ADAMS OPERATIONS IN TOPOLOGICAL HOCHSCHILD HOMOLOGY

GRÉGORY GINOT

ABSTRACT. The aim of this paper is to define and study Adams operations on topological Hochschild homology. They are analogous to the standard Adams operations defined in Hochschild homology of a ring. We compare them with classical operations and prove that they are compatible with the product structure and the standard Bökstedt spectral sequence.

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Adams operations on the Hochschild (co)homology of a commutative algebra were first defined by Gerstenhaber and Schack [13] and Loday [20]. These operations, often called λ-operations, yield an algebraic structure, due to Grothendieck, called a γ-ring. They proved to be very usefull in Hochschild and cyclic homology theory. For instance they provide a nice spliting of Hochschild homology groups by (higher) André-Quillen ones in characteristic zero. In the general case they induce an interesting filtration closely related to other well known theories such as Harrison homology. Moreover most of these results extend to the cyclic homology of a commutative algebra and its variants.

In the mid 1980's, Bökstedt (following ideas of Goodwillie) defined and studied topological Hochschild homology $THH(F)$ of a functor F with smash product ([2],

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[3]) mainly to extend standard Hochschild homology to the case of rings up to homotopy. Later on, Bökstedt, Hsiang and Madsen [4] defined topological cyclic homology, a topological generalisation of negative cyclic homology. These theories have been intensively studied and applied for computations of algebraic K-theory.

Since the pioneering work of Bökstedt, many constructions of categories of spectra equipped with smash product having nice associative and commutative properties at the space level (and not only the homotopy level) have arised, see [12], [24], [31], [21], [29] for example. The aim was to provide a good setting for "brave new algebras", that is to say topological generalisations of algebraic constructions to homotopy ring spaces. The main goal of this paper is to extend the standard algebraic λ-operations in Hochschild homology to topological Hochschild homology in this context, to study some of their properties and compare them with their classical level. Loday [20] construction of Adams operations makes sense not only for commutative algebras but also for any functor F from the category of finite sets to the category of modules over a commutative ring k. McCarthy [23] gave a geometric description of Loday's λ-operations which we follow to define analogous operations Φ^k on topological Hoschild homology.

It is to be noted that many constructions of topological Hochschild homology are avalaible in many competiting categories of structured spectra. Each of them have their own advantages. For example, Bökstedt's topological Hochschild homology construction is, so far, the only one that leads to a nontrivial topological cyclic homology theory. On the other hand, the very natural model in the category of S-algebras given in [12] leads to relative version of topological Hochschild homology (that is over a "ground ring" which is not the sphere spectrum) generalizing the algebraic ones. Thus it is important to have Adams operations for the different models that are equivalent when pasing to stable homotopy groups.

In the first three Sections we work in the framework of Γ-spaces and $\mathbb{S}$-algebras first introduced by Segal [31] and developped in [21], [29]. It is a very quick way to get a strict monoidal symmetric category homotopy equivalent to (connective) ring spectra. For example Eilenberg-Mac Lane spectra are easy to define in this category. Moreover we mainly work with Bökstedt initial THH model for topological Hochschild homology in these sections. There is two main reasons for that: first it is the model which leads to topological cyclic homology and we hope that the Adams operations we build can be generalized to that context someday. Moreover this situation has been widely studied and is wellsuited in the context of linear categories and Mac Lane homology; we use this in subsections 3.3, 3.4 . More precisely, in Section 1 we make some recollections on $\mathbb{S}$-algebras. In Section 2 we build the Adams operations shows that they are multiplicative and start studying the induced γ-ring structure on homotopy groups. In section 3 we compare these operations with standard operations on Hochschild homology of discrete rings and with some operations introduced by McCarthy in Mac Lane homology. There is a classical spectral sequence, originally due to Böktedt,

$$HH_*(H_*(A, F_p)) \Rightarrow H_*(THH(A), F_p)$$

computing the topological Hochschild homology. We show that it is a spectral sequence of γ-rings. It gives both a way to compute Adams operations and an additionnal structure that should respect this usefull spectral sequence. Also we give the construction of operations on the thh model of [12] in order to state relative results.

A very different category of ring spectra was constructed in Elmendorff, Kriz, Mandell and May monograph [12]: the category of S-modules. In section 4, we give Adams operations on the thh model in this category. It is the model which is the closer from the algebraic standard complex computing Hochschild homology for flat rings. We also show that another classical spectral sequence from [12] computing $\pi_*(thh(A))$ is a spectral sequence of γ-rings. The Harrison homology is related to the filtration of the Hochschild homology induced by the Adams operations. Adams operations on the *thh*-model lead to a definition of topological Harrison homology which is related to topological Hocshchild homology in a way similar to the algebraic relationship. Finally we compare our constructions to the one introduced by McClure, Schwänzl and Vogt on the model $A \otimes S^1$.

An important feature of the various constructions of both, strict monoidal symmetric categories of ring spectra and models for topological Hochschil homology is that they give the same result when passing to homotopy. The Appendix A is dedicated to recall this result in details together with the fact that our various Adams operations constructions give the same result in the homotopy category. For convenience of the reader we have recall a few basic facts about γ-rings in Appendix B.

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Notations: For the remainder of the paper we let *Fin* be the the category of finite sets $k_+ = \{0, 1, ..., k\}, k \geq 0$ and any maps. Following Segal [31], we let Γ be the category of finite pointed sets $k_+ = \{0, 1, ..., k\}, k \geq 0$ and pointed maps *i.e.* maps fixing 0 (actually, this category is the opposite of Segal's one). We use the notation Δ for the usual simplicial category and *Simp'* will be the category of simplicial pointed sets (also called spaces). Given a simplicial object X_* in a category, we denote $|X|$ its geometric realisation.

The category Top will be the category of compactly generated topological spaces (see [25]) and we will denote Δ_+^r the standard r-simplex with based point.

The capital letter K will usually stands for a field and the letter k for a commutative unital ring. Additionally, for any field K or commutative ring k and spectrum X, we will denote $H_*(X, K)$, $H_*(X, k)$ the spectrum homology $HK_*(X)$ and $Hk_*(X)$.

Throughout the paper we shall make no distinctions between the expressions "λ-operations" and "Adams operations". λ-operations on a ring without unit I are a family of operations $(\Phi^k : I \to I)_{k\geq 0}$ that induce a structure of a γ-ring on I. We refer to [1], [15] and [18] for definitions and properties of λ-rings and have collected a few definitions and results about them in appendix B.

When a, b are objects of a category $\mathcal{C}$, $\mathcal{C}(a, b)$ is the set of morphims from a to b.

1. A few facts about Γ-spaces and $\mathbb{S}$-algebras

Γ-spaces were first studied by Segal [31] (but our treatment is based on [21], [29]). A Γ-*space* X is a functor $X : \Gamma \to Simp'$. We write Γsp for the category of Γ-spaces. The *underlying space* of X is $X(1_+)$. The category Γ admits inner operations $\vee$ (wedge sum), $\wedge$ (smash product) defined by $k_+ \vee \ell_+ = (k + \ell)_+$ and $k_+ \wedge \ell_+ = (k\ell)_+$ (with lexicographic order).

A Γ-space X naturally extends into a functor *Simp'* $\to$ *Simp'*. It will still be denoted by X (and $\mathbb{P}X$ in the appendix; the only place where confusion could arise). First, X induces a functor X : *Sets'* $\to$ *Simp'* (where *Sets'* stands for the category of all pointed sets and pointed maps) defined by

$$X(E) = \operatorname*{colim}_{k_+ \to E} X(k_+).$$

Next, if $K \in$ *Simp'*, we let $X(K) \in$ *Simp'* be the simplicial space whose p-simplices are

$$X(K)_p = X(K_p)_p.$$

For $L \in$ *Simp'*, there is a natural application $X(K) \wedge L \to X(K \wedge L)$. Taking K to be the m-sphere S^m, we see that X induces a connective spectrum $X(S) : n \mapsto X(S^n)$. By definition, *the homotopy groups of* X are the homotopy groups of this spectrum, that is

$$\pi_k(X) = \pi_k(X(S)) = \operatorname*{colim}_n \pi_{k+n} X(S^n).$$

The smash product of two Γ-spaces X, Y was defined by Lydakis [21] as the following Γ-space:

$$X \wedge Y = \left(k_+ \mapsto \operatorname*{colim}_{m_+ \wedge n_+ \to k_+} X(m_+) \wedge Y(n_+)\right).$$

There is a particular Γ-space (called the sphere Γ-space) $\mathbb{S} : \Gamma \hookrightarrow$ *Simp'* defined by

$$k_+ \mapsto (n \mapsto k_+).$$

The induced extension *Simp'* $\xrightarrow{\mathbb{S}}$ *Simp'* is the identity and the associated spectrum is the sphere spectrum $n \mapsto S^n$.

The category $(\Gamma sp, \wedge, \mathbb{S})$ is a symmetric monoidal category for the smash product. A $\mathbb{S}$-*algebra* is a monoid in this category; hence a $\mathbb{S}$-algebra is a Γ-space A together with an associative product $\mu : A \wedge A \to A$ and a compatible unit $\eta : \mathbb{S} \to A$. The $\mathbb{S}$-algebra A is said to be commutative if $\mu \circ T = \mu$ (where T is the twist morphism). This means that the following diagram is commutative for all $k, \ell \geq 0$

$$\begin{array}{ccc} A(k_+) \wedge A(\ell_+) & \xrightarrow{\mu} & A(k_+ \wedge \ell_+) \\ T \downarrow & & \downarrow A(T') \\ A(\ell_+) \wedge A(k_+) & \xrightarrow{\mu} & A(\ell_+ \wedge k_+) \end{array}$$

where $T' : k_+ \wedge \ell_+ \to \ell_+ \wedge k_+$ is the isomorphism exchanging the two ordering relations.

Notice that a $\mathbb{S}$-algebra A also induces a connective functor with smash product (see [2] for example). For all $K, L \in$ *Simp'*, there is a natural product $A(K) \wedge A(L) \to A(K \wedge L)$ and the associated spectrum is a ring-spectrum.

Given a $\mathbb{S}$-algebra A, a left A-*module* is a Γ-space M together with an action $A \wedge M \to M$. A right A-module is a Γ-space M together with an action $M \wedge A \to M$. The coequalizer map

$$M \wedge A \wedge N \rightrightarrows M \wedge N \to M \wedge_A N$$

in the category Γsp of yields a smash product for a left A-module N and right A-module M, denoted by $M \wedge_A N$, for which A is a unit. Moreover when A is a commutative $\mathbb{S}$-algebra, the categories of left and right modules are isomorphic so that we can simply speak of A-modules. In that case, we denote by A-mod the category of A-modules. This category is symmetric. We call A-algebras the monoids in the symmetric monoidal category $(A\text{-mod}, \wedge_A, A)$. We will also need to consider bimodules. Such is a left A-module M together with a compatible right action of A.

2. Adams operations on Bökstedt model $THH(A)$

2.1. **Preliminaries on $THH(A)$.** We will follow Bökstedt [2]. However, as we mainly deal with commutative $\mathbb{S}$-algebras, we use Brun's indexing category which is more convenient to handle products and the operations we will construct in this framework (because the concatenation on the indexing categories $J(k)$ is strictly commutative). This does not change the homotopy type by Bökstedt [2] approximation lemma, see 2.1 below.

Following Brun [6], we denote J the category of finite subsets of $\{1, 2, \dots\}$ with morphims the maps which are injective. We still denote $x \vee y$ the (disjoint) union of two elements x, y of J. For $k \geq 1$, let $J(k)$ be the subcategory of J^k consisting of tuples of mutually disjoint subsets. We identify the sphere S^1 with the simplicial set $\Delta^1_+/\partial\Delta^1_+$ where Δ^1_+ is the standard 1-simplex and $\partial\Delta^1_+$ is its boundary. We denote $+$ its base point. For $y \in J$, the "y"-sphere is the simplicial set

$$S^y = Simp'(y \vee \{0\}, S^1_*)/\{\alpha_*/+ \in \alpha_k(y),\, k \geq 1\}.$$

More generally, for $x = (x_0, ..., x_q) \in J(q+1)$, we set $S^x = S^{x_0} \wedge ... \wedge S^{x_q}$.

Consider a $\mathbb{S}$-algebra A with associative product μ. For an object $x = (x_0, ..., x_q) \in J(q+1)$, we define $G(A, x)$ to be the Γ-space $k_+ \mapsto G(A, x)(k_+)$, where $G(A, x)(k_+)$ is the simplicial mapping space, that is to say the set whose p-simplices are given by

$$G(A, x)_p(k_+) = Simp'(S^{x_0} \wedge ... \wedge S^{x_q} \wedge \Delta^p_+ \,;\, \mathbb{TS}|A(S^{x_0}) \wedge ... \wedge A(S^{x_q}) \wedge k_+|)$$

where $\mathbb{TS}$ is the singular complex functor. Henceforth we will simply write $\mathrm{Map}(-;-)$ for the simplicial mapping space. When no confusion may occur, for $x \in J(q+1)$ we will simply write $S^x = S^{x_0} \wedge ... \wedge S^{x_q}$ and $A^{\wedge q+1}(S^x)$ for

$$A(S^{x_0}) \wedge A(S^{x_1}) \wedge ... \wedge A(S^{x_q}).$$

With these notations, $G(A, x)$ could be written as

$$k_+ \mapsto \mathrm{Map}(S^x;\ A^{\wedge q+1}(S^x) \wedge k_+).$$

This defines a functor $x \mapsto G(A, x)$ from $J(q+1)$ to Γ-spaces.

There is a cyclic Γ-space $THH(A)_* = (q \mapsto THH(A)_q)$ defined (*cf.* [2], [3], [6] by

$$THH(A)_q = \operatorname*{hocolim}_{x \in J(q+1)} G(A, x).$$

The cyclic structure of $THH(A)_*$ is given by faces $d_i : THH(A)_n \to THH(A)_{n-1}$, degeneracies $s_j : THH(A)_n \to THH(A)_{n-1}$ and cyclic permutations $t : THH(A)_n \to THH(A)_n$. The map t is induced by the cyclic permutation

$$\begin{array}{rcl} \tau : J(n+1) & \to & J(n+1) \\ (x_0, ..., x_n) & \mapsto & (x_n, x_0, ..., x_{n-1}). \end{array}$$

There is a functor $\partial_i : J(n+1) \to I^n$ given by

$$\partial_i(x_0, ..., x_n) = \begin{cases} (x_0, x_1 ..., x_i \vee x_{i+1}, ..., x_n) & \text{if } i < n \\ (x_n \vee x_0, x_1, ..., x_n) & \text{if } i = n. \end{cases}$$

The map $d_i : THH(A)_n \to THH(A)_{n-1}$ is the map induced by

$$\widetilde{d_i} : G(A, x) \to G(A, \partial_i(x))$$

defined, for every map $f \in G(A, x)$ and $i < n$, by

$$\widetilde{d_i}(f) = \mathrm{id}_{A(S^{x_0}) \wedge ... \wedge A(S^{x_{i-1}})} \wedge \mu \wedge \mathrm{id}_{A(S^{x_{i+1}}) \wedge ... \wedge A(S^{x_n})} \circ f \circ \gamma_i$$

where γ_i is induced by the isomorphism $S^{x_i \vee x_{i+1}} \cong S^{x_i} \wedge S^{x_{i+1}}$. For $i = n$,

$$\widetilde{d_n}(f) = (\mu \wedge \mathrm{id}_{A(S^{x_2}) \wedge ... \wedge A(S^{x_{n-1}})}) \circ t \circ f \circ \gamma_1 \circ \tau$$

Degeneracies are defined in a similar way. We denote $THH(A) = |THH(A)_*|$ the geometric realisation of $THH(A)_*$.

