THE FINITE VOLUME METHOD BASIS

The Finite Volumes Basis - F. Benkhaldoun

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Summary of the talk:

- Introduction
- Physical Models considered
- Exact solutions for Scalar Conservatin Laws - Shoks and Entropy
- Exact solutions for Systems of Conservatin Laws
- Application to Dam Break problem
- The Finite Volume Method for 1D Scalar problems
- The Finite Volume Method for 2D problems

The goal of this presentation is to propose some basic ideas behind the development of a complete finite volume code for the numerical simulation of multifimensional hydrodynamic and flow problems.
In particular, Finite Volume method relay on the hypothesis of a piecewise constant approximate solution. This implies that at each time level one has to solve a collection of local Riemann Problems.
This short course aims to explain the properties of these Riemann Problem, how to solve them both in the case of scalar conservation laws and the case of nonlinear hyperbolic systems, and from this to devise the general concept of conservative, stable and convergent Finite Volume schemes.

MAIN PHYSICAL MODELS

Non Linear Systems Considered

We are intersted by fluid flow problems described by such non linear systems:

$$[\partial W/\partial t] + [\partial F(W)/\partial x] + [\partial G(W)/\partial y] + [\partial H(W)/\partial z] = S(W) \tag{1}$$

Examples:

Homogeneous Hyperbolic systems

$$[\partial W/\partial t] + [\partial F(W)/\partial x] + [\partial G(W)/\partial y] + [\partial H(W)/\partial z] = 0 \quad (2)$$

Euler equations in one space dimension

$$\begin{cases} [\partial\rho/\partial t] & + \quad [\partial(\rho u)/\partial x] & = 0 \\ [\partial(\rho u)/\partial t] & + \quad \left[\partial\left(\rho u^2 + P\right)/\partial x\right] & = 0 \\ [\partial E/\partial t] & + \quad [\partial\left[u\left(E + P\right)\right]/\partial x] & = 0 \end{cases} \qquad (3)$$

with the perfect gas equation of state:
$p = (\gamma - 1)\left(E - [1/2]\,\rho u^2\right)$, where ρ is the fluid density, u the velocity, E the energy and p the pressure.

We consider water flow in a configuration where the water depth is neglectible when compared to the characteristic length of the domain. [4]).
If the bottom is flat, and the friction neglectible, the problem is described by the following system:

$$\left\{ \begin{array}{lcll} [\partial h/\partial t] & + & [\partial (hu)/\partial x] & = 0 \\ [\partial (hu)/\partial x] & + & [\partial/\partial x] \left(hu^2 + [1/2]\, gh^2 \right) & = 0 \end{array} \right. \qquad (4)$$

h being the water depth, u the velocity, and g the gravity constant.

Exact solutions and properties for 1D scalar problems

Consider the 1D scalar problem:

$$\begin{cases} [\partial u/\partial t] + [\partial f(u)/\partial x] = 0 \quad \text{in } \mathbb{R}\times]0,T[\\ u = u(x,t) \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{5}$$

In the sequel, note $X = \mathbb{R} \times [0, T[$.

Example, Burger's equation: $[\partial u/\partial t] + [\partial/\partial x]\left(\left[u^2/2\right]\right) = 0$

Fundamental theorem

Theorem

The 3 following assertions are equivalent:

i) u is a weak solution of problem (5) , i.e:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\left[\partial\varphi/\partial t\right] + f(u)\left[\partial\varphi/\partial x\right]\right) dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)dx = 0,$$

$\forall \varphi \in D\left(\mathbb{R} \times [0,+\infty[\right)$

ii) $\forall R = [x_1, x_2] \times [t_1, t_2] \subset \Omega = \mathbb{R} \times [0, T]$,

$$\int_{\partial R} \left[u.n_t + f(u).n_x\right] d\sigma = 0$$

Theorem

iii) If u is C^1, *u is classical solution of* $[\partial u/\partial t] + [\partial/\partial x] f(u) = 0$, *and on a shoc curve* $\Gamma(u_l, u_r)$, *the solution is governed by the jump condition:* $[f(u)] = s[u]$.
One defines the jump $[u] = u_r - u_l$, *and the curve* $\Gamma(u_l, u_r)$, *which equation is:* $[dx/dt] = s$, *separates the left and right states* u_l *and* u_r

The jump condition is called the Rankime-Hugoniot condition in gas dynamics.

Remark

- *i) says that we are considering solutions in the distribution sens*
- *iii) is used to develop exact solutions*
- *ii) is the basic property to write finite volume schemes*

Smooth solution

If $u \in C^1(X)$, one has: (5) $\Longrightarrow [\partial u/\partial t] + f'(u)\,[\partial(u)/\partial x] = 0$ then in the frame (x, t), u is constant on the characteristic curve given by:

$$\begin{cases} [dx(t)/dt] = f'\,[u\,(x(t), t)] \\ \\ x(t = 0) = x_0 \end{cases} \tag{6}$$

One deduce the solution u:

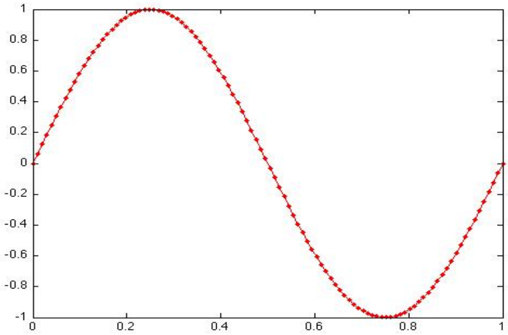
$$u\,(x(t), t) = u\,(x(0), 0) = u\,(x_0, 0) = u_0(x_0)$$

