DERIVED DEFORMATION THEORY OF ALGEBRAIC STRUCTURES

GRÉGORY GINOT, SINAN YALIN

Abstract. The main purpose of this article is to develop an explicit derived deformation theory of algebraic structures at a high level of generality, encompassing in a common framework various kinds of algebras (associative, commutative, Poisson...) or bialgebras (associative and coassociative, Lie, Frobenius...), that is algebraic structures parametrized by props.

A central aspect is that we define and study moduli spaces of deformations of algebraic structures *up to quasi-isomorphisms* (and not only up to isomorphims or ∞-isotopies). To do so, we implement methods coming from derived algebraic geometry, by encapsulating these deformation theories as classifying (pre)stacks with good infinitesimal properties and derived formal groups. In particular, we prove that the Lie algebra describing the deformation theory of an object in a given ∞-category of dg algebras can be obtained equivalently as the tangent complex of loops on a derived quotient of this moduli space by the homotopy automorphims of this object.

Moreover, we provide explicit formulae for such derived deformation problems of algebraic structures up to quasi-isomorphisms and relate them in a precise way to other standard deformation problems of algebraic structures. This relation is given by a fiber sequence of the associated dg-Lie algebras of their deformation complexes. Our results provide simultaneously a vast generalization of standard deformation theory of algebraic structures which is suitable (and needed) to set up algebraic deformation theory both at the ∞-categorical level and at a higher level of generality than algebras over operads.

In addition, we study a general criterion to compare formal moduli problems of different algebraic structures and apply our formalism to E_n-algebras and bialgebras.

Contents

Introduction

Deformations of algebraic structures of various kind, both classical and homotopical, have played a central role in mathematical physics and algebraic topology since the pioneering work of Drinfeld [17, 18] in the 80s as well as the work of Kontsevich [58, 59] or Chas-Sullivan [13] in the late 90s. For instance, in classical deformation quantization, a star-product is a deformation of the commutative algebra of functions to an associative algebra while a quantum group is a deformation of the cocommutative bialgebra structure of a universal envelopping algebra.

In most applications, one consider deformations of algebraic structures up to some equivalence relations, usually called gauge equivalences. In particular, different gauge equivalences on the same algebraic structure lead to *different* deformation theories. This data is organized into a moduli *space* of deformations whose connected components are the gauge equivalence classes of the deformed structure. Their higher homotopy groups encode (higher) symmetries which are becoming increasingly important in modern applications. By the Deligne philosophy, now a deep theorem by Lurie [67] and Pridham [80] such a moduli space is equivalent to the data of a homotopy Lie algebra.

The emergence of derived/higher structures techniques allows not only to consider general moduli spaces of deformations (derived formal moduli problems), but also to consider deformations of algebraic structures more general than those given by Quillen model categories of algebras over operads. In particular, it allows to consider bialgebraic structures, that is algebras over *props*, in *high generality*.

The main goal of this paper is to exploit these techniques to prove several new results about deformation theory of algebraic structures. In particular, we seek to provide appropriate extension of classical algebraic deformation theory simultaneously in two directions:

(1) By considering very general kinds of algebraic structures parametrized by props, which are of crucial importance in various problems of topology, geometry and mathematical physics where such structures appear;
(2) By considering derived formal moduli problems controlling the deformation theory of algebras *in the ∞-category of algebras*, that is *up to quasi-isomorphism*, contrary to the setting of standard operadic deformation theory which considers deformations *up to ∞-isotopies* (see § 0.2 below and section 5,6.1, 6.2 as well for detailed comparison and examples)[1].

Both directions require to work out new methods:

(1) by getting rid of the standard use of Quillen model structures to describe model categories of algebras, which does not make sense anymore for algebras over props: one *has to work directly at an ∞-categorical level.*
(2) by replacing the classical gauge group action and classical deformation functors by appropriate derived moduli spaces of algebraic structures and *derived formal groups* of homotopy automorphisms.

We now explain in more details the motivations and historical setting for our work in § 0.1 and then our contributions and main results in § 0.2.

---

[1]for instance, algebraic structures up to quasi-isomorphisms form precisely Kontsevich setting encompassing deformation of functions into star-products in the analytic or algebraic geometry context as well as for smooth manifolds, where it boils down to ”up to isomorphism“

0.1. **Motivations.** As already mentioned, many algebraic structures of various types play a key role in algebra, topology, geometry and mathematical physics. This is the case of associative algebras, commutative algebras, Lie algebras, and Poisson algebras to name a few. All these kinds of algebra share a common feature, being defined by operations with several inputs and one single output (the associative product, the Lie bracket, the Poisson bracket). The notion of operad is a unifying approach to encompass all these structures in a single formalism, and has proven to be a very powerful tool to study these structures, both from a combinatorial perspective and in a topological or dg-context[2]. The first historical examples, of topological nature, are the operads of little n-disks discovered in the study of iterated loop spaces in the sixties. Algebras governed by (versions of) these operads, as well as their dg-cousin formed by (shifted) Poisson algebras and their deformation theory play a prominent role in a variety of topics such as the study of iterated loop spaces, Goodwillie-Weiss calculus for embedding spaces, deformation quantization of Poisson manifolds and Lie bialgebras, factorization homology and derived symplectic/Poisson geometry [58, 59, 66, 69, 12, 29, 28, 36, 44, 48, 55, 60, 71, 79, 85, 92].

However, algebraic structure governed by operations with several inputs *and several outputs* also appear naturally in a variety of topics related to the same fields of mathematics. Standard example are associative and coassociative bialgebras and Lie bialgebras, which are central in various topics of algebraic topology, representation theory and mathematical physics [17, 18, 5, 26, 27, 38, 74, 75]. Here the formalism of props, which actually goes back to [70], is the convenient unifying framework to handle such structures. Props plays a crucial role in the deformation quantization process for Lie bialgebras, as shown by Etingof-Kazdhan ([26], [27]), and more generally in the theory of quantization functors [23, 76]. Props also appear naturally in topology, for example the Frobenius bialgebra structure on the cohomology of compact oriented manifolds coming from Poincaré duality, and the involutive Lie bialgebra structure on the equivariant homology of loop spaces on manifolds, which lies at the heart of string topology ([14],[15]) and are also central in symplectic field theory and Lagrangian Floer theory by the work of Cielebak-Fukaya-Latsheev [16]. Props also provide a concise way to encode various field theories such as topological quantum field theories and conformal field theories, and have recently proven to be the kind of algebraic structure underlying the topological recursion phenomenom, as unraveled by Kontsevich and Soibelman in [61] (see [9] for connections with mathematical physics and algebraic geometry).

A meaningful idea to understand the behavior of these various structures and, accordingly, to get more information about the mathematical objects on which they act, is to organize all the possible deformations of a given structure into a single *geometric object* which encapsulates not only the deformations but also an equivalence relation between these deformations. That is, to define a *formal moduli problem.* Such ideas goes back to the pioneering work of Kodaira-Spencer in geometry and the work of Gerstenhaber on associative algebras and Hochschild cohomology. In the eighties supported by Deligne and Drinfeld, a groundbreaking principle emerged, asserting that any formal moduli problem corresponds to a certain differential graded Lie algebra which parametrizes algebraically the corresponding deformation theory. The deformations correspond to special elements

[2] where the strict algebraic structure are no longer invariant under the natural equivalence of the underlying object and need to be replaced by their homotopy enhancement

of this Lie algebra called the Maurer-Cartan elements, and equivalences of deformations are determined by a quotient under the action of a gauge group. This principle had major applications among which one can pick deformation theory of complex manifolds, representation spaces of fundamental groups of projective varieties in Goldman-Millson's theory, and Kontsevich deformation quantization of Poisson manifolds.

The theory of "classical" or "underived" formal moduli problems was not sufficient to made this principle completely precise and had several limitations (like impossibility to consider weak equivalences or getting non equivalent dg-Lie algebras describing the same moduli problem)

These difficulties were solved by considering higher structured geometric moduli problem using ∞-*category theory* and *derived algebraic geometry*. The appropriate formalism is then the theory of *derived formal moduli problems*, which are simplicial presheaves over augmented artinian cdgas satisfying some extra properties with respect to homotopy pullbacks (a derived version of the Schlessinger condition). Precisely, Lurie and Pridham [66, 80] proved that (derived) formal moduli problems and dg Lie algebras are equivalent as ∞-categories.In fact, a given formal moduli problem controlling the infinitesimal neighbourhood of a point on a moduli space corresponds to a dg Lie algebra called the deformation complex of this point.

In this paper, we use rather systematically these ideas of derived formal moduli problems and derived techniques to study deformation theory of *algebraic structures*. In particular, we give a conceptual explanation of the differences between various deformation complexes appearing in the literature by explaining which kind of derived moduli problem each of these complexes controls. A key part of our study is that we study algebras over *very general props* and that we consider moduli spaces of deformations of algebraic structures up to *quasi-isomorphisms*.

0.2. **Main results.** We study moduli spaces of algebraic structures and formal moduli problems controlling their deformations. In the differential graded setting, algebraic structures are deformed as algebraic structures *up to homotopy*. A convenient formalism to deal with such at a high level of generality, encompassing not only algebras but also bialgebras, is the notion of (dg-)properad [94]. Briefly, to any complex X, on can associate its endomorphism properad $End_X(m,n) = Hom(X^{\otimes m}, X^{\otimes n})$. Then, given a properad P, a P-algebra structure on X is given by a properad morphism

$$P \to End_X.$$

There are several possible natural notions for defining deformations of (possibly homotopy) algebraic structures and we consider and compare several of them.