Now if M is a bimodule with left action $\ell : A \wedge M \to M$ and right action $r : M \wedge A \to M$, there is a simplicial Γ-space $THH(A, M)_*$ obtained by replacing the first factor $A(S^{x_0})$ by $M(S^{x_0})$ in the definition of $THH(A)_*$. We will write $THH(A,M)$ for its simplicial realization. Precisely

$$THH(A,M)_q = \operatorname*{hocolim}_{x\in J(q+1)} G(A,M,x)$$

where $G(A,M,x)$ is the Γ-space defined by

$$G(A,M,x)(k_+) = \mathrm{Map}(S^x \ ; \ M(S^{x_0}) \wedge A(S^{x_1}) \ldots \wedge A(S^{x_q}) \wedge k_+).$$

The simplicial structure is the same except for $\widetilde{d_n}(f)$, $\widetilde{d}_0(f)$ which have to be replaced by

$$\widetilde{d_n}(f) = (\ell \wedge \mathrm{id}_{A(S^{x_2})\wedge\ldots\wedge A(S^{x_{n-1}})}) \circ t \circ f \circ \gamma_1 \circ \tau,$$
$$\widetilde{d}_0(f) = (r \wedge \mathrm{id}_{A(S^{x_2})} \wedge \ldots \wedge A(S^{x_{n-1}})) \circ f.$$

Of course, if M is the canonical bimodule A, then $THH(A,M)_* = THH(A)_*$. The use of bimodules different from A is needed in Section 3.3.

We finish by recalling Bökstedt fundamental approximation Lemma [2].1.5 (also see [6].2.5.2).

Lemma 2.1 (Böksdedt). *If $F : \mathcal{C} \to Simp'$ is a functor from a monoidal category to simplicial pointed sets such that the unit for the monoidal structure is an initial object and every morphism $c_1 \to c_2$ in $F_i\mathcal{C}$ is $\lambda(i)$-connected, then the inclusion map*

$$F(c) \to \operatorname*{\mathit{hocolim}}_{c\in\mathcal{C}} F$$

is $\lambda(i) - 1$-connected.

This lemma in particular implied that the $THH(A)$ model using the indexing category J is weakly equivalent to the one using the category I, see [6].4.5. We recall the argument here because it applies to any monoidal category and to the "limits of a mapping space like" functor used in Proposition A.6. Recall that I is the full subcategory of J with objects $n_+, n \in \mathbb{N}$. There is a functor $\widetilde{G}(A,-)$: $J(q+1) \times I^{q+1} \to \Gamma sp$ given by

$$k_+ \mapsto \widetilde{G}(A,x,y) = \mathrm{Map}(S^x \times S^y;\ A(S^{x_0} \wedge S^{y_0}) \wedge \ldots A(S^{x_q} \wedge S^{y_q}) \wedge k_+)$$

and simplicial natural transformations

$$THH(A)_q \to \operatorname*{hocolim}_{(x,y)\in J(q+1)\times I^{q+1}} \widetilde{G}(A,x,y) \leftarrow \left(k_+ \mapsto \operatorname*{hocolim}_{y\in I^{q+1}} \mathrm{Map}(S^y; A^{q+1}(S^y) \wedge k_+)\right)$$

induced by the inclusions $J^{q+1} \to J(q+1) \times I^{q+1} \leftarrow I^{q+1}$. The approximation lemma 2.1 implies that the two natural transformations are weak equivalences hence the equivalence between $THH(A)$ and Bökstedt model.

2.2. **McCarthy's operations φ^k.** Recall that there is a functor $\mathrm{sd}_r : \Delta \to \Delta$ (the edgewise subdivision functor) defined by

$$\mathrm{sd}_r([n]) = [n] \cup [n] \cup \ldots \cup [n] \quad (r \text{ factors}).$$

When X is a simplicial Γ-space (*a fortiori* a cyclic one) we denote $\mathrm{sd}_r(X) = X \circ \mathrm{sd}_r$. We identify the simplex Δ_+^{rn-1} with the r-fold join of Δ_+^{n-1} with itself. There is a map $D_r :|\, \mathrm{sd}_r(X)\,| \to |\, X\,|$ induced by

$$\begin{array}{ccc} X_{rn-1} \times \Delta_+^{n-1} & \longrightarrow & X_{rn-1} \times \Delta_+^{rn-1} \\ (x,u) & \mapsto & (x, \frac{u}{r} \oplus \ldots \oplus \frac{u}{r}) \quad (r \text{ factors}) \end{array}$$

which is a homeomorphism (*cf.* [4] Lemma 1.1).

We now recall the construction of a natural system from McCarthy [23]. Such is a topological analog of a γ-ring. *A natural system* on X is a family of simplicial maps $(\varphi^k : \mathrm{sd}_k(X) \to X)_{k\geq 0}$ such that $\varphi^0 = *$, $\varphi^1 = \mathrm{id}$ and the diagram (2.2.1) is commutative up to homotopy

$$\begin{tikzcd} \mathrm{sd}_{rs}(X) \arrow[r, "\mathrm{sd}_s(\varphi^r)"] \arrow[dr, "\varphi^{rs}"'] & \mathrm{sd}_s(X) \arrow[d, "\varphi^s"] \\ & X. \end{tikzcd} \qquad (2.2.1)$$

For the remainder of the paper we will denote Φ^r the composite map $\Phi^r = \mid \varphi^r \mid \circ D_r^{-1}$. A natural system is said to be *cyclic* if $\varphi^r_{n-1} \circ t = t \circ \varphi^r_{n-1}$. A map of natural system is a simplicial transformation $f : X \to Y$ that commutes with the family $(\varphi^k)_{k\geq 0}$ up to homotopy:

$$\begin{tikzcd} \mathrm{sd}_k(X) \arrow[r, "\varphi^k"] \arrow[d, "\mathrm{sd}_k f"'] & X \arrow[d, "f"] \\ \mathrm{sd}_k(Y) \arrow[r, "\varphi^k"] & Y. \end{tikzcd}$$

There exists a functor $\Theta : \Delta^{op} \to \Gamma$ (*cf.* [20]) which is the identity on the objects and satisfies $\Theta(f)(i) = j$ if there exists j such that $f(j-1) < i \leq f(j)$ and is 0 if not. Henceforth a Γ-space is considered a simplicial space by Θ. This functor Θ obviously factors through the category *Fin* of finite sets.

McCarthy [23] gave a "universal"construction of natural systems Γ-spaces as follows. We define a family of applications $\varphi^r_n \in Hom_\Gamma(rn + r - 1, n)$ by setting $\varphi^r_n(p) = p$ modulo $n+1$.

Lemma 2.2. (McCarthy [23]**)** *If X is a Γ-space (respectively a Fin-space), then the familly $(X \circ \varphi^r_n)$ defines a natural system (resp. a cyclic one).*

If $X = \widetilde{X} \circ \Theta$, $Y = \widetilde{Y} \circ \Theta$ are simplicial spaces associated to Γ-spaces through the functor Θ, they both have natural system by the previous lemma. Assume there is a simplical map $f : X \to Y$ induced by a map $\tilde{f}$ of Γ-spaces. Then $\tilde{f} \circ (X \circ \varphi^r_n) = (Y \circ \varphi^r_n) \circ \tilde{f}$ and we get the following easy but usefull Lemma to compare natural systems.

Lemma 2.3. *Let $X = \widetilde{X} \circ \Theta$, $Y = \widetilde{Y} \circ \Theta$ be simplicial spaces factorizing through the category Γ. If $f : X \to Y$ is the transformation induced by a natural transformation $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ then f is a map of natural systems (with the natural systems from 2.2 on X and Y).*

2.3. **Operations for commutative $\mathbb{S}$-algebras.** We have seen that, for any $\mathbb{S}$-algebra, there exists a functor $THH(A) : \Delta^{op} \longrightarrow \Gamma sp$. We wish to apply lemma 2.2 to define λ-operations on $THH(A)$ when A is commutative. So we need to prove that in that case there exists a factorisation for $THH(A) : \Delta^{op} \to \Gamma sp$ through *Fin*, i.e.

$$THH(A) := \qquad \Delta^{op} \xrightarrow{\Theta} Fin \xrightarrow{\widetilde{T(A)}} \Gamma sp.$$

Lemma 2.4. *Such a factorisation exists if A is a commutative $\mathbb{S}$-algebra. If moreover M is a symmetric A-bimodule, then there is a factorisation*

$$THH(A, M) := \qquad \Delta^{op} \xrightarrow{\Theta} \Gamma \xrightarrow{\widetilde{T(A,M)}} \Gamma sp.$$

Proof: On objects we define $\widetilde{T(A)}_n = THH(A)_n$. For a morphism $\delta : [n] \to [m]$ and a map

$$f : S^{x_0} \wedge \ldots \wedge S^{x_n} \wedge \Delta^p_+ \to A(S^{x_0}) \wedge \ldots \wedge A(S^{x_n}) \wedge k_+$$

we wish to construct a map

$$\delta(f) : S^{y_0} \wedge ... \wedge S^{y_m} \wedge \Delta^p_+ \to A(S^{y_0}) \wedge ... \wedge A(S^{y_m}) \wedge k_+.$$

For $0 \leq i \leq m$, define $y_i = x_{i_1} \vee ... \vee x_{i_k}$ when $\delta^{-1}(i) = \{i_1, ..., i_k\}$ and $y_i = 0_+$ if $\delta^{-1}(i) = \emptyset$. We denote by k_i the cardinality of the set $\delta^{-1}(i)$. We write x^i_j for x_{i_j}. Let π be the unique permutation that reindexes $x_0 \vee ... \vee x_n$ in the form

$$x^0_0 \vee x^0_1 \vee ... \vee x^0_{k_0} \vee ... \vee x^m_0 \vee ... \vee x^m_{k_m}.$$

Then $\delta(f)$ is the composite map

$$\begin{aligned} S^{y_0} \wedge ... \wedge S^{y_m} \wedge \Delta^p_+ &\longrightarrow S^{x^0_0} \wedge ... \wedge S^{x^0_{k_0}} \wedge ... \wedge S^{x^m_{k_m}} \wedge \Delta^p_+ \\ &\xrightarrow{\pi} S^{x_0} \wedge ... \wedge S^{x_n} \wedge \Delta^p_+ \xrightarrow{f} A(S^{x_0}) \wedge ... \wedge A(S^{x_n}) \wedge k_+ \\ &\xrightarrow{\circ\pi} A(S^{x^0_0} \wedge ... \wedge S^{x^0_{k_0}}) \wedge ... \wedge A(S^{x^n_0} \wedge ... \wedge S^{x^n_{k_n}}) \wedge k_+ \\ &\longrightarrow A(S^{y_0}) \wedge ... \wedge A(S^{y_m}) \wedge k_+. \end{aligned}$$

This construction is compatible with the morphisms of the category $J(q+1)$.

We have to prove that this functor is well defined and that the desired factorisation holds. In particular we have to check that $\Theta(d_0) = \Theta(d_1)$, where d_0, d_1 are the two morphisms $[1] \to [0]$ in Δ^{op}. Let f be a map :

$$S^{x_0} \wedge S^{x_1} \wedge \Delta^p_+ \to A(S^{x_0}) \wedge A(S^{x_1}) \wedge k_+.$$

We denote x the subset $x_0 \vee x_1 = x_1 \vee x_0$. The map $d_0(f)$ is the composite map

$$S^x \wedge \Delta^p_+ \cong S^{x_0} \wedge S^{x_1} \wedge \Delta^p_+ \xrightarrow{f} A(S^{x_0}) \wedge A(S^{x_1}) \wedge k_+ \to A(S^{x_0} \wedge S^{x_1}) \wedge k_+.$$

With the notations of section 1, $d_1(f)$ is the composite map

$$\begin{aligned} S^x \wedge \Delta^p_+ \quad &\cong \quad S^{x_1} \wedge S^{x_0} \wedge \Delta^p_+ \xrightarrow{f \circ T} A(S^{x_0}) \wedge A(S^{x_1}) \wedge k_+ \\ &\xrightarrow{T} A(S^{x_1}) \wedge A(S^{x_0}) \wedge k_+ \to A(S^{x_1} \wedge S^{x_0}) \wedge k_+. \end{aligned}$$

Denoting i, j the two canonical identifications $i : S^{x_0} \wedge S^{x_1} \xrightarrow{\sim} S^{x_0 \vee x_1} = S^x$, $j : S^{x_1} \wedge S^{x_0} \xrightarrow{\sim} S^{x_1 \vee x_0} = S^x$, we have and $T' = j \circ i^{-1}$ the resulting identification $S^{x_0 \vee x_1} \cong S^{x_1 \vee x_0}$, we see that the following diagram (2.4.1) is commutative for all maps f when A is commutative :

$$\begin{tikzcd} S^{x_0 \vee x_1} \wedge \Delta^p_+ \arrow[r, "f"] & A(S^{x_0}) \wedge A(S^{x_1}) \wedge k_+ \arrow[r, "T"] \arrow[d, "\mu"] & A(S^{x_1}) \wedge A(S^{x_0}) \wedge k_+ \arrow[d, "\mu"] \\ & A(S^{x_0} \wedge S^{x_1}) \wedge k_+ \arrow[r, "A(T')"] \arrow[dr, "A(j)"] & A(S^{x_1} \wedge S^{x_0}) \wedge k_+ \arrow[d, "A(j)"] \\ & & A(S^x) \wedge k_+. \end{tikzcd} \quad (2.4.1)$$

Hence $d_0(f) = d_1(f)$. Checking the other simplicial identities is done in an analogous way and relies on the commutativity of the "concatenation" functor in $J(r)$.

Defining $\widetilde{T(A,M)}_n = THH(A,M)_n$ on objects, the previous construction applies *mutatis mutandis* to the case of a bimodule. The symmetry condition gives that the analog of diagram 2.4.1 is commutative. □

Remark : We will see later on that a reciprocical assertion holds if we use a different model for topological Hochschild homology 3.1.

The lemma implies that there is a cyclic natural system $\varphi^k : \mathrm{sd}_k THH(A)_* \to THH(A)_*$ and operations $\Phi^k =: |\varphi^k| \circ D^{-1} : THH(A) \to THH(A)$. The same is true for $THH(A,M)$ without the cyclic property. As in [23], there is an explicit description of the φ^k operations. A r-simplex in $(\mathrm{sd}_q(THH(A)))_{n-1}$ is given by a chain

$$X^0 \leftarrow X^1 \leftarrow ... \leftarrow X^r = x = (x_0, ..., x_{q(n+1)-1})$$

where each $X^i \in J(qn)$ together with a map

$$f : S^{x_0} \wedge ... \wedge S^{x_{q(n)-1}} \wedge \Delta^p_+ \to A(S^{x_0}) \wedge ... \wedge A(S^{x_{q(n)-1}}) \wedge k_+.$$

Writing $S^{x_0} \wedge ... \wedge S^{x_{q(n)-1}}$ as a "matrix"

$$\begin{pmatrix} S^{x_0} & S^{x_1} & \dots & S^{x_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ S^{x_{(q-1)n}} & S^{x_{(q-1)n+1}} & \dots & S^{x_{qn-1}} \end{pmatrix},$$

such a map has a "matrix form"

$$\left(S^{x_{(n-1)i+j}}\right)_{\substack{0 \le i \le q-1 \\ 0 \le j \le n-1}} \wedge \Delta^p_+ \xrightarrow{f} \left(A(S^{x_{(n-1)i+j}})\right)_{\substack{0 \le i \le q-1 \\ 0 \le j \le n-1}} \wedge k_+.$$

With this notation, for $0 \le i \le n-2$, faces of $\mathrm{sd}_q(THH(A))$

$$\begin{pmatrix} S^{x_0} & \dots & S^{x_i \vee x_{i+1}} & \dots & S^{x_{n-1}} \\ \vdots & \dots & \vdots & \dots & \vdots \\ S^{x_{q_0}} & \dots & S^{x_{q_i} \vee x_{q_{i+1}}} & \dots & S^{x_{q_{n-1}}} \end{pmatrix} \wedge \Delta^p_+$$

$$\xrightarrow{d_i f} \begin{pmatrix} A(S^{x_0}) & \dots & A(S^{x_i \vee x_{i+1}}) & \dots & A(S^{x_{n-1}}) \\ \vdots & \dots & \vdots & \dots & \vdots \\ A(S^{x_{q_0}}) & \dots & A(S^{x_{q_i} \vee x_{q_{i+1}}}) & \dots & A(S^{x_{q_{n-1}}}) \end{pmatrix} \wedge k_+.$$

(with $q_i = (q-1)n + i$) are given by multiplications of the adjacent columns and degeneracies by the insertion of a column of unit maps. The "last face" operator d_{n-1} is given by first cyclically rotating the last column and then multiplying it with the first one.