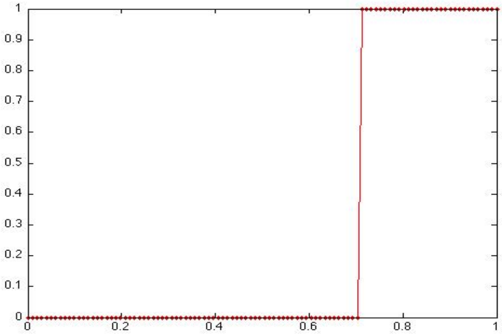
1. Linear case

$$f(u) = cu : [\partial u/\partial t] + c\,[\partial u/\partial x] = 0 \Longrightarrow a(u) = c$$

characteristic curve: $x = x_0 + tc \Leftrightarrow x_0 = x - ct$

Solution: $u(x,t) = u_0(x - ct)$

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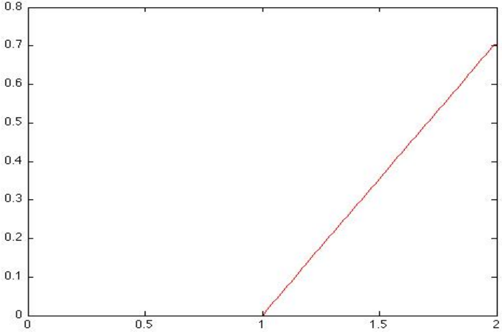
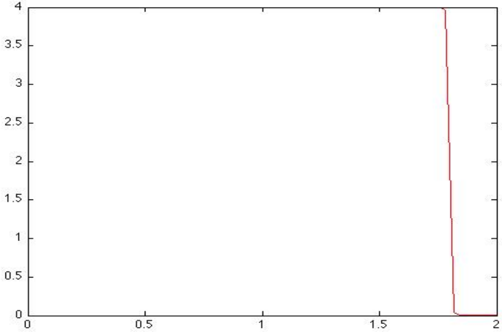
2. Burger's equation

Consider the non linear equation:
$[\partial u/\partial t] + [\partial/\partial x]\left[u^2/2\right] = 0,$
here $f(u) = \left[u^2/2\right]$, then $a(u) = f'(u) = u$
The characteristic curve is given by: $x = x_0 + tu_0(x_0)$

Continuous solution of Burgers equation



Discontinuous solution of Burgers equation



Non Unicity of weak solutions

Example: Consider the Burger's equation:

$$[\partial u/\partial t] + [\partial/\partial x]\left[u^2/2\right] = 0 \qquad (7)$$

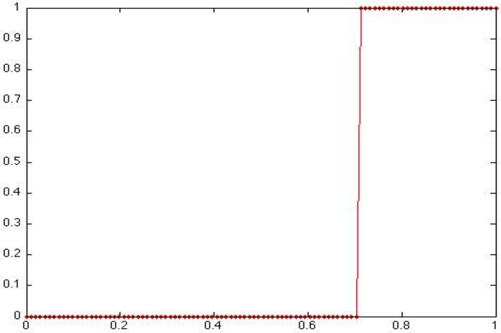
and the initial condition : $u_0(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$

First possibiliy: a weak solution with the shoc $\Gamma(0,1)$.

The jump condition gives:

$[f(u)] = s\,[u] \Longrightarrow \left[\left[u_r^2/2\right] - \left[u_l^2/2\right]\right] = s\,[u_r - u_l] \Longrightarrow s =$

$[1/2] \Longrightarrow u(x,t) = \begin{cases} 0 & \text{si} \quad [x/t] < [1/2] \\ 1 & \text{si} \quad [x/t] \geq [1/2] \end{cases}$



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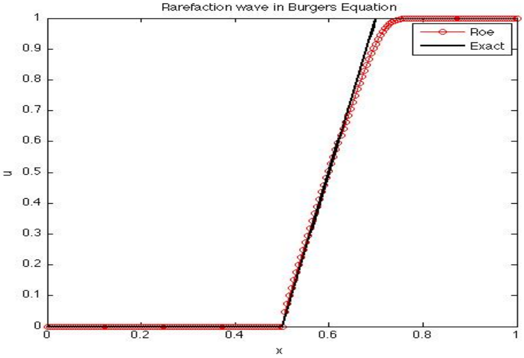
Second possibility: a continuous weak solution.

$$u(x,t) = \begin{cases} 0 & \text{si} \quad [x/t] < 0 \\ [x/t] & \text{si} \quad 0 \leq [x/t] < 1 \\ 1 & \text{si} \quad [x/t] \geq 1 \end{cases}$$



We come to the fact that one needs a specific criterium to select, among the above two weak solutions, the unique physical one.

The entropy solution

Definition

A smooth convex function U, is said to be an entropy of the problem, if there exists an entropy flux F such that: $U'(u)f'(u) = F'(u)$.

Definition

a weak solution u of (5) is said entropy solution if $\forall \varphi \in D\left(\mathbb{R}\times]0,T]\right)$:

$$\int_0^T \int_{\mathbb{R}} \left(U(u)\left[\partial\varphi/\partial t\right] + F(u)\left[\partial\varphi/\partial x\right]\right) dxdt \geq 0,$$ where U is an entropy of the problem, and F its entropy flux.

Theorem

(Kruzkov 1970) Under some regularity assumptions on u_0*, there exists a unique entropy weak solution of problem* (5).

Proposition

A piecewise C^1 function u, is an entropy weak solution of (5) if and only if:
i) u is a classical solution in (x, t) regions where u is C^1
ii) On an shoc curve Γ, u satisfies $[F(u)] \leq s\,[U(u)]\,, \forall\,(U, F)$ a couple of entropy and antropy flux.

Corollaire

1) If f is strictly convex, then a shoc is entropic if and only if:
$f'(u_r) < s < f'(u_l)$

Corollaire

2) If f is strictly convex, then a shoc is entropic if and only if:
$u_r < u_l$

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Application : the first weak solution in example (7) is non entropic, and hence non admissible.
The second weak solution is the unique entropy solution.