A standard *(pr)operadic approach* to define a deformation complex of those structure is as follows. Given a properad P, the notion of homotopy P-algebra (or P-algebra up to homotopy) can be defined properly by considering cofibrant resolutions of properads. That is, by considering P_∞-algebra where P_∞ is a *cofibrant resolution* of P in the model category of properads. To any P_∞-algebra structure $\psi : P_\infty \to End_X$ on a complex X, there is a formal moduli problem $\underline{P_\infty\{X\}}^\psi$ controlling the deformation theory of the properad morphism ψ. The associated deformation complex is an explicit dg Lie algebra noted $g^\psi_{P,X}$. This is a rather standard approach.

However, we can also construct a derived formal moduli problem controling the deformation theory of a P_∞-algebra A directly *in the ∞-category $P_\infty - Alg$* of P_∞-algebras (with quasi-isomorphisms as weak equivalences). This is not the same as deforming the morphism ψ (in a way precised below, the Maurer-Cartan elements are the same in both cases but *the gauge equivalence relation differs*).

To set up the appropriate framework for such a deformation theory, we introduce in Section 2 the notion of derived prestack group, which can be thought as a family of homotopy formal groups parametrized by a base space and apply this formalism to the deformation theory of algebras over properads. Briefly, one associates to A its *derived prestack group of homotopy automorphisms* which is the ∞-functor

$$\begin{array}{rcl} G_P(A) : CDGA_{\mathbb{K}} & \to & E_1\text{–}\mathrm{Alg}^{gp}(\mathrm{Spaces}) \\ R & \longmapsto & haut_{P_\infty - Alg(Mod_A)}(A \otimes R) \end{array}$$

where $haut_{P_\infty - Alg(Mod_A)}(A\otimes R)$ is the ∞-group of self equivalences of $A\otimes R$ in the ∞-category of A-linear P_∞-algebras.Taking homotopy fibers over augmented Artinian cdgas, we obtain a derived formal group (see Section 2.2):

$$\widehat{G_P(A)}_{id}(R) = hofib(G_P(A)(R) \to G_P(A)(\mathbb{K}))$$

whose values at an augmented Artinian cdga R is the space of R-deformations of A. Precisely, we prove

Theorem 0.1 (See Theorem 2.22)**.** *The simplicial presheaf $G_P(A)$ defines a grouplike E_1-monoid object in the ∞-category of infinitesimally cohesive simplicial ∞-presheaves. In particular $\widehat{G_P(A)}_{id}$ is a derived formal group.*

By the *equivalence between derived formal groups and derived formal moduli problems*, these deformations are parametrized by a dg Lie algebra $Lie(\widehat{G_P(A)}_{id})$.

Two natural questions arise from these constructions.

- First, can we relate the classical deformation theory of the morphism $\psi : P_\infty \to End_X$, controled by $g^{\psi}_{P,X}$, to the deformation theory of (X,ψ) in $P_\infty - Alg$, controled by $Lie(\widehat{G_P(X,\psi)})$?
- Second, is there an explicit formula computing $Lie(\widehat{G_P(X,\psi)})$ for general P and (X,ψ) ?

The answer to the first question is the following natural homotopy fiber sequence relating these two deformation complexes :

Theorem 0.2 (See Theorem 2.26)**.** *There is a fiber sequence of L_∞-algebras*

$$g^{\psi}_{P,X} \longrightarrow Lie(\widehat{G_P(X,\psi)}) \longrightarrow Lie(\underline{haut}(X))$$

where $Lie(\underline{haut}(X))$ is the Lie algebra of homotopy automorphisms of X as a complex.

To illustrate concretely how this fiber sequence explains the difference between $g^{\psi}_{P,X}$ and $Lie(\widehat{G_P(X,\psi)})$, let us start with the following observation. One should note that the deformation complex $g^{\psi}_{P,X}$ does *not* give exactly the usual cohomology theories of algebras. As a motivating example, let us consider the case of the Hochschild cochain complex of a dg associative algebra A which can be written as $\mathrm{Hom}(A^{\otimes *}, A)$. This Hochschild complex is bigraded, with a cohomological grading induced by the grading of A and a weight grading given by the tensor powers $A^{\otimes \bullet}$.

It turns out that the classical deformation complex $g^{\psi}_{Ass,A}$ is $\mathrm{Hom}(A^{\otimes>1}, A)$ and in particular misses the summand $\mathrm{Hom}(A, A)$ of weight 1; which is precisely the one allowing to consider algebras up to (quasi-)isomorphisms.

The Lie algebra $g^{\psi}_{P,X}$ can be described very explicitly in terms of a convolution algebra associated to the properad P_∞ (Proposition 1.14). In section 4, we provide a similar properadic description of the Lie algebra of the formal moduli of homotopy automophisms $\widehat{G_P(X,\psi)}$. To do so, we use the "plus" construction $g^{\psi^+}_{P^+,X}$ which is a functorial construction to modifying any dg-Prop to get the right cohomology theory.This gives us an explicit model of the deformation complex of (X, ψ) in the ∞-category of P_∞-algebras *up to quasi-isomorphisms* and thus answers the second question:

Theorem 0.3 (See Theorem 4.18)**.** *There is an equivalence of L_∞-algebras*

$$Lie(\widehat{G_P(X,\psi)}) \simeq g^{\psi}_{P,X} \rtimes^h End(X) \simeq g^{\psi^+}_{P^+,X}.$$

Note that the middle term of this equivalence exhibits $Lie(\widehat{G_P(X,\psi)})$ as a homotopical semi-direct product of $g^{\psi}_{P,X}$ with the Lie algebra $End(X)$ of endomorphisms of X (equipped with the commutator of the composition product as Lie bracket). This is proved in Section 3 where we reinterpret the deformation complex $Lie(\underline{haut}_{P_\infty-Alg}(X,\psi))$ as the tangent Lie algebra of a homotopy quotient of $\underline{P_\infty\{X\}}$ by the ∞-action of $\underline{haut}(X)$.

To summarize, the conceptual explanation behind this phenomenon is as follows. On the one hand, the L_∞-algebra $g^{\psi}_{P,X}$ controls the deformations of the P_∞-algebra structure over a *fixed complex* X, that is, the deformation theory of the properad morphism ψ. On the other hand, we built a *derived formal group* $\widehat{\underline{haut}_{P_\infty}(X,\psi)}_{id}$ whose corresponding L_∞-algebra $Lie(\widehat{\underline{haut}_{P_\infty}(X,\psi)}_{id})$ describes *another derived deformation problem*: an R-deformation of a P-algebra A in the ∞-category of P_∞-algebras *up to quasi-isomorphisms* is a an R-linear P_∞-algebra $\tilde{A} \simeq A \otimes R$ with a $\mathbb{K}$-linear P_∞-algebra quasi-isomorphism $\tilde{A} \otimes_R \mathbb{K} \xrightarrow{\sim} A$. The later L_∞-algebra admits two equivalent descriptions

$$Lie(\widehat{\underline{haut}_{P_\infty}}(A)_{id}) \simeq g^{\varphi}_{P,X} \ltimes_{hol} End(X) \simeq g^{\varphi^+}_{P^+,X}$$

where the middle one exhibits this moduli problem as originating from the homotopy quotient of the space of P_∞-algebra structures on X by the homotopy action of self-quasi-isomorphisms $haut(X)$, that is, deformations of the P_∞-algebra structure *up to self quasi-isomorphisms* of X, and the right one encodes this as simultaneous compatible deformations of the P_∞-algebra structure *and* of the differential of X. We will go back to this in full details in Sections 3 and 6.

Returning to the Hochschild complex example, we now see the role of the weight 1 part $\mathrm{Hom}(A, A)$. Indeed, in the case of a an associative dg algebra A, the complex $g^{\psi^+}_{Ass^+,A} \cong Hom(A^{\otimes>0}, A)[1]$ computes the reduced Hochschild cohomology of A, where the right hand side is a sub-complex of the standard Hochschild cochain complex shifted down by 1 equipped with its standard Lie algebra structure.The complex $g^{\psi}_{Ass,A} \cong \mathrm{Hom}(A^{\otimes>1}, A)[1]$ is the one controlling the formal moduli problem of deformations of A with fixed differential[3], where the right hand side is the subcomplex of the previous shifted Hochschild cochain complex where we have removed the $Hom(A, A)$ component[4].

In addition, in Section 2, we prove a general criterion to compare formal moduli problems induced by algebras :

Theorem 0.4 (see Theorem 2.27)**.** *Let F be an equivalence of presheaves of ∞-categories*

$$F : \underline{P_\infty - Alg} \xrightarrow{\sim} \underline{Q_\infty - Alg}.$$

Then F induces equivalences of of derived formal moduli problems

$$\underline{P_\infty\{X\}}^{\psi} \simeq \underline{Q_\infty\{F(X)\}}^{F(\psi)}$$

(and their associated formal groups) where $F(\psi)$ is the Q_∞-algebra structure on the image of (X, ψ) under F.