Lemma 2.5. *With this presentation of* $(\mathrm{sd}_q(THH(A)))_{n-1}$, *operations* φ^k *are given by*

$$\begin{pmatrix} S^{x_0} & S^{x_1} & \dots & S^{x_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ S^{x_{(q-1)n}} & S^{x_{(q-1)n+1}} & \dots & S^{x_{qn-1}} \end{pmatrix} \wedge \Delta^p_+$$

$$\xrightarrow{f} \begin{pmatrix} A(S^{x_0}) & A(S^{x_1}) & \dots & A(S^{x_{n-1}}) \\ \vdots & \vdots & \dots & \vdots \\ A(S^{x_{(q-1)n}}) & A(S^{x_{(q-1)n+1}}) & \dots & A(S^{x_{qn-1}}) \end{pmatrix} \wedge k_+$$

$$\xrightarrow{\wedge \dots \wedge} A(S^{x_0 \vee x_n \vee \dots \vee x_{(q-1)n}}) \wedge ... \wedge A(S^{x_{n-1} \vee x_{2n-} \vee \dots \vee x_{qn-1}})$$

where the last map is the iterated multiplication followed by identification of spheres.

Proof: Formally the proof is analogous to the computation in [23] Section 5. $\square$

There are operations $\Phi^k = \varphi^k \circ D_k^{-1}$ on $THH(A,M)$ thanks to Lemma 2.2 and Lemma 2.4. Recalll that to every γ-ring (R, Φ^k) we can associate a natural filtration (see [15] , [18] or the appendix B) $F^\gamma_n R$ defined by

$$F^\gamma_p X = \langle \gamma^{p_1}(x_1) ... \gamma^{p_s}(x_s) \, ; \, x_1, .., x_s \in R \text{ and } p_1 + ... + p_s \ge p \rangle$$

where $\gamma^k = \sum_{i=0}^{k-1}(-1)^i\binom{k-i}{i}\Phi^{k-i}$. The notation $\langle y_1...y_p\rangle$ stands for the abelian group generated by monomials $y_1...y_p$.

Theorem 2.1. *Let A be a commutative $\mathbb{S}$-algebra and M a symmetric A-bimodule.*

- **i):** *The abelian group $\pi_*(THH(A,M))$ equipped with the trivial multiplication and the operations Φ^k is a λ-ring.*
- **ii):** *For every commutative ring k, the λ-ring structure induces a canonical filtration $\widetilde{F_n^\gamma} = F_n^\gamma\pi_*(THH(A,M))$ on $\pi_*(THH(A,M))$ and also a filtration $F_n^\gamma = F_n^\gamma H_*(THH(A,M),k)$ on $H_*(THH(A,M),k)$ with*
$$\widetilde{F_1^\gamma} = \pi_*(THH(A)), \qquad F_1^\gamma = H_*(THH(A,M),k).$$
- **iii):** *One has $F_{n+2}^\gamma(H_n(THH(A),K)) = 0$ if K is a field.*
- **iv):** *For the field $\mathbb{Q}$ of rational numbers there is a natural decomposition*
$$\pi_n(THH(A,M))\otimes\mathbb{Q} = \pi_n^{(1)}(THH(A,M),\mathbb{Q})\oplus ...\oplus\pi_n^{(n)}(THH(A,M),\mathbb{Q}).$$

A stronger result than *iii)* is obtained in Corollary 3.3 after the study of these operations in the case of discrete rings.

Proof:

- **i):** It follows from Lemma 2.4 that the operations Φ^k define a natural system on $THH(A,M)$. The definition of a natural system implies that the following diagram (2.1.1) is commutative
$$\begin{tikzcd}
\mid THH(A,M)\mid & \mid \mathrm{sd}_s THH(A,M)\mid \arrow[l, "D_s"'] \arrow[r, "\varphi^s"] & \mid THH(A,M)\mid \\
& \mid \mathrm{sd}_{rs} THH(A,M)\mid \arrow[ul, "D_{rs}"] \arrow[u, "D_r"'] \arrow[r, "\mathrm{sd}_r\varphi^s"] \arrow[dr, "\varphi^{rs}"'] & \mid \mathrm{sd}_r THH(A,M)\mid \arrow[u, "D_r"] \arrow[d, "\varphi^r"] \\
& & \mid THH(A,M)\mid
\end{tikzcd} \tag{2.1.1}$$
and we finally have $\Phi^r\Phi^s = \Phi^{rs}$ on π_*. As φ^0 is trivial and φ^1 is the identity, this implies that $(\pi_*(THH(A,M)), 0, (\Phi^k)_{k\geq 0})$ is a λ-ring.
- **ii):** The filtration $\widetilde{F_n^\gamma}$ is the canonical decreasing filtration associated to the λ-ring structure on $\pi_*(THH(A,M))$. There is also a λ-ring structure on $H_*(THH(A,M),K)$ (the proof is analogous to (i)) that induces the filtration F_n^γ. The statement that $\widetilde{F_1^\gamma}$, F_1^γ are the whole γ-rings is a standard result of γ-rings with trivial multiplication theory [1].4.1.
- **iii):** It is well known that the skeleton filtration on $THH(A,M)$ induces a converging spectral sequence (*cf.* [3] or [27] for a published version)
$$G^2_{r,s} = HH_r(H_*(A,K),H_*(M,K))_s \Rightarrow H_{r+s}(THH(A,M),K),$$
where, for a graded ring R_* and R_*-bimodule M_*, $HH_r(R_*,M_*)_s$ means the subgroup of the Hochschild homology $HH_r(R_*,M_*)$ generated by tensors of total degree s in $M_*\otimes R_*^{\otimes *}$.

  More precisely, since there is a well-known stable homotopy equivalence [2] (it is a corollary of approximation Lemma 2.1)
$$THH(A,M)_r \cong A\wedge A\wedge ...\wedge A \ (r+1 \text{ factors}),$$
the term G^1 of the skeleta filtration spectral sequence is
$$G^1_{r,*} \quad = \quad H_*(THH(A,M)_r,K)$$

and Künneth's theorem implies

$$G^1_{r,*} = H_*(THH(A,M)_r, K) \cong H_*(M,K) \otimes H_*(A,K)^{\otimes r}.$$

The cyclic structure of $THH(A)$ induces the structure of a cyclic space $q \mapsto H_*(A,K)^{\otimes q+1}$. This cyclic structure is the one defining the standard Hochschild complex $C_*(H_*(A,K), H_*(M,K))$ as a K-algebra. The simplicial structure induced on $H_*(M,K) \otimes H_*(A,K)^{\otimes *}$ by the simplicial structure of $THH(A,M)$ is exactly the one defining the standard Hochschild complex. It is known [20] that $C_*(B,M)$ factors through the Γ-abelian group

$$\mathcal{L}(B,M) = \left(q \mapsto M \otimes B^{\otimes q}\right).$$

With this identification, the induced operations $\widetilde{\Phi^k}$ on the level 1 of the spectral sequence $G^1_{r,*} = H_*(THH(A,M)_r, K)$ are defined by

$$(C_*(H_*(A,K), H_*(M,K)) \circ \varphi^r_s) \circ D_r^{-1}.$$

McCarthy [23] Example 3.8 has proved that these operations coincide up to the sign $(-1)^{k-1}$ with Loday's standard operations λ^k on the Hochschild complex (*cf.* [20]). Then it follows from [20], Theorem 3.5 that

$$F_{n+2}(HH_r(H_s(A,K))) = 0 \text{ when } r \leq n.$$

Consequently, for $r+s \leq n$, one has $F_{n+2}(HH_r(H_s(A,K))) = 0$ and we finally get $F^{\gamma}_{n+2}(H_n(THH(A,M),K)) = 0$ see B.2.

iv): The theory of λ-rings ([15] , [18]) implies that, when $k = \mathbb{Q}$, there is a decomposition in eigenspaces of the Adams operations

$$\pi_n(THH(A,M)) \otimes \mathbb{Q} = \bigoplus_{j \geq 1}^{\infty} \pi_n^{(j)}(THH(A,M), \mathbb{Q}).$$

But, as $\pi_n(THH(A,M)) \otimes \mathbb{Q} \cong H_n(THH(A,M), \mathbb{Q})$, the conclusion is an immediate consequence of *iii*).

□

Remark : (1) In particular Theorem 2.1 holds for $M = A$, that is for $THH(A)$.

(2) The property *ii*) of Theorem 2.1 in fact holds for any homology theory E_* (the proof is the same).

(3) Let X be any spectrum. We denote by $[X\,,THH(A)]$ the group of homotopy classes of spectra maps $X \to THH(A)$., There is a product $THH(A) \wedge THH(A) \xrightarrow{m} THH(A)$ (see section 2.4) inducing a ring structure on $[X, THH(A)]_*$. Mimicking [18], Section 5, one can prove that the operations Φ^k induce a λ-ring structure on $[X, THH(A)]_*$. Moreover the ring structure is trivial when each X_r is a co-H-space.

2.4. **Ring structure and Adams operations.** Product structures on $THH(A)$ have been first studied by Hesselholt and Madsen ([14]). Here we still follow Brun's presentation ([6]). When A is a commmutative $\mathbb{S}$-algebra, then the Γ-space $THH(A)$ naturally becomes an $\mathbb{S}$-algebra, *i.e.* there exist a product $m : THH(A) \wedge THH(A) \to THH(A)$.

Theorem 2.2. *Let A be a commutative $\mathbb{S}$-algebra. The following diagram is commutative for all $k \geq 0$:*

$$\begin{tikzcd} THH(A) \wedge THH(A) \arrow[r, "m"] \arrow[d, "\Phi^k \wedge \Phi^k"'] & THH(A) \arrow[d, "\Phi^k"] \\ THH(A) \wedge THH(A) \arrow[r, "m"] & THH(A). \end{tikzcd}$$

Proof: It is enough to check that the following diagram (2.2.1) is commutative for all $k, r \geq 1$.

$$\begin{tikzcd} THH(A)_r \wedge THH(A)_r \arrow[r, "m_r"] & THH(A)_r \\ \mathrm{sd}_k THH(A)_r \wedge \mathrm{sd}_k THH(A)_r \arrow[r, "\mathrm{sd}_k m_r"] \arrow[u, "D_k \wedge D_k"] \arrow[d, "\varphi^k \wedge \varphi^k"'] & \mathrm{sd}_k THH(A)_r \arrow[u, "D_k"'] \arrow[d, "\varphi^k"'] \\ THH(A)_r \wedge THH(A)_r \arrow[r, "m_r"] & THH(A)_r . \end{tikzcd} \tag{2.2.1}$$

For the upper square of the diagram, the commutativity follows easily from the naturality of D_*. The product m (see [6]) is given by the composition

$$\begin{array}{rcl} m : \operatorname*{hocolim}_{x\in J(r)} G(A,x) \wedge \operatorname*{hocolim}_{y \in J(r)} G(A,y) & \xrightarrow{i} & \operatorname*{hocolim}_{(x,y)\in J(r)\times J(r)} G(A,x) \wedge G(A,y) \\ & \xrightarrow{\widetilde{\mu}} & \operatorname*{hocolim}_{z\in J(r)} G(A,z). \end{array}$$

The map i is induced by the smash product of Γ-spaces. The map $\widetilde{\mu}$ is defined as follows. There is a product map $j : J(r) \times J(r) \to J(r)$ which sends the tupple $\big((x_0,\dots,x_{r-1}),(y_0,\dots,y_{r-1})\big)$ to $\big(x_0 \vee (N+y_0), \dots x_{r-1} \vee (N+y_{r-1})\big)$ where $N = \max(x_0,\dots,x_{r-1})$. There is a map $\check{\mu} : G(A,x) \wedge G(A,y) \to G(A, j(x,y))$ which, to any map

$$f : S^x \wedge S^y \wedge \Delta^m_+ \to A(S^x) \wedge A(S^y) \wedge s_+,$$

associates the composite map

$$\begin{array}{rcl} S^{x_0 \vee N+y_0} \wedge \dots \wedge S^{x_{r-1}\vee N + y_{r-1}} \wedge \Delta^m_+ & \xrightarrow{T} & S^x \wedge S^y \wedge \Delta^r_+ \\ & \xrightarrow{f} & A(S^{x_0}) \wedge \cdots \wedge A(S^{y_0}) \wedge \dots \\ & & \cdots \wedge A(S^{y_{r-1}}) \wedge s_+ \\ & \xrightarrow{A(T^{-1})\circ \mu^k} & A(S^{x_0 \vee N + y_0}) \wedge \dots \\ & & \cdots \wedge A(S^{x_{r-1}\vee N+y_{r-1}}) \wedge s_+ \end{array}$$

where T is the identification of sphere induced by j. This map induces a map

$$\widetilde{\mu} : \operatorname*{hocolim}_{(x,y)\in J(r)\times J(r)} G(A,x)\wedge G(A,y) \to \operatorname*{hocolim}_{z \in J(r)} G(A,z).$$

The maps i clearly commutes with Φ^k (for all $k \geq 0$). Hence, the commutativity of the lower square of (2.2.1) will follow from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{sd}_k THH(A)_k \wedge \mathrm{sd}_k THH(A)_k \arrow[r, "\mathrm{sd}_k \widetilde{\mu}"] \arrow[d, "\varphi^k \wedge \varphi^k"'] & \mathrm{sd}_k THH(A)_r \arrow[d, "\varphi^k"'] \\ THH(A)_k \wedge THH(A)_k \arrow[r, "\widetilde{\mu}"] & THH(A)_k \end{tikzcd}$$

which is a consequence of the commutativity of the $\mathbb{S}$-algebra A as in the proof of Lemma 2.4. □

When passing to homology, Theorem 2.2 implies that $\pi_*(THH(A))$ has a second ring structure compatible with the Adams operations (which are related to the trivial multiplication on $\pi_*(THH(A))$). With the notation of the appendix B, we have.

Corollary 2.6. *$(\pi_*(THH(A)), \varphi^k, m)$ is a multiplicative γ-ring.*

Remark : Of course, $\pi_*(THH(A))$ can be replaced by the homology $E_*(THH(A))$ for any spectrum E in Corollary 2.6; we are particularly interested in $E = HK$.

3. Discrete rings

In this section we specialize to the important case where the $\mathbb{S}$-algebra come from a ring.

3.1. **Operations on the model $thh^R(A, M)$.** Schwede [29] proved that the category of $\mathbb{S}$-algebra is a cofibrantly generated model category. This property enables to build a model for topological Hochschild homology mimicking the standard complex in algebra following the ideas of [12]. The main advantage is that it leads to a definition of relative topological Hochschild homology which relates nicely to the classical algebraic Hochschild homology of discrete rings as we will see and need in the next subsection 3.2.ii). The main drawback here is the lack of explicit cofibrant $\mathbb{S}$-algebras. In this subsection, we build Adams operations for this model.

Suppose given (for the remainder of the section) a commutative $\mathbb{S}$-algebra L and a cofibrant commutative L-algebra B (*i.e.* a Γ-space together with a commutative product $\mu : B \wedge_L B \to B$ and a unit $\eta : L \to B$). When the cofibrancy condition for B is not satisfied, then one replace B by a cofibrant approximation $\mathbb{C}B \xrightarrow{\sim} B$. There exists a cyclic L-module $thh^L(B)_*$ such that

$$thh^L(B)_q = B^{\wedge_L (q+1)}$$

for all $q \geq 0$ and the structural maps are the faces

$$d_i = \begin{cases} \mathrm{id}^i \wedge \mu \wedge \mathrm{id}^{q-i} \text{ si } 0 \leq i \leq q-1, \\ \mu \wedge \mathrm{id}^q \circ \tau \text{ si } i = q, \end{cases}$$

degeneracies $s_j = \mathrm{id}^j \wedge \eta \wedge \mathrm{id}^{q-j+1}$ and the permutation t on the $q+1$ factors of $thh^L(B)_q$. We denote $thh^L(B)$ the geometric realisation of $thh^L(B)_*$. Theorem 2.2 of [29] implies that the functor $B \mapsto thh^L(B)$ preserves weak equivalences (using the arguments of [12].IX.2). The results of section 2.3 extend without difficulty to $thh^L(B)$

Lemma 3.1. *There exists a factorisation of $thh^L(B)$ of the form*

$$thh^L(B)_* : \Delta^{op} \xrightarrow{\Theta} Fin \xrightarrow{\widetilde{t(B)}} \Gamma sp$$

if and only if B is commutative.

Proof: If B is commutative, the proof is analogous to the proof of Lemma 2.4. In particular, the functor $\widetilde{thh(B)}$ is given by $\widetilde{thh(B)}_n = thh^L(B)_n$. For any map $\delta : [n] \to [m]$, we define σ as the permutation which sends the ordered set $\{0, \dots, n\}$ to $\{j_1^0, \dots, j_{k_0}^0, j_1^1, \dots j_{k_1}^1, \dots, j_1^m, \dots j_{k_m}^m\}$ where for $0 \leq i \leq m$, we denote $\delta^{-1}(i) = \{j_1^i, \dots, j_{k_i}^i\}$ (if it is not empty). Let $n(\delta)$ be the number of nonempty subsets of the form $\delta^{-1}(i), 0 \leq i \leq m$.

