Cohomology of Lie 2-groups

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Abstract

In this paper we study the cohomology of (strict) Lie 2-groups. We obtain an explicit Bott-Shulman type map in the case of a Lie 2-group corresponding to the crossed module $A \to 1$. The cohomology of the Lie 2-groups corresponding to the universal crossed modules $G \to \mathrm{Aut}\,(G)$ and $G \to \mathrm{Aut}^+(G)$ is the abutment of a spectral sequence involving the cohomology of $GL(n, \mathbb{Z})$ and $SL(n, \mathbb{Z})$. When the dimension of the center of G is less than 3, we compute explicitly these cohomology groups. We also compute the cohomology of the Lie 2-group corresponding to a crossed module $G \xrightarrow{i} H$ for which $\mathrm{Ker}(i)$ is compact and $\mathrm{Coker}(i)$ is connected, simply connected and compact and apply the result to the *string* 2-group.

1 Introduction

This paper is devoted to the study of Lie 2-group cohomology. A Lie 2-group is a Lie groupoid $\Gamma_2 \rightrightarrows \Gamma_1$, where both the space of objects Γ_1 and the space of morphisms are Lie groups and all the groupoid structure maps are group morphisms. This is what is usually refer to as "groupoids over groups". It is well known that Lie 2-groups are equivalent to crossed modules [11, 2]. By a

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crossed module, we mean a Lie group morphism $G \xrightarrow{i} H$ together with a right action of H on G by automorphisms satisfying certain compatibility conditions. In this case, $\mathrm{Ker}\, i$ is called the kernel, while $H/i(G)$ is called cokernel of the crossed module.

Lie 2-groups arise naturally in various places in mathematical physics, for instance, in higher gauge theory [4]. They also appeared in the theory of non-abelian gerbes. As was shown by Breen [8, 9] (see also [15]), a G-gerbe is equivalent to a 2-group principal bundle in the sense of Dedecker [13], where the structure 2-group is the one corresponding to the crossed module $G \xrightarrow{i} \mathrm{Aut}\,(G)$ with i denoting the map to the inner automorphisms.

As in the 1-group case, associated to any Lie 2-group Γ, there is a simplicial manifold $N_\bullet\Gamma$, called the nerve of the 2-group. Thus one defines the cohomology of a Lie 2-group Γ with trivial coefficients $\mathbb{R}$ as the cohomology of this simplicial manifold $N_\bullet\Gamma$ with coefficients $\mathbb{R}$. The latter can be computed using a double de Rham chain complex. A very natural question arises as to whether there is a Bott-Shulman type map [6, 7] for such a Lie 2-group. Unfortunately, the answer seems to be out of reach in general. However, we are able to describe a class of cocycles in $\Omega^{3r}([G \xrightarrow{i} H])$ generated by elements in $S\big((\mathfrak{g}^*)^{\mathfrak{g},H}[3]\big)$, the symmetric algebra on the vector space $\big((\mathfrak{g}^*)^{\mathfrak{g},H}\big)$ with degree 3. Here we denote by $[G \xrightarrow{i} H]$ the Lie 2-group corresponding to the crossed module $G \xrightarrow{i} H$. As a consequence, we explicitly describe, for any abelian group A, cocycles in $\Omega^\bullet([A \to 1])$ which generate the cohomology group $H^\bullet([A \to 1])$. These cocycles are given by skew-symmetric polynomial functions on the Lie algebra $\mathfrak{a}$ of A. Such an explicit map is also obtained in the case when the cokernel of $G \to H$ is finite. Our approach is based on the following idea. A Lie 2-group $[G \to H]$ induces a short exact sequence of Lie 2-groups:

$$1 \to [\mathrm{Ker}\, i \to 1] \to [G \to H] \to [1 \to \mathrm{Coker}\, i] \to 1$$

which in turn induces a fibration of 2-groups. As a consequence, we obtain a Leray-Serre spectral sequence. Discussions on these topics occupy Sections 4-5.

We also use the spectral sequence to compute the cohomology of a 2-group $[G \xrightarrow{i} H]$ with connected and simply connected compact cokernel $\mathrm{Coker}(i) \cong H/i(G))$ and compact kernel $\mathrm{Ker}(i)$. In general, the cohomology of $[G \xrightarrow{i} H]$ depends on a transgression homomorphism

$$T : H^3([\mathrm{Ker}(i) \to 1]) \to H^4([1 \to H/i(G)]).$$

An example of such 2-group is given by the string 2-group [3] for which we recover computations also idependently due to Baez and Stenvenson [5].

Next we apply our result to study the cohomology of particular classes of 2-groups: $[G \xrightarrow{i^+} \mathrm{Aut}^+(G)]$ and $[G \xrightarrow{i} \mathrm{Aut}\,(G)]$, where $\mathrm{Aut}^+(G)$ is the orientation preserving automorphism group of G. If G is a semi-simple Lie group, the result is immediate since both kernel and cokernel are finite groups. However when G is a general compact Lie group, the situation becomes much subtler. This is due to the fact that the connected component of the center $Z(G)$ is a torus T^n, and therefore $\mathrm{Out}^+(G)$ and $\mathrm{Out}\,(G)$ are no longer finite groups. Indeed they are closely related to $SL(n,\mathbb{Z})$ and $GL(n,\mathbb{Z})$, whose cohomology groups are in general very difficult to compute, and still remains an open question for large n. Nevertheless, we obtain a spectral sequence involving cohomology of these groups, converging to the cohomology of the 2-group. For $n \leq 3$, using a result of Soulé [25], we are able to compute the cohomology groups explicitly.

One of the main motivations for studying cohomology of 2-groups is to study characteristic classes of gerbes. Since G-gerbes correspond to principal $[G \xrightarrow{i} \mathrm{Aut}\,(G)]$-bundles, any nontrivial cohomology class in $H^\bullet([G \xrightarrow{i} \mathrm{Aut}\,(G)])$ defines a universal characteristic class for G-gerbes. And a Bott-Shulman type cocycle allow one to express such a universal characteristic class in terms of geometric data such as connections just like in the usual Chern-Weil theory. This will be discussed in detail in [15].

Note that the constructions in this paper can be defined in the more general context of weak Lie 2-groups as defined by Henriques in [16] since the cohomology and homotopy groups are defined using the nerve.

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Notations : Given a (graded) vector space V we denote by $V[k]$ the graded vector space with shifted grading $\left(V[k]\right)^n = V^{n-k}$. Thus if V is concentrated in degree 0, $V[k]$ is concentrated in degree k. The graded symmetric (or free commutative) algebra on a graded vector space V will be denoted by $S(V)$. We write $S(V)^q$ for the subspace of homogeneous elements of total degree q, that is, $S(V)^q = \{x_1 \ldots x_r \in S^r(V) \,/\, r \geq 0 \text{ and } |x_1| + \cdots + |x_r| = q\}$. In particular, if $x \in S(V)^p$ and $y \in S(V)^q$, on has $x \cdot y = (-1)^{pq} y \cdot x$. Thus if V is concentrated in even degrees, $S(V)$ is a polynomial algebras. On the other hand, if V is concentrated in odd degrees, $S(V)$ is an exterior algebra.

Unless otherwise stated, all cohomology groups are taken with real coefficients.

2 Crossed modules

A *crossed module* of Lie groups is a Lie group morphism $G \xrightarrow{i} H$ together with a right H-action $(h, g) \to g^h$ of H on G by Lie group automorphisms satisfying:

1. for all $(h, g) \in G \times H$, $i(g^h) = h^{-1} i(g) h$;
2. for all $(x, y) \in G \times G$, $x^{i(y)} = y^{-1} x y$.

A (strict) morphism $\left(G_2 \xrightarrow{i_2} H_2\right) \to \left(G_1 \xrightarrow{i_1} H_1\right)$ of crossed modules is a pair $(\phi : G_2 \to G_1, \psi : H_2 \to H_1)$ of Lie group morphisms such that $\psi \circ i_2 = i_1 \circ \phi$ and $\phi(g)^{\psi(h)} = \phi(g^h)$ for all $g \in G_2$, $h \in H_2$.

There is a well known equivalence of categories between the category of crossed modules and the category of (strict) Lie 2-groups [11]. Recall that a Lie 2-group is a group object in the category of Lie groupoids meaning it is a Lie groupoid $\Gamma_2 \rightrightarrows \Gamma_1$ where, both, Γ_2 and Γ_1 are Lie group and all structure maps are Lie group morphisms. Such a 2-group will be denoted by $\Gamma_2 \rightrightarrows \Gamma_1 \rightrightarrows \{*\}$. The crossed module $G \xrightarrow{i} H$ gives rise to the 2-group $G \rtimes H \rightrightarrows H \rightrightarrows \{*\}$. The groupoid $G \rtimes H \rightrightarrows H$ is the transformation groupoid: the source and target maps $s, t : G \rtimes H \to H$ are respectively given by $s(g,h) = h$, $t(g,h) = h \cdot i(g)$. The (so called vertical) composition is $(g,h) \star (g', h \cdot i(g)) = (gg', h)$. The group structure on H is the usual one while the group structure (the so called horizontal composition) on $G \rtimes H$ is the semi-direct product of Lie groups: $(g,h) * (g',h') = (g^{h'}g', hh')$. Conversely, there is a crossed module associated to any Lie 2-group [11]. In the sequel we make no distinctions between crossed modules and 2-groups. We use the short notation $[G \xrightarrow{i} H]$ for the Lie 2-group corresponding to a crossed module $G \xrightarrow{i} H$.