Linear systems

Let A be a constant square matrix of ordre p, $W_0 \in L^\infty(\mathbb{R})^p$, and the system:

$$\begin{cases} [\partial W/\partial t] + A\,[\partial W/\partial x] &= 0 \\ W(x,0) &= W_0(x) \end{cases} \tag{8}$$

We assume that the system is strictly hyperbolic (i.e. A is $\mathbb{R}$-diagonalizable and has p distinct eigenvalues).

Hyperbolic Systems

Hyperbolic system means $A = R\Lambda R^{-1}$ with
$\Lambda = Diag(\lambda_1, ..., \lambda_j, ..., \lambda_p)$ and $\lambda_1 < \lambda_2 < ...\lambda_j < ... < \lambda_p$.
$R = [r_1, ..., r_j, ..., r_p]$ is the eigenvectors matrix, i.e.:

$$A\, r_j = \lambda_j\, r_j$$

The rows l_i of the inverse matrix R^{-1} are the left eigenvalues of the system, i.e.:

$$l_i\, A = \lambda_i\, l_i, \quad \text{and} \quad l_i \,.\, r_j = \delta_{ij}$$

and

$$R^{-1} = \begin{bmatrix} & l_1 & \\ . & . & . \\ & l_i & \\ . & . & . \\ & l_p & \end{bmatrix}$$

Proposition

The solution of the linear system is given by:

$$W(x,t) = \sum_{j=1}^{p} [l_j \cdot W_0(x - \lambda_j t] \cdot r_j.$$

Proof: Let's make the change of variable

$$V = R^{-1} W, \qquad (v_j = l_j W).$$

Thus,

$$W = RV = \sum_{j=1}^{p} v_j r_j.$$

The linear system becomes

$$
\begin{aligned}
[\partial V/\partial t] + \Lambda\,[\partial V/\partial x] &= 0, \qquad\qquad x \in \mathbb{R}, \quad t > 0,\\
V(x,0) &= R^{-1} W_0(x) = V_0(x), .
\end{aligned}
$$

where

$$v_j(x,t) = v_j(x - \lambda_j t, 0) = l_j W_0(x - \lambda_j t),$$

and

$$W(x,t) = RV(x,t) = \sum_{j=1}^{p} v_j(x,t) r_j.$$

Proposition

The solution of the Riemann problem

$$\begin{aligned} [\partial W/\partial t] + [\partial/\partial x]\, F(W(x,t)) &= 0, \\ W(x,0) &= W_0(x) = \begin{cases} W_L, & \text{if} \quad x \leq 0, \\ W_R, & \text{if} \quad x > 0, \end{cases} \end{aligned}$$

is self-similar i.e.,

$$W(x,t) = H\left([x/t]\right).$$

Proof:

For $\alpha > 0$ let $y = \alpha x$ and $\tau \alpha t$. Let

$$U(y,\tau) = W(x,t) = W\left([y/\alpha], [\tau/\alpha]\right).$$

Remark that

$$U(y,\tau = 0) = W_0\left([y/\alpha]\right) = \begin{cases} W_L, & \text{if} \quad y \leq 0, \\ W_R, & \text{if} \quad y > 0, \end{cases},$$

$$[\partial U/\partial \tau] = [1/\alpha]\,[\partial W/\partial t], \quad \text{and} \quad [\partial F(U)/\partial y] = [1/\alpha]\,[\partial F(W)/\partial x].$$

Hence,

$$[\partial U/\partial \tau] + [\partial/\partial y]\, F(U) = 0.$$

Thus,

$$U(y,\tau) = W(y,\tau) = W(\alpha x, \alpha t) = W(x,t),$$

and W is constant on the rays $[x/t] = cst$,

$$W(x,t) = H\left([x/t]\right)$$

The Riemann Problem

Consider the initail value problem:

$$\begin{aligned} [\partial W/\partial t] + A\,[\partial W/\partial x] &= 0, \\ W(x,0) &= W_0(x) = \begin{cases} W_L, & \text{if} \quad x \leq 0, \\ W_R, & \text{if} \quad x > 0, \end{cases} \end{aligned} \tag{9}$$

Note that if $W_L = \sum_{k=1}^{p} \alpha_k r_k$ and $W_R = \sum_{k=1}^{p} \beta_k r_k$

$$W_R - W_L = \sum_{k=1}^{p} (\alpha_k - \beta_k)\, r_k.$$

Proposition

The solution of the problem (9) is made of constant states, separated by characteristic curves $C_k : [x/t] = \lambda_k$ *in the frame* (x, t)*. The solution shows a jump*

$$[W]_k = (\beta_k - \alpha_k)\, r_k,$$

across the k*-characteristic* C_k*.* λ_k *is the speed of propagation of the discontinuity* $[W]_k$ *(also called k-wave).*

Proof:

Remark that $v_{0,k}(x) = l_k . \sum_{j=1}^{p} \gamma_j(x)\, r_j$,

where $\gamma_j(x) = \alpha_j$ if $x < 0$, and $\gamma_j = \beta_j$ if $x > 0$.

But $l_k . r_j = \delta_{k,j}$, hence $v_k(x,t) = \gamma_k(x - \lambda_k\, t)$.