Then the induced map $\overline{\delta} : thh^L(B)_n \to thh^L(B)_m$ is the composite

$$thh^L(B)_n \xrightarrow{\overline{\sigma}} thh^L(B)_{n(\delta)} \xrightarrow{\overline{\eta}} thh^L(B)_m.$$

The map $\overline{\sigma}$ is the composite of the permutation covering σ followed by the the wedge of the iterated multiplications $\mu^{k_i} : B^{\wedge_L k_i} \to B$. The map $\overline{\eta}$ is given by composition with the wedge of the unit map $\eta : L \to B$ for each i such that $\delta^{-1}(i)$ is empty.

The reciprocical assertion follows from the fact that the two maps

$$B \wedge_L B \xrightarrow{\mu} B, \quad \text{and} \quad B \wedge_L B \overset{\mu \circ \mathrm{twist}}{\longrightarrow} B$$

should be equals. □

Theorem 3.1. *Suppose given a cofibrant commutative L-algebra B. The operations*

$$\Phi^k = (thh^L(B) \circ \varphi^k) \circ D_k^{-1} : thh^L(B) \to thh^L(B)$$

induce a structure of λ-ring on $\pi_*(thh^L(B))$ equipped with trivial multiplication.

Proof: Lemma 4.1 and Lemma 2.2 imply that the operations $\varphi^k = (thh(B) \circ \varphi^k)$ define a (cyclic) natural system on $thh^L(B, N)$ $(THH^L(B, N))$. The remainder of the proof proceeds as for Theorem 2.1.(i). □

Remark : We have already seen in the proof of Theorem 2.1 that a classical computation of Bökstedt implies that each level of the simplicial spaces $THH_*(A)$, $thh_*(A)$ are stably equivalent. It is a result of Shipley [32] in the category of symmetric spectra that both model are homotopy equivalent. In Appendix A we show that both models are equivalent upon passing to homotopy categories as γ-rings and that it is true in various categories of structured ring spectra.

3.2. **Adams operation for discrete rings.** Let R be a discrete ring. There is a natural Γ-space HR defined by

$$HR : \begin{cases} k_+ \mapsto R \oplus \ldots \oplus R \ (k \text{ factors}) \\ \\ (f : k_+ \to \ell_+) \mapsto HA(f)(a_1, \ldots, a_k) = (b_1, \ldots, b_\ell) \text{ with } b_j = \sum\limits_{f(i)=j} a_i. \end{cases}$$

The space HR is called the *Eilenberg-Mac Lane Γ-space associated to R.* The product map $R \otimes R \to R$ and the unit $k \to R$ give a $\mathbb{S}$-algebra structure on HR which is commutative when R is.

We define the *topological Hochschild homology* of the ring R to be the space $THH(HR)$, often simply denoted $THH(R)$. When HR is a L-algebra, with L a commutative $\mathbb{S}$-algebra, we also write $thh^L(R)$ for the Γ-space $thh^L(\mathbb{C}(HR))$ with $\mathbb{C}$ a cofibrant replacement functor, that is the model introduced in section 3.1 for the (relative) topological Hochschild homology .

There exist operations λ^k and $\Phi^k = (-1)^{k-1}\lambda^k$ (known as λ-operations) defined on the Hochschild homology group of a commutative algebra (*cf.* [20], [19], [23]). Given a commutative flat k-algebra R, there exist well known isomorphims ([12] in the category of S-modules for example) $\pi_n(thh^{Hk}(HR)) \cong HH_n^k(R)$ where $HH^k(R)$ is the Hochschild homology of R in the category of k-algebra. The following theorem extends this into a γ-ring isomorphism. Recall that, for R graded, we denote $HH_r^k(R)_s$ the Hochschild homology groups where s is the internal degree and r the homology degree.

Theorem 3.2. **i):** *For a discrete commutative ring R which contains $\mathbb{Q}$ the following diagram is commutative for all $n \geq 1$*

$$\begin{array}{ccc} \pi_n(THH(HR)) \otimes \mathbb{Q} & \xrightarrow{\simeq} & HH_n(R) \\ \Phi^k \downarrow & & \downarrow \Phi^k \\ \pi_n(THH(HR)) \otimes \mathbb{Q} & \xrightarrow{\simeq} & HH_n(R). \end{array}$$

ii): *Suppose R is a flat commutative k-algebra where k is a commutative ring with unit. Then the following diagram is commutative for all $n \geq 1$:*

$$\begin{array}{ccc} \pi_n(thh^{Hk}(R)) & \xrightarrow{\simeq} & HH_n(R) \\ \Phi^k \downarrow & & \downarrow \Phi^k \\ \pi_n(thh^{Hk}(R)) & \xrightarrow{\simeq} & HH_n(R). \end{array}$$

iii): *If A is a commutative $\mathbb{S}$-algebra and K is any field, then there is a converging spectral sequence of multiplcative γ-rings*

$$G^2_{r,s} = HH^K_r(H_*(A,K))_s \Rightarrow H_{r+s}(THH(A),K)$$

such that the operations Φ^k induced on the term $G^2_{*,*}$ coincide with the standard λ-operations on the Hochschild homology. The same result holds with $thh(A)$ instead of $THH(A)$.

Proof: We proceed as in the proof of Theorem 2.1.iii). The space $THH(A)_*$ is the cyclic Γ-space $q \mapsto THH(A)_q$. It is well known ([2]) that the filtration of $THH(A)$ by its skeleta gives rise to a first quadrant spectral sequence converging towards $H_{r+s}(THH(A),K)$ (for any field K) whose first term is

$$G^1_{r,*} = H_*(A,K)^{\otimes r+1}.$$

We have seen that the induced cyclic structure is Hochschild standard one and that the operations Φ^k induced on the Hochschild complex $C_*(H_*(A,K))$ coincide with the usual ones of [20]. The same holds for $thh(A)_* = \mathbb{C}(A)^{\wedge *+1}$ because $\mathbb{C}(A) \to A$ is a weak equivalence. This proves (iii).

Now, if $A = HR$ with $\mathbb{Q} \subset R$, one has $H_0(HR,\mathbb{Q}) = R \otimes \mathbb{Q} = R$ and $H_n(HR,\mathbb{Q}) = 0$ for $n \geq 1$. Hence the spectral sequence collapses and we have $HH_n(R) = HH^{\mathbb{Q}}_n(R) \cong H_n(THH(HR),\mathbb{Q})$. But, since R contains $\mathbb{Q}$, we get

$$\pi_n(THH(HR)) \otimes \mathbb{Q} \cong H_n(THH(R),\mathbb{Q}) \cong HH_n(R)$$

which proves (i).

Finally, there is a simplicial map $f : thh^{Hk}(R)_* \to H\big(R^{\otimes_k *+1}\big)$ given by the composite

$$\mathbb{C}(HR)^{\wedge_{Hk} *+1} \longrightarrow HR^{\wedge_{Hk} *+1} \longrightarrow H(R^{\otimes_k *+1}).$$

which is a Fin-map when R is commutative. The right map above folows from [29].1.2 and universal property of the smash product over Hk. Hence by Lemma 2.3, f commutes with the operations Φ^k when passing to π_*. When R is a flat k-algebra, there is an isomorphism $HR^{\wedge_{Hk} r+1} \cong HR^{\otimes_k r+1}$ (*cf.* [28],[29]). Therefore the map $f^1 : G^1_{*,*} \to H^1_{*,*}$ at the level 1 of the spectral sequences coming from the skeleta filtrations of $thh^{Hk}(R)_*$ and $H\big(R^{\otimes_k *+1}\big)$ is an isomorphism. Moreover it is an algebra map. Thus $G^1_{*,*}$ is :

$$G^1_{r,0} = R \otimes_k \ldots \otimes_k R \ (r+1 \text{ factors}) \text{ and } G^1_{r,s} = 0 \text{ if } s \geq 1$$

and we have the γ-ring map $f_* : \pi_n(thh^{Hk}(R)) \to HH_n(R)$ is an isomorphism. $\square$

Example : Bökstedt [3] proved that $\pi_n(THH(\mathbb{Z}/p)) = \mathbb{Z}/p$ when n is even and is 0 for n odd. We now compute the operations φ^k acting on $\pi_*(THH(\mathbb{Z}/p))$. The proof of Bökstedt relies on the fact that $THH(\mathbb{Z}/p)$ is an Eilenberg-Mac Lane spectrum and the analysis of the spectral sequence $G^2_{*,*}$ of theorem 3.2.(iii) converging to

$H_*(THH(\mathbb{Z}/p), F_p)$, where F_p denotes the field with p elements. In fact, for all $q \geq 0$, there is equivalence

$$HF_p \wedge THH_q(\mathbb{Z}/p) \cong HF_p \wedge (HF_p)^{q+1} \cong (HF_p \wedge HF_p) \wedge_{HF_p} (HF_p)^{q+1}$$

hence (by [28].12.2) the spectral sequence takes the form

$$G^2_{*,*} = HH_*(H_*(HF_p, F_p)) \Longrightarrow H_*(HF_p, F_p) \otimes_{F_p} \pi_*(THH(\mathbb{Z}/p)).$$

It is well known that $H_*(HF_p, F_p)$ is the symmetric graded $\mathbb{Z}/p$-algebra on generators $(\xi_i)_{i\geq 1}$ of degrees $2^i - 1$ if $p = 2$ and $(\xi_i)_{i\geq 1}$, $(\tau_j)_{j\geq 0}$ of respective degree $2p^i - 2$, $2p^j - 1$. Bökstedt shows that in the case $p = 2$, the spectral sequence collapses at level $G^2_{*,*}$ and that, for p odd, the term $G^{\infty}_{*,*}$ is generated by the elements $d\tau_j$ in $HH_1(H_*(HF_p, F_p)) \cong \Omega_{H_*(HF_p,F_p)}$ (the module of Kähler differential).

But the standard Adams operations λ^k on Hochschild homology acts on $d\tau_j \in HH_1(H_*(HF_p, F_p))$ by multiplication by k. Then Theorem 3.2.(iii) insures that the operation λ^k acts on $\pi_{2n}(THH(\mathbb{Z}/p))$ by multiplication by $k^{c_1+\cdots+c_{\ell(n)}}$ where the c_j are the digits of the unique decomposition $n = c_1 p^{i_1} + \ldots c_{\ell(n)} p^{i_{\ell(n)}}$ of n in base p. Moreover $F^{\gamma}_{c_1+\cdots+c_{\ell(n)}+2}\pi_{2n}(THH(\mathbb{Z}/p)) = 0$.

3.3. **The case of Γ-simplicial abelian groups.** As this section deals with discrete rings and their Eilenberg-Mac Lane $\mathbb{S}$-algebras, it is of interest to restrict one's attention to Γ-spaces that factor through functors $\Gamma \to s\mathcal{A}b$ (where $s\mathcal{A}b$ stands for the category of simplicial abelian groups), that is to say to Γ-simplicial abelian groups (see [6], [9] for details). Henceforth we denote the category of Γ-simplicial abelian groups by $\Gamma s\mathcal{A}b$. The Eilenberg-MacLane functor $H : s\mathcal{A}b \to \Gamma\mathrm{sp}$ factors through the forgetful functor $U : \Gamma s\mathcal{A}b \to \Gamma\mathrm{sp}$ to give a functor $\bar{H} : s\mathcal{A}b \to \Gamma s\mathcal{A}b$. The category $\left(\Gamma s\mathcal{A}b, \otimes, \bar{H}\mathbb{Z}\right)$ is symmetric monoidal with unit $\bar{H}\mathbb{Z}$. An $\bar{H}\mathbb{Z}$-algebra is a monoid in this category. When R is a commutative ring, $\bar{H}R$ is a commutative $\bar{H}\mathbb{Z}$-algebra.

Proposition 3.2. *Let A be an $\bar{H}\mathbb{Z}$-algebra and M a A-bimodule. Then the filtration induced by the operations Φ^k on $\pi_n(THH(A, M))$ satifies*

$$F^{\gamma}_{n+2}\pi_n(THH(A, M)) = 0.$$

Proof: Taking a functorial replacement functor if necesseray, we can assume A takes flat values. Replacing smash product by tensor product in the definition of THH gives an algebraic analogous theory for $\bar{H}\mathbb{Z}$-algebras (*cf.* [6], [9].IV.1.3) as follows. The Γ-simplicial abelian group $AG(A, M, x)$, $x \in J(q + 1)$, has n-simplices given by :

$$AG(A, M, x)_n(k_+) = s\mathcal{A}b\left(\mathbb{Z}(S^{x_0})\otimes\ldots\otimes\mathbb{Z}(S^{x_q})\wedge\Delta^n_+; M(S^{x_0})\otimes\ldots\otimes A(S^{x_q})\otimes\mathbb{Z}(k_+)\right)$$

whith $\mathbb{Z}(X) = \mathbb{Z}[X]/\mathbb{Z}*$. For a commutative $\bar{H}\mathbb{Z}$-algebra A and a bimodule M, we define

$$\underline{HH}^{\mathbb{Z}}(A, M)_q = \operatorname*{hocolim}_{x\in J(q+1)} AG(A, M, x)_q$$

where the colimit is taken in simplicial abelian groups. It gives a simplicial-$\Gamma s\mathcal{A}b$ object with the same simplicial structure maps than those of $THH(A, M)$. The proof of Lemma 2.4 can be mimicked to show that there is a factorisation of the functor $\Delta^{\mathrm{op}} \xrightarrow{\underline{HH}^{\mathbb{Z}}(A,M)} \Gamma s\mathcal{A}b$ through Γ^{op} :

$$\underline{HH}^{\mathbb{Z}}(A, M) := \qquad \Delta^{\mathrm{op}} \xrightarrow{\Theta} \Gamma^{\mathrm{op}} \xrightarrow{\widetilde{\underline{HH}}^{\mathbb{Z}}_*(A,M)} \Gamma s\mathcal{A}b.$$

Hence Lemma 2.2 gives Adams operations Φ^r on $\underline{HH}^{\mathbb{Z}}(A,M)$. It is well known (see [9].IV.1.3.3 for example) that the inclusion

$$M(S^{x_0}) \wedge \ldots \wedge A(S^{x_q}) \wedge k_+ \mapsto M(S^{x_0}) \otimes \ldots \otimes A(S^{x_q}) \otimes \mathbb{Z}(k_+)$$

induces an equivalence $THH(A,M) \cong \underline{HH}^{\mathbb{Z}}(\mathbb{Z}(A),M)$. It is straightforward from Lemma 2.3 to check that the previous map commutes with λ-operations.

For any $k_+ \in \Gamma$, $\widetilde{\underline{HH}}^{\mathbb{Z}}_*(A,M)(k_+)$ is a Γ-simplicial abelian group. Therefore, Loday's explicit combinatorial operations λ^k [20] (lying in $\mathbb{Z}[\Sigma_n]$) are defined on $\underline{HH}^{\mathbb{Z}}_*(A,M)$. But the computations in [23] Section 3, Lemma 2.5 and [23] 3.9 imply that Loday's λ^k operations coincide (up to a sign) with the previous operations Φ^k defined on $\underline{HH}^{\mathbb{Z}}(A,M)$. Then the combinatorial computation in [20] Theorem 3.9 (also see [19] 6.4.5) yields the desired result. □

Remark :

1. The Proposition 3.2 applies to $A = \bar{H}R$ and $M = \bar{H}R$, with R a (simplicial) ring, hence the result holds for $\pi_* THH(HR)$.
2. The Proposition 3.2 holds with π_* replaced by any homology theory E_*.