Definition 2.1. Let $(\phi, \psi) : \big(G_2 \xrightarrow{i_2} H_2\big) \to \big(G_1 \xrightarrow{i_1} H_1\big)$ be a morphism of crossed modules with ψ being a submersion. The *kernel* of the map (ϕ,ψ) is, by definition (see [21]), the crossed module $(G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1)$ where $\tilde{i}$ is the natural group morphism induced by i_2 and ϕ. The $H_2 \times_{H_1} G_1$-action on G_2 is induced by the H_2-action: $g_2^{(h_2,g_1)} = g_2^{h_2}$. The structure map $H_2 \times_{H_1} G_1 \to H_2$ induces a natural crossed module morphism $(G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1) \to \big(G_2 \xrightarrow{i_2} H_2\big)$.

A Lie group G can be seen as a Lie 2-group with trivial 2-arrows, *i.e.* as the Lie 2-group $G \rightrightarrows G \rightrightarrows \{*\}$. The associated crossed module is $1 \to G$. It yields an embedding of the category of Lie groups in the category of Lie 2-groups. Similar to the case of a group, associated to a Lie 2-group $\Gamma : \Gamma_2 \rightrightarrows \Gamma_1 \rightrightarrows \{*\}$, there is a simplicial manifold $N_\bullet\Gamma$, called its (geometric) nerve. It is the nerve of the underlying 2-category as defined by Street [26]. In particular, $N_0\Gamma = \{*\}$, $N_1\Gamma = \Gamma_1$ and $N_2\Gamma$ consists of 2-arrows of Γ_2 fitting in a commutative square:

$$\begin{tikzcd} & A_1 \arrow[dr, "f_0"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ A_0 \arrow[ur, "f_2"] \arrow[rr, "f_1"'] & & A_2 \end{tikzcd} \tag{2.1}$$

$N_2\Gamma$ is naturally a submanifold of $\Gamma_2 \times \Gamma_1 \times \Gamma_1 \times \Gamma_1$. For $p \geq 3$, an element of $N_p\Gamma$ is a p-simplex (labelled by arrows of Γ) such that each subsimplex of dimension 3 is a commutative tetrahedron, whose faces are given by elements of $N_2\Gamma$ (see (3.5) below or [21, 20, 26]). Also see Remark 3.7 below.

The nerve $N_\bullet$ defines a functor from the category of Lie 2-groups to the category of simplicial manifolds. The nerve of a Lie group considered as a Lie 2-group is isomorphic to the usual (1-)nerve [23]. Taking the fat realization of the nerve defines a functor from Lie 2-groups to topological spaces. In particular, the homotopy groups of a Lie 2-group can be defined as the homotopy groups of its nerve.

Note that Lie 2-groups embed evidently in the category of weak Lie 2-groupoids (see for instance [2] and [16]). There is a notion of fibration for (weak) Lie 2-groups due to Henriques [16, Section 2 and 4] (see also [27, 28] as well). We also refere the reader to [20, 21] for an excellent exposition in the case of discrete 2-groups. In the present paper, however, we only use a special kind of fibrations, which is given by the following lemma:

Lemma 2.2. *Let* $(\phi,\psi) : \big(G_2 \xrightarrow{i_2} H_2\big) \to \big(G_1 \xrightarrow{i_1} H_1\big)$ *be a morphism of crossed modules with* ϕ *and* ψ *being surjective submersions. Then* $(\phi,\psi) : [G_2 \xrightarrow{i_2} H_2] \to [G_1 \xrightarrow{i_1} H_1]$ *is a fibration of Lie 2-groups. The kernel of the morphism* (ϕ,ψ) *(as in Definition 2.1),* i.e. *the Lie 2-group* $[G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1]$*, is a homotopy fiber of* (ϕ,ψ) *and is equivalent to* $[ker(\phi) \xrightarrow{i_2} ker(\psi)]$*.*

Proof. Let Γ_1 and Γ_2 be the Lie 2-groups corresponding to the crossed modules $(G_1 \xrightarrow{i_1} H_1)$ and $(G_2 \xrightarrow{i_2} H_2)$ respectively, and $\Phi : \Gamma_2 \to \Gamma_1$ the map induced by $(\phi,\psi) : \big(G_2 \xrightarrow{i_2} H_2\big) \to \big(G_1 \xrightarrow{i_1} H_1\big)$. Since ϕ and ψ are surjective submersions, $N_m\Phi : N_m\Gamma_2 \to N_m\Gamma_1$ is a surjective submersion for all m. Since Γ_2 and Γ_1 are (strict) Lie 2-groups, their nerves $N_\bullet\Gamma_2$ and $N_\bullet\Gamma_1$ are simplicial manifolds satisfying the Kan condition for simplicial manifolds as in [16, Definiton 1.2 and Definition 1.4]. Thus, for all m,j, the canonical map $N_m\Gamma_2 = Hom(\Delta^m_\bullet, N_\bullet\Gamma_2) \to Hom(\Lambda[m,j]_\bullet, N_\bullet\Gamma_2)$ are surjective submersions for $m \leq 2$ and diffeomorphisms for $m > 2$. Here $\Delta^m_\bullet$ is the simplicial m-simplex and $\Lambda[m,j]_\bullet$ its jth-horn, *i.e.*, the subcomplex generated by all facets containing the jth-vertex. The same results holds when Γ_2 is replaced by Γ_1.

The map $N_m\Gamma_2 = Hom(\Delta^m_\bullet, N_\bullet\Gamma_2) \to Hom(\Lambda[m,j]_\bullet, N_\bullet\Gamma_2)$ and the map $N_m\Gamma_2 = Hom(\Delta^m_\bullet, N_\bullet\Gamma_2) \to Hom(\Delta^m_\bullet, N_\bullet\Gamma_1) = N_m\Gamma_1$ induced by $\Phi : \Gamma_2 \to \Gamma_1$ yields, for all j, a smooth map from $N_m\Gamma_2$ to the space $C[m,j]$, which consists of the commutative squares

$$\begin{tikzcd}
\Lambda[m,j]_\bullet \arrow[r] \arrow[d] & N_\bullet\Gamma_2 \arrow[d, "N_\bullet\Phi"] \\
\Delta^m_\bullet \arrow[r] & N_\bullet\Gamma_1
\end{tikzcd}$$

See [16, Definition 2.3]. Note that $C[m,j]$ can be identified with with the fiber product $Hom(\Lambda[m,j]_\bullet, N_\bullet\Gamma_2) \times_{Hom(\Lambda[m,j]_\bullet, N_\bullet\Gamma_1)} Hom(\Delta^m_\bullet, N_\bullet\Gamma_1)$. By the definition of a fibration [16, Definition 2.3], it suffices to prove that (for all m,j) the map $N_m\Gamma_2 \to C[m,j]$ is a surjective submersion . For $m > 2$, $Hom(\Lambda[m,j]_\bullet, N_\bullet\Gamma_2) \cong N_m\Gamma_2$. Thus $C[m,j] \cong N_m\Gamma_2$ and we are done. For $m = 1$, $C[1,j] \cong H_1$ and the map $N_1\Gamma_2 \cong H_2 \to C[1,j] \cong H_1$ is ψ. For $m = 2$, $C[2,j]$ is identified with $H_2^{\times 2} \times G_1$ and the map $N_2\Gamma_2 \to C[2,j]$becomes $H_2^{\times 2} \times G_2 \xrightarrow{id^{\times 2}\times\phi} H_2^{\times 2} \times G_1$. The latter is a surjective submersion since ϕ is a submersion.

The fiber $F_\bullet$ of $N_\bullet\Phi$ is the pullback $pt_\bullet \times_{N_\bullet\Gamma_1} N_\bullet\Gamma_2$ where $pt_\bullet = N_\bullet[1 \to 1]$ is the point (viewed as a constant simplicial manifold). Thus, $F_\bullet$ is the nerve of the Lie 2-groups $[ker(\phi) \xrightarrow{i_2} ker(\psi)]$. Here the crossed module structure of $ker(\phi) \xrightarrow{i_2} ker(\psi)$ is induced by that of $G_2 \xrightarrow{i_2} H_2$. The inclusions $ker(\phi) \hookrightarrow G_2$ and the map $ker(\psi) \hookrightarrow H_2 \xrightarrow{id\times 1} H_2 \times_{H_1} G_1$ yield a crossed module homomorphism $\big(ker(\phi) \xrightarrow{i_2} ker(\psi)\big) \to \big(G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1\big)$, which is an equivalence of crossed modules. See [2, 20, 21, 27, 15] for the definition of equivalence of crossed modules or Lie 2-groups. It follows that $N_\bullet[G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1]$ is weakly homotopic to $F_\bullet$. Furthemore the following natural diagram

$$\begin{tikzcd}
{[G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1]} \arrow[drr] & & \\
{[ker(\phi) \xrightarrow{i_2} ker(\psi)]} \arrow[u] \arrow[rr] & & {[G_2 \xrightarrow{i_2} H_2]}
\end{tikzcd}$$

is commutative. Thus $[G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1]$ is a homotopy fiber of the map $(\phi,\psi) : [G_2 \xrightarrow{i_2} H_2] \to [G_1 \xrightarrow{i_1} H_1]$. q.e.d.