And as $W(x,t) = \sum_{k=1}^{p} v_k(x,t)\, r_k$,

Finaly $\forall (x,t)$ such that $[x/t] \neq \lambda_k$:

$$W(x,t) = \sum_{[x/t]<\lambda_k} \alpha_k\, r_k + \sum_{[x/t]>\lambda_k} \beta_k\, r_k$$

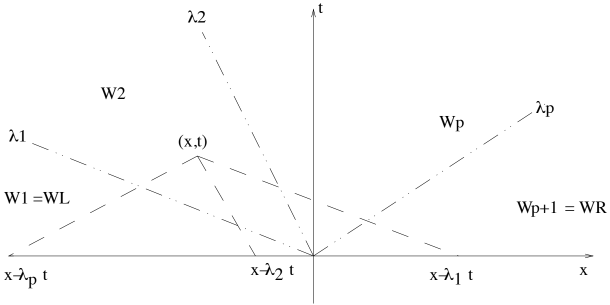


Figure: Riemann Problem Solution

Phase frame

One has:

$$W(x,t) = W_L + \sum_{[x/t] > \lambda_k} (\beta_k - \alpha_k)\, r_k,$$

or

$$W(x,t) = W_R + \sum_{[x/t] < \lambda_k} (\beta_k - \alpha_k)\, r_k,$$

Remark: Solving the Rieman Problem consists in a decomposition of the initial discontinuity into several jumps:

$$W(x,t) = W_L + \sum_{1}^{P} (\beta_k - \alpha_k)\, r_k,$$

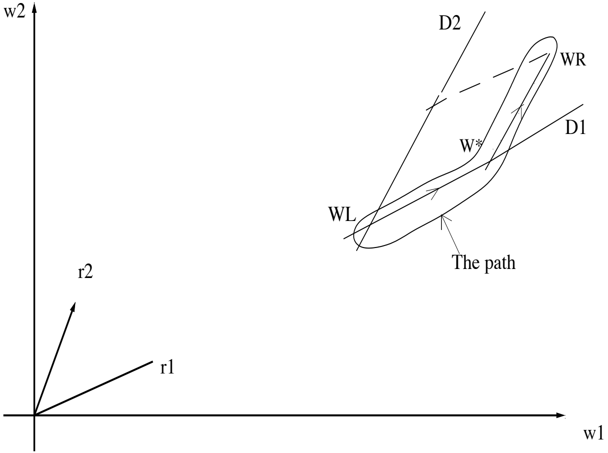


Figure: Riemann Problem Solution in Phase frame

Lines $D1$ and $D2$ give the location of all the states that can be connected to W_L by a 1-wave or a 2-wave family.

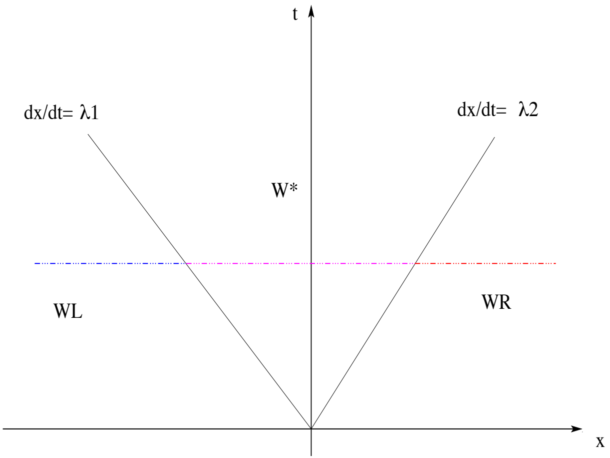


Figure: Riemann Problem Solution in (x,t) frame

Non Linear Riemann Problem

Consider the problem:

$$\begin{cases} [\partial W/\partial t] + [\partial F(W)/\partial x] & = & 0 \\ W(x,0) & = & W_0(x) \\ & = & \begin{cases} W_L, \text{if} \quad x \leq 0, \\ W_R, \text{if} \quad x > 0, \end{cases} \end{cases}$$

We assume that $F'(W) = A(W)$ is strictly diagonisable in $\mathbb{R}$, with: $\lambda_1(W) < \lambda_2(W) < ... < \lambda_p(W)$.

Hugoniot Locus

Goal: Construct a weak solution made of m discontinuities propagationg at speeds:
$s_1 < s_2 < ... < s_m$.

Consider the discontinuity $(\hat{W}, \tilde{W})$ with speed s.

The jump condition (Rankine-Hugoniot) writes:
$F(\hat{W}) - F(\tilde{W}) = s(\hat{W} - \tilde{W})$

This gives m equation with $m + 1$ unknowns.

$\longrightarrow$ A one-parameter family solution.

In analogy with the linear case, one writes:

$$\tilde{W}_l = \tilde{W}_l(u, \hat{W}) = \hat{W} + u r_l, \quad u \in \mathbb{R}$$

$$s_l = s_l(u, \hat{W})$$

$\tilde{W}_l$ is connected to $\hat{W}$ by a l-wave.

Proposition

The curve (Hugoniot Locus) $\tilde{W}_l$ is tangent to the vector r_l at $W = \hat{W}$.

Proof: Remark that $\tilde{W}_l(0, \hat{W}) = \hat{W}$, then derive the Rankine Hugoniot relation with respect to u and put $u = 0$.
One gets $m = p$, $s_l(0, \hat{W}) = \lambda_l$, and $\left[d\tilde{W}/du\right] = \alpha r_l(\hat{W})$.

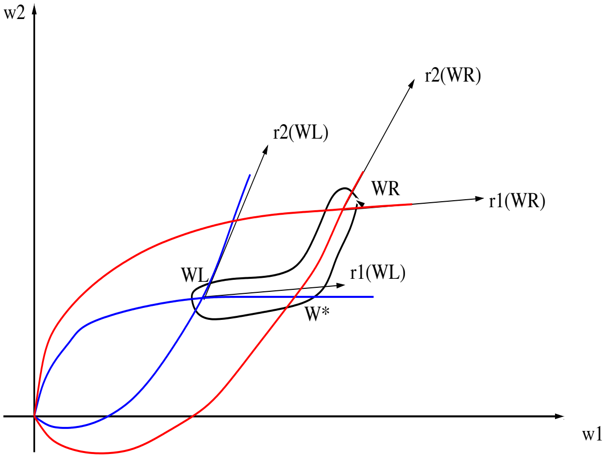


Figure: Non Linear Riemann Problem Solution in phase frame

Application: exact solution for Dam break problem

$$\begin{cases} [\partial h/\partial t] & + \quad [\partial (hu)/\partial x] & = 0 \\ [\partial (hu)/\partial x] & + \quad [\partial/\partial x]\left(hu^2 + [1/2]\, gh^2\right) & = 0 \end{cases} \qquad (10)$$