In Section 5, we apply our machinery to derived deformation theory of n-shifted Poisson algebras (that is Poisson algebras with a Poisson bracket of degree $1 - n$) and E_n-algebras:

Theorem 0.5 (See Corollary 5.7)**.** *(1) The* Tamarkin deformation complex[5] [87] controls deformations of A in $Pois_{n,\infty} - Alg[W^{-1}_{qiso}]$, that is, in homotopy dg-$Pois_n$-algebras up to quasi-isomorphisms. *It is thus equivalent to the tangent Lie algebra $g^{\psi^+}_{Pois^+_n,A}$ of $G_{Pois_n}(A)$.*

(2) For $n \geq 2$ the Tamarkin deformation complex of A is equivalent, as an L_∞-algebra, to the E_n-tangent complex of A seen as an E_n-algebra via the formality of E_n-operads.

To the best of the authors knowledge, the proof that this complex is indeed a deformation complex in the precise meaning of formal derived moduli problems is new, as well as the concordance with the L_∞-structure induced by the higher Deligne conjecture (which provides an E_{n+1}-algebra structure on the E_n-tangent complex of an E_n-algebra). We also prove that the deformation complex $g^{\psi}_{Pois_n,A}$ of the formal moduli problem $\underline{Pois_{n\infty}\{A\}}^{\psi}$ of homotopy n-Poisson algebra structures deforming ψ is given by the L_∞-algebra $CH^{(\bullet>1)}_{Pois_n}(A)[n]$, which is a further truncation of $CH_{Pois_n}(A)[n]$.

---

[3] Thus, when A is an ordinary, non dg, vector space, the complex $g^{0}_{Ass,A}$ parametrizes the moduli space of associative algebra structures on A, while $g^{0^+}_{Ass^+,A}$ parametrizes the moduli space of asociative algebra structures up to isomorphism of algebras

[4] there is also a third complex, the full shifted Hochschild complex $\mathrm{Hom}(A^{\otimes\geq 0}, A)[1]$, which controls not the deformations of A itself but the linear deformations of its dg category of modules Mod_A [57, 79]

[5] which we denote $CH^{(\bullet>0)}_{Poisn}(A)[n]$ since it is the part of positive weight in the full Poisson complex [11]

Concerning bialgebras, we obtain the first theorem describing precisely why (a suitable[6] version of) the Gerstenhaber-Schack complex

$$C^*_{GS}(B,B) \cong \prod_{m,n\geq 1} Hom_{dg}(B^{\otimes m}, B^{\otimes n})[-m-n]$$

is the appropriate deformation complex of a dg bialgebra *up to quasi-isomorphisms* in terms of derived moduli problems:

Theorem 0.6 (See Theorem 5.9)**.** *The Gerstenhaber-Schack complex is quasi-isomorphic to the L_∞-algebra controlling the deformations of dg bialgebras up to quasi-isomorphisms:*

$$C^*_{GS}(B,B) \cong g^{\varphi^+}_{Bialg^+_\infty,B} \simeq Lie(\widehat{\underline{haut}_{Bialg_\infty}}(B)_{id}).$$

Note that our theorem 0.3 implies that the L_∞-algebra structure induced on $C^*_{GS}(B,B)$ contains as a sub L_∞-algebra the Merkulov-Vallette deformation complex [73].

Finally, in Section 6, we give an overview and comparison of various (derived or not) deformation problems of algebraic structures arising in our work and the litterature.

Remark 0.7. A natural candidate for the deformation ∞-functor of a P_∞-algebra A in the ∞-category of P_∞-algebras (localized with respect to quasi-isomorphisms) is defined as follows. One associate, to any augmented artinian dg algebra R, the simplicial nerve $\mathcal{N}wP_\infty - Alg(Mod_R)$ of the subcategory of weak equivalences of P_∞-algebras in R-modules. The augmentation $R \to \mathbb{K}$ induces a simplicial map

$$\mathcal{N}wP_\infty - Alg(Mod_R) \to \mathcal{N}wP_\infty - Alg(Ch_\mathbb{K}).$$

The evaluation of the classifying presheaf of deformations of A on an augmented artinian dg algebra R is the homotopy fiber of the map above at the base point A. In other words, it is the formal completion $\underline{\mathcal{N}w\widehat{P_\infty - Alg}}_A$ of the functor $R \mapsto \mathcal{N}wP_\infty - Alg(Mod_R)$ at A. We detail its construction and properties in Section 2.3. In the *operadic* setting, such a functor has been studied by Hinich in [49].

The space $\underline{\mathcal{N}w\widehat{P_\infty - Alg}}_A(R)$ is homotopy equivalent to the maximal ∞-subgroupoid of the ∞-category $P_\infty - Alg(Mod_R)[W^{-1}_{qiso}]$ generated by R-linear P_∞-algebras B such that $B \otimes_R \mathbb{K} \simeq A$. So it encapsulates the whole deformation theory of A in the ∞-category $P_\infty - Alg[W^{-1}_{qiso}]$ as we can think of it, that is, an R-deformation of A is an R-linear P_∞-algebra whose restriction modulo R is quasi-isomorphic to A, and equivalences between R-deformations are defined by compatible R-linear quasi-isomorphisms whose restriction modulo R is homotopic to Id_A. However, in general, such a simplicial presheaf ***does not provide a derived formal moduli problem***. Even in the operadic case, one needs A and P to be non-positively graded to describe it as the nerve of dg Lie algebra (see [49, Section 4.3]).

The relationship between this classifying presheaf of algebras and the derived formal group of homotopy automorphisms in the neighbourhood of the identity is given by

$$\widehat{\underline{haut}_{P_\infty}}(A)_{id} = \Omega_* \underline{\mathcal{N}w\widehat{P_\infty - Alg}}_A$$

[6]there are several closely related versions of the Gerstenhaber-Schack, depending on how we truncate them

where Ω_* is the loop space for pointed functors as explained in Section 2. By the general formalism explained in Section 2 we have

$$\mathbb{T}_{\underline{haut}_{\widehat{P_\infty}}(A)_{id}} = Lie(L(\underline{\mathcal{N}w\widehat{P_\infty - Alg}}_A))$$

where L is the completion of $\underline{\mathcal{N}w\widehat{P_\infty - Alg}}_A$ in a derived formal moduli problem. Another way to state this is that in general $\underline{\mathcal{N}w\widehat{P_\infty - Alg}}_A$ *is a* 1*-proximate moduli functor in the sense of* [66].

0.3. **Further applications and perspectives.** A first major application appeared earlier in our preprint [46], where some of the results of the present article were announced. Our article provides complete proofs of these results and add some new ones as well. In [46], we use them crucially to prove longstanding conjectures in deformation theory of bialgebras and E_n-algebras as well as in deformation quantization. We prove a conjecture stated by Gerstenhaber and Schack (in a wrong way) in 1990 [38], whose correct version is that the Gerstenhaber-Schack complex forms an E_3-algebra, hence unraveling the full algebraic structure of this complex which remained mysterious for a while. It is a "differential graded bialgebra version" of the famous Deligne conjecture for associative differential graded algebras (see for instance [85] and [58]). The second one, enunciated by Kontsevich in his celebrated work on deformation quantization of Poisson manifolds [59] in 2000, is the formality, as an E_3-algebra, of the deformation complex of the symmetric bialgebra which should imply as a corollary Drinfeld's and Etingof-Kazdhan's deformation quantization of Lie bialgebras (see [17], [26] and [27]). We solve both conjectures actually at a greater level of generality than the original statements. Moreover, we deduce from it a generalization of Etingof-Kadhan's celebrated deformation quantization in the homotopical and differential graded setting.

The new methods developed here to approach deformation theory and quantization problems have several possible continuations. In particular, we aim to investigate in future works how our derived algebraic deformation theory could be adapted to provide new deformation theoretic approach, formality statements and deformation quantization of shifted Poisson structures in derived algebraic geometry. This problem can help understand quantum invariants of various moduli spaces of G-bundles over algebraic varieties and topological manifolds, which are naturally shifted Poisson stacks.

Moreover, our framework for derived algebraic deformation theory shall also be useful to study deformation problems related to the various kinds of (bi)algebras structures mentionned in this introduction, occuring in mathematical physics, algebraic topology, string topology, symplectic topology and so on.

Acknowledgement. The authors wish to thank V. Hinich, S. Merkulov, P. Safranov and T. Willwacher for their useful comments. They were also partially supported by ANR grants CHroK and CatAG and the first author benefited from the support of Capes-Cofecub project 29443NE and Max Planck Institut fur Mathematik in Bonn as well.

Notations and conventions

The reader will find below a list of the main notations used at several places in this article.