As far as this paper is concerned, it is sufficient to consider Lemma 2.2 as a definition of a fibration of Lie 2-groups. In particular all fibrations of Lie 2-groups in this paper arise as in Lemma 2.2. That is, they are induced by a morphism (ϕ,ψ) of crossed modules with both ϕ and ψ being surjective submersions.

Example 2.3. The main examples of interest in this paper are obtained as follows (see Section 3). Let $G \xrightarrow{i} H$ be a crossed module and $\psi : H \to K$ be a Lie group morphism such that $\psi(i(g)) = 1$ for all $g \in G$. Then the map $(1,\psi) : [G \xrightarrow{i} H] \to [1 \to K]$ is a map of 2-groups and it is a fibration if ψ is a surjective submersion. The kernel of the map $(1,\psi)$ (as defined in Definition 2.1) is the Lie 2-group $[G \xrightarrow{i} ker(\psi)]$, which is equal to the Lie 2-group $[ker(1) \xrightarrow{i} ker(\psi)]$.

Remark 2.4. Let us recall that a 2-group is a group object in the category of groupoids. Then the Lie 2-group $[G_2 \xrightarrow{\tilde{i}} H_2 \times_{H_1} G_1]$ is the (weak) fiber product (of Lie groupoids, see [19]) $[1 \to 1] \times_{[G_1 \xrightarrow{i_1} H_1]} [G_2 \xrightarrow{i_2} H_2]$. In particular it is the correct fiber product to look at if one is interested in group stacks rather than Lie 2-groups.

3 Cohomology of Lie 2-groups

The de Rham cohomology groups of a Lie 2-group Γ are defined as the cohomology groups of the bicomplex $(\Omega^\bullet(N_\bullet\Gamma), d_{\mathrm{dR}}, \partial)$, where $d_{\mathrm{dR}} : \Omega^p(N_q\Gamma) \to$

$\Omega^{p+1}(N_q\Gamma)$ is de Rham differential and $\partial : \Omega^p(N_q\Gamma) \to \Omega^p(N_{q+1}\Gamma)$ is induced by the simplicial structure on $N_\bullet\Gamma$: $\partial = (-1)^p \sum_{i=0}^{q+1}(-1)^i d_i^*$ where $d_i : N_\bullet\Gamma \to N_{\bullet-1}\Gamma$ are the face maps. We use the shorter notation $\Omega^\bullet_{\text{tot}}(\Gamma)$ for the associated total complex. Hence $\Omega^n_{\text{tot}}(\Gamma) = \bigoplus_{p+q=n} \Omega^p(N_q\Gamma)$ with (total) differential $d_{\text{dR}} + \partial$. We denote by $H^\bullet(\Gamma)$ the cohomology of Γ. It is well-known that $H^\bullet(\Gamma)$ is naturally isomorphic to the cohomology of the fat realization of its nerve $N_\bullet\Gamma$ (for instance see [7]).

The simplicial structure of the nerve $N_\bullet\Gamma$ of a Lie 2-group Γ gives rise to a structure of cosimplicial algebra on the space of de Rham forms $\Omega^\bullet_{\text{tot}}(\Gamma)$. Thus, there exists an associative cup-product $\cup : \Omega^\bullet_{\text{tot}}(\Gamma) \otimes \Omega^\bullet_{\text{tot}}(\Gamma) \to \Omega^\bullet_{\text{tot}}(\Gamma)$ making $(\Omega^\bullet_{\text{tot}}(\Gamma), d_{\text{dR}} + \partial, \cup)$ into a differential graded algebra and, therefore, $(H^\bullet(\Gamma), \cup)$ is a graded commutative algebra. The same holds for singular cohomology.

If $G \xrightarrow{i} H$ is a crossed module, we denote by $\Omega^\bullet_{\text{tot}}([G \xrightarrow{i} H])$ the total complex of the corresponding Lie 2-group. A map of Lie 2-groups $f : \Gamma \to G$ induces a simplicial map $N_\bullet\Gamma \to N_\bullet G$, and by pullback, a map of cochain complexes $\Omega^\bullet(N_\bullet G) \xrightarrow{f^*} \Omega^\bullet(N_\bullet\Gamma)$. A similar construction, replacing the de Rham forms by the singular cochains with coefficient in a ring R, yields the singular cochain functor $C^\bullet([G \xrightarrow{i} H], R)$ of the Lie 2-group $[G \xrightarrow{i} H]$ whose cohomology $H^\bullet([G \xrightarrow{i} H], R)$ is the singular cohomology with coefficients in R. If $R = \mathbb{R}$, the singular cohomology groups coincides with the de Rham cohomology groups. The cohomology of a Lie group considered as a Lie 2-group is the usual cohomology of its classifying space since, in that case, the 2-nerve is isomorphic to the 1-nerve of the Lie group [26].

Given a crossed module $G \xrightarrow{i} H$ of Lie groups, $i(G)$ is a normal subgroup of H. Hence, the projection $H \to H/i(G)$ induces a Lie 2-group morphism

$$[G \xrightarrow{i} H] \longrightarrow [1 \to H/i(G)] \tag{3.1}$$

which is a fibration (by Lemma 2.2) with the fiber being the 2-group $[G \xrightarrow{i} i(G)]$. The canonical morphism of crossed modules $(\text{Ker}(i) \to 1) \to (G \xrightarrow{i} i(G))$ is an equivalence (see [2, 20, 21, 27, 15] for the equivalence of crossed modules or Lie 2-groups) and in particular, the Lie 2-group $[G \xrightarrow{i} i(G)]$ and $[\text{Ker}(i) \to 1]$ have weakly homotopic nerves. It follows that there is a Leray-Serre spectral sequence.

Lemma 3.1. *There is a converging spectral sequence of algebras*

$$L_2^{p,q} = H^p([1 \to H/i(G)], \mathcal{H}^q([\text{Ker}(i) \to 1])) \Longrightarrow H^{p+q}([G \xrightarrow{i} H]) \tag{3.2}$$

where $\mathcal{H}^q([\text{Ker}(i) \to 1])$ *is the de Rham cohomology viewed as a local coefficient system on* $[1 \to H/i(G)]$.

Proof. It follows from the main theorem of [1] that the realization of the map $[G \xrightarrow{i} H] \to [1 \to H/i(G)]$ is a quasi-fibration with homotopy fiber being the (fat) realization of $[\mathrm{Ker}(i) \to 1]$ (since this fat realization is homotopic to the fat realization of $[G \xrightarrow{i} i(G)]$). In fact, one can show that this quasi-fibration is indeed a fibration. The spectral sequence (3.2) is the Leray-Serre spectral sequence of this (quasi)fibration. q.e.d.

By the same argument, it also follows that there is a long exact sequence of homotopy groups [1]

$$\dots \pi_1([1 \to H/i(G)]) \to \pi_0([\mathrm{Ker}(i) \to 1]) \to \pi_0([G \xrightarrow{i} H]) \to \pi_0([1 \to H/i(G)]) \to 0. \quad (3.3)$$

Remark 3.2. The algebra structures in Lemma 3.1 are induced by the algebra structure on the singular or de Rham cohomology of the respective Lie 2-groups.

Remark 3.3. A similar proof implies that if $[G_2 \xrightarrow{i_2} H_2] \to [G_1 \xrightarrow{i_1} H_1]$ is a fibration of 2-groups with fiber F, then there is a Leray spectral sequence

$$L_2^{p,q} = H^p([G_1 \xrightarrow{i_1} H_1], \mathcal{H}^q(F)) \Longrightarrow H^{p+q}([G_2 \xrightarrow{i_2} H_2]).$$

Remark 3.4. In the special case of discrete 2-groups, the Leray-Serre spectral sequence (3.2) has been studied in [12]. In this rather different context, the higher differentials in the spectral sequence are related to the k-invariant of the crossed module.