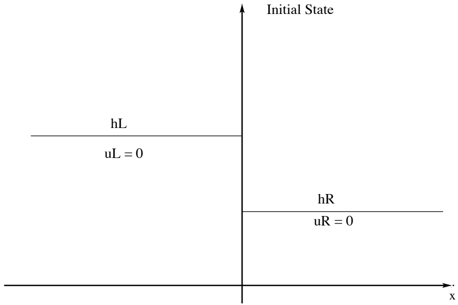


Figure: Initial condition for Dam break problem

Here $r_1(W) = [1, \lambda_1(W)]^T$, $r_2(W) = [1, \lambda_2(W)]^T$, where $\lambda_1(W) = u - c$ and $\lambda_2(W) = u + c$.

The Rankine Hugoniot relation gives:

$$\tilde{u} = \epsilon(\hat{h} - \tilde{h})\sqrt{[g/2]\left(\left[1/\hat{h}\right] + \left[1/\tilde{h}\right]\right)},\ \text{where } \epsilon = 1 \text{ or } \epsilon = -1.$$

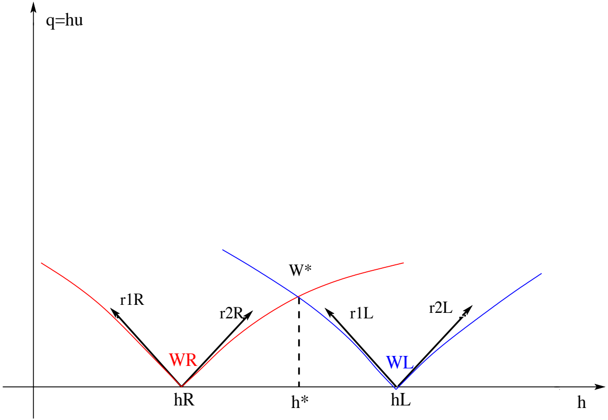


Figure: Dam Break Solution in phase frame based on Rankine Hugoniot relation

One then gets:

$$u^\star = (h_L - h^\star)\sqrt{[g/2]\left([1/h_L] + [1/h^\star]\right)}$$

$$s_1 = [-u^\star h^\star/(h_L - h^\star)]$$

$$u^\star = (h^\star - h_R)\sqrt{[g/2]\left([1/h_R] + [1/h^\star]\right)}$$

$$s_2 = [u^\star h^\star/(h^\star - h_R)]$$

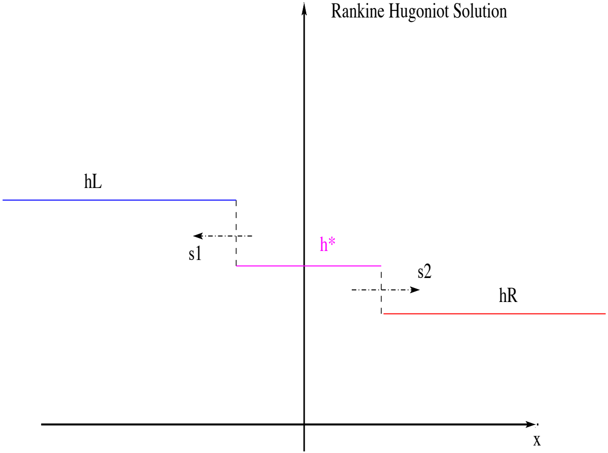


Figure: Dam Break Solution in phase frame with the hypothesis of two shocks

Lax Entropy Condition

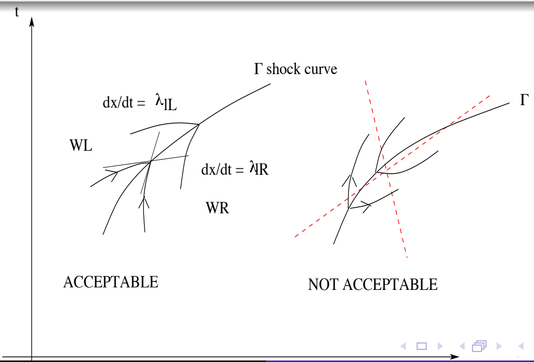
Proposition

A shock in a l-wave family is admissible if:

$$\lambda_l(W_L) > s_l > \lambda_l(W_R)$$

or

$$\lambda_l^{-1}(W_L) < s_l^{-1} < \lambda_l^{-1}(W_R)$$



Rarefaction Wave

Proposition

Let $W(x,t) = H([x/t])$ the solution of the Riemann Problem in a smooth region. Then H is the solution of the following ODE:

$$H'(\mu) = [\lambda_l(H(\mu)).r_l(H(\mu))]^{-1} \cdot r_l(H(\mu))$$

$$\mu_1 < \mu < \mu_2$$

$$H(\mu_1) = W_L$$

Application to SW system gives the solution:

$$h = [1/9g]\left(2\sqrt{gh_L} - [x/t]\right)^2$$

$$u = [2/3]\left(\sqrt{gh_L} + [x/t]\right)$$

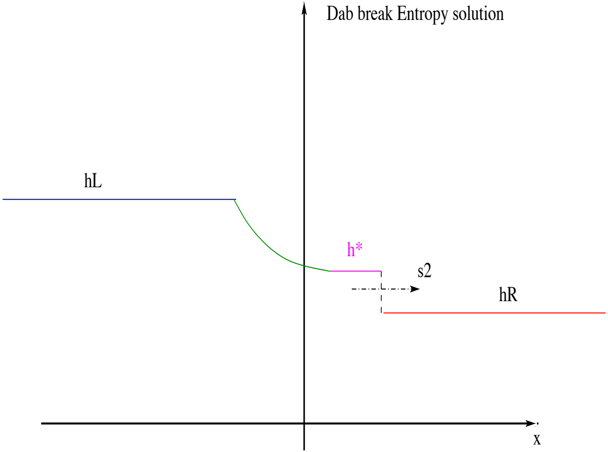
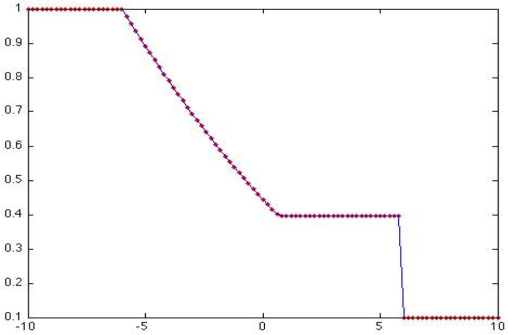


Figure: Dam Break Entropy Solution



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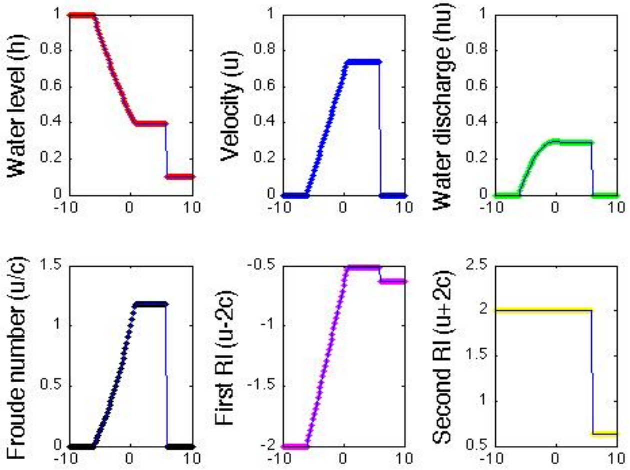
Water level (h)
Velocity (u)
Water discharge (hu)
Froude number (u/c)
First RI (u-2c)
Second RI (u+2c)
1
0.8
0.6
0.4
0.2
0
1.5
0.5
-0.5
-1
-1.5
-2
2.5
2
-10
10

Riemann problem and self similar solution

Definition

We call Riemann problem a system of a scalar conservation law and a discontinuous initial condition:

$$\begin{cases} [\partial u/\partial t] + [\partial f(u)/\partial x] = 0 \\ u_0(x) = \left| \begin{array}{lll} u_l & \text{si} & x < 0 \\ u_r & \text{si} & x > 0 \end{array} \right. \end{cases} \tag{11}$$

Lemma

The solution of the Riemann problem is self similar. i.e.: there exists a function g such that $u(x,t) = g([x/t])$.

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proof of the lemma:

consider the change of variables: $X = \alpha x$, and $\tau = \alpha t$, $\alpha > 0$

One has then $u(x,t) = u\left(x(X,\tau), t(X,\tau)\right) = U(X,\tau)$

One shows that the fuction u and U are solutions of the same problem, and hence $U(X,\tau) = u(\alpha x, \alpha t) = u(x,t)$, which gives $u(x,t) = g([x/t])$.

Proposition

The solution of the Riemann problem writes: $u(x,t) = g([x/t])$, *with* $u(x,t) = u_l$ *for* $[x/t] < \alpha_1$, *and* $u(x,t) = u_r$ *pour* $[x/t] > \alpha_2$.

Remark

At the location of the original discontinuity $x = 0$, *the solution does not depend upon time:* $u(0,t) = g(0) = Riem(u_l, u_r)$.

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We discretize the spacial domain: $\mathbb{R}$: $x_j = jh$, $j \in \mathbb{Z}$, and the temporal domain: $[0, T[$: $t_n = n\tau$.
Let $u_j^n = u^n(x_j)$ be the approximate solution of u at (x_j, t_n),.
Introduce the piecewise constant function $u_\tau(x,t) = u^n(x)$ if $t \in T_n = [t_n, t_{n+1}[$, where $u^n(x) = u_j^n$ if $x \in I_j =]x_{j-[1/2]}, x_{j+[1/2]}[$.
Recall that if u_τ is a weak solution of the problem, one has (property *ii*)):

$$\forall R \subset \mathbb{R} \times [0, T] : \quad \int_{\partial R} [u_\tau . n_t + f(u_\tau) . n_x] \, d\sigma = 0$$

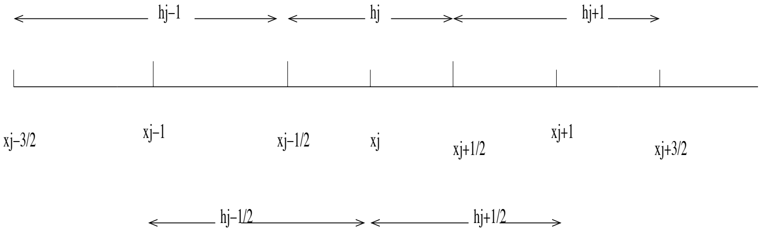

Figure: 1D discretization

Using $R = R_j^n = I_j \times T_n$ and the approximation:

$$u_j^{n+1} = [1/h] \int_{x_{j-[1/2]}}^{x_{j+[1/2]}} u_\tau(x, t_{n+1}) dx$$

one gets:

$$u_j^{n+1} = u_j^n - [\tau/h] \left[f\left(Riem(u_j^n, u_{j+1}^n)\right) - f\left(Riem(u_{j-1}^n, u_j^n)\right) \right]$$

or

$$u_j^{n+1} = u_j^n - r \left[g^G(u_j^n, u_{j+1}^n) - g^G(u_{j-1}^n, u_j^n) \right]$$

g^G is the numerical flux of Godunov.