Corollary 3.3. *For any $\mathbb{S}$-algebra A, the following identities hold:*

$$F_1^{\gamma}(\pi_n(THH(A)) = \pi_n(THH(A)), \qquad F^{\gamma}_{n+2}\pi_n(THH(A)) = 0.$$

Proof: First equality has already been seen. By results of Dundas, Goodwillie and McCarthy [9].IV.1.4, we know that there exists an equivalence

$$THH(A) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}-\emptyset} THH((A)_S)$$

where $\mathcal{P}$ is the set of finite partitions of $\{1,2,...\}$ and each $\mathbb{S}$-algebra $(A)_S$ is equivalent to an Eilenberg-Mac Lane Γ-space $H(R)_S$ for some simplicial ring $(R)_S$. The construction of the $\mathbb{S}$-algebra $(A)_S$ is based on the iteration of the adjunction $A \mapsto U\mathbb{Z}(A)$. As already seen in the proof of Theorem 3.2, the λ-operations are compatible with the two functors U and $\mathbb{Z}(-)$. Hence, thanks to B.2 it suffices to prove the result for each $H(R)_S$ which follows from Proposition 3.2. □

3.4. **Comparison with Mac Lane homology.** It is known from Pirashvili and Waldhausen [27] that, for any discrete ring R, there is an isomorphism $\pi_*(THH(R)) \cong H_*^{ML}(R,R)$ where $H_*^{ML}(R,R)$ is the Mac Lane homology of the ring R. Recall that for any objects c,d of a category $\mathcal{C}$, we use the notation $\mathcal{C}(c,d)$ for the set of morphism from c to d in $\mathcal{C}$.

Let $\mathcal{P}_R$ be the additive category of finitely generated projective modules over the commutative ring R. The Mac Lane homology of R is the homology of the simplicial module

$$q_+ \mapsto D_q(R) = \bigoplus_{c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q} \mathcal{P}_R(c_0, c_q)$$

with faces and degeneracies defined as for the Hochschild complex (*cf.* [17]). McCarthy defined λ-operations on Mac Lane homology in [23], Section 6 in the following way: let T^r be a linear functor from $\mathcal{P}_R^r$ to $\mathcal{P}_R$ such that, for any $p_1, \cdots, p_r \in \mathcal{P}_R$, we have

$$T^r(p_1,\ldots,p_r) = p_1 \otimes \ldots \otimes p_r \qquad \text{and} \qquad T^r(R,\ldots,R) = R.$$

We denote τ a natural isomorphism $T^r \to T^r \circ t$ where $t(p_1,\ldots,p_r) = (p_r,...,p_1)$. Now, for any $= (f_0,\ldots...,\ldots f_{r(q-1)}) \in \mathrm{sd}_r(D_*(R))_{q-1}$ (with $f_i \in \mathcal{P}_R(c_i,c_{i-1})$, let define

$$\lambda^r(f_0,\ldots...,\ldots f_{r(q-1)}) =$$

$$\left(\tau\circ T^r(f_0,\ldots,f_{(r-1)q}), T^r(f_1,\ldots,f_{(r-1)q+1}),\ldots,T^r(f_{q-1},\ldots,f_{(qr-1)})\right)$$

These maps define a natural system on $D_*(R)$ (see [23]), hence yield Adams operations Φ^k on $H_*^{ML}(R,R)$.

Theorem 3.3. *There is a commutative diagram*

$$\begin{tikzcd} H_*^{ML}(R,R) \arrow[r, "\simeq"] \arrow[d, "\Phi^k"'] & \pi_*(THH(HR)) \arrow[d, "\Phi^k"] \\ H_*^{ML}(R,R) \arrow[r, "\simeq"] & \pi_*(THH(HR)). \end{tikzcd}$$

Proof: The map $\lambda^r : \mathrm{sd}_r D_*(R) \to D_*(R)$ above also induces a natural system on the simplicial abelian group with q-simplices

$$C\mathcal{P}_q(R) = \bigoplus_{c_0\leftarrow c_1\leftarrow\cdots\leftarrow c_q} \mathcal{P}_R(c_0,c_q)\otimes\bigotimes_{i=1}^{q}\mathbb{Z}\left(\mathcal{P}_R\right)(c_i,c_{i-1}).$$

Again the faces and degeneracies are defined as for Hochschild homology. There is a map $\alpha : D_*(R) \to C\mathcal{P}_*(R)$ sending $c_0 \leftarrow \cdots \leftarrow c_q$ to its class in $C\mathcal{P}_q(R)$; it induces an isomorphism in homology (*cf.* [10].1 and [17].1). By construction, α commutes with the maps λ^r.

Topological Hochschild homology can be defined for any Γ-simplicial category $\mathcal{C}$ (see [9].IV for example). The definition is like the one for topological Hochschild homology of a $\mathbb{S}$-algebra with $A(S^{x_0})\wedge\cdots\wedge A(S^{x_q})$ replaced by the coproduct

$$\bigvee_{c_0,\ldots,c_q\in\mathcal{C}} \mathcal{C}(c_0,c_q)(S^{x_0})\wedge\cdots\wedge\mathcal{C}(c_q,c_{q-1})(S^{x_q})$$

in the definition of $G(A,x)$ (Section 2.3).

We denote $\mathcal{P}_R^\vee$ the Γ-simplicial category with the same objects as $\mathcal{P}_R$ and morphism $\mathcal{P}_R^\vee(c,d)(k_+) = \mathcal{P}_R(c,\bigvee^k d)$. We denote $\mathbb{Z}(S^x) = \mathbb{Z}(S^{x_0})\otimes\ldots\otimes\mathbb{Z}(S^{x_q})$, from [9].IV.2.4 and [7].IV, we know that there is a simplicial abelian group $\mathcal{R}(R)$ with q-simplices $\mathcal{R}_q(R)$ given by

$$\operatorname*{holim}_{x\in J(q+1)} s\mathcal{A}b\left(\mathbb{Z}(S^x);\bigoplus_{c_0,\ldots,c_q\in\mathcal{P}_R}\mathcal{P}_R^\vee(c_0,c_q)(S^{x_0})\otimes\bigotimes_{i=1}^{q}\mathbb{Z}\left(\mathcal{P}_R^\vee\right)(c_i,c_{i-1})(S^{x_i})\otimes -\right)$$

and equivalences

$$C\mathcal{P}(R)\xrightarrow{\simeq}\mathcal{R}(R)\xleftarrow{\simeq}THH(H\mathcal{P}_R). \tag{3.3.1}$$

The last map is similar to the equivalence $\underline{HH}^{\mathbb{Z}}(\mathbb{Z}(\bar{H}R),\bar{H}R)\cong THH(HR)$ in the proof of Proposition 3.2 noticing that there is a homotopy equivalence $H\mathcal{P}_R(c,d)\cong\mathcal{P}_R^\vee(c,d)$ [8]. Moreover by Morita equivalence and [9].IV one has $THH(H\mathcal{P}_R)\cong THH(R)\cong\underline{HH}^{\mathbb{Z}}(\mathbb{Z}(\bar{H}R,\bar{H}R))$ (the last isomorphism has already been seen in the proof of Proposition 3.2). The natural system λ^k above yields a natural system on $\mathcal{R}_*(R)$ such that $C\mathcal{P}(R)\xrightarrow{\simeq}\mathcal{R}(R)$ is a γ-ring map. This natural system is defined, for $x\in J(kq)$ and

$$f:\mathbb{Z}(S^x)\to\bigoplus_{c_0,\ldots,c_{kq-1}\in\mathcal{P}_R}\mathcal{P}_R^\vee(c_0,c_{kq-1})(S^{x_0})\otimes\bigotimes_{i=1}^{kq-1}\mathbb{Z}\left(\mathcal{P}_R^\vee\right)(c_i,c_{i-1})(S^{x_i})\otimes -$$

by $\lambda^k_{\mathcal{R}}(f)$ is the composite

$$\begin{aligned}
\mathbb{Z}(S^x) &\to \mathbb{Z}\big(S^{x_0}\wedge\cdots\wedge S^{x_{(k-1)q}}\big)\otimes\cdots\otimes\mathbb{Z}\big(S^{x_{q-1}}\wedge\cdots\wedge S^{x_{kq-1}}\big)\\
&\xrightarrow{f} \bigoplus_{c_0,\ldots,c_{kq-1}\in\mathcal{P}_R} \mathcal{P}_R^\vee(c_0,c_{kq-1})(S^{x_0})\otimes\bigotimes_{i=1}^{kq-1}\mathbb{Z}\left(\mathcal{P}_R^\vee\right)(c_i,c_{i-1})(S^{x_i})\\
&\xrightarrow{\lambda^k} \bigoplus_{c_0,\ldots,c_{kq-1}\in\mathcal{P}_R} \mathcal{P}_R^\vee(\tau\circ T^k(c_0,\ldots,c_{(k-1)q}),T^k(c_{q-1},\ldots,c_{kq-1}))\big(S^{x_{0,\ldots,(k-1)q}}\big)\\
&\quad\otimes\bigotimes_{i=1}^{q}\mathbb{Z}\left(\mathcal{P}_R^\vee\right)(T(c_i,\ldots,c_{(k-1)q+i}),T(c_{i+1},\ldots,c_{(k-1)q+i+1}))(S^{x_{i,\ldots,(k-1)q+i}})\\
&\to \mathcal{R}_{q-1}(R).
\end{aligned}$$

with $S^{x_{i_1,\ldots,i_r}}=S^{x_{i_1}}\wedge\cdots\wedge S^{x_{i_r}}$. We are left to give a γ-ring equivalence between $\underline{HH}^{\mathbb{Z}}(\mathbb{Z}(\bar{H}R,\bar{H}R))_*$ and $\mathcal{R}_*(R)$.

Recall that a Γ-abelian group $\underline{HH}^{\mathbb{Z}}(\mathbb{Z}(\bar{H}R),\bar{H}R)$ has been defined in the proof of Proposition 3.2. The ring R is a projective module over itself. Thus, there is a map

$$\underline{HH}^{\mathbb{Z}}(\mathbb{Z}(\bar{H}R),\bar{H}R)\xrightarrow{\beta}\mathcal{R}(R).$$

This map β is an equivalence by the left equivalence of (3.3.1), Morita equivalence of $THH(HR)$ (see [11], 2.5) and the equivalence $\underline{HH}^{\mathbb{Z}}(\mathbb{Z}(\bar{H}R),\bar{H}R)\cong THH(R)$[9].IV. The formula above for the natural system $\lambda^k_{\mathcal{R}}$ applied to the free rank one module R and the formula of Lemma 2.5 for the natural system φ^k defined in Proposition 3.2 imply that the map β is also a map of natural systems. Thus the result. □

4. Adams operations in the category of S-modules

There exist other strict monoidal categories of symmetric spectra in which several definitions for topological Hochschild homology are possible. In this section we show how our previous results can be done in the category of S-modules build by Elmendorff, Kriz, Mandell and May [12]. Moreover we compare our constructions with some of their results. The objects of the category $S-$ mod of S-modules are much more complicated to describe than the Γ-spaces and we refer to [12] for definitions and notations. The letter S (not to be confused with $\mathbb{S}$) stands for the representant of the sphere spectrum in this category. The category $S-$ mod has a very rich structure: it is a symmetric monoidal topologically enriched closed model category. We denote $S-$ alg the subcategory of monoids. Unlike the Γ-spaces, the homotopy category of $S-$ mod is equivalent to the homotopy category of spectra (without connectivity condition). Moreover Eilenberg-Mac Lane spectra can be build as cell S-modules.

4.1. **The $thh^L(B)$-model.** First we give the thh-model (as in section 3.1) for the category $S-$mod. It is more efficient as there are more explicit cofibrant S-modules (like the cell S-modules). This model was first studied in [12] Section IX.2. Suppose given (for the remainder of the section) a q-cofibrant commutative S-algebra L and a q-cofibrant commutative L-algebra B (*i.e.* $\mu : B\wedge_L B\to B$, $\eta : L\to B$). This condition ensures that the construction respects weak equivalences of S-modules. Let N be a B-bimodule with structure maps $\ell : B\wedge_L N\to N$, $r : N\wedge_L B\to N$. There exists a simplicial L-module $thh^L(B,N)_*$ such that

$$thh^L(B,N)_q = N\wedge_L B^{\wedge_L q}$$

for all $q \geq 0$ and the structural maps are the faces

$$d_i = \begin{cases} \mathrm{id}^i \wedge \mu \wedge \mathrm{id}^{q-i} \text{ si } 1 \leq i \leq q-1, \\ \ell \wedge \mathrm{id}^q \circ \tau \text{ si } i = q, \\ r \wedge \mathrm{id}^q \text{ si } i = 0, \end{cases}$$

and degeneracies $s_j = \mathrm{id}^j \wedge \eta \wedge \mathrm{id}^{q-j+1}$. If $N = B$, $thh^L(B)_* := thh^L(B,B)_*$ is cyclic thanks to the permutation action t on the $q+1$ factors of $thh^L(B)_q$. We denote $thh^L(B,N)$ the geometric realisation of $thh^L(B,N)_*$. The results of section 2.3 can be made stronger without difficulty for $thh^L(B)$.

Lemma 4.1. *There exists a factorisation of $thh^L(B)$ of the form*

$$thh^L(B)_* : \Delta^{op} \xrightarrow{\Theta} Fin' \xrightarrow{\widetilde{t(B)}} \Gamma sp$$

if and only if B is commutative. If B is commutative, there exists a factorisation

$$thh^L(B,N)_* : \Delta^{op} \xrightarrow{\Theta} Fin' \xrightarrow{\widetilde{t(B,N)}} \Gamma sp$$

if and only if N is symmetric.

Proof: The proof of Lemma 3.1 applies *verbatim*. □

Theorem 4.1. *Suppose given a commutative L-algebra B and a symmetric B-module N. The operations*

$$\Phi^k = (thh^L(B,N) \circ \varphi^k) \circ D_k^{-1} : thh^L(B,N) \to thh^L(B,N)$$

induce a structure of λ-ring on $\pi_*(thh^L(B,N))$ equipped with trivial multiplication. This structure is compatible with the S-algebra structure on $thh^L(B)$ when $N = B$.

Proof: Lemma 4.1 and Lemma 2.2 imply that the operations $\varphi^k = (thh(B) \circ \varphi^k)$ define a natural system on $thh^L(B,N)$, cyclic if $N = B$. The remainder of the proof proceeds as for Theorem 2.1.(i). There is a structure of commutative S-algebra on $thh^L(B,N)$ [12].X.2.2 induced by the fact that $q \mapsto thh_q^L(B)$ is a simplicial commutative S-algebra when B is commutative. Corollary 4.7 below implies that this product makes $(\pi_*(thh^L(B,N)), (\Phi^k)_{k\geq 0}, *)$ a multiplicative γ-ring. It is also easy to see that map $\varphi^k : \mathrm{sd}_k thh_*^L(B) \to thh_*^L(B)$ commute with the product by straightforward formal computation. □

Remark : The results of sections 2.3, 3.2 for THH apply to $thh(B,M)$ when B and M model connective spectrum thanks to the result of Appendix A. For more general S-algebras, the results of [12] Section $X.2$ enable to mimick the proofs of 3.2, 2.1 to get similar statements. We list below the only difference:

1. Given any field K, the γ-ring filtration induced by Theorem 4.1 satisfies $F_1^\gamma(H_n(thh^L(B,N),K)) = H_n(thh^L(B,N),K)$ and there is an induced splitting
$$\pi_*(thh^L(B,M)) \otimes \mathbb{Q} = \bigoplus_{n\geq 0} \pi_*^{(n)}(thh^L(B,N),\mathbb{Q}).$$
Howewer the vanishing property for F_{n+2} are no longer true in general.
2. Suppose R is a flat commutative k-algebra where k is a commutative ring with unit and M a R-bimodule (all of them discrete, non-graded). Then there is an isomorphism of (multiplicative if $M = R$) γ-rings
$$\pi_*(thh^{Hk}(HR,HM)) \cong HH_*^k(R,M).$$

3. If B is a commutative S-algebra, N a B-bimodule and K is any field, then there is a spectral sequence of γ-rings
$$G^2_{r,s} = HH^K_r(H_*(B,K), H_*(N,K))_s \Rightarrow H_{r+s}(thh^L(B,N),K).$$
Howewer the spectral sequence is no more concentrated in the first quadrant (but first half-plane).

There are other spectral sequences converging to $\pi_*(thh^L(R,M))$ [12] Theorem X.1.6 which we prove to be γ-rings spectral sequences.

Proposition 4.2. *Let B be a cofibrant commutative L-algebra, N a B-bimodule and assume that B_* is L_*-flat, then there is a spectral sequence of (multiplicative if $N = B$) γ-rings*
$$HH^{L_*}_*(B_*, N_*) \Rightarrow \pi_*(thh^L(B,N)).$$
When $L = Hk$ with k a discrete ungraded commutative ring, then there is an isomorphism of multiplicative γ-rings $HH^k_*(A,M) \cong \pi_*(thh^{Hk}(HA, HM))$.