- We work over a field of characteristic zero denoted $\mathbb{K}$.
- We work with cochain complexes and a cohomological grading and denote $Ch_{\mathbb{K}}$ the category of $\mathbb{Z}$-graded *cochain* complexes over $\mathbb{K}$.
- Let $(\mathcal{C}, W_{\mathcal{C}})$ be a relative category, also called a category with weak equivalences. Meaning $\mathcal{C}$ is a category and $W_{\mathcal{C}}$ its subcategory of weak equivalences. The *hammock localization* (see [20]) of such a category with respect to its weak equivalences is denoted $L^H(\mathcal{C}, W_{\mathcal{C}})$, and the *mapping spaces* of this simplicial localization are noted $L^H(\mathcal{C}, W_{\mathcal{C}})(X, Y)$.
- We will note $L(\mathcal{M})$ the ∞-category associated to a model category $\mathcal{M}$, that is the coherent nerve of its simplicial localization.
- Given a relative category (M, W), we denote by $M[W^{-1}]$ its ∞-categorical localization. Further, we will write $\mathcal{N}W$ for the coherent nerve of the subcategory of weak-equivalences W.
- Several categories of algebras and coalgebras will have a dedicated notation: *cdga* for the category of commutative differential graded algebras, *dgArt* for the category of Artinian cdgas, *dgCog* for the category of dg coassociative coalgebras and *dgLie* for the category of dg Lie algebras.
- Given a cdga A, the category of A-modules is noted Mod_A. More generally, if $\mathcal{C}$ is a symmetric monoidal category tensored over $Ch_{\mathbb{K}}$, the category of A-modules in $\mathcal{C}$ is noted $Mod_A(\mathcal{C})$.
- Given a dg Lie algebra g, its Chevalley-Eilenberg algebra is noted $C^*_{CE}(g)$ and its Chevalley-Eilenberg coalgebra is noted $C^{CE}_*(g)$.
- More general categories of algebras and coalgebras over operads or properads will have the following generic notations: given a properad P, we will note $P - Alg$ the category of dg P-algebras and given an operad P we will note $P - Cog$ the category of dg P-coalgebras.
- Given a properad P, a *cofibrant resolution* of P is noted P_∞.
- When the base category is a symmetric monoidal category $\mathcal{C}$ other than $Ch_{\mathbb{K}}$, we note $P - Alg(\mathcal{C})$ the category of P-algebras in $\mathcal{C}$ and $P - Cog(\mathcal{C})$ the category of P-coalgebras in $\mathcal{C}$.
- Algebras over properads form a relative category for the weak equivalences defined by chain quasi-isomorphisms. The subcategory of *weak equivalences of* $P - Alg$ is noted $wP - Alg$.
- Given a properad P and a complex X, we will consider an associated convolution Lie algebra noted $g_{P,X}$ which will give rise to two deformation complexes: the deformation complex $g^{\varphi}_{P,X}$ controling the formal moduli problem of deformations of a P-algebra structure φ on X, and a variant $g^{\varphi^+}_{P^+,X}$ whose role will be explained in Section 3.
- We will consider various moduli functors in this paper, defined as simplicial presheaves over Artinian augmented cdgas: the simplicial presheaf of P_∞-algebra structures on X noted $\underline{P_\infty\{X\}}$, the formal moduli problem of deformations of a given P_∞-algebra structure φ on X noted $\underline{P_\infty\{X\}^{\varphi}}$, and the derived prestack group of homotopy automorphisms of (X, φ) noted $\underline{haut}_{P_\infty}(X, \varphi)$. The derived prestack group of automorphisms of X as a chain complex will be denoted $\underline{haut}(X)$. More generally, when we extend a simplicial set Q into a simplicial presheaf over Artinian cdgas, we will use the underlined notation $\underline{Q}$ for that extension.

1. Formal moduli problems and algebraic structures

Formal moduli problems arise when one wants to study the deformation theory of an object in a category, of a structure on a given object, of a point in a given moduli space (variety, scheme, stack, derived stack). The general principle of moduli problems is that the deformation theory of a given point in its formal neighbourhood (that is, the formal completion of the moduli space at this point) is controlled by a certain tangent dg Lie algebra.

However, if one does not work in a derived geometric/higher categoricla context, there are several well known issues with this principle:

- Lie algebras which are not quasi-isomorphic can nevertheless describe the same moduli problem (a famous example is the deformation theory of a closed subscheme, seen either as a point of a Hilbert scheme or as a point of a Quot scheme). Even worse, their is no systematic recipe to build a Lie algebra out of a moduli problem;
- Deformation problems for which the equivalence relation is given by weak equivalences (say, quasi-isomorphisms between two deformations of a dg algebra) do not fit in the framework of classical algebraic geometry (that is, deformations which manifests a non trivial amount of homotopy theory);
- There is no natural interpretation of the obstruction theory in terms of the corresponding moduli problem.

To overcome these difficulties encountered when working in underived deformation theory, in particular in the correspondence between deformation functors and dg Lie algebras, one has to consider moduli problems in a derived setting. The rigorous statement of an equivalence between (derived) formal moduli problems and dg Lie algebras was proved independently by Lurie in [66] and by Pridham in [80]. In this paper, what *we will call moduli problems are actually derived moduli problems.*

1.1. Formal moduli problems and (homotopy) Lie algebras.

We start by an overview of the notion of formal moduli problem and relevant L_∞-algebras.

Definition 1.1. Formal moduli problems are functors $F : dgArt_{\mathbb{K}}^{aug} \to sSet$ from augmented Artinian commutative differential graded algebras to simplicial sets satisfying the following conditions:

(1) The functor F preserves weak equivalences (that is, quasi-isomorphisms of cdgas are sent to weak equivalences of simplicial sets);
(2) There is a weak equivalence $F(\mathbb{K}) \simeq pt$;
(3) The functor F is **infinitesimally cohesive**: Given any (homotopy) pullback diagram
$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$
in $dgArt_{\mathbb{K}}$ such that the induced maps $\pi_0(A) \to \pi_0(B)$ and $\pi_0(B') \to \pi_0(B)$ are surjective, the induced diagram
$$\begin{array}{ccc} F(A') & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ F(B') & \longrightarrow & F(B) \end{array}$$
is a (homotopy) pullback in $sSet$.

Remark 1.2. Condition (3) is a derived version of the classical *Schlessinger condition* introduced in [66] and developped in [67] The notion of (infinitesimally) cohesive generalizes to any functor from connective dg-commutative algebras to $sSet$. In that general setting functors satisfying condition (3) in Definition 1.1 are called cohesive, while infinitesimally cohesive stands for those functors satisfying this condition only when the the maps $\pi_0(A) \to \pi_0(B)$ and $\pi_0(B') \to \pi_0(B)$ are further required to have nilpotent kernels. For Artinian cdgas, the latter condition is automatic and therefore infinitesimally cohesive and cohesive are the same. We stick to the longer name to recall that special property of the Artinian context.

The value $F(\mathbb{K})$ corresponds to the point of which we study the formal neighbourhood, the evaluation $F(\mathbb{K}[t]/(t^2))$ on the algebra of dual numbers encodes infinitesimal deformations of this point, and the $F(\mathbb{K}[t]/(t^n))$ are polynomial deformations of a higher order, for instance.

Formal moduli problems form a full sub-∞-category noted $FMP_{\mathbb{K}}$ of the ∞-category of simplicial presheaves over augmented Artinian cdgas. By [66, Theorem 2.0.2], this ∞-category is equivalent to the ∞-category $dgLie_{\mathbb{K}}$ of dg Lie algebras. Moreover, one side of the equivalence is made explicit, and is equivalent to the nerve construction of dg Lie algebras studied thoroughly by Hinich in [47]. The homotopy invariance of this nerve relies on nilpotence conditions on the dg Lie algebra. In the case of formal moduli problems, this nilpotence condition is always satisfied because one tensors the Lie algebra with the maximal ideal of an augmented Artinian cdga.

In patcice it is oftten convenient to work with homotopy Lie algebras, that is, L_∞-algebras, rather than strict dg-Lie algebras:

Definition 1.3. (1) An L_∞-algebra is a graded vector space $g = \{g_n\}_{n\in\mathbb{Z}}$ equipped with maps $l_k : g^{\otimes k} \to g$ of degree $2-k$, for $k \geq 1$, satisfying the following properties:

- $l_k(..., x_i, x_{i+1}, ...) = -(-1)^{|x_i||x_{i+1}|} l_k(..., x_{i+1}, x_i, ...)$
- for every $k \geq 1$, the generalized Jacobi identities
$$\sum_{i=1}^{k} \sum_{\sigma\in Sh(i,k-i)} (-1)^{\epsilon(i)} l_k(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(k)}) = 0$$
where σ ranges over the $(i, k-i)$-shuffles and
$$\epsilon(i) = i + \sum_{j_1<j_2, \sigma(j_1)>\sigma(j_2)} (|x_{j_1}||x_{j_2}| + 1).$$

It is standard that the above definition is equivalent to the following:

(2) An L_∞-algebra structure on a graded vector space $g = \{g_n\}_{n\in\mathbb{Z}}$ is exactly the data of a coderivation $Q : Sym^{\bullet\geq 1}(g[1]) \to Sym^{\bullet\geq 1}(g[1])$ of degree 1 of the cofree cocommutative coalgebra $Sym^{\bullet\geq 1}(g[1])$ such that $Q^2 = 0$.

The bracket l_1 is in particular a differential that makes g a cochain complex. The dg-coalgebra of (2) is called the reduced *Chevalley-Eilenberg chain complex* of the L_∞-algebra g, denoted $\overline{C}^{CE}_*(g)$. The dg-algebra $\overline{C}^*_{CE}(g)$ obtained by dualizing the dg coalgebra of (2) is called the (reduced) *Chevalley-Eilenberg cochain algebra* of g.

Definition 1.4. A L_∞ algebra g is *filtered* if it admits a decreasing filtration
$$g = F_1 g \supseteq F_2 g \supseteq ... \supseteq F_r g \supseteq ...$$

compatible with the brackets: for every $k \geq 1$,

$$l_k(F_r g, g, ..., g) \in F_r g.$$

We require moreover that for every r, there exists an integer $N(r)$ such that $l_k(g, ..., g) \subseteq F_r g$ for every $k > N(r)$.