Definition

A finite volume scheme is said under conservative form when it can be written:

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A conservative finite volume numerical scheme is consistent if $g(u_{j-q+1}, ..., u_{j+q})$ tends to $f(u)$ when u_{j+i} tends to u, whith $-(q-1) \leq i \leq q$.

Definition

The numerical flux g is said Lipschitz continuous if $|(g(v_{j-q+1}, ..., v_{j+q}) - f(v)| < K \max_{-(q-1)\leq i\leq q} |v_{j+i} - v)|$

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Fundamental Lax-Wendroff theorem

Theoreme

Considere $X = \mathbb{R} \times [0, T[$

Suppose S_τ *is conservative with a Lipschitz continuous flux and*

$u^0 = [1/h] \int_{I_j} u_0(x)dx$, *if:*

i) $\|u_\tau\|_{L^\infty(X)} \leq C$

ii) $u_\tau \longrightarrow u$, *when* $\tau \longrightarrow 0$, *almost everyware (a.e.) in* $L^1(X)$

then u is a weak solution of the problem.

Lemme

The set $BV(X) = \left\{v \in L^1(X) \text{ such that } VT(v) < R, \text{ and } supp(v(.,t)) \subset [-A, A] \subset \mathbb{R}\right\}$, *is a compact subset of* $L^1(X)$.

Definition

The scheme S_τ is TV-stable if $\exists\ \tau_0 > 0, /\ \forall\ \tau < \tau_0,\ u_\tau \in BV(X)$.

Remarque

$$VT(u_\tau) = \sum_{n=0}^{T/\tau} \sum_{j=-\infty}^{+\infty} \left(\tau \left|u_{j+1}^n - u_j^n\right| + h \left|u_j^{n+1} - u_j^n\right|\right)$$

Proposition

If S_τ *is TV-stable, then it is convergent. i.e.* $u_\tau \longrightarrow u$ *a.e. in* $L^1(X)$.

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If S_τ *is TV-stable, then it is convergent. i.e.* $u_\tau \longrightarrow u$ *a.e. in* $L^1(X)$.

Proposition

A convergent conservative finite volume scheme with lipschitz continuous numerical flux, which is entropy consistent, converges to the unique entropy solution.

Incremental form of a finite volume scheme

Definition

A finite volume scheme is under incremental form if one can writes:

$$u_j^{n+1} = u_j^n + C_{j+[1/2]}^n \Delta u_{j+[1/2]}^n - D_{j-[1/2]}^n \Delta u_{j-[1/2]}^n \quad (12)$$

with:

$$\Delta u_{j+[1/2]}^n = u_j^{n+1} - u_j^n$$

Proposition

Under the conditions: $\forall j \in \mathbb{Z}, \forall n \geq 0$

$$C^n_{j+[1/2]} \geq 0, \quad D^n_{j+[1/2]} \geq 0 \tag{13}$$

$$C^n_{j+[1/2]} + D^n_{j+[1/2]} \leq 1 \tag{14}$$

The finite volume scheme under incremental form is TVD stable.

Proposition

Under the conditions: $\forall j \in \mathbb{Z}, \forall n \geq 0$

$$C^n_{j+[1/2]} \geq 0, \quad D^n_{j-[1/2]} \geq 0 \tag{15}$$

$$C^n_{j+[1/2]} + D^n_{j-[1/2]} \leq 1 \tag{16}$$

The finite volume scheme under incremental form is L_∞ *stable.*

FINITE VOLUMES FOR 2D PROBLEMS

$$W_t + F(W)_x + G(W)_y - \tilde{F}(W)_x - \tilde{G}(W)_y = S(W)$$
$$W = (h, hu, hv, hC)^T$$
$$F(W) = (hu, hu^2 + \frac{g}{2}h^2, huv, huC)^T$$
$$G(W) = (hv, huv, hv^2 + \frac{g}{2}h^2, hvC)^T \tag{17}$$
$$\tilde{F}(W) = (0, 0, 0, hD_x\frac{\partial C}{\partial x})^T \quad \tilde{G}(W) = (0, 0, 0, hD_y\frac{\partial C}{\partial y})^T$$
$$S(W) = (0, gh(S_{ox} - S_{fx}), gh(S_{oy} - S_{fy}), 0)^T$$

where g is the gravity acceleration. S_{ox}, S_{oy} and S_{fx}, S_{fy} are respectively the bed slopes and friction terms. C means the average pollutant concentration, D_x and D_y are dispersion coefficients considered equal in the sequel.

The integration of equation (17) is done over a finite volume T_i. We will denote by ∂T_i the boundary of cell T_i, by Γ_{ij} the interface between cells T_i and T_j, and $E(i)$ is the set of triangles that have common edge with volume T_i.

The Finite Volume method

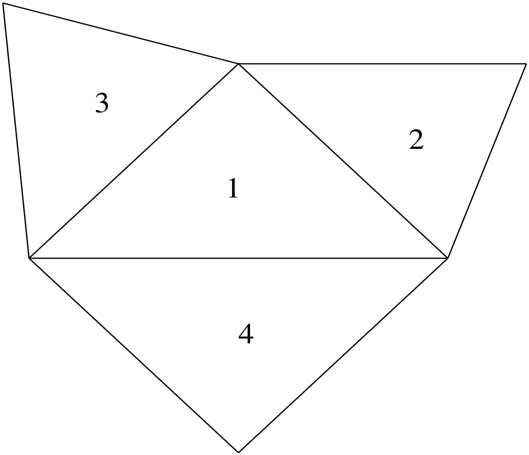
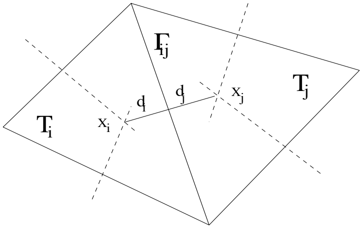