Proof: Theorem X.1.6 and X.2.6 of [12] give all the result but the γ-rings compatibility assertion. Let $B^e = B \wedge_L B^{op}$ the envelopping L-algebra of B. The spectral sequence is obtained in the following way ([12].IV.5). Given a free B^e_*-resolution of B_*
$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} B_* \to 0$$
one build wedges K_p of $p+s$ spheres (one for each basis element of F_p of degree s) and, inductively, cofiber sequences
$$\mathbb{F}K_p \xrightarrow{k} M_p \xrightarrow{i} M_{p+1} \xrightarrow{j} \Sigma\mathbb{F}K_p$$
starting with $M_0 = B$ and where $\mathbb{F}$ is the free B^e-modules functor. The sequence is such that $\pi_*(M_p) \cong \Sigma^p\mathrm{Ker}(d_{p-1})$, j_* and k_* realizes the canonical inclusion and epimorphism on $\Sigma^* F_*$ and i_* is trivial in π_*. This yields an exact couple $E^1_{p,q} = \pi_{p+q}(N \wedge_{B^e} \mathbb{F}K_p)$, $D^1_{p,q} = \pi_{p+q+1}(N \wedge_{B^e} M_{p+1})$ whose term $E^2_{p,q}$ is $\mathrm{Tor}^{B^e_*}_{p,q}(N_*, B_*)$. Consider the Bar construction of B as a L-algebra. It is a simplicial B^e-module $B^{ar}_*(B) = B \wedge_L B^{\wedge_L *} \wedge_L B$ weakly equivalent to B. The algebraic Bar construction gives a simplicial resolution $B^{ar}_*(B_*) = B_* \otimes_{L_*} B^{\otimes_{L_*}*} \otimes_{L_*} B_*$ of B_* as a $B_* \otimes_{L_*} B_*$-module. By the flatness hypothesis it is a flat resolution, hence $N_* \otimes_{B^e_*} B^{ar}_*$ computes $\mathrm{Tor}^{B^e_*}_{p,q}(N_*, B_*)$. Let $f : F_* \to B^{ar}_*(B_*)$ be a map of resolutions. Lemma [12].X.2.4 ensures that $thh^L(B,N)_* \cong N \wedge_{B^e} B^{ar}_*(B)$. The skeletal filtration of $B^{ar}_*(B)$ yields an exact couple

$$\begin{tikzcd} F^1_{*,*} \arrow[rr, "\bar{i}"] & & F^1_{*,*} \arrow[dl, "\bar{j}"] \\ & G^1_{*,*} \arrow[ul, "\bar{k}"] & \end{tikzcd}$$

with $G^1_{p,q} = \pi_q(N \wedge_{B^e} B^{ar}_p(B))$ and, denoting sB^{ar}_p the image of $\bigvee_{0 \le j \le p} B^{ar}_j(B) \times \Delta^j_+$ in $|B^{ar}_*(B)|$, $F^1_{p,q} = \pi_q(N \wedge_{B^e} sB^{ar}_p)$. The map $f : F_* \to B^{ar}_*(B_*)$ and the freeness properties of F_* yields a map $\mathbb{F}K_p \to \Sigma^p B^{ar}_p(B)$. Starting with $M_1 \to \Sigma\mathbb{F}K_0 \to B^{ar}_0(B) \cong \Sigma sB^{ar}_0$, by induction and cofiber sequences arguments we get maps $M_{p+1} \to \Sigma sB^{ar}_p$ inducing a map of exact couples $(D^*_{*,*}, E^*_{*,*}) \to (F^*_{*,*}, G^*_{*,*})$. Hence there is a map of spectral sequences $E^*_{*,*} \to G^*_{*,*}$, which, by flatness hypothesis, is the isomorphism $\mathrm{Tor}^{B^e_*}_{p,q}(B_*, B_*) \cong HH^{L_*}_{p,q}(M_*, B_*)$ at the level 2. The statement now follows from the fact that $G^*_{*,*}$ is a spectral sequence of γ-rings.

This has been checked, for the skeletal filtration, in the proofs of Theorem 2.1 and 3.2 for example. □

4.2. **Topological Harrison homology.** In the algebraic setting, the Adams operations ϕ^k acting on the Hochschild complex $C_*(A,A)$ induces a filtration on this complex whose weight 1-term gives the standard complex computing the Harrison homology. In this part, we give a topological analog of this situation. In what follows, B is a cofibrant commutative L-algebra.

The simplicial structure yields an augmentation $thh^L(B)_* \xrightarrow{j} thh^L(B)_0 = B$ which is a map of L-algebras. We denote $I(B)_*$ the following homotopy pullback

$$\begin{tikzcd} I(B)_* \arrow[r, "i"] \arrow[d] & thh^L(B)_* \arrow[d, "j"] \\ \mathrm{pt} \arrow[r] & B \end{tikzcd} \tag{4.1}$$

in the Reedy category of simplicial L-algebras, that is to say the homotopy fiber of j. We write $\mu : I(B)_* \wedge I(B)_* \to I(B)_*$ the multiplication.

Definition 4.3. *The topological Harrison homology $thar^L(B)_*$ of a commutative L-algebra B is the homotopy pushout*

$$\begin{tikzcd} I(B)_* \wedge I(B)_* \arrow[r, "i\circ\mu"] \arrow[d] & thh^L(B)_* \arrow[d, "p"] \\ \mathrm{pt} \arrow[r] & thar^L(B). \end{tikzcd}$$

We denote $thar^L(B)$ its realization.

This definition mimicks the algebraic construction. When $L = Hk$, $B = HR$ we have an analog of Theorem 3.2. Write $I(R)$ for the augmentation complex

$$0 \longrightarrow I(R) \longrightarrow C_*^K(R) \longrightarrow R \longrightarrow 0$$

where $C_*^K(R)$ is the standard Hochschild complex of the K-algebra R. Let $CHar^K(R)$ be the standard Harrison complex of R and $Har_*^k(R)$ its homology. There is a natural complex morphism $C_*^k(R) \overset{p^{\mathrm{alg}}}{\to} CHar^k(R)$ identifying $CHar^k(R)$ with the quotient of the Hochschild complex by $I(R).I(R)$ (where the product is the shuffle product). See [19].IV.2 for example.

Proposition 4.4. *Let K be a field and R a commutative K-algebra. There is a commutative diagram*

$$\begin{tikzcd} \pi_*(thh^{HK}(HR)) \arrow[r, "p"] \arrow[d, "\cong"'] & \pi_*(thar^L(HR)) \arrow[d, "\cong"] \\ HH_*^K(R) \arrow[r, "\mathrm{palg}"] & Har_*^K(R). \end{tikzcd}$$

Proof: The map $C_*^K(R) \to R$ is a fibration and p^{alg} is a cofibration. The isomorphism of algebras $\pi_*(thh^{HK}(HR)) \cong HH_*^K(R)$ and the homotopy fiber sequence 4.1 ensures that $\pi_*(I(HR)) = H_*(I(R))$. Moreover the Eilenberg-Zilber theorem identifies the product on $\pi_*(I(HR))$ with the one induced by the shuffle product on $H_*(I(R))$ through this isomorphism. Thus the sequence

$$I(HR) \wedge I(HR) \xrightarrow{i\circ\mu} thh^{HK}(HR) \xrightarrow{p} thar^{HK}(HR)$$

gives that $\pi_*(thar^{HK}(HR))$ is isomorphic to the Harrison homology groups $Har^K_*(R)$. □

Let E_* be a homology theory, we say that a simplicial S-module X is E_*-proper if there is a proper S-module Y and a E_*-equivalence between Y and X, see Definition [12].X.2.1.

Proposition 4.5. *If B is a commutative $\mathbb{S}$-algebra, K a field and $I(B)$ is HK_*-proper, there is a converging spectral sequence*

$$G^2_{r,s} = Har_r(H_*(A,K))_s \Longrightarrow H_{r+s}(thar^S(A),K).$$

Proof: As cofibrations are preserved by pushout, the simplicial spectra $tha(B)$ which is the pushout

$$\begin{tikzcd} X_* \arrow[r] \arrow[d] & thh^S(B)_* \arrow[d] \\ \mathrm{pt} \arrow[r] & tha(B)_* \end{tikzcd}$$

is proper. Here X_* is a proper HK_*-approximation of $I(B)\wedge I(B)$ obtained from the one of $I(B)$ by wedge and Künneth Theorem. The induced map $tha(B)\to thar^S(B)$ is a HK_*-equivalence. Hence, skeleta filtration yields a strongly converging spectral sequences

$$E^1_{r,s} = H_s(tha(B)_r,K) \Longrightarrow H_{r+s}(tha(B),K) \cong H_{r+s}(thar^S(A),K)$$

and $H_s(X_r,K) \Longrightarrow H_{r+s}(X,K)$ see [12].X.2.9. By hypothesis, $H_s(X_r,K)\cong H_s(I(B)\wedge I(B),K)$. Thus a proof similar to the one of Proposition 4.4 applied to the cofiber sequence defining $tha(B)$ ensures that the term $E^1_{*,*}$ of the spectral sequence computing $H_*(thar^S(A),K)$ is

$$E^1_{p,*} \cong \begin{cases} CHar_p(H_*(B,K),H_*(B,K)) \text{ if } p>0 \\ 0 \text{ if } p=0 \end{cases}.$$

Hence the second term of the spectral sequence is isomorphic to the Harrison homology $Har_p(H_*(B,K))$. □

Example : We want to compute $\pi_*(thar^S(HF_p))$. We proceed as in the example following Theorem 3.2. We detail the case $p=2$. Let J_* be the simplicial spectrum defined by $J_0 = *$ and $J_{n\geq 1} = HF_p^{\wedge n+1}$ with simplicial structure induced by the multiplication and unit map. It is a proper spectrum by [12].VII.7.5. Its asssociated spectral sequence collapses at level 2 as for the skeleta filtration of $thh^S(H\mathbb{Z}/p)$ see [3]. Hence $H_*(J,K)\cong H_*(I(H_*(HF_p,F_p)))\cong H_*(I(HF_p),F_p)$. Thus $I(HF_p)$ is HF_{p*}-proper. By Proposition 4.5, the skeleta filtration induces a spectral sequence

$$Harr_*(H_*(HF_p,F_p)) \Longrightarrow H_*(thar^S(HF_p),F_p) \cong H_*(HF_p,F_p)\otimes\pi_*(thar^S(HF_p)).$$

But for $A=\mathbb{Z}/p$, $H_*(A,F_p)$ is graded polynomial, hence $Harr_p(H_*(A,F_p))=0$ for $p\geq 2$ and the spectral sequence collapses. Hence, the topological Harrison homology group of $\mathbb{Z}/2$ are

$$\pi_{2^i}(thar^S(\mathbb{Z}/2)) = \mathbb{Z}/2 \text{ for } i\geq 0 \quad\text{and}\quad \pi_n(thar^S(\mathbb{Z}/2)) = 0 \text{ if } n\neq 2^i.$$

In the case of a prime $p>2$, a similar computation using that $H_*(thh^S(HF_p),F_p)$ is generated by the elements $d\tau_j$ (see the example after Theorem 3.2) yields

$$\pi_{2p^i-1}(thar^S(\mathbb{Z}/p)) = \pi_{2p^i}(thar^S(\mathbb{Z}/p)) = \mathbb{Z}/p \text{ for } i\geq 0$$

and $\pi_n(thar^S(\mathbb{Z}/p)) = 0$ if $n\neq 2p^i-1$ or $2p^i$.

In the algebraic setting, the Harrison homology is closely related to the weight 1 part of the natural filtration on Hochschild homology coming from the γ-ring structure. From the topological point of view we have.

Corollary 4.6. *Let B be a commnutative L-algebra.*

- **i):** *There is a natural system on $thar^L(B)$ such that $p : thh^L(B) \to thar^L(B)$ is a map of natural system.*
- **ii):** *Let B be a connective commutative S-algebra, K a field and $I(B)$ is HK_*-proper. Then $F_3^\gamma H_*(thar^S(B), K) = 0$. In particular*

$$p\big(F_3^\gamma H_*(thh^S(B), K)\big) = 0.$$

Remark : The result of 4.6.ii) obviously holds for $thar^{HK}(HR)$ by proposition 4.4.1.

Proof:

- **i):** The constant simplicial L-algebra B and the map $j : thh^L(B)_* \to B$ obviously factor through the category Fin. Thus $\pi_*(B)$ has a canonical γ-ring (with trivial multiplication) structure which satisfies $F^\gamma_{n \geq 2}\pi_*(B) = 0$. Moreover j is a map of natural system. As diagram 4.1 is homotopy cartesian, there is an induced natural system $\varphi^k : \mathrm{sd}_k I(B) \to I(B)$. Similarly the homotopy pushout $thar^L(B)$ admits a natural system such that p is a map of natural system.
- **ii):** By Proposition 4.5, there is a first quadrant spectral sequence

$$G^2_{r,s} = Har_r^K(H_*(B,K))_s \Longrightarrow H_{r+s}(thar^S(B), K).$$

It is a spectral sequence of γ-rings because p is a map of natural system. The Adams operations induced on $Har_*^K(H_*(B,K))$ are induced by Loday's ones on $HH_*(H_*(B,K))$ via the epimorphism $p^{\mathrm{alg}} : HH_*(H_*(B,K)) \to Har_*^K(H_*(B,K))$. Hence $F_3^\gamma Har_r^K(H_*(B,K)) = 0$ implies the result by Proposition B.2.

□

Remark : It is possible to define topological Harrison homology in the same way in any symmetric monoidal category of spectra. The method of Appendix A ensures that the constructions of $thar^L(B)$ in the categories of Γ-spaces, symmetric spectra and S-modules are equivalent.

4.3. **The $B \otimes S^1$-model.** Now we turn attention to the $B \otimes S^1$-model which was first introduced by McClure, Schwänzl and Vogt [24]. We show that our construction on $thh^L(B)$ is compatible with the maps introduced in [24] (even without looking at homotopy category as in the Appendix A). We the use it to give a new proof of the multiplicativity of Adams operations on $thh^L(B)$.

The categories of commutative $\mathbb{S}$-algebras or L-algebras (in the sense of [12]) is tensored (see [24]) over the category of topological spaces. This means that, for all commutative L-algebras A, B, there are natural homeomorphisms

$$Hom_L(A \otimes S^1, B) \cong \mathrm{Top}(S^1, Hom_L(A,B))$$

where $Hom_{\mathbb{S}}$ stands for the set of morphisms in the category of commutative $\mathbb{S}$-algebras.

Let B be a L-algebra. There exists a natural isomorphism of L-algebras (see [12].X.3)

$$thh^L(B) \cong B \otimes S^1.$$

Precisely, there is a simplicial isomorphism $thh^L(B)_* \cong B \otimes S^1_*$ (*cf.* [24], [12]) which induces the isomorphism $thh^L(B) = B \otimes S^1$ after realisation. Here, we identify S^1_* with the simplicial set $\Delta(1)_*/\partial\Delta(1)_*$ where $\Delta(1)_*$ is the standard 1-simplex and $\partial\Delta(1)_*$ is its boundary.

The maps $\Psi^k : S^1 \to S^1$, $k \geq 0$, defined by
$$\Psi^k(e^{2i\pi t}) = e^{2i\pi k t}.$$
induce Adams operations, denoted $\mathrm{id} \otimes \Psi^k$, on $B \otimes S^1$ (*cf.* [24]).

Theorem 4.2. *The following diagram is commutative*

$$\begin{tikzcd}
B \otimes S^1 \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\otimes\Psi^k"'] & thh^L(B) \arrow[d, "\Phi^k"] \\
B \otimes S^1 \arrow[r, "\simeq"] & thh^L(B).
\end{tikzcd}$$

Proof: We have to prove that the diagram (4.2.1) below is commutative

$$\begin{tikzcd}
thh^L(B) & B \otimes S^1 \arrow[l, "\simeq"'] & B\otimes \mid S^1_* \mid \arrow[l, "B\otimes -"'] & B\otimes \mid S^1_* \mid \arrow[l, "\mathrm{id}"'] \arrow[dd, "\mathrm{id}\otimes\Psi^k"] \\
\mid \mathrm{sd}_k thh^L(B)_* \mid \arrow[u, "D_k"] \arrow[d, "\varphi^k"] & \mid \mathrm{sd}_k B \otimes S^1_* \mid \arrow[l, "\simeq"] \arrow[u, "D_k"] \arrow[d, "\varphi^k"] & B\otimes \mid \mathrm{sd}_k S^1_* \mid \arrow[u, "\mathrm{id}\otimes D_k"] \arrow[d, "\mathrm{id}\otimes\varphi^k"] & \\
thh^L(B) & B \otimes S^1 \arrow[l, "\simeq"] & B\otimes \mid S^1_* \mid \arrow[l, "B\otimes -"] & B\otimes \mid S^1_* \mid . \arrow[l, "\mathrm{id}"]
\end{tikzcd} \tag{4.2.1}$$

The left squares of (4.2.1) commute by naturality of D and transfer of structure. The middle rectangle is commuting in view of [24], Proposition 4.3. Finally, a computation analogous to [23], Lemma 1.4 insures the commutativity of the right rectangle of (4.2.1). □

Theorem 4.2 give a new proof of the compatibility with the product structure when B is indeed cofibrant.