We now give a more explicit description of the complex $\Omega^\bullet_{\mathrm{tot}}([G \xrightarrow{i} H])$ in degree ≤ 4 which will be needed in Sections 4 and 7. Until the end of this Section, we denote by Γ the 2-group $G \rtimes H \rightrightarrows H \rightrightarrows \{*\}$ associated to the crossed module $G \xrightarrow{i} H$. One has $N_0\Gamma = *$ and $N_1\Gamma = H$. Since there is only one object in the underlying category, all 1-arrows can be composed. Thus, a triangle as in equation (2.1) is given by a 2-arrow $\alpha \in G \rtimes H$ and a 1-arrow f_0. Hence, $N_2\Gamma \cong (G \times H) \times H$. With this choice, for $(g, h, f) \in N_2\Gamma$, the corresponding 2-arrow α and 1-arrows f_0, f_1, f_2 in equation (2.1) are respectively given by

$$\alpha = (g, h), \qquad f_0 = f, \qquad f_1 = h \cdot i(g), \qquad f_2 = h \cdot f^{-1}. \quad (3.4)$$

The three face maps $d_i : N_2\Gamma \to N_1\Gamma$ $(i = 0, 1, 2)$ are given by $d_i(g, h, f) = f_i$, $i = 0, 1, 2$ (see equation (3.4)).

Remark 3.5. Of course the choice of f_0 is a convention, we could have equivalently chosen to work with f_2.

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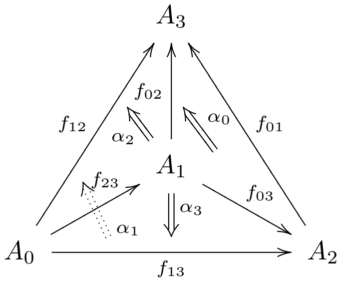
[1] The sequence should not be confused with the long exact sequence of simplicial homotopy groups in [16, Section 3].

$N_3\Gamma$ is the space of commutative tetrahedron labelled by objects and arrows of Γ



(3.5)

The commutativity means that one has $(\alpha_3 * f_{01}) \star \alpha_1 = (f_{23} * \alpha_0) \star \alpha_2$, where $\star$ is the vertical multiplication of 2-arrows and $*$ is the horizontal multiplication. Since there is only one object, such a tetrahedron is given by $\alpha_0, f_{01}, \alpha_2$ and α_3 satisfying $s(\alpha_3) = s(\alpha_2).(t(\alpha_0))^{-1}.s(\alpha_0).f_{01}^{-1}$. Thus $N_3\Gamma \cong G^3 \times H^3$. The face maps d_i ($i = 0...4$) are given by the restrictions to the triangle which doesn't contain A_i as a vertex. Thus, given $(g_0, g_2, g_3, h_0, f_{01}, h_2) \in G^3 \times H^3$, one has

$$d_0(g_0, g_2, g_3, h_0, f_{01}, h_2) = (g_0, h_0, f_{01}) \tag{3.6}$$

$$d_1(g_0, g_2, g_3, h_0, f_{01}, h_2) = ((g_3^{-1})^{f_{01}} g_0^{h_2 i(g_0^{-1}) h_0^{-1}} g_2, h_2, f_{01}) \tag{3.7}$$

$$d_2(g_0, g_2, g_3, h_0, f_{01}, h_2) = (g_2, h_2, h_0.i(g_0)) \tag{3.8}$$

$$d_3(g_0, g_2, g_3, h_0, f_{01}, h_2) = (g_3, h_2.i(g_0^{-1}) f_{01}^{-1}, h_0.f_{01}^{-1}). \tag{3.9}$$

Remark 3.6. The choice of indices in $(g_0, g_2, g_3, h_0, f_{01}, h_2)$ is reminiscent of the tetrahedron (3.5). That is the two arrow $\alpha_0 = (g_0, h_0) \in G \times H$, the 1-arrow from A_2 to A_3 is f_{01} and so on ... For instance, the 2-arrow $\alpha_1 = (g_1, h_1) \in G \times H$ is given by Equation (3.7), i.e, $g_1 = (g_3^{-1})^{f_{01}} g_0^{h_2 i(g_0^{-1}) h_0^{-1}} g_2$ and $h_1 = h_2$.

Applying the differential form functor, we get

(a) $\Omega^0_{\mathrm{tot}}([G \xrightarrow{i} H]) = \Omega^0(*) \cong \mathbb{R}$,

(b) $\Omega^1_{\mathrm{tot}}([G \xrightarrow{i} H]) = \Omega^0(H)$,

(c) $\Omega^2_{\mathrm{tot}}([G \xrightarrow{i} H]) = \Omega^1(H) \oplus \Omega^0(G \times H \times H)$. The differentials from $\Omega^1_{\mathrm{tot}}([G \xrightarrow{i} H])$ to $\Omega^2_{\mathrm{tot}}([G \xrightarrow{i} H])$ are given by $d_{\mathrm{dR}} : \Omega^0(H) \to \Omega^1(H) \subset \Omega^2_{\mathrm{tot}}([G \xrightarrow{i} H])$ and $\partial = d_0^* - d_1^* + d_2^* : \Omega^0(H) \to \Omega^0(G \times H \times H) \subset \Omega^2_{\mathrm{tot}}([G \xrightarrow{i} H])$.

(d) $\Omega^3_{\mathrm{tot}}([G \xrightarrow{i} H]) = \Omega^2(H) \oplus \Omega^1(G \times H \times H) \oplus \Omega^0(G^3 \times H^3)$. The differential are similar to the previous ones.

(e) $\Omega^4_{\mathrm{tot}}([G \xrightarrow{i} H]) = \Omega^3(H) \oplus \Omega^2(G \times H \times H) \oplus \Omega^1(G^3 \times H^3) \oplus \Omega^0(N_4\Gamma)$.

Remark 3.7. For $p \geq 4$, an element in $N_p\Gamma$ is a commutative p-simplex labelled by arrows of Γ whose faces of dimension 2 are elements of $N_2\Gamma$ with compatible edges. Denoting $A_0, \ldots, A_p$ the vertices of the p-simplex, the commutativity implies that it is enough to know all the 2-faces containing A_0. Reasoning as for $N_3\Gamma$, it follows that $N_p\Gamma \cong G^{\frac{p(p-1)}{2}} \times H^p$. Details are left to the reader.

Let $\mathfrak{g}$ be the Lie algebra of G. There is an obvious map $(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow \Omega^1(G)$ which sends $\xi \in \mathfrak{g}^*$ to its left invariant 1-form ξ^L. By composition we have a map

$$(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow \Omega^1(G) \xrightarrow{p_1^*} \Omega^1(G \times H \times H) \hookrightarrow \Omega^3_{\text{tot}}([G \xrightarrow{i} H]) \tag{3.10}$$

where $p_1 : G \times H \times H \to G$ is the projection.

The action of H on G induces an action of H on $\mathfrak{g}$, and therefore an action on $\mathfrak{g}^*$. The map I above clearly restricts to $(\mathfrak{g}^*)^{\mathfrak{g},H}$, the subspace of $\mathfrak{g}^*$ consisting of elements both $\mathfrak{g}$ and H invariant. Assigning the degree 3 to elements of $(\mathfrak{g}^*)^{\mathfrak{g},H}$, *i.e.*, replacing $(\mathfrak{g}^*)^{\mathfrak{g},H}$ by $(\mathfrak{g}^*)^{\mathfrak{g},H}[3]$, we have the following

Proposition 3.8. *The map* $I : \big((\mathfrak{g}^*)^{\mathfrak{g},H}[3], 0\big) \to \big(\Omega^{\bullet}_{\text{tot}}([G \xrightarrow{i} H]), d_{\text{dR}} + \partial\big)$ *is a map of cochain complexes,* i.e., $(d_{\text{dR}} + \partial)(I) = 0$.

Proof. We have $(\mathfrak{g}^*)^{\mathfrak{g}} \cong \mathfrak{z}(\mathfrak{g})^*$, where $\mathfrak{z}(\mathfrak{g})$ is the center of the Lie algebra $\mathfrak{g}$. Since the de Rham differential vanishes on $(\mathfrak{g}^*)^{\mathfrak{g}}$, it remains to prove that $\partial \circ I = 0$. For any $\xi \in (\mathfrak{g}^*)^{\mathfrak{g}}$ and left invariant vector fields $\overleftarrow{X}$, $\overleftarrow{Y}$,

$$\begin{aligned} m^*(\xi^L)(\overleftarrow{X}_g, \overleftarrow{Y}_h) = \xi^L(m_*(\overleftarrow{X}_g, \overleftarrow{Y}_h)) &= \xi^L(\overleftarrow{X}_{gh}, \overleftarrow{\mathrm{Ad}_g Y}_{gh}) \\ &= \xi(X) + \xi(Y) \\ &= \big(p_1^*(\xi^L) + p_2^*(\xi^L)\big)(\overleftarrow{X}_g, \overleftarrow{Y}_h), \end{aligned}$$

where $m, p_1, p_2 : G \times G \to G$ are respectively the product map and the two projections. If, moreover, $\xi \in (\mathfrak{g}^*)^{\mathfrak{g},H}$, then $\tilde{m}^* = \tilde{p_2}^*$, where $\tilde{m}, \tilde{p_2} : G \times H \to G$ are respectively the action map and the projection. Since $I(\xi) \in \Omega^1(G) \subset \Omega^1(G \times H \times H) \subset \Omega^3_{\text{tot}}([G \xrightarrow{i} H])$, the result follows from a simple computation using formulas (3.6)-(3.9). q.e.d.