A filtered L_∞ algebra g is *complete* if the canonical map $g \to lim_r g/F_r g$ is an isomorphism.

In particular a nilpotent L_∞-algebra is complete and, if $\mathfrak{m}$ is the augmentation ideal of an Artinian CDGA, then $g \otimes \mathfrak{m}$ is also complete for any L_∞-algebra g.

The completeness of a L_∞ algebra allows to define properly the notion of Maurer-Cartan element:

Definition 1.5. (1) Let g be a complete L_∞-algebra and $\tau \in g^1$, we say that τ is a Maurer-Cartan element of g if

$$\sum_{k \geq 1} \frac{1}{k!} l_k(\tau, ..., \tau) = 0.$$

The set of Maurer-Cartan elements of g is noted $MC(g)$.

(2) The simplicial Maurer-Cartan set is then defined by

$$MC_\bullet(g) = MC(g \hat{\otimes} \Omega_\bullet),$$

, where $\Omega_\bullet$ is the Sullivan cdga of de Rham polynomial forms on the standard simplex $\Delta^\bullet$ (see 7.5 and [84]) and $\hat{\otimes}$ is the completed tensor product with respect to the filtration induced by g.

The simplicial Maurer-Cartan set is a Kan complex, functorial in g and preserves quasi-isomorphisms of complete L_∞-algebras. The *Maurer-Cartan moduli set* of g is $\mathcal{MC}(g) = \pi_0 MC_\bullet(g)$: it is the quotient of the set of Maurer-Cartan elements of g by the homotopy relation defined by the 1-simplices. When g is a *complete dg Lie algebra*, it turns out that this homotopy relation is equivalent to the action of the gauge group $exp(g^0)$ (a prounipotent algebraic group acting on Maurer-Cartan elements), so in this case this moduli set coincides with the one usually known for Lie algebras. We refer the reader to [98] for more details about all these results. We also recall the notion of

Definition 1.6 (Twisting by a Maurer-Cartan element)**.** . The twisting of a complete L_∞ algebra g by a Maurer-Cartan element τ is the complete L_∞ algebra g^τ with the same underlying graded vector space and new brackets l_k^τ defined by

$$l_k^\tau(x_1, ..., x_k) = \sum_{i \geq 0} \frac{1}{i!} l_{k+i}(\underbrace{\tau, ..., \tau}_{i}, x_1, ..., x_k)$$

where the l_k are the brackets of g.

Let us explain briefly why Lurie's equivalence [66, Theorem 2.0.2] lifts from the ∞-category of dg Lie algebras $dgLie$ to the ∞-category of L_∞-algebras $L_\infty - Alg$. Let $p : L_\infty \xrightarrow{\sim} Lie$ be the cofibrant resolution of the operad Lie encoding L_∞-algebras. This morphism induces a functor $p^* : dgLie \to L_\infty - Alg$ which associates to any dg Lie algebra the L_∞-algebra with the same differential, the same bracket of arity 2 and trivial higher brackets in arities greater than 2. This functor is a right Quillen functor belonging to a Quillen equivalence

$$p_! : L_\infty - Alg \leftrightarrows dgLie : p^*,$$

since p is a quasi-isomorphism of Σ-cofibrant operads (see [31, Theorem 16.A]). Quillen equivalences induce equivalences of the ∞-categories associated to these model categories. Therefore, we have a commutative triangle of ∞-categories

$$\begin{array}{ccc} L_\infty - Alg & & \\ {\scriptstyle p^*}\uparrow & \searrow^{\tilde{\psi}} & \\ dgLie & \xrightarrow[\psi]{} & FMP_{\mathbb{K}} \end{array}$$

where ψ maps a Lie algebra to its nerve functor, and $\tilde{\psi}$ maps an L_∞-algebra to its Maurer-Cartan space defined as $dgArt^{aug}_{\mathbb{K}} \ni R \mapsto MC_\bullet(g \otimes \mathfrak{m}_R)$ (where $\mathfrak{m}_R$ is the maximal ideal of R).

The maps p^* and ψ are weak equivalences of ∞-categories (the model of quasi-categories is used in [68], but actually any model works). By the two-out-of-three property of weak equivalences, this implies that $\tilde{\psi} : L_\infty - Alg \to FMP_{\mathbb{K}}$ is a weak equivalence of ∞-categories.

Definition 1.7. Let F be a formal moduli problem. We denote $F \mapsto \mathcal{L}_F \in L_\infty - Alg$ an inverse of the equivalence $\tilde{\psi} : L_\infty - Alg \to FMP_{\mathbb{K}}$.

To conclude, let us say a word about *formal deformations*. Although the ring of formal power series in one variable $\mathbb{K}[[t]]$ is not Artinian, given a formal moduli problem F, one can properly define the notion of formal deformation, or deformation over $\mathbb{K}[[t]]$, by setting

$$F(\mathbb{K}[[t]]) \ := \ \underset{i}{\underleftarrow{lim}} \ F(\mathbb{K}[t]/t^i).$$

(where we consider a homotopy limit in the ∞-category of simplicial sets). Let us note g_F the dg Lie (or L_∞) algebra of F via the Lurie-Pridham correspondence. By [93, Corollaire 2.11] (or [66]), there is an natural weak-equivalence

$$F(\mathbb{K}[[\hbar]]) \ \simeq \ Map(\mathbb{K}[-1], g_{\mathcal{F}})$$

where $\mathbb{K}[-1]$ is the one dimensional Lie algebra concentrated in degree 1 with trivial Lie bracket. Here Map denotes the derived mapping space in the ∞-category of dg Lie algebras, which can be explicited in the corresponding model category by taking a cofibrant resolution of $\mathbb{K}[-1]$. We refer the reader to [93, Section 1.1] for example, to see an explicit construction of such a cofibrant resolution. The main point of interest for us here, is that the space of Lie morphisms from such a resolution is equivalent to the space of Maurer-Cartan elements in formal power series without constant terms, that is

$$F(\mathbb{K}[[t]]) \simeq MC_\bullet(tg_F[[t]]).$$

This means that formal deformations of a point can be explicitely described in terms of the corresponding Lie algebra.

1.2. **Moduli spaces of algebraic structures and their formal moduli problems.** We now explain a first (prop)eradic approach to moduli of algebraic structures.

Moduli spaces of algebraic structures were originally defined by Rezk as simplicial sets, in the setting of simplicial *operads* [81]. This notion can be extended to algebras over differential graded *props* as follows (see [98]):

Definition 1.8. Let P_∞ be a cofibrant prop and X be a complex. The *moduli space of P_∞-algebra structures* on X is the simplicial set $P_\infty\{X\}$ defined by

$$P_\infty\{X\} = Mor_{Prop}(P_\infty, End_X \otimes \Omega_\bullet),$$

where the prop $End_X \otimes \Omega_\bullet$ is defined by

$$(End_X \otimes \Omega_\bullet)(m,n) = End_X(m,n) \otimes \Omega_\bullet$$

and $\Omega_\bullet$ is the Sullivan cdga of the standard simplex $\Delta^\bullet$ (see 7.5).

Given a *cofibrant* properad P_∞ and any properad Q, we will denote

$$Map_{Prop}(P,Q) := Mor_{Prop}(P, Q \otimes \Omega_\bullet)$$

the *mapping space of properads morphisms.*

Indeed, the aritywise tensor product $(-) \otimes \Omega_\bullet$ forms a functorial simplicial resolution in the model category of dg props [98, Proposition 2.5]. This simplicial set enjoys the following key properties, see [98]:

Proposition 1.9. *(1) The simplicial set $P_\infty\{X\}$ is a Kan complex and*

$$\pi_0 P_\infty\{X\} = [P_\infty, End_X]_{Ho(Prop)}$$

is the set of homotopy classes of P_∞-algebra structures on X.

(2) Any weak equivalence of cofibrant props $P_\infty \xrightarrow{\sim} Q_\infty$ induces a weak equivalence of Kan complexes $Q_\infty\{X\} \xrightarrow{\sim} P_\infty\{X\}$.

We can extend the moduli space of P_∞-structure to a simplicial presheaf by base change from $\mathbb{K}$ to any Artinian cdga.

Definition 1.10. Let P_∞ be a cofibrant properad and X be a complex. We define a simplicial presheaf $\underline{P_\infty\{X\}} : dgArt_{\mathbb{K}}^{aug} \rightarrow sSet$ by the formula

$$\underline{P_\infty\{X\}} : A \in dgArt_{\mathbb{K}}^{aug} \mapsto P_\infty \otimes A\{X \otimes A\}_{Mod_A}$$

where $P_\infty \otimes A\{X \otimes A\}_{Mod_A}$ is the mapping space of dg props in A-modules $Map(P_\infty \otimes A, End_{X\otimes A}^{Mod_A})$ and $End_{X\otimes A}^{Mod_A}$ is the endomorphism prop of $X \otimes A$ taken in the category of A-modules.

In other words, $\underline{P_\infty\{X\}}(A)$ is the *simplicial moduli space* of P_∞-algebra structures on $X \otimes A$ in the category of A-modules. Indeed, since Mod_A is tensored over $Ch_{\mathbb{K}}$, on can make P_∞ act on A-modules either by morphisms of dg props in A-modules from $P_\infty \otimes A$ to the endomorphism prop defined by the internal hom of Mod_A, or by morphisms of dg props from P_∞ to the endomorphism prop defined by the external hom of Mod_A. See for instance [98, Lemma 3.4].