Corollary 4.7. *The operations Φ^k commute with the product on $thh^L(B)$.*

Proof: The product on $B \otimes S^1$ is induced by the composite (see [24])
$$(B \otimes S^1) \wedge (B \otimes S^1) \xrightarrow{\simeq} B \otimes (S^1 \sqcup S^1) \xrightarrow{\mathrm{id}\otimes f} B \otimes S^1$$
where $f : S^1 \sqcup S^1 \to S^1$ is the codiagonal map. It is easy to check that
$$f \circ (\Psi^k \sqcup \Psi^k) = \Psi^k \circ f.$$
Hence the following diagram is commutative :

$$\begin{tikzcd}
(B \otimes S^1) \wedge (B \otimes S^1) \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\otimes\Psi^k \wedge \mathrm{id}\otimes\Psi^k"'] & B \otimes (S^1 \sqcup S^1) \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "\mathrm{id}\otimes(\Psi^k \sqcup \Psi^k)"'] & B \otimes S^1 \arrow[d, "\mathrm{id}\otimes\Phi^k"] \\
(B \otimes S^1) \wedge (B \otimes S^1) \arrow[r, "\simeq"] & B \otimes (S^1 \sqcup S^1) \arrow[r, "\mathrm{id}\otimes f"] & B \otimes S^1.
\end{tikzcd}$$

□

Appendix A.

This appendix is devoted to show why our different constructions are equivalent. It is now well known that the various approach to a category of structured ring spectra are Quillen equivalent [26], [30]. In any of these categories there are few possible models for topological Hochschild homology of a monoid. It is folklore result, partially dispatched in several papers, that these constructions are equivalent. We recall here how [26], [30] imply that the constructions we use for topological

Hochschild homology are equivalent and show in addition that the Adams operations we built coincide. Hence, in practice, one can always choose the more suitable model to deal with a particular case without restrictions. More precisely, we investigate the equivalences between the THH and thh-model in 3 simplicial sets-based model categories for spectra and 4 topological spaces based ones.

Many symmetric monoidal categories of spectra are categories of diagram spectra [26]. We first deal with them before turning attention to the S-modules of [12]. Let $\mathcal{D}$ be either the category Σ of symmetric groups, $\mathcal{W}$ of pointed spaces homeomorphic to a finite CW complex or Γ the opposite of Segal category [31]. Let $\mathcal{D}Top$ be the category of functors from $\mathcal{D}$ to spectra over based compactly generated topological spaces. Recall from [26].III that $\mathcal{D}$-spectra are the same than $\mathcal{D}$-spaces and $\mathcal{D}$-functors with smash product for $\mathcal{D} = \Gamma, \mathcal{W}$. We use this identification without further comment henceforth. For $\mathcal{D} = \Gamma, \Sigma$, we note $\mathcal{D}sp$ the category of functors from $\mathcal{D}$ to spectra over based simplicial sets. A simplicial analog of $\mathcal{W}Top$ is the category $\mathcal{S}sp$ of simplicial functors [22], that is functors from the category of based finite simplicial sets to pointed simplicial sets. One notice that Σsp is the category of symmetric spectra of the fundamental paper [16].

In the first part of [26], it is stated that there are adjoint pair of functors

$$\mathbb{P} : \Gamma Top \rightleftarrows \mathcal{W}Top : \mathbb{U}, \qquad \mathbb{P} : \Sigma Top \rightleftarrows \mathcal{W}Top : \mathbb{U}$$

where $\mathbb{U}$ are lax monoidal forgetful functors and $\mathbb{P}$ are their left adjoint strong monoidal prolongation functor.

Theorems 0.4, 0.5 and 0.6 in [26] asserts that the adjunction $P : \Sigma Top \rightleftarrows \mathcal{W}Top : U$ induces a Quillen equivalence of their underlying categories of monoids and also, for any monoid $W \in \mathcal{W}Top$ between the categories of W-modules (resp. W-algebras) and $\mathbb{U}W$-modules (resp $\mathbb{U}W$-algebras). For any cofibrant monoid $R \in \Sigma Top$, there are also Quillen equivalences between the categories of R-modules (resp. R-algebras) and $\mathbb{P}R$-modules (resp $\mathbb{P}R$-algebras). The relevant model categories structures here are the stable ones.

Remark : actually, the adjunction between ΣTop and $\mathcal{W}Top$ in [26] factored through the category of orthogonal spectra and there are Quillen equivalences between the relevant categories built from monoids. Every result below concerning ΣTop can be striaghtforwardly translated to orthogonal spectra and related to ΣTop, the proofs going on *mutatis mutandis*.

There is also an absolute stable model structure on $\mathcal{W}Top$ [26] section 17 and the identity $\mathcal{W}Top \to \mathcal{W}Top$ is a left adjoint of a Quillen equivalence between the stable structure and the absolute stable structure. We will also denote $(\mathbb{U}, \mathbb{P})$ the functors of these Quillen equivalence.

When the category $\mathcal{W}Top$ is equiped with its absolute model structure, the adjunction $\mathbb{P} : \Gamma Top \rightleftarrows \mathcal{W}Top : \mathbb{U}$ yields connective Quillen equivalence between the categories of W-modules (resp. W-algebras) and $\mathbb{U}W$-modules (resp $\mathbb{U}W$-algebras) and, for any cofibrant monoid $R \in \Gamma Top$, between the categories of R-modules (resp. R-algebras) and $\mathbb{P}R$-modules (resp $\mathbb{P}R$-algebras), see [26] 0.11, 0.12, 0.13.

The $thh^L(B, N)$ model of sections 3.1 and 4.1 makes sense without a change in any cofibrantly generated symmetric monoidal model category of spectra. It is also possible to define the classical Bökstedt functor $THH(A, M)$ for a monoid A in $\mathcal{D}sp$ and a A-bimodule M (see [32] for $\mathcal{D} = \Sigma$). However, as we want our models to have Adams operations for A commutative, we have to work with J as an indexing category. The definition given in section 2.3 for $\mathcal{D} = \Gamma$ can be done for $\mathcal{D} = \Sigma$ and $\mathcal{S}sp$ with the following slight modification in the definition of $G(A, M, x)$:

- For $\mathcal{S}sp$, $G(A, x)$ is made a simplicial functor by the formula

$$G(A,M,x)(X) = \mathrm{Map}(S^x, M(S^{x_0}) \wedge A(S^{x_1} \wedge \dots A(S^{x_q}) \wedge X)$$

where Map is the simplicial mapping space.

- For $\mathcal{D} = \Sigma$, and A a monoid in Σsp, we denote by $A(S^y)$ the prolungation $\mathbb{P}A$ applied to the CW-complex S^y $(y \in J)$ and we made $G(A,x)$ a symmetric spectrum by the formula

$$G(A,M,x)_n = \mathrm{Map}(S^x, M(S^{x_0}) \wedge A(S^{x_1}) \wedge \dots A(S^{x_q}) \wedge S^n).$$

If $x_0 = \{1, \dots, n_0\}, \dots x_q = \{1, \dots, n_q\}$, we remark that this coincide with the definition of [32].Section 4.

Of course, the change of index category from I to J does not change the weak homotopy type by the approximation Lemma 2.1.

Replacing the simplicial mapping space by the topologized one in the definition of $G(A,x)$ yields a topological Hochschild homology functor *à la Bökstedt* in ΓTop, ΣTop and $\mathcal{W}Top$ as above.

We first compare diagram spectra of topological spaces and simplicial sets when $\mathcal{D} = \Gamma^{op}$ or Σ. Let $\mathbb{GR} : Simp' \rightleftarrows Top' : \mathbb{TS}$ be the adjunction between the Geometric Realization and the Total Singular complex functors. These functors applied levelwise induce adjoint functors on the categories of $\mathcal{D}$-spectra and their various subcategories of rings and modules with $\mathbb{GR}$ being strong monoidal. Theorem 19.5 of [26] ensures that these functors induce Quillen equivalences between the categories of R-modules and $\mathbb{GR}R$-modules when R is a monoid in the category of symmetric spectra or Γ-spaces based on simplicial sets.

Proposition A.1. *Let B be a cofibrant monoid in $\Sigma\mathcal{S}$ or in Γsp. The functor $\mathbb{GR}$ yields an isomorphism*

$$\alpha : \mathbb{GR}thh^B(A,M) \cong thh^{\mathbb{GR}B}(\mathbb{GR}A, \mathbb{GR}M)$$

for any cofibrant B-algebra A and A-bimodule M which is a morphism of γ-rings if A is commutative and M symmetric. It is also multiplicative if $M = A$.

Proof: First, $\mathbb{GR}$ is Quillen left adjoint, hence preserves cofibrant and cofibrant replacement. Moreover, as $\mathbb{GR}$ is a strong monoidal, it induces a simplicial isomorphism

$$\alpha_* : \mathbb{GR}(M \wedge_B A \wedge_B \dots \wedge_B A) \cong \mathbb{GR}M \wedge_{\mathbb{GR}B} \mathbb{GR}A \wedge_{\mathbb{GR}B} \dots \wedge_{\mathbb{GR}B} \mathbb{GR}A.$$

Hence there is a natural isomorphism $\alpha : \mathbb{GR}thh^B(A,M) \cong thh^{\mathbb{GR}B}(\mathbb{GR}A, \mathbb{GR}M)$. As A is commutative, M symmetric and $\mathbb{GR}$ preserves these symetries, the simplicial morphism α_* factors through the category Γ. By Lemma 2.3 this implies that α is a γ-ring map. When $M = A$, the multiplicativity property is straightforward to check. $\square$

As the geometric realization is functorial and left adjoint there are natural isomorphisms of simplicial objects

$$(q \mapsto \mathbb{GR}THH(A,M)_q) \cong \big(q \mapsto THH(\mathbb{GR}A, \mathbb{GR}M)_q\big)$$

for the categories Γ and Σ. Again, when A is commutative and M symmetric, it factors as an isomorphism of Γ-object and we have:

Proposition A.2. *Let A be a monoid in Σsp or in Γsp and M a A-module. The functor $\mathbb{GR}$ yields an isomorphism*

$$\alpha : \mathbb{GR}THH(A,M) \cong THH(\mathbb{GR}A, \mathbb{GR}M),$$

which is a morphism of γ-rings if A is commutative and M symmetric, multiplicative if $M = A$.

We will now give analogs of propositions A.1, A.2 for $\mathcal{S}sp$ and $\mathcal{W}Top$ (with its absolute stable model structure). Let $\mathcal{V}$ be the category with objects the pointed finite simplicial sets and maps $\mathcal{V}(K_*, L_*)$ the geometric realization of the usual simplicial mapping spaces $\mathrm{Map}(K_*, L_*)$. Recall from [26] 19.11 that there is a diagram of Quillen equivalences

$$\begin{tikzcd} \mathcal{S}sp \arrow[r, shift left, "\mathbb{GR}"] \arrow[d, shift left, "\mathbb{U}"] & \mathcal{V}Top \arrow[l, shift left, "\mathbb{TS}"] \arrow[r, shift left, "\mathbb{P}"] & \mathcal{W}Top \arrow[l, shift left, "\mathbb{U}"] \arrow[d, shift left, "\mathbb{U}"] \\ \Sigma sp \arrow[u, shift left, "\mathbb{P}"] \arrow[rr, shift left, "\mathbb{GR}"] & & \Sigma Top \arrow[ll, shift left, "\mathbb{TS}"] \arrow[u, shift left, "\mathbb{P}"] \end{tikzcd}$$

with $\mathbb{P}$ being monoidal left adjoints and $\mathbb{U}$ lax monoidal and right adjoints.

Proposition A.3. *Given a commutative monoid R in $\mathcal{S}sp$, A a R-algebra and M a A-bimodule there are natural isomorphisms*

$$\mathbb{PGR}thh^R(A, M) \cong thh^{\mathbb{PGR}R}(\mathbb{PGR}A, \mathbb{PGR}M) \text{ if } A,\ R \text{ are cofibrants,}$$

$$\mathbb{PGR}THH(A, M) \cong THH(\mathbb{PGR}A, \mathbb{PGR}M)$$

which are γ-ring maps if A is commutative and M symmetric, multiplicative if $M = A$.

Proof: The composite functor $\mathbb{PGR}$ is a Quillen left adjoint and strong monoidal. Mimicking the proofs of propositions A.1, A.2 yields the result. □

Remark : Recall from section 1 the well known natural prolongation functor from Γ-spaces to endofunctors of pointed simplicial sets. In particular the prolongation functor $\mathbb{P} : \Gamma sp \to \mathcal{S}sp$ which identifies Γ-spaces with special kinds of simplicial functors [22].5.8. It has the obvious restriction functor as right adjoint and is strong monoidal. We have a diagram of adjoint pairs

$$\begin{tikzcd} \mathcal{S}sp \arrow[r, shift left, "\mathbb{GR}"] \arrow[d, shift left, "\mathbb{U}"] & \mathcal{V}Top \arrow[l, shift left, "\mathbb{TS}"] \arrow[r, shift left, "\mathbb{P}"] & \mathcal{W}Top \arrow[l, shift left, "\mathbb{U}"] \arrow[d, shift left, "\mathbb{U}"] \\ \Gamma sp \arrow[u, shift left, "\mathbb{P}"] \arrow[rr, shift left, "\mathbb{GR}"] & & \Gamma Top \arrow[ll, shift left, "\mathbb{TS}"] \arrow[u, shift left, "\mathbb{P}"] \end{tikzcd}$$

which is commutative for the right adjoints, hence commutative up to isomorphism for the left adjoints. The top and bottom lines are Quillen equivalences by [26] Theorem 19.11, Theorem 19.4 and the right one is a connective Quillen equivalence (aka a Quillen pair inducing an equivalence on the homotopy categories of connective objects). Moreover direct inspection shows that the functor $\mathbb{U} : \mathcal{S}sp \to \Gamma sp$ preserves fibrations and acyclic fibrations. Therefore one has the following result.

Proposition A.4. *There is a connective Quillen equivalence $\mathbb{P} : \Gamma sp \rightleftarrows \mathcal{S}sp : \mathbb{U}$ with $\mathbb{P}$ strong monoidal.*

Having related the topological Hochschild homology functor between simplicial based and topological based diagram categories, we turn to the comparison of this functor between $\mathcal{D}$-spaces and $\mathcal{D}'$-spaces.

Proposition A.5. *Given any commutative monoid B in $\mathcal{D}Top$ or $\mathcal{D}sp$, R a B-algebra and M a R-bimodule, then there are natural isomorphisms*

$$\mathbb{P}thh^B(R, M) \cong thh^{\mathbb{P}B}(\mathbb{P}R, \mathbb{P}M) \text{ for cofibrant monoids } B \text{ and } R$$

$$\text{and } \mathbb{P}THH(R,M) \cong THH(\mathbb{P}R, \mathbb{P}M).$$

These isomorphims are γ-ring map if R is commutative, M symmetric and multiplicative if $M = R$.

Proof: The functors $\mathbb{P}$ are strong monoidal and preserve cofibrants (they are left Quillen adjoint), hence as in the proof of A.1 we get the statement for the thh-model.