By Proposition 3.8, the images of the map $I : (\mathfrak{g}^*)^{\mathfrak{g},H}[3] \to \Omega^3_{\text{tot}}([G \xrightarrow{i} H])$ are automatically cocycles. Recall that $S\big((\mathfrak{g}^*)^{\mathfrak{g},H}[3]\big)$ is the free graded commutative algebra on the vector space $(\mathfrak{g}^*)^{\mathfrak{g},H}[3]$ which is concentrated in degree 3. Thus $S\big((\mathfrak{g}^*)^{\mathfrak{g},H}[3]\big)$ is indeed an exterior algebra. By the universal property of free graded commutative algebras, we obtain :

Corollary 3.9. *The map* $I : (\mathfrak{g}^*)^{\mathfrak{g},H}[3] \to H^3([G \xrightarrow{i} H])$ *extends uniquely to a morphism of graded commutative algebras*

$$I : S\big((\mathfrak{g}^*)^{\mathfrak{g},H}[3]\big)^{\bullet} \to H^{\bullet}([G \xrightarrow{i} H]).$$

In fact, the class $I(\xi_1 \cdot \dots \cdot \xi_r)$, $\xi_1, \dots, \xi_r \in (g^*)^{\mathfrak{g}}$, *is represented by the cocycle* $I(\xi_1) \cup \dots \cup I(\xi_r) \in \Omega^{3r}([G \xrightarrow{i} H])$.

4 Cohomology of $[A \to 1]$

The following lemma is well-known.

Lemma 4.1. *The nerve $N_\bullet([S^1 \to 1])$ is a $K(\mathbb{Z}, 3)$-space.*

Proof. Since $\mathbb{Z}$ is discrete, $[\mathbb{Z} \to 1]$ is a $K(\mathbb{Z}, 2)$-space (see for instance [18, 20, 21]). Furthermore, $[\mathbb{R} \to 1]$ is homotopy equivalent to $[1 \to 1]$. Thus the result follows from the fibration of 2-groups $[\mathbb{Z} \to 1] \longrightarrow [\mathbb{R} \to 1] \longrightarrow [S^1 \to 1]$. q.e.d.

Let A be an abelian compact Lie group with Lie algebra $\mathfrak{a}$. Then $[A \to 1]$ is a crossed module. By Corollary 3.9, we have a map $I : S(\mathfrak{a}^*[3]) \to H^\bullet([A \to 1])$.

Proposition 4.2. *Let A be an abelian compact Lie group with Lie algebra $\mathfrak{a}$. The map $I : S(\mathfrak{a}^*[3])^\bullet \to H^\bullet([A \to 1])$ is an isomorphism of graded algebras.*

Proof. Since our cohomology groups have real coefficients, it is sufficient to consider the case where A is a torus T^k. Indeed, writing $A_0 \cong T^k$ for the connected component of the identity in A, we have a fibration:

$$[A_0 \to 1] \to [A \to 1] \to [A/A_0 \to 1].$$

Since A is compact, A/A_0 is a finite group. Thus $N_\bullet[A/A_0 \to 1]$ is a $K(A/A_0, 2)$-space and in particular is simply connected. Then the Leray spectral sequence (Lemma 3.1 and Remark 3.3) simplifies as

$$L_2^{\bullet,\bullet} = H^\bullet(K(A/A_0, 2)) \otimes H^\bullet([A_0 \to 1]) \Longrightarrow H^\bullet([A \to 1]).$$

Since A/A_0 is finite, $H^{i>0}(K(A/A_0, 2)) \cong 0$. Hence $H^\bullet([A \to 1]) \xrightarrow{\sim} H^\bullet([A_0 \to 1])$.

Now, assume $A = T^k$. The Künneth formula implies that $H^\bullet([A \to 1]) \cong \big(H^\bullet([S^1 \to 1])\big)^{\otimes k}$ as an algebra. Since I is a morphism of algebras, it is sufficient to consider the case $k = 1$, i.e., $A = S^1$.

Lemma 4.1 implies that $H^\bullet([S^1 \to 1]) \cong S(x)$, where x is of degree 3. It remains to prove that the map (3.10)

$$I : \mathbb{R} \to \Omega^1(S^1) \to \Omega^3([S^1 \to 1])$$

generates the degree 3 cohomology of $[S^1 \to 1]$, *i.e.*, that $I(1)$ is not a coboundary in $\Omega^3([S^1 \to 1])$. Clearly $I(1)$ is the image of the fundamental 1-form on S^1 by the inclusion $\Omega^1(S^1) \hookrightarrow \Omega^3([S^1 \to 1])$. By Section 3, it is obvious that $\Omega^2([S^1 \to 1]) \cong \Omega^0(S^1)$ and that the only component of the coboundary operator $\delta : \Omega^2([S^1 \to 1]) \to \Omega^3([S^1 \to 1])$ lying in $\Omega^1(S^1) \hookrightarrow \Omega^3([S^1 \to 1])$ is the de Rham differential $d_{\mathrm{dR}} : \Omega^0(S^1) \to \Omega^1(S^1)$. Since the fundamental 1-form is not exact, the result follows. q.e.d.

5 The case of a finite cokernel

In this section, we consider the particular case of a Lie 2-group $[G \xrightarrow{i} H]$ with finite cokernel.

Theorem 5.1. *Let $[G \xrightarrow{i} H]$ be a Lie 2-group with finite cokernel $C := H/i(G)$ and compact kernel $\mathrm{Ker}(i)$. Let $\mathfrak{k}$ be the Lie algebra of $\mathrm{Ker}(i)$. There is an isomorphism of graded algebras*

$$H^\bullet([G \xrightarrow{i} H]) \cong \big(S\big(\mathfrak{k}^*[3]\big)^\bullet\big)^C .$$

In particular, the cohomology is concentrated in degree $3q$, $q \geq 0$.

Proof. The 2-group $[1 \to C]$ is naturally identified with the 1-group C. Thus its nerve $N_\bullet[1 \to C]$ coincides with the classifying space BC of C. Furthermore, since C is finite (thus discrete), the cohomology (with local coefficients) $H^\bullet([1 \to C], \mathcal{H}^q([\mathrm{Ker}(i) \to 1]))$ is isomorphic to the usual group cohomology $H^\bullet(C, H^q([\mathrm{Ker}(i) \to 1])$, where the C-module structure on $H^q([\mathrm{Ker}(i) \to 1]))$ is induced by the C-action on $\mathrm{Ker}(i)$. Since $H^q([\mathrm{Ker}(i) \to 1])$ is an $\mathbb{R}$-module and C is finite, the cohomology $H^\bullet(C, H^q([\mathrm{Ker}(i) \to 1]))$ is concentrated in degree zero so that the spectral sequence of Lemma 3.1 collapses. Hence

$$H^\bullet([G \xrightarrow{i} H]) \cong H^0(C, H^q([\mathrm{Ker}(i) \to 1])) \cong H^q([\mathrm{Ker}(i) \to 1]))^C \cong S\left(\left((\mathfrak{k}^*)^{\mathfrak{k}}[3]\right)^q\right)^C$$

According to Proposition 4.2, they are also isomorphic as algebras due to the multiplicativity of the spectral sequence and the freeness of $S\big((\mathfrak{k}^*)^{\mathfrak{k}}[3]\big)$. q.e.d.

Remark 5.2. One can find explicit generators for the cohomology $H^\bullet([G \to H])$ as follows. For all $y \in \mathrm{Ker}(i)$, $x \in G$, $y^{-1}xy = x^{i(y)} = x$. Thus $K \subset Z(G)$ and $\mathfrak{z}(\mathfrak{g})$ splits as a direct sum $\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{k} \oplus \mathfrak{n}$. We denote by J the map $\mathfrak{g}^{\mathfrak{g}} \cong \mathfrak{z}(\mathfrak{g}) \to \mathfrak{k}$. The composition of $J^* : \mathfrak{k}^* \to \mathfrak{g}^*$ with the map (3.10) is the map

$$\tilde{I} : \mathfrak{k}^* \xrightarrow{J^*} (\mathfrak{g}^*)^{\mathfrak{g}} \to \Omega^1(G \times H^2) \subset \Omega^3_{\mathrm{tot}}([G \to H]).$$

If $x_1 \ldots . x_q \in S^{q \geq 2}\big(\mathfrak{k}^*[3]\big)$, then $I(x_1) \cup \cdots \cup I(x_q)$ lies in $\Omega^q(N_{2q}([G \to H])) \subset \Omega^{3q}_{\mathrm{tot}}([G \to H])$. Note that the action of $h \in H$ on K depends only on the class of h in C. Since C is finite, it follows that, for any $x \in S^q(\mathfrak{k}^*[3])$, $\tilde{I}(x)$ is a cocycle if and only if x is C-invariant. Let $\check{I}(x) = \tilde{I}\big(\sum_{c \in C} x^c\big)$. Then $\check{I}(x)$ is indeed a cocycle and $\check{I}\big(\mathfrak{k}^*\big)$ generates the cohomology $H^\bullet([G \xrightarrow{i} H])$.