By Proposition 1.9, the simplicial set $\underline{P_\infty\{X\}}(A)$ classifies $P_\infty \otimes A$-algebra structures on $X \otimes A$. However, the simplicial presheaf $\underline{P_\infty\{X\}}$ is not a formal moduli problem, since $\underline{P_\infty\{X\}}(\mathbb{K})$ is in general not contractible.

Definition 1.11. The *formal moduli problem* $\underline{P_\infty\{X\}}^\psi$ *controlling (a certain type of) formal deformations of a* P_∞*-algebra structure* $\psi : P_\infty \to End_X$ *on* X *is* defined, on any augmented Artinian cdga A, as *the homotopy fiber*

$$\underline{P_\infty\{X\}}^\psi(A) = hofib(\underline{P_\infty\{X\}}(A) \to \underline{P_\infty\{X\}}(\mathbb{K})) \tag{1.1}$$

taken over the base point ψ, the map being induced by the augmentation $A \to \mathbb{K}$.

The moduli spaces of algebraic structures and its associated formal moduli problem are encoded by L_∞-algebras according to Lurie - Pridham Theorem. We now explain how those L_∞-structures can be described explicitly using *dg-properads* following [72] and [98].

Cofibrant resolutions of a properad P can always be obtained as a cobar construction $\Omega(C)$ on some coproperad C (which is usually the bar construction or the Koszul dual if P is Koszul). Given a cofibrant resolution $P_\infty := \Omega(C) \xrightarrow{\sim} P$ of P and another properad Q, one constructs the convolution dg Lie algebra $Hom_\Sigma(\overline{C}, Q)$:

Definition 1.12. Let C be an augmented coproperad and Q be a properad. Their associated *convolution* dg Lie algebra is the dg $\mathbb{K}$-module

$$Hom_\Sigma(\overline{C}, Q)$$

of morphisms of Σ-biobjects from the augmentation ideal of C to Q endowed with the differential induced by the internal ones of C and Q. It is equipped with the Lie bracket given by the antisymmetrization of the convolution product.

This convolution product is defined similarly to the convolution product of morphisms from a coalgebra to an algebra, using the infinitesimal coproduct of C and the infinitesimal product of Q.

The total complex $Hom_\Sigma(\overline{C}, Q)$ is a complete dg Lie algebra. More generally, if P is a properad with minimal model $(\mathcal{F}(s^{-1}\overline{C}), \partial) \xrightarrow{\sim} P$ for a certain homotopy coproperad C (see [72, Section 4] for the definition of homotopy coproperads), and Q is any properad, then *the complex* $Hom_\Sigma(\overline{C}, Q)$ *is a complete dg* L_∞ *algebra* (which is not a dg-Lie algebra in general).

The simplicial mapping space of morphisms $P_\infty \to Q$ is computed by the convolution L_∞-algebra $Hom_\Sigma(\overline{C}, Q)$ thanks to the following theorem:

Theorem 1.13. *(cf.* [98, Theorem 2.10,Corollary 4.21]*) Let* P *be a dg properad equipped with a semi-free resolution* $P_\infty := (\mathcal{F}(s^{-1}\overline{C}), \partial) \xrightarrow{\sim} P$ *and* Q *be a dg properad. The simplicial presheaf*

$$\underline{Map}(P_\infty, Q) : A \in dgArt_{\mathbb{K}}^{aug} \mapsto Map_{Prop}(P_\infty, Q \otimes A)$$

is equivalent to the simplicial presheaf

$$\underline{MC_\bullet}(Hom_\Sigma(\overline{C}, Q)) : A \in dgArt_{\mathbb{K}}^{aug} \mapsto MC_\bullet(Hom_\Sigma(\overline{C}, Q) \otimes A)$$

associated to the complete L_∞*-algebra* $Hom_\Sigma(\overline{C}, Q)$*.*

Note that by [97, Corollary 2.4], the tensor product $MC_\bullet(Hom_\Sigma(\overline{C}, Q) \otimes A)$ does not need to be completed because A is Artinian. In order to get a fromal moduli problem, we also consider the simplicial presheaf

$$\underline{MC_\bullet^{fmp}}(Hom_\Sigma(\overline{C}, Q)) : A \in dgArt_{\mathbb{K}}^{aug} \mapsto MC_\bullet(Hom_\Sigma(\overline{C}, Q) \otimes m_A),$$

where m_A is the maximal ideal of A. This presheaf is a formal moduli problem associated to $Hom_\Sigma(\overline{C}, Q)$. In the case where $Q = End_X$, Theorem 1.13 implies that the the simplicial presheaf $\underline{MC_\bullet}(Hom_\Sigma(\overline{C}, End_X))$ is equivalent to $\underline{P_\infty\{X\}}$ (definition 1.10).

This theorem applies in particular to the case of a Koszul properad, which includes for instance Frobenius algebras, Lie bialgebras and their variants such as involutive Lie bialgebras in stSS:DefALgStructClassicring topology. It applies also to more general situations such as the properad *Bialg* encoding associative and coassociative bialgebras, which is homotopy Koszul [72, Proposition 41].

We now describe the L_∞-algebra structure encoding this formal moduli problem. It is given by twisting the convolution Lie algebra as follows. The *twisting* of $Hom_\Sigma(\overline{C}, End_X)$ by a properad morphism $\psi : P_\infty \to End_X$ is often called the *deformation complex*[7] of ψ, and we have an isomorphism

$$g^\psi_{P,X} = Hom_\Sigma(\overline{C}, End_X)^\psi \cong Der_\psi(\Omega(C), End_X)$$

where the right-hand term is the complex of derivations with respect to ψ [73, Theorem 12].

Proposition 1.14. *The tangent L_∞-algebra of the formal moduli problem $\underline{P_\infty\{X\}}^\psi$ is given by*

$$g^\psi_{P,X} = Hom_\Sigma(\overline{C}, End_X)^\psi.$$

Proof. Let A be an augmented Artinian cdga. By Theorem 1.13, we have the homotopy equivalences

$$\begin{aligned} \underline{P_\infty\{X\}}^\psi(A) &\simeq hofib(\underline{MC_\bullet}(g_{P,X})(A) \to \underline{MC_\bullet}(g_{P,X})(\mathbb{K})) \\ &= hofib(MC_\bullet(g_{P,X} \otimes A) \to MC_\bullet(g_{P,X})) \\ &\simeq MC_\bullet(hofib_{L_\infty}(g_{P,X} \otimes A \to g_{P,X})) \end{aligned}$$

where $hofib_{L_\infty}(g_{P,X} \otimes A \to g_{P,X})$ is the homotopy fiber, over the Maurer-Cartan element ψ, of the L_∞-algebra morphism $g_{P,X} \otimes A \to g_{P,X}$ given by the tensor product of the augmentation $A \to \mathbb{K}$ with $g_{P,X}$. This homotopy fiber is nothing but $g^\varphi_{P,X} \otimes m_A$, where m_A is the maximal ideal of A, so there is an equivalence of formal moduli problems

$$\underline{P_\infty\{X\}}^\psi \simeq \underline{MC^{fmp}_\bullet}(g^\psi_{P,X}).$$

By Lurie's equivalence theorem, this means that $g^\psi_{P,X}$ is the Lie algebra of the formal moduli problem $\underline{P_\infty\{X\}}^\psi$. □

2. Derived formal groups of algebraic structures and associated formal moduli problems

In this section, we explain how the theory of formal moduli problems is related to derived formal groups, and how this allows to state the correspondence between formal groups and Lie algebras at a higher (and derived) level of generality. This correspondence is suitable for us to define a natural deformation problem of homotopy P-algebras structures on a complex X up to quasi-isomorphisms and to understand how it relates to those associated moduli space of algebraic structures from section 1.2.

[7]Proposition 1.14 belows justifying the name, though of course one has to be careful about which kind of deformation it encodes

Definition 2.1. Let $\mathcal{C}$ be a stable ∞-category. We denote by $Mon^{gp}_{E_1}(\mathcal{C})$ the ∞-category of grouplike E_1-monoids in $\mathcal{C}$, that is the subcategory of grouplike objects in the ∞-category of E_1-algebras in $\mathcal{C}$ equipped with the cartesian monoidal structure. here an E_1-monoid G is said to be *grouplike* if the two canonical maps $(\mu, \pi_i) : G \times G \to G \times G$ (induced by the multiplication $\mu : G \times G \to G$ and the the two canonical projections π_1, $\pi_2 : G \times G \to G$) are equivalences.

A *group object* of $\mathcal{C}$ is an object of $Mon^{gp}_{E_1}(\mathcal{C})$.

Example 2.2. Loop spaces provide the main source of examples of group objects in topology (which are also called H-groups in this particular setting). A topological monoid M is said to be grouplike if $\pi_0 M$ is a group, and since any grouplike topological monoid is equivalent to a loop space, grouplike topological monoids model group objects in the ∞-category of topological spaces in the sense of Definition 2.1. The same holds true for grouplike simplicial monoids, which model group objects in the ∞-category of simplicial sets and will be especially useful for us to study homotopy automorphisms of algebras.

