$(1) \Rightarrow 1 = \nabla \lambda_p[H(\xi)]\, \alpha(\xi)\, \vec{r_p}[H(\xi)]$$

So $\underline{\alpha(\xi) = [\nabla \lambda_p[H(\xi)] \cdot \vec{r_p}[H(\xi)]]^{-1}}$

Application to the Rarefaction
in the 1-wave family

$$\lambda_1(W) = u - c = \frac{q}{h} - \sqrt{gh} \;;\; r_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$$

$$\nabla \lambda_1 \cdot r_1 = -\frac{3}{2}\sqrt{\frac{g}{h}} \Rightarrow$$

$$\begin{cases} h'(\xi) = -\frac{2}{3}\sqrt{\frac{h}{g}} \\ q'(\xi) = -\frac{2}{3}\sqrt{\frac{h}{g}}\left(\frac{q}{h} - \sqrt{gh}\right) = \frac{2}{3}\left(h - \frac{q}{c}\right) \end{cases}$$

with $\xi_1 = \xi(W_L) = \lambda_1(W_L) = 0 - \sqrt{gh_L}$

$h(\xi_1) = h_L \quad q(\xi_1) = 0$

Solution of the O.D.E :

$$\begin{cases} h = \frac{1}{9g}\left(2\sqrt{gh_L} - \frac{x}{t}\right)^2 \\ u = \frac{2}{3}\left(\sqrt{gh_L} + \frac{x}{t}\right) \end{cases} \Big/ \frac{x}{t} = \xi$$

How to compute h^* and u^*?

Remark : $\xi_2 = \lambda_1(W^*) = u^* - \sqrt{gh^*}$

gives
$$\begin{cases} h^* = \dfrac{1}{9g}\left(2\sqrt{gh_L} + \sqrt{gh^*} - u^*\right)^2 \\ u^* = \dfrac{2}{3}\left(\sqrt{gh_L} - \sqrt{gh^*} + u^*\right) \end{cases}$$

$$\Rightarrow \quad u^* = 2\left(\sqrt{gh_L} - \sqrt{gh^*}\right) \qquad (1)$$

And from the 2-shoc (W^*, W_R)

$$u^* = (h^* - h_R)\sqrt{\frac{g}{2}\left(\frac{1}{h_R} + \frac{1}{h^*}\right)} \qquad (2)$$

(1) + (2) $\longrightarrow$ a non linear equation to solve, e.g. by Newton iterations

NON DIMENSIONALIZED FORM
OF THE S-W- Equations

note $\overline{W}$ the variables with dimension

$$(SWD)\begin{cases} \dfrac{\partial \bar h}{\partial \bar t} + \dfrac{\partial}{\partial \bar x}(\bar h \bar u) = 0 \\ \dfrac{\partial}{\partial \bar t}(\bar h \bar u) + \dfrac{\partial}{\partial \bar x}\left(\bar h \bar u^2 + \dfrac{g}{2}\bar h^2\right) = 0 \end{cases}$$

put $h = \bar h / h_L \iff \bar h = h_L . h$

$u = \bar u / \bar U$ with $\bar U = \sqrt{g h_L}$

$x = \bar x / h_L$ $\quad t = \bar t / t_0$ / $t_0 = \sqrt{\dfrac{h_L}{g}}$

Then (SWD) becomes

$$(SWND)\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0 \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{h^2}{2}\right) = 0 \end{cases}$$