As the functors $\mathbb{P}$ are left adjoint, they commute with colimits and we are left to consider the simplicial object $\mathbb{P}G_*(A,M,x)$. For any $q \geq 0$, $G_q(A,M,x)$ is the $\mathcal{D}$-object obtained from the simplicial (resp. topological) mapping space $\mathrm{Map}(S^x, \mathbb{P}M(S^{x_0}) \wedge \mathbb{P}A(S^{x_q}))$ by prolongation. Moreover, by definition, the simplicial sets (resp. topological spaces) $\mathbb{P}M(S^{x_0})$ and $\mathbb{P}A(S^{x_i})$ are exactly the same than $M(S^{x_0})$ and $A(S^{x_i})$. The natural isomorphism $\beta : \mathbb{P}THH(R,M) \cong THH(\mathbb{P}R, \mathbb{P}M)$ follows. If in addition R is commutative and M symmetric, the transformation β becomes a Γ-natural transformation and we conclude as for A.1. $\square$

We now compare the Adams operations between the THH model and the thh one in any diagram categories. Shipley [32] 4.2.8 has shown the equivalence in the category Σsp. From her result we get the next proposition.

Proposition A.6. *Let $\mathcal{D}$ be $\Gamma, \Sigma, \mathcal{W}$ (resp. $\Gamma, \Sigma, \mathcal{S}$) and R be a cofibrant monoid in $\mathcal{D}Top$ (resp. $\mathcal{D}sp$) and M a R-bimodule, then there is a natural isomorphisms in the homotopy category* $\mathrm{Ho}\mathcal{D}Top$ *(resp.* $\mathrm{Ho}\mathcal{D}sp$*)*

$$thh(R,M) \cong THH(R,M)$$

of γ-rings if R is commutative and M symmetric, multiplicative if $M = R$.

Proof: As Quillen equivalences give equivalences of homotopy categories, the propositions A.1,A.2, A.3 and A.6 shows that it is enough to prove the result in one category. We do it for $\mathrm{Ho}\Sigma sp$.

We know from Shipley [32] Theorem 4.2.8 that for a cofibrant commutative monoid R and M a R-bimodule, there is a zigzag of stable equivalences between $thh(R,M)$ and $THH(R,M)$. This zigzag is induced by the following zigzag of simplicial symmetric spectra:

$$R^{\wedge(*+1)} \xrightarrow{i} L(R^{\wedge(*+1)}) \xrightarrow{j} ML(R^{\wedge(*+1)}) \xleftarrow{h} DL(R^{\wedge(*+1)}) \xleftarrow{Di} D(R^{\wedge(*+1)})$$

together with $THH(R)_* \xrightarrow{\theta_*} D(R^{\wedge(*+1)})$.

The functors L, M and D are defined in [32] section 3 (replacing the index category I by the category J in the definitions whenever it is necessary):

- L is the fibrant replacement functor,
- $DX = (DX_n)_{n \in \mathbb{N}}$ is the symmetric spectrum
$$D(X)_n = \operatorname*{hocolim}_{y \in J} \mathrm{Map}(S^y, X(S^y) \wedge S^n),$$
- MX is the symmetric spectrum $M(X)_n = \operatorname*{hocolim}_{y \in J} \mathrm{Map}(S^y, X(S^y \wedge S^n))$.

The simplicial functor $M \wedge R^{\wedge *}$ factors through the category Γ hence $M \wedge R^{\wedge *}$, $L(M \wedge R^{\wedge *})$, $ML(M \wedge R^{\wedge *})$, $DL(M \wedge R^{\wedge *})$ and $D(M \wedge R^{\wedge *})$ inherit Adams operations and the previous zigzag is a zigzag of γ-rings by naturality thanks to Lemmas 2.2 and 2.3. The only difficulty is for the transformation $D \to M$ with our indexing category J. But the compatibility is direct by inspection of the prolungation functors.

Denoting $\mu : J^{\times *} \to J$ the concatenation, the map Θ_* is the composition

$$\operatorname*{hocolim}_{x \in J^{q+1}} G(R,M,x) \to \operatorname*{hocolim}_{x \in J^{q+1}} G((M \wedge R^{\wedge q}, \mu(x)) \to D(M \wedge R^{\wedge q}).$$

The arguments of the proof of proposition 2.2 shows that Θ_* is a γ-ring map whenever R is commutative and M symmetric, multiplicative if $M = R$. As the zigzag with index category I is a zigzag of weak equivalences, see [32] Theorem 4.2.8, application of the approximation lemma 2.1 yields that the same is true with index category J. The result in $\mathrm{Ho}\Sigma sp$ is then immediate. □

Finally we bring in the category of S-modules of [12] in our comparison. It was proved by Schwede in [30] section 5 that there exists a Quillen equivalence $\Lambda : \Sigma sp \leftrightarrows S - \mathrm{mod} : \Psi$ (with Λ strong monoidal) inducing equivalences between the categories of monoids and the categories of modules over cofibrant symmetric ring spectra. The relevant model structures are the positive ones for symmetric spectra which are Quillen monoidal equivalent to stable ones with the identity Id as strong monoidal left adjoint see [30], [26] section 14. Mimicking the proof of proposition A.5 we get

Proposition A.7. *Given any commutative cofibrant monoid B in Σsp, R a cofibrant B-algebra and M a R-bimodule, there are natural isomorphisms*

$$\Lambda thh^B(R, M) \cong thh^{\Lambda B}(\Lambda R, \Lambda M), \qquad \mathrm{Id}\left(thh^B(R, M)\right) \cong thh^{\mathrm{Id}(B)}(\mathrm{Id}\,(R)\,, \mathrm{Id}\,(M))$$

which are γ-ring maps if R is commutative, M symmetric, multiplicative if $M = R$.

The category Σsp is tensored over topological spaces too, hence there is a $R \otimes S^1$ model for the topological Hochschild homology in Σsp as in section 4.3.

Corollary A.8. *Given any commutative cofibrant monoid B in Σsp there is a natural isomorphism of γ-rings $\Lambda B \otimes S^1 \cong \Lambda B \otimes S^1$.*

Proof: As Λ is left adjoint it commutes with colimit hence:

$$|\Lambda(B \otimes S^1_*)| \cong \Lambda|B \otimes S^1_*| = \Lambda(B \otimes S^1) \text{ and}$$
$$\Lambda B \otimes S^1 = |\Lambda B \otimes S^1_*| \cong |\Lambda(B \otimes S^1_*)|$$

by proposition A.7. □

Combining the previous results of this appendix we had the following "meta" corollary

Corollary A.9. *The various models describing the γ-rings structure of topological Hochschild homology of a commutative monoid in a category of structured ring spectra and a symmetric bimodule are isomorphic in homotopy categories.*

Remark : In any symmetric monoidal closed model category of spectra, one can simply define topological Hochschild homology as the derived object $\mathbb{C}(M) \wedge_{A \wedge A^{op}} A$ with $\mathbb{C}(M)$ a cofibrant replacement of M as a $A \wedge A^{op}$-module. We hav not considered them because we have no model for the Adams operations on it. Nevertheless the proofs of A.1, A.2, A.3 shows that the previous prolungation functors $\mathbb{P}$ and Λ induced isomorphisms for this model which are already known to be equivalent to the thh-model ones (see [12].IX.2, [32] Section 4).

Appendix B.

In this appendix we recall a few results and notations about γ-rings. We refer to [15], [18] for detailed treatment.

Adams operations on a ring R are a family $(\varphi^k)_{k \geq 0}$ of self maps $R \to R$ satisfying various relations of compatibility with the ring structure of R. We are only interested in the case of a ring with the zero mutliplication, the one relevant for the algebraic Hochschild complex see [20]. Moreover we follow the "more geometric" sign convention introduced in [23]. In this context we can make the following definition:

a family of set maps $(\varphi^k : R \to R)_{k\geq 0}$ on a ring R with trivial multiplication gives to R a structure of *γ-ring* if

a): $\varphi^0 = 0$ and $\varphi^1 = \mathrm{id}$,
b): $\varphi^k(x+y) = \varphi^k(x) + \varphi^k(y)$ for any $k \geq 0,\ , x, y \in R$,
c): $\varphi^k(\varphi^\ell) = \varphi^{k\ell}$ for any $k, l \geq 0$.

We will also write that the maps $(\varphi^k)_{k\geq 0}$ are *Adams operations* for the ring with trivial multiplication R.

To every γ-ring (R, Φ^k) we can associate a natural filtration (see [15] , [18]) $F_n^\gamma R$ $(n \geq 0)$ defined by

$$F_p^\gamma X = \langle \gamma^{p_1}(x_1)...\gamma^{p_s}(x_s)\ ;\ x_1,..,x_s \in R \text{ and } p_1 + ... + p_s \geq p\rangle$$

where $\gamma^k = \sum_{i=0}^{k-1}(-1)^i\binom{k-i}{i}\Phi^{k-i}$. The notation $\langle y_1...y_p\rangle$ stands for the abelian group generated by monomials $y_1...y_p$. This filtration has the property that, for $n \geq 1$, if $x \in F_n^\gamma R$, then $\varphi^k(x) \equiv -k^{n-1}x$ modulo $F_{n+1}^\gamma R$.

When the ring R admits another (non-trivial in general) product structure we will say that a family of Adams operations $(\varphi^k)_{k\geq 0}$ makes R a *multiplicative γ-ring* (with trivial mutliplication) if (R, φ^k) is a γ-ring with trivial multiplication and the maps φ^k are ring maps for the second multiplication. The standard example is the Hochschild cochain complex with the maps λ^k of [20] together with the shuffle product. In this paper, a γ-ring structure for a spectra X means a γ-ring structure on $\pi_*(X)$ (with trivial multiplication). If X is a ring spectra, this structure is multiplicative if the multiplication of X induced a multiplicative γ-ring structure on $\pi_*(X)$.

A spectral sequence of γ-rings is a spectral sequence E^*_{**} for which each level E^p_{**} is a γ-ring with Adams operations $(\varphi^{q,k})_{k\geq 0}$ and with the property that the operations $(\varphi^{q+1,k})_{k\geq 0}$ are induced by the operations $(\varphi^{q,k})_{k\geq 0}$ when passing to homology. It converges as a γ-ring if the abutment has a γ-ring structure which is the one of E^∞_{**} on the associated graded. Let $C_* = F_0C_* \supset F_1C_* \supset \dots$ be a filtred complex endowed with Adams operations φ^k which are complex maps and assume $\varphi^k F_qC_* \subset F_qC_*$ for any $k, q \geq 0$. As the differential is compatible with the maps φ^k, the homology of C is also a γ-ring with structures self-maps given by the homology of the φ^k. Moreover the associated graded of C_* inherit a structure of γ-ring and we have.

Lemma B.1. *Let C be a complex endowed with Adams operations φ^k compatible with the differential. Given any filtration of C_* compatible with the Adams operations, the associated spectral sequence $E^*_{**} \Rightarrow H_*(C)$ is a spectral sequence of γ-rings.*

Similarly an exact couple

$$\begin{tikzcd} D^r_{**} \arrow[rr, "i"] & & D^r_{**} \arrow[dl, "j"] \\ & E^R_{**} \arrow[ul, "k"] & \end{tikzcd}$$

of γ-rings where i, j, k are γ-rings maps gives a spectral sequence of γ-rings.

Vanishing conditions for the natural filtration F^γ of a γ-ring on the pieces of a spectral sequence of γ-rings can be transfered to the abutment in some cases.

Proposition B.2. *Let $E^r_{p,q} \Longrightarrow H_{p+q}$ be a converging bounded below spectral sequence of γ-rings. Assume that*

$$\forall n \geq 0\, \exists M_n \geq 0 \text{ such that } \forall m \geq M_n\ \ \oplus_{p+q=n} F_m^\gamma E^\infty_{p,q} = 0.$$

Then, for all $m \geq M_n$ one has $F_m^\gamma H_n = 0$.

Proof: Write $H_*^{(p)}$ for the filtration of H_* corresponding to the spectral sequence. Assume $x \in H_n^{(p)}$ and $[x]$ is its class in $E^\infty_{p,n-p}$. If $x \in F^\gamma_m H_n$ then for all $k \geq 1$, one has

$$\varphi^k(x) = k^m x \text{ modulo } F^\gamma_{m+1} H_n.$$

and then $[x] \in F^\gamma_m E^\infty_{p,n-p}$ for all $k \geq 1$, hence is trivial by hypothesis. It implies that $x \in H_n^{(p-1)}$ and by induction one get the statement. □

References

[1] F. Atiyah; D. Tall, *Group representations, λ-rings and the J-homomorphism*, . Topology 8 1969 253–297.

[2] M. Bökstedt, *Topological Hochschild homology*, Preprint Bielefeld, 1986.

[3] M. Bökstedt, *Topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$*, Preprint Bielefeld, 1986.

[4] M. Böstedt; W. Hsiang; I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. 111 (1993), no. 3, 465–539.

[5] A. Bousfield; D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[6] M. Brun, *Topological Hochschild homology of $\mathbb{Z}/p^n$*, J. Pure Appl. Algebra 148 (2000), no. 1, 29–76. Topology 29 (1990), no. 3, 317–344.

[7] B. Dundas, *The homology of additive categories*, Preprint, avalaible at http://www.math.ntnu.no/~dundas

[8] B. Dundas, *The Cyclotomic trace for symmetric monoidal categories* in "Geometry and Topology": Aarhus, Contemp. Math. 258, p. 121–144.

[9] B. Dundas, T. Goodwillie, R. McCarthy, *The local structure of algebraic K-theory*, avalaible at http://www.math.ntnu.no/~dundas

[10] B. Dundas, R. McCarthy, *Stable K-theory and topological hochschild homology*, Ann. of Math. 140 (1994) 685–701.

[11] B. Dundas, R. McCarthy, *Topological Hochschild homology of ring functors and exact categories*, J. Pure Appl. Algebra 109 (1996), no. 3, 231–294.

[12] A. Elmendorf; I. Kriz; M. Mandell; J.P.May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, 47. American Mathematical Society, Providence, RI, 1997.

[13] M. Gerstenhaber; S. Schack, *A Hodge-type decomposition for commutative algebra cohomology*, J. Pure Appl. Algebra 48 (1987), no. 3, 229–247.

[14] L. Hesselholt; I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology 36 (1997), no. 1, 29–101.

[15] H. Hiller, *λ-rings and algebraic K-theory.* J. Pure Appl. Algebra 20 (1981), no. 3, 241–266.

[16] M. Hovey; B. Shipley; J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. 13 (2000), no. 1, 149–208.

[17] M. Jibladze, T. Pirashvili, *Cohomology of algebraic theories*, Journal of ALgebra, 137(2) (1991) 253-296.

[18] C. Kratzer, *λ-structure en K-théorie algébrique.* Comment. Math. Helv. 55 (1980), no. 2, 233–254.

[19] J.L. Loday, *Cyclic homology*, Springer-Verlag, Berlin, 1998.

[20] J.L. Loday, *Opérations sur l'homologie cyclique des algèbres commutatives.* Invent. Math. 96 (1989), no. 1, 205–230.

[21] M. Lydakis, *Smash products and Γ-spaces*, Math. Proc. Cambridge Philos. Soc. 126 (1999), no. 2, 311–328.

[22] M. Lydakis, *Simplicial functors and stable homotopy theory*, preprint

[23] R. McCarthy, *On operations for Hochschild homology*, Comm. Algebra 21 (1993), no. 8, 2947–2965.

[24] J. McClure; R. Schwänzl; R. Vogt, *$THH(R) \cong R \otimes S^1$ for E_∞ ring spectra*, J. Pure Appl. Algebra 121 (1997), no. 2, 137–159.

[25] M. McCord; *Classifying spaces and infinite symmetric products* Trans. Amer. Math. Soc. 146 (1969), 273–298.

[26] M. Mandell; J.P. May; S. Schwede; B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) 82 (2001), no. 2, 441–512.

[27] T. Pirashvili; F. Waldhausen, *Mac Lane homology and topological Hochschild homology*, J. Pure Appl. Algebra 82 (1992), no. 1, 81–98.

[28] A. Robinson, *Derived tensor products in stable homotopy theory*, Topology 22 (1983), no. 1, 1–18.

[29] S. Schwede, *Stable homotopical algebra and Γ-spaces*, Math. Proc. Cambridge Philos. Soc. 126 (1999), no. 2, 329–356.
[30] S. Schwede, *S-modules and symmetric spectra*, Math. Ann. 319 (2001), no. 3, 517–532
[31] G. Segal, *Categories and cohomology theories*, Topology 13 (1974), 293–312.
[32] B. Shipley, *Symmetric spectra and topological Hochschild homology*, *K*-Theory 19 (2000), no. 2, 155–183.

Grégory Ginot, Centre de Mathématiques et de Leurs Applications, ENS Cachan, and Laboratoire Analyse Géométrie et Applications, Université Paris 13, 61, av du Psdt Wilson, 94230 Cachan Cedex, France
E-mail address: ginot@cmla.ens-cachan.fr