2.1. **Generalities on derived formal groups.** First, let us remark that the category of formal moduli problems is pointed. In fact, we have :

Lemma 2.3. *Let $SPsh^{pt}((dgArt^{aug}_{\mathbb{K}})^{op})$ be the full sub-∞-category of $SPsh((dgArt^{aug}_{\mathbb{K}})^{op})$ consisting of those ∞-functors F from augmented dg Artinian algebras to simplicial sets such that $F(\mathbb{K})$ is contractible. This ∞-category is pointed.*

Proof. Let us first note pt the ∞-functor sending any augmented Artinian cdga to the simplicial set generated by a single vertex. Now let R and R' be two augmented Artinian cdgas, let us write $\eta_R, \eta_{R'}$ for their respective unit morphisms and $\epsilon_R, \epsilon_{R'}$ their respective augmentations. Let $f : R \to R'$ be a morphism of augmented Artinian cdgas. A morphism of augmented Artinian cdgas commutes with units, so the diagram

$$\begin{array}{ccccc} pt(R) & \xrightarrow{\sim} & F(\mathbb{K}) & \xrightarrow{F(\eta_R)} & F(R) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle f} \\ pt(R') & \xrightarrow{\sim} & F(\mathbb{K}) & \xrightarrow{F(\eta_{R'})} & F(R') \end{array}$$

commutes as well, hence we get a unique morphism of ∞-functors $pt \to F$. A morphism of augmented Artinian cdgas commute with augmentations, so the diagram

$$\begin{array}{ccccc} F(R) & \xrightarrow{F(\epsilon_R)} & F(\mathbb{K}) & \xrightarrow{\sim} & pt(R) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} & & {\scriptstyle =}\downarrow \\ F(R') & \xrightarrow{F(\epsilon_{R'})} & F(\mathbb{K}) & \xrightarrow{\sim} & pt(R') \end{array}$$

commutes as well, hence a unique morphism $F \to pt$. ☐

Consequently, one can form the *pointed loop space functor* as the homotopy pullback

$$\Omega_* F := pt \times^h_F pt \tag{2.1}$$

in $SPsh^{pt}((dgArt^{aug}_{\mathbb{K}})^{op})$. Let us note that since $dgArt^{aug}_{\mathbb{K}}$ and $sSet$ are presentable, the ∞-category of pointed ∞-functors is presentable as well. Therefore $SPsh^{pt}((dgArt^{aug}_{\mathbb{K}})^{op})$ is a presentable pointed ∞-category, and $FMP_{\mathbb{K}}$ is a presentable pointed sub-∞-category of it. Therefore the universal property of homotopy pushouts makes $\Omega_* F$ into a group-like E_1-monoid in simplicial presheaves (see 2.5.(1) below).

Moreover, the inclusion

$$i : FMP_{\mathbb{K}} \hookrightarrow SPsh^{pt}((dgArt_{\mathbb{K}}^{aug})^{op})$$

commutes with small homotopy limits, and since small homotopy limits in $SPsh^{pt}((dgArt_{\mathbb{K}}^{aug})^{op})$ are determined pointwise, we have proved

Lemma 2.4. *For any derived formal moduli problem F and any augmented Artinian cdga R, we have $(\Omega_* F)(R) \cong \Omega_{\eta_R} F(R)$ where the base point of $F(R)$ is given by the morphism $F(\eta_R) : pt \simeq F(\mathbb{K}) \to F(R)$ induced by the unit η_R of R.*

The base point of $F(R)$ corresponds to the "trivial R-deformation" of the unique point of $F(\mathbb{K})$. It is important to mention that $FMP_{\mathbb{K}}$ is presentable [66, Remark 1.1.17] and that the inclusion of $FMP_{\mathbb{K}}$ in pointed ∞-functors admits a left adjoint (applying the ∞-categorical adjoint functor theorem)

$$L : SPsh^{pt}((dgArt_{\mathbb{K}}^{aug})^{op}) \to FMP_{\mathbb{K}}$$

making a simplicial presheaf canonically into a formal moduli problem. When F is a formal moduli problem, then $LF \cong F$, otherwise LF is the (best) formal moduli problem approximating the pointed ∞-functor F. The functor L is hard to understand explicitely in general, but is related to the (standard) pointwise classifying space functor, in the sense that we have a natural equivalence

$$L(B\Omega_* F) \cong LF \tag{2.2}$$

where B is given by applying objectwise the classifying space functor from E_1-monoids in spaces to spaces.

The loop space functor enjoys the following properties (as a consequence of Lurie's work [66], see for example [8, Proposition 2.15] for a proof):

Proposition 2.5.
(1) *Let $\mathcal{C}$ be a pointed presentable ∞-category. The pointed loop space ∞-functor lifts to a (∞-categorical) limit preserving functor*

$$\Omega_* : \mathcal{C} \to Mon_{E_1}^{gp}(\mathcal{C})$$

where $Mon_{E_1}^{gp}(\mathcal{C})$ (1.4) is the ∞-category of grouplike E_1-monoids in $\mathcal{C}$.
(2) *In the case $\mathcal{C} = FMP_{\mathbb{K}}$, the loop space functor is an equivalence.*

Definition 2.6. A *derived formal group* is an object of $Mon_{E_1}^{gp}(FMP_{\mathbb{K}})$, that is a group object in the stable pointed ∞-category of formal moduli problems.

By proposition 2.5.(2), the functor Ω_* has a left adjoint

$$B_{fmp} : Mon_{E_1}^{gp}(FMP_{\mathbb{K}}) \longrightarrow FMP_{\mathbb{K}}. \tag{2.3}$$

The functor B_{fmp} is obtained as a generalized bar construction given by the realization of a simplicial object in derived formal moduli problems, hence a homotopy colimit corresponding to a classifying space ∞-functor for derived formal groups (see [8, Lemma 2.16] and [69, Remark 5.2.2.8]). Composing equivalence (2) with Lurie's equivalence theorem [66] result into the equivalence

$$Mon_{E_1}^{gp}(FMP_{\mathbb{K}}) \cong FMP_{\mathbb{K}} \cong L_\infty - Alg$$

between the ∞-category of dg-Lie algebras and derived formal groups.

This equivalence is an analogue to the classical correspondence between formal/algebraic/Lie groups and Lie algebras. This equivalence holds true not only in the commutative case but also for iterated loop spaces and noncommutative moduli problems, see [8, Proposition 2.15].

Remark 2.7. Note that B_{fmp} is *not* defined pointwise by the standard classifying space. If it was so, then, given a formal moduli problem F, for any Artinian cdga R, there would be an equivalence $B\Omega_{\eta_R}F(R) \simeq F(R)$, which would imply that $F(R)$ is connected. This is not the case, since $F(R)$ is equivalent to the nerve of the dg Lie algebra $\mathfrak{L}_F \otimes m_R$ (where $\mathfrak{L}_F$ is the dg Lie algebra of F via Lurie-Pridham correspondence), and the connected components of the later are the equivalence classes of Maurer-Cartan elements of $\mathfrak{L}_F$.

The tangent complex of a formal moduli problem F has a canonical Ω-spectrum structure. Indeed, for any integer n, one has a homotopy pullback of augmented Artinian cdgas

$$\begin{array}{ccc} \mathbb{K}\oplus\mathbb{K}[n] & \longrightarrow & \mathbb{K} \\ \downarrow & & \downarrow \\ \mathbb{K} & \longrightarrow & \mathbb{K}\oplus\mathbb{K}[n+1] \end{array}$$

where $\mathbb{K}\oplus\mathbb{K}[n]$ is the square zero extension of $\mathbb{K}$ by $\mathbb{K}[n]$. Such a square satisfies the conditions required to apply the infinitesimal cohesiveness property of F, and moreover $F(\mathbb{K})$ is contractible, hence inducing a weak equivalence of simplicial sets

$$F(\mathbb{K}\oplus\mathbb{K}[n]) \xrightarrow{\sim} pt \times^h_{\mathbb{K}\oplus\mathbb{K}[n+1]} pt \simeq \Omega_* F(\mathbb{K}\oplus\mathbb{K}[n+1]).$$

We recognize here the structure of an Ω-spectrum $\mathcal{T}_F$, whose associated complex is $\mathbb{T}_F$, the tangent complex of F (at its unique points). Now, recall that the pointed loop space functor for formal moduli problems is determined pointwise by the standard pointed loop space, so $\mathcal{T}_{\Omega_* F} \simeq \Omega\mathcal{T}_F$, which means that

$$T_{\Omega_* F} \;\cong\; T_F[-1] \tag{2.4}$$

for the corresponding complexes.

Remark 2.8. Since $(\Omega_* F)(R) \simeq \Omega_{\eta_R}F(R)$ (lemma 2.4), the derived formal group of a formal moduli problem F seems to retain, for any R, only the informations about the connected component of the trivial R-deformation. However, all the information of the deformation problem is in fact contained here, since its tangent complex gives the dg Lie algebra controling it. To understand how this is possible, let us remind that by infinitesimal cohesiveness of F we have, for example, equivalences

$$F(\mathbb{K}\oplus\mathbb{K}[n]) \simeq \Omega_* F(\mathbb{K}\oplus\mathbb{K}[n+1])$$

which means that the space of $\mathbb{K}\oplus\mathbb{K}[n]$-deformations is equivalent to the space of self-equivalences of the trivial $\mathbb{K}\oplus\mathbb{K}[n+1]$-deformation. For example, deformations over the algebra of dual numbers $\mathbb{K}[t]/(t^2)$ are recovered as loops over the trivial $\mathbb{K}[\epsilon]/(\epsilon^2)$-deformation with ϵ of degree 1.