Let $1 \to A \to G \xrightarrow{p} H \to 1$ be a Lie group central extension. Since A is central, there is a canonical action of H on G. It is easy to see that $G \xrightarrow{p} H$ is a crossed module.

Corollary 5.3. *Let $[G \xrightarrow{p} H]$ be the Lie 2-group corresponding to a central extension of H by a compact abelian group A. There is an isomorphism of graded algebras*

$$H^\bullet([G \xrightarrow{p} H]) \cong S(\mathfrak{a}^*[3])^\bullet$$

where $\mathfrak{a}$ is the Lie algebra of A.

Recall that $S(\mathfrak{a}^*[3])^\bullet$ is a graded commutative algebra generated by degree 3 generators (given by any basis of $\mathfrak{a}^*$).

Proof. Since $G \xrightarrow{p} H$ is a surjective submersion, the cokernel $H/p(G) \cong \{*\}$ is trivial. Moreover the kernel of $[G \xrightarrow{p} H]$ is $[A \to 1]$. Hence the conclusion follows from Theorem 5.1. q.e.d.

Remark 5.4. Identifying the crossed module $A \to 1$ with the kernel of $G \xrightarrow{p} H$ yields a canonical morphism of 2-groups $\rho : [A \to 1] \to [G \to H]$. It follows from the proof of Theorem 5.1 that the isomorphism $H^\bullet([G \xrightarrow{p} H]) \xrightarrow{\sim} S(\mathfrak{a}[3])^\bullet$ is given by the composition

$$S(\mathfrak{a}[3])^\bullet \xrightarrow{\sim} H^\bullet([A \to 1]) \underset{\sim}{\xrightarrow{\rho^*}} H^\bullet([G \xrightarrow{p} H]).$$

Example 5.5. Let G be a compact Lie group. It is isomorphic to a quotient of $Z \times G'$ by a central finite subgroup. Here G' is the commutator subgroup of G. Hence there is a map $G \to \mathrm{Aut}\,(G')$ yielding a Lie 2-group $[G \to \mathrm{Aut}\,(G')]$ through the action of $\mathrm{Aut}\,(G')$ on G' (see Section 7 below). Theorem 5.1 implies that

$$H^\bullet([G \to \mathrm{Aut}\,(G')]) \cong S\big((\mathfrak{g}^*)^{\mathfrak{g}}[3]\big)^\bullet.$$

6 The case of a connected compact cokernel

The results of Section 3 can be applied to a more general type of 2-groups $[G \xrightarrow{i} H]$, where G and H are Fréchet Lie groups (thus possibly infinite dimensional). See [3] for more details on Fréchet Lie 2-groups. In such a case, instead of de Rham cohomology, singular cohomology with real coefficients can be used.

We start with the following lemma.

Lemma 6.1. *Let G and H be Fréchet Lie groups. Assume that $C = H/i(G)$ is a connected compact Lie group, and $\mathrm{Ker}(i)$ is compact. Then the third page $L_3^{\bullet,\bullet}$ of the Leray spectral sequence (3.2) is concentrated in bidegree $(p, 3q)$ $(p, q \geqslant 0)$, and*

$$L_4^{p,3q} = H^p(BC) \otimes S^q(\mathfrak{a}^*[3]). \tag{6.1}$$

Here BC is the classifying space of $C = H/i(G)$, and $\mathfrak{a}$ is the Lie algebra of $A = \mathrm{Ker}(i)$.

Note that, since $S(\mathfrak{a}^*[3])$ is a graded commutative algebra[2] generated by elements of degree 3, it lies in degree $3q$ (where $0 \leq q \leq \dim(\mathfrak{a})$).

Proof. Note that C is the cokernel $[1 \to C]$ of $[G \xrightarrow{i} H]$ (see Section 2). Since $C = H/i(G)$ is connected, its classfying space BC is simply connected. It follows

[2] $S(\mathfrak{a}^*[3])$ is in fact an exterior algebra, since it is generated by odd degree generators.

that the $L_2^{i,j}$ term of the Leray spectral sequence in Lemma 3.1 is isomorphic to

$$L_2^{i,j} \cong H^i(BC) \otimes H^j([A \to 1])$$

as an algebra. By Proposition 4.2, $H^\bullet([A \to 1]) \cong S(\mathfrak{a}^*[3])^\bullet$ is concentrated in degree $3q$ ($q \geq 0$). Since the differential $d_2 : L_2^{i,j} \to L_2^{i+2,j-1}$ is a derivation, it follows that $d_2 = 0$ for degree reason. Similarly, $d_3 = 0$. Thus $L_4^{\bullet,\bullet} \cong L_3^{\bullet,\bullet} \cong L_2^{\bullet,\bullet}$.
q.e.d.

The (higher) differential $d_4 : L_4^{i,j} \to L_4^{i+4,j-3}$ induces a transgression homomorphism

$$T : \mathfrak{a}^* \cong L_4^{0,3} \xrightarrow{d_4} L_4^{4,0} \cong H^4(BC). \tag{6.2}$$

Proposition 6.2. *Under the same hypothesis as in Lemma 6.1, there is a natural linear isomorphism*

$$H^\bullet([G \xrightarrow{i} H]) \cong \big(H^\bullet\big(BC\big)/(\mathrm{Im}(T)\big) \otimes S\big(\,\mathrm{Ker}(T)[3]\big)^\bullet$$

which is an algebra isomorphism if we assume furthermore that $C = H/i(G)$ is simply connected.

Here BC is the classifying space of C and $\mathrm{Im}\, T$ is the ideal generated by the image of T.

Proof. Since $d_4 : L_4^{i,j} \to L_4^{i+4,j-3}$ is a derivation, it is uniquely determined by T. From Lemma 6.1, it follows that

$$L_5^{\bullet,\bullet} \cong \big(H^\bullet\big(BC\big)/(\mathrm{Im}\, T)\big) \otimes S\big(\,\mathrm{Ker}(T)[3]\big)^\bullet.$$

For degree reasons, $d_r = 0$ for all $r \geqslant 5$. Thus $L_5^{\bullet,\bullet} \cong L_\infty^{\bullet,\bullet}$ as an algebra and the linear isomorphism $H^\bullet([G \xrightarrow{i} H]) \cong \big(H^\bullet\big(BC\big)/(\mathrm{Im}(T)\big) \otimes S\big(\,\mathrm{Ker}(T)[3]\big)^\bullet$ follows since our ground ring is a field. If C is further simply connected, then $H^\bullet(BC)$ is a polynomial algebra with generators x_i of even degree $|x_i| = 2i$, $i \geqslant 2$. In particular, $H^4(BC)$ has no decomposable elements, thus $L_\infty^{\bullet,\bullet}$ is a polynomial algebras with graded generators. It follows that $L_\infty^{\bullet,\bullet} \cong H^\bullet([G \xrightarrow{i} H])$ as an algebra.
q.e.d.

As an application, below we compute the cohomology of *string 2-group* [3] *String*(G). Let G be a connected and simply connected compact simple Lie group. There is a unique left invariant closed 3-form ν on G, which generates $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$. By transgression, the form ν corresponds to a class $[\omega] \in H^4(BG, \mathbb{Z})$, which determines the basic central extension [22, 3]

$$1 \to S^1 \to \widetilde{\Omega G} \xrightarrow{\tilde{p}} \Omega G \to 1$$

of the based (at identity) loop group ΩG of G. Associated to ν is a (homotopy class of) map $\Omega BG \to G \to K(\mathbb{Z}, 3) \cong [S^1 \to 1]$ which induces an isomorphism on π_3. Let PG denote the space of paths $f : [0, 1] \to G$ starting at the identity.