Figure: cell and neighbors

The Finite Volume method



Let us write:

$$\begin{aligned}\mathcal{F}(W,\vec{n}) &= n_x F(W) + n_y G(W),\\ \tilde{\mathcal{F}}(W,\vec{n}) &= n_x \tilde{F}(W) + n_y \tilde{G}(W),\end{aligned}$$

where (n_x, n_y) are components of the outward unit normal to ∂T_i.
Due to Green's formula, this leads to:

$$\text{Meas}(T_i)\frac{\partial W_i}{\partial t} + \int_{\partial T_i} \mathcal{F}(W,\vec{n})\, d\sigma - \int_{\partial T_i} \tilde{\mathcal{F}}(W,\vec{n})\, d\sigma = \int_{T_i} S(W)\, dV \tag{18}$$

one has to evaluate the convection and diffusion flux $\mathcal{F}(W,\vec{n})$, $\tilde{\mathcal{F}}(W,\vec{n})$ over the three borders of the cell T_i.

Let us split the boundary of the cell T_i in a union of partial boundaries associated to each edge.

$$\partial T_i = \bigcup_{j \in E(i)} \Gamma_{ij}$$

eqn(18) gives at time t_{n+1},

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{A(T_i)} \left\{ \sum_{j \in E(i)} \left(\int_{\Gamma_{ij}} \mathcal{F}(W, \vec{n})\, d\sigma - \int_{\Gamma_{ij}} \tilde{\mathcal{F}}(W, \vec{n})\, d\sigma \right) \right\}$$

$$- \frac{\Delta t}{A(T_i)} \int_{T_i} S(W)\, dV$$

Convection flux approximation

We seek an approximation of

$$\int_{\Gamma_{ij}} \mathcal{F}(W, \vec{n})\, d\sigma \;=\; \Phi(W_i, W_j, \vec{n}_{ij})\, meas(\Gamma_{ij})$$

where Φ is the numerical flux, W_i and W_j are respectively the values of W at cells T_i and T_j.
P.L. Roe proposed a particular choice of Φ based upon the resolution of approximate linear Riemann problems,

$$\begin{aligned} \Phi(W_i, W_j, \vec{n}_{ij}) &= \frac{1}{2}[\mathcal{F}(W_i, \vec{n}_{ij}) + \mathcal{F}(W_j, \vec{n}_{ij})] \\ &- \frac{1}{2}|A^*(W_i, W_j, \vec{n}_{ij})|(W_j - W_i) \end{aligned} \qquad (19)$$

with specific requirements about the matrix A^*.

Second Order using MUSCL method

The scheme described above is first order accurate for the convective part. It can be easily extended to second order accuracy upon non-structured meshes, by using MUSCL technique intoduced by Van Leer[3]. We can split this technique in two steps: First, in order to increase the accuracy of the scheme, one approximates the state W in the set of linear piecewise functions. At the interface Γ_{ij}, we define left and right states given by linear interpolation,

$$
\begin{aligned}
W_{ij}^{-} &= W_i + \frac{1}{2}\vec{\nabla} W_i . \overrightarrow{G_i G_j}, \\
W_{ij}^{+} &= W_j - \frac{1}{2}\vec{\nabla} W_j . \overrightarrow{G_i G_j},
\end{aligned}
\tag{20}
$$

where $\vec{\nabla}$ denotes the gradient operator, G_i and G_j are respectively the barycenters of cells T_i and T_j.

Second Order using MUSCL method

The remaining problem is to evaluate the gradient upon the cell considered. In our case, $\dfrac{\partial W_i}{\partial x}$ and $\dfrac{\partial W_i}{\partial y}$ are evaluated as the minimum points of the following quadratic function,

$$\Psi_i(X, Y) = \sum_{j \in K(i)} \left| W_i + (x_j - x_i) X + (y_j - y_i) Y - W_j \right|^2$$

where $K(i)$ is the indices set of neigbourhood triangles that have common edge or vertex with the triangle T_i, (x_i, y_i) are barycenter coordinates of cell T_i.

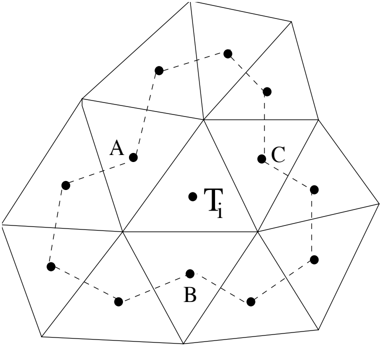


Figure: Neighboring cells for the Least Square approximation in MUSCL method

Unfortunatly, this method does not garantie the monotonicity preserving of the scheme, to overcome this difficulty, one uses limitations techniques. A general two-dimensional MinMod limiter is obtained by,

$$\frac{\partial^{\lim} W_i}{\partial x} = \frac{1}{2}\left[\min_{j\in K(i)} \; sgn\left(\frac{\partial W_j}{\partial x}\right) + \max_{j\in K(i)} \; sgn\left(\frac{\partial W_j}{\partial x}\right)\right] \min_{j\in K(i)} \left|\frac{\partial W_j}{\partial x}\right|$$

$\frac{\partial^{\lim} W_i}{\partial y}$ is evaluated in the same way. Then, interpolated left and right values are obtained by replacing in eqn(20) $\vec{\nabla} W_i$ and $\vec{\nabla} W_j$ respectively with $\vec{\nabla}^{\lim} W_i$ and $\vec{\nabla}^{\lim} W_j$. Afterward, Roe numerical flux is calculated using W_{ij}^- and W_{ij}^+.

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