The conjugation action of PG on ΩG lifts to $\widetilde{\Omega G}$. The string 2-group (see [3]) is the Fréchet 2-group corresponding to the crossed module

$$String(G) := [\widetilde{\Omega G} \xrightarrow{p} PG],$$

where p is the composition

$$p : \widetilde{\Omega G} \xrightarrow{\tilde{p}} \Omega G \hookrightarrow PG.$$

By construction, $\mathrm{Ker}(p) \cong S^1$, $PG/p(\widetilde{\Omega G}) \cong G$ and, also $\pi_3(String(G)) = 0$ (as follows from [3, Theorem 3]). Recall that the cohomology $H^\bullet(G)$ is the exterior algebra on generators $x_1, \dots, x_r$, where x_i is of degree $2e_i + 1$ and $e_1, \dots, e_r$ are the exponents of G. Note that we can choose $x_1 = \nu$. Similarly $H^\bullet(BG)$ is the polynomial algebra on generators $y_1, \dots, y_r$ of degree $|y_i| = 2e_i$, where y_1 can be taken to be $[\omega]$. To apply Proposition 6.2, it suffices to compute the transgression homomorphism $T : \mathbb{R} \to H^4(BG) \cong \mathbb{R}$, where the domain $\mathbb{R}$ is identified with the Lie algebra of S^1. Since $[\omega] \in H^4(BG)$ is obtained by the transgression from $[\nu] \in H^3(G) \cong H^3(N\Omega G)$, it follows that $T(1)$ is the generator of $H^4(BG)$. Indeed, there is a commutative diagram of Fréchet 2-groups fibrations

$$\begin{tikzcd}
 & {[1 \to \Omega G]} \arrow[r] \arrow[d, "j"'] & {[1 \to PG]} \arrow[r, "ev"] \arrow[d] & {[1 \to G]} \arrow[d, equal] \\
{[S^1 \to 1]} \arrow[r] & {[\widetilde{\Omega G} \xrightarrow{\tilde{p}} \Omega G]} \arrow[r] & String(G) \arrow[r, "ev"] & {[1 \to G]}
\end{tikzcd}$$

where the right horizontal arrows are induced by $ev : PG \to G$, $f \mapsto f(1)$ and the canonical inclusion $[S^1 \to 1] \to [\widetilde{\Omega G} \xrightarrow{\tilde{p}} \Omega G] = \mathrm{Ker}(ev)$ is an equivalence of Fréchet 2-groups. Thus the transgression map T is the composition $T' \circ j^*$ where $T' : H^3(G) \cong H^3([1 \to \Omega G]) \to H^4([1 \to G]) \cong H^4(BG)$ is the transgression map associated to the fibration $[1 \to PG] \xrightarrow{ev} [1 \to G]$. Since PG is contractible, $T'(\nu)$ is a generator[3] of $H^4(BG)$. It also follows from the exact sequence (3.3) that j is an isomorphism on π_3 and so is

$$j^* : \mathbb{R} \cong H^3([S^1 \to 1]) \to H^3([1 \to \Omega G]) \cong H^3(B\Omega G) \cong H^3(G).$$

Hence $T = T' \circ j* : \mathbb{R} \to H^4(BG) \cong \mathbb{R}$ is an isomorphism. Thus, we recover the following result of Baez-Stevenson [5]:

Proposition 6.3.

$$H^\bullet(String(G)) \cong S(y_2, \dots, y_r) \cong H^\bullet(BG)/([\omega]),$$

where the y_is are the generators of $H^4(BG)$.

[3]as for the case of the "universal" fibration $G \to EG \to BG$

7 The case of $[G \to \mathrm{Aut}^+(G)]$ and $[G \to \mathrm{Aut}\,(G)]$

Let G be a compact Lie group. There is a canonical morphism $G \xrightarrow{i} \mathrm{Aut}\,(G)$ given by inner automorphisms which is also a crossed module. Since inner automorphisms are orientation preserving, we also have a crossed module $G \xrightarrow{i^+} \mathrm{Aut}^+(G)$ where $\mathrm{Aut}^+(G)$ is the group of orientation preserving automorphisms.

Now, assume G is a semi-simple Lie group. Then both $\mathrm{Out}\,(G)$ and $\mathrm{Out}^+(G)$ are finite groups. Moreover, $\mathrm{Ker}(i)$ and $\mathrm{Ker}(i^+)$ are finite too. Thus, by Theorem 5.1, we obtain

Proposition 7.1. *Let G be a semi-simple Lie group.*

$$H^n([G \to \mathrm{Aut}\,(G)]) \cong H^n([G \to \mathrm{Aut}^+(G)]) \cong \begin{cases} 0 & \text{if } n > 0, \\ \mathbb{R} & \text{if } n = 0. \end{cases}$$

For general compact Lie groups, the cohomology of $[G \xrightarrow{i^+} \mathrm{Aut}^+(G)]$ and $[G \xrightarrow{i} \mathrm{Aut}^+(G)]$ can be computed with the help of spectral sequences.

Theorem 7.2. *If G is a compact Lie group, there are converging spectral sequences of graded commutative algebras*

$$E_2^{+\,p,q} = H^p\big(SL(n,\mathbb{Z}), S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q\big) \Longrightarrow H^{p+q}\big([G \xrightarrow{i^+} \mathrm{Aut}^+(G)]\big) \quad (7.1)$$

$$E_2^{p,q} = H^p\big(GL(n,\mathbb{Z}), S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q\big) \Longrightarrow H^{p+q}\big([G \xrightarrow{i} \mathrm{Aut}\,(G)]\big) \quad (7.2)$$

where $n = \dim((\mathfrak{g}^*)^{\mathfrak{g}})$ is the dimension of $(\mathfrak{g}^*)^{\mathfrak{g}}$, and the $SL(n,\mathbb{Z})$-action (or $GL(n,\mathbb{Z})$-action) on $S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q$ is induced by the natural action on $(\mathfrak{g}^*)^{\mathfrak{g}} \cong \mathbb{R}^n$.

In particular the spectral sequences are concentrated in bidegrees $(p, 3k)$ (p and $k \geq 0$) and

$$E_2^{+\,0,q\neq 0,3n} \cong 0 \quad \text{and} \quad E_2^{+\,0,0} \cong E_2^{+\,0,3n} \cong \mathbb{R} \quad (7.3)$$

$$E_2^{0,q>0} \cong 0 \quad \text{and} \quad E_2^{0,0} \cong \mathbb{R} \quad (7.4)$$

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{z}(\mathfrak{g})$ the Lie algebra of its center $Z(G)$. Then $\mathfrak{z}(\mathfrak{g})^* = (\mathfrak{g}^*)^{\mathfrak{g}}$. Since the kernel of $G \xrightarrow{i} \mathrm{Aut}\,(G)$ is $Z(G)$, we have the fibration

$$[Z(G) \to 1] \xrightarrow{j} [G \xrightarrow{i} \mathrm{Aut}\,(G)] \longrightarrow [1 \to \mathrm{Out}\,(G)], \quad (7.5)$$

where j is the inclusion map. By Lemma 3.1, we have a spectral sequence $H^p([1 \to \mathrm{Out}\,(G)], \mathcal{H}^q([Z(G) \to 1])) \Longrightarrow H^{p+q}([G \xrightarrow{i} \mathrm{Aut}\,(G)])$ and similarly for $[G \xrightarrow{i^+} \mathrm{Aut}^+(G)]$. According to Proposition 4.2, $H^q([Z(G) \to 1]) \cong S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q$. Since G is compact, the group $\mathrm{Out}\,(G)$ is discrete. Thus, the

$E_2^{p,q}$ and $E_2^{+p,q}$-terms of the spectral sequences become the group cohomology groups $H^p(\mathrm{Out}\,(G), S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q)$ and $H^p(\mathrm{Out}^+(G), S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q)$ respectively. Note that the center of G is stable under the action by any automorphism. Hence, there are canonical group morphisms $\mathrm{Out}\,(G) \to \mathrm{Out}\,(Z(G))$ and $\mathrm{Out}^+(G) \to \mathrm{Out}^+(Z(G))$.

First assume that $G \cong Z(G) \times G'$, where $Z(G) = S^1 \times \cdots \times S^1$ is a torus of dimension n and $G' = [G,G]$ is semi-simple. Then the canonical map $\mathrm{Out}\,(G) \to \mathrm{Out}\,(Z(G))$ has an obvious section $\mathrm{Out}\,(Z(G)) \to \mathrm{Out}\,(G)$ given by $\phi \mapsto \phi \times \mathrm{id}_{G'}$. Since G' is the commutator subgroup of G, it is also stable under automorphisms. It follows that $\mathrm{Out}\,(G) \cong GL(n,\mathbb{Z}) \times \mathrm{Out}\,(G')$ and $\mathrm{Out}^+(G) \cong SL(n,\mathbb{Z}) \times \mathrm{Out}^+(G')$ since $\mathrm{Aut}\,(Z(G)) \cong GL(n,\mathbb{Z})$. We now need to find out the $\mathrm{Out}\,(G)$ and $\mathrm{Out}^+(G)$-actions on $H^\bullet([Z(G) \to 1]) \cong S((\mathfrak{g}^*)^{\mathfrak{g}}[3])^q$. If $t_1, \ldots, t_n$ are coordinates on $Z(G)$, then $(\mathfrak{g}^*)^{\mathfrak{g}} \cong \mathbb{R}dt_1 \oplus \cdots \oplus \mathbb{R}dt_n$ and, according to Proposition 4.2, the elements $I(dt_1), \ldots, I(dt_n) \in \Omega^1(S^1 \times \cdots \times S^1) \cong \Omega^3([Z(G) \to 1])$ form a basis of $H^\bullet([Z(G) \to 1])$. It follows that the $\mathrm{Out}\,(G)$ and $\mathrm{Out}^+(G)$-actions on $H^\bullet([Z(G) \to 1])$ reduce to the standard $GL(n,\mathbb{Z})$ and $SL(n,\mathbb{Z})$-actions on the vector space $\mathbb{R}dt_1 \oplus \cdots \oplus \mathbb{R}dt_n$. Since $\mathrm{Out}\,(G')$ and $\mathrm{Out}^+(G')$ are finite and act trivially on $H^\bullet([Z(G) \to 1])$, the spectral sequences 7.1 and 7.2 follow from the Künneth formula.

In general, since G is compact, it is isomorphic to the quotient ($G \cong Z \times G')/\Delta$, where Z is the connected component of the center $Z(G)$ and $\Delta = Z(G) \cap G'$ is finite central. Let $\tilde{G}'$ be the universal cover of G', which is a compact Lie group, and $p : Z \times \tilde{G}' \to G$ be the covering of G given by the composition $Z \times \tilde{G}' \to Z \times G' \to G$. Let $f \in \mathrm{Aut}\,(G)$, then $f \circ p : Z \times \tilde{G}' \to G$ is a Lie group morphism. There is a unique lift

$$\begin{tikzcd} Z \times \tilde{G}' \arrow[r, dotted, "\tilde{f}"] \arrow[d, "p"'] & Z \times \tilde{G}' \arrow[d, "p"] \\ G \arrow[r, "f"] & G \end{tikzcd}$$

of the map $f \circ p : Z \times \tilde{G}' \to G$ into a map $\tilde{f} : Z \times \tilde{G}' \to Z \times \tilde{G}'$ preserving the unit. Indeed, for this, it is sufficient to check that $\big(f_* \circ p_*\big)(\pi_1(Z \times \tilde{G}')) \subset p_*(\pi_1(Z \times \tilde{G}'))$. Clearly $p_*(\pi_1(Z \times \tilde{G}')) \cong p_*(\mathbb{Z}^n)$ is the non-torsion part of $\pi_1(G)$. It is thus stable by any automorphism, therefore by $f_* : \pi_1(G) \to \pi_1(G)$. Since p is a group morphism and $\tilde{f} \in \mathrm{Aut}\,(Z \times \tilde{G}')$, it follows that any automorphism of G lifts uniquely into an automorphism of $Z \times \tilde{G}'$. We are thus back to the previous case.

By the above discussions, we already know that the action of $SL(n,\mathbb{Z})$ and $GL(n,\mathbb{Z})$ on $(\mathfrak{g}^*)^{\mathfrak{g}} \cong \mathbb{R}^n$ is the standard one. Since the symmetric algebra on odd generators is isomorphic to an exterior algebra, $E_2^{0,q}$ and $E_2^{+0,q}$ are respectively isomorphic to $\Lambda^k\big(\mathbb{R}^n\big)^{GL(n,\mathbb{Z})}$ (as a $GL(n,\mathbb{Z})$-module) and $\Lambda^k\big(\mathbb{R}^n\big)^{SL(n,\mathbb{Z})}$ (as a $SL(n,\mathbb{Z})$-module). Furthermore, if $q \neq 3k$, $E_2^{0,q}$ and $E_2^{+0,q}$ vanish for degree reasons. In particular, the $GL(n,\mathbb{Z})$-action is trivial for $k=0$ and, for $k=n$, it reduces to the multiplication by the determinant on $\Lambda^n(\mathbb{R}^n) \cong \mathbb{R}$. For $0<k<n$, $SL(n,\mathbb{Z})$ (and thus $GL(n,\mathbb{Z})$ too) has no fixed points in $\Lambda^k(\mathbb{R}^n)$ besides 0. The last assertion follows. q.e.d.

In general, the description of the group cohomology of $GL(n,\mathbb{Z})$ and $SL(n,\mathbb{Z})$ with arbitrary coefficients for general n is still an open question unless $n \leq 4$ (for instance see [25, 17]).

Corollary 7.3. *Let G be a compact Lie group. Assume that $n = \dim(\mathfrak{g}^{\mathfrak{g}}) \leq 3$.*

$$H^p([G \to \mathrm{Aut}^+(G)]) \cong \begin{cases} \mathbb{R} & \text{if } p = 0,\, 3n \\ 0 & \text{otherwise.} \end{cases}$$

$$H^p([G \to \mathrm{Aut}\,(G)]) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Proof. If $n=0$, it reduces to Proposition 7.1. For $n=1$, $GL(1,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $SL(1,\mathbb{Z}) = \{1\}$. The spectral sequences of Theorem 7.2 are concentrated in bidegrees $(0,0)$ and $(0,3)$, hence collapse.

For $n=2$, $SL(2,Z)$ is an amalgamated sum $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ over a tree [24]. For any $SL(2,\mathbb{Z}))$-module M, the action of $SL(2,\mathbb{Z})$ on this tree yields an exact sequence

$$\ldots \to H^i(\mathbb{Z}/4\mathbb{Z}, M) \oplus H^i(\mathbb{Z}/6\mathbb{Z}, M) \to H^i(\mathbb{Z}/2\mathbb{Z}, M) \to H^{i+1}(SL(2,\mathbb{Z}), M) \to \ldots$$

Since the cohomology of a finite group acting on an $\mathbb{R}$-vector space vanishes in positive degrees, the only non trivial terms in the spectral sequence $E_2^{+p,q}$ are for $p=0$. It follows that the spectral sequence collapses and the result is given by Equation (7.3) in Theorem 7.2. A similar computation gives the result for $GL(2,\mathbb{Z}) \cong SL(2,\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$.

For $n=3$, one uses the fundamental domain introduced by Soulé in [25]. Let M^q be the $SL(3,\mathbb{Z})$-module $S^q((\mathfrak{g}^*)^{\mathfrak{g}}[3]) \cong \Lambda^q(\mathbb{R}^3)$ $(q = 0 \ldots 3)$. Since M^0 and M^3 are isomorphic to $\mathbb{R}$ with trivial action, the groups $H^{p>0}(SL(3,\mathbb{Z}), M^q)$ are trivial for $q=0,3$. Now assume $q=1$ or $q=2$. The group $SL(3,\mathbb{Z})$ acts by conjugation on the projective space of symmetric positive definite 3×3 matrices. Let D_3 be the subset of such matrices whose diagonal coefficients are all the same. The orbit $X_3 = D_3 \cdot SL(3,\mathbb{Z})$ of D_3 under $SL(3,\mathbb{Z})$ is a homotopically trivial triangulated space of dimension 3 [25]. Let Σ_i be the set of equivalence classes of cells of dimension i modulo the $SL(3,\mathbb{Z})$-action. For $\sigma \in \Sigma_i$, we denote by $SL(3,\mathbb{Z})_\sigma$ the stabilizer of the cell σ and M^q_σ for M^q endowed with the induced action of $SL(3,\mathbb{Z})_\sigma$ twisted by the orientation character $SL(3,\mathbb{Z})_\sigma \to \{\pm 1\}$. There is a spectral sequence $E_1^{i,j} = \bigoplus_{\sigma\in\Sigma_i} H^j\big(SL(3,\mathbb{Z})_\sigma, M^q_\sigma\big)$ converging to $H^{i+j}\big(SL(3,\mathbb{Z}), M^q\big)$ (see [10] Section VII.7). The stabilizers $SL(3,\mathbb{Z})_\sigma$ are described in [25] Theorem 2. They are all finite. Thus the spectral sequence reduces to $E_1^{i,0} = \bigoplus_{\sigma\in\Sigma_i} \left(M_\sigma^q\right)^{SL(3,\mathbb{Z})_\sigma}$. Direct inspection using Theorem 2 in [25] shows that $E_1^{i\leq 1,0} = 0$, $E_1^{3,0} \cong \left(M^q\right)^4$ and

$$E_2^{2,0} \cong \left(M^q\right)^4 \oplus \left(M^{qA}\right)^3 \oplus \left(M^{qB}\right) \oplus \left(M^{qC}\right)^2$$

where A, B, C are respectively the matrices

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The term d_1 of the spectral sequences is described in [10] Section VII.8. In our case, since the stabilizers of cells of dimension 3 are trivial, the differential d_1 is induced by the inclusions $\left(M_\sigma^q\right)^{SL(3,\mathbb{Z})_\sigma} \hookrightarrow M_\tau^q$ for each 3-dimensional cell $\tau \in \Sigma_3$ with $\sigma \subset \tau$ a subface of dimension 2. It follows that $E_2^{i,j} \cong 0$. Hence the result follows for $[G \to \mathrm{Aut}^+(G)]$. The case for $[G \to \mathrm{Aut}\,(G)]$ follows using the Künneth formula since $GL(3,\mathbb{Z}) \cong SL(3,\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$. q.e.d.

Remark 7.4. For $n = \dim((\mathfrak{g}^*)^{\mathfrak{g}}) = 4$, it should be possible to compute explicitly $H^\bullet([G \to \mathrm{Aut}^+(G)])$ and $H^\bullet([G \to \mathrm{Aut}\,(G)])$ using Theorem 7.2 and the techniques and results of [17]. For $n = 5, 6$, the results of [14] suggest that the cohomology groups $H^\bullet([G \to \mathrm{Aut}^+(G)])$ and $H^\bullet([G \to \mathrm{Aut}\,(G)])$ should be non trivial. For larger n, it seems a difficult question to describe explicitly the spectral sequences of Theorem 7.2.

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