More generally, if F is a pointed ∞-functor such that $\Omega_* F$ is a derived formal group (e.g. a 1-proximate moduli problem in the sense of [66, Definition 5.1.5], see also cite[Lemma 2.11]BKP), then

$$T_{\Omega_* F} \;\simeq\; T_{LF}[-1].$$

This comes from [66, Lemma 5.1.12]. In other words, *The derived formal group $\Omega_* F$ controls the deformations parametrized by the formal moduli completion of F.*

Note that many functors are not representable by a derived stack via Lurie's representability theorem [67], but produce nethertheless derived formal moduli problems when restricted to Artinian cdgas, so one can associate a Lie algebra to them without any representability condition.

Example 2.9. A case of interest for us is when F is an infinitesimally cohesive (in the sense of [67, Definition 2.1.1]) simplicial ∞-presheaf over $(dgArt_{\mathbb{K}}^{aug})^{op}$. That is a simplicial presheaf preserving weak equivalences and satisfying the derived Schlessinger condition (1.1), but such that $F(\mathbb{K})$ is not (necessarily) contractible. Then one can nethertheless attach to any $\mathbb{K}$-point $x \in F(\mathbb{K})$ a derived formal moduli problem $\widehat{F_x}$ by setting

$$\widehat{F_x}(R) = hofib_x(F(R) \to F(\mathbb{K})),$$

where the map is induced by the augmentation $R \to \mathbb{K}$ of the Artinian cdga R (see the proof of 2.11). Thus, one attaches to any $x \in F(\mathbb{K})$ a derived formal group by taking the pointed loop space of the construction above. Hence, such a F parametrizes a family of derived formal moduli problems over $F(\mathbb{K})$.

2.2. **Derived prestack group and their tangent L_∞-algebras.** We will now study families of derived formal groups, which we call derived prestack groups. These are analogues of Lie groups but in the context of infinitesimally cohesive prestacks instead of manifolds. In particular, they have an associated L_∞-algebra given by their tangent space at the neutral element.

Let us denote by $SPsh_\infty^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op})$ the ∞-category of infinitesimally cohesive ∞-functors on $dgArt_{\mathbb{K}}^{aug}$ with values in simplicial sets. Here we note $SPsh$ for simplicial presheaves and $infcoh$ for the infinitesimal cohesiveness of the corresponding ∞-functors. We can consider its ∞-category of group objects (Definition 2.1); that is we introduce the following definition:

Definition 2.10. A *derived prestack group* is a group object in the ∞-category $SPsh_\infty^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op})$. More precisely, the ∞-category of derived prestack groups is $Mon_{E_1}^{gp}(SPsh_\infty^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op}))$.

The relevance of the definition is given by the following

Lemma 2.11. *Let G be a derived prestack group. For any $x \in G(\mathbb{K})$, the completion*

$$\widehat{G_x} := \Big(R \mapsto hofib_x\big(G(R) \to G(\mathbb{K})\big)\Big)$$

is a formal derived group.

Proof. The map $G(R) \to G(\mathbb{K})$ is induced by the augmentation $R \to \mathbb{K}$ of R. Since the homotopy fiber is an ∞-limit, it preserves the infinitesimally cohesive condition and weak equivalences. By definition, the homotopy fiber computed for $R = \mathbb{K}$ is a point and therefore $\widehat{G_x}$ is a formal moduli problem according to definition 1.1. □

In other words, a derived prestack group G is a family of derived formal groups parametrized by $G(\mathbb{K})$. In what follows, we will by especially interested in the formal completion at the neutral element. The pointed loop space construction commutes with homotopy fibers, so for any $F \in SPsh_{\infty}^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op})$ and any $x \in F(\mathbb{K})$, we have

$$\widehat{(\Omega_x F)}_e = \Omega_* \widehat{F_x}.$$

Hence the derived formal group associated to the derived prestack group $\Omega_x F$ by completion at the constant loop is the derived formal group corresponding to the formal moduli problem $\widehat{F_x}$.

Remark 2.12. One cannot expect the formal completion of any derived stack at a point to produce a derived formal group and a corresponding tangent Lie algebra, because of the lack of cohesiveness. However, any derived Artin stack (that is, geometric for smooth morphisms) is in particular cohesive, see for instance [65, Corollary 6.5] and [67, Lemma 2.1.7].

To legitimate constructions we are going to use in the next section, it is worth mentionning the following properties of infinitesimally cohesive simplicial presheaves:

Lemma 2.13. *(1) The ∞-category $SPsh_{\infty}^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op})$ is stable under small limits.*

(2) If $\mathcal{C}$ and $\mathcal{D}$ are two equivalent ∞-categories, the ∞-categories $SPsh_{\infty}^{infcoh}(\mathcal{C}^{op})$ and $SPsh_{\infty}^{infcoh}(\mathcal{D}^{op})$ are equivalent as well.

Proof. (1) Follows from the definition of infinitesimally cohesive ∞-functors [67, Remark 2.1.11].

(2) This is just a particular case of an equivalence of ∞-categories of sheaves induced by an equivalence of their ∞-sites, here with the discrete Grothendieck topology. □

Definition 2.14. *(Tangent Lie algebra of derived groups)*

- Let $\widehat{G}$ be a derived formal group (2.6). Its *tangent homotopy Lie algebra* is
$$Lie(\widehat{G}) := \mathfrak{L}_{B_{fmp}(\widehat{G})} \in Lie_{\infty} - Alg$$
where B_{fmp} is the equivalence (2.3) and $\mathfrak{L}_{(-)}$ the one of 1.7.
- Let G be a derived prestack group (2.10). Its *tangent homotopy Lie algebra* is
$$Lie(G) := Lie(\widehat{G}_1)$$
where $\widehat{G}_1$ is the formal completion at the unit of G (2.11).

The following result shows that the tangent at the identity of a derived prestack group inherits a canonical structure of homotopy Lie algebra (which completely determines it if it is actually a derived formal group).

Proposition 2.15. *Let G be a derived prestack group.*

1. *There is an equivalence of underlying complexes $Lie(G) \cong (T_G)_1$ between its Lie algebra and its tangent space at 1.*
2. *If F is a formal moduli problem and $G \cong \Omega F$, then $Lie(G) = \mathfrak{L}_F$.*
3. *For any point x in $G(\mathbb{K})$, $(T_G)_x \cong (T_G)_1$.*

Proof. By Proposition 2.5 and (2.4) we have equivalences of complexes

$$T_{\widehat{G}_1} \cong T_{\Omega B_{fmp}\widehat{G}_1} \cong T_{B_{fmp}\widehat{G}_1}[-1] \cong \mathfrak{L}_{B_{fmp}\widehat{G}_1}$$

where the first equivalence follows from the fact that ΩB_{fmp} is equivalent to the identity, the second equivalence from 2.4 and the third equivalence from Lurie's result [67] asserting that the underlying complex of the Lie algebra $\mathfrak{L}_F$ of a formal moduli problem F is equivalent to $T_F[-1]$. The first claim follows then from Definition 2.14.

The second claim follows from the fact that ΩB_{fmp} is the identity and Lemma 2.4, using the sequence of equivalences

$$Lie(\Omega F) = \mathfrak{L}_{B_{fmp}\Omega F} \cong \mathfrak{L}$$

and that $B_{fmp}\Omega$ is equivalent to the identity.

To conclude, since G is a grouplike monoid object, the map $G \to G$ induces by multiplication by x is an equivalence which proves the last statement. □

Example 2.16. Easy examples of derived prestack groups G are given by infinitesimally cohesive ∞-functors

$$G : dgArt^{aug}_{\mathbb{K}} \to \Omega\text{-}Spaces$$

where Ω-*Spaces* is the ∞-category $Mon^{gp}_{E_1}(Top)$ of grouplike E_1-monoids in spaces, i.e., group objects in topological spaces. By May's recognition principle, the latter are (weakly) equivalent to loop spaces, hence the terminology. Our examples of interests will take place in the ∞-category of grouplike simplicial monoids $sMon^{gl}$ as a model for $Mon^{gp}_{E_1}(Top)$ (i.e. we use the equivalence between the model categories of topological spaces and simplicial sets and strictification to model Ω-*Spaces*). As we explained, a derived prestack group G gives rises to a family of derived formal groups parametrized by $G(\mathbb{K})$.

In the next section we will focus the formal neighbourhood of the identity in homotopy automorphism groups, and see how this formalism applies to homotopy automorphisms of algebras over properads.

2.3. **Prestacks of algebras and derived groups of homotopy automorphisms.** We now define our second type of moduli of algebraic structures build on automorphisms of the structure.

First we recall that the self equivalences of an object in an ∞-category are canonically a group object in space (as in example 2.16). When the ∞-category comes from a model category, strict models for those self equivalences are given by simplicial monoids of homotopy automorphisms. We refer the reader to [36, Section 2.2] for a detailed account on simplicial monoids of homotopy automorphisms in model categories and to [19, 20, 21] for the generalization to homotopy automorphisms in the simplicial localization of any relative category.

Definition 2.17. Let X be a chain complex. Let P be a properad, P_∞ a cofibrant resolution of P, and $(X, \psi : P_\infty \to End_X)$ be a P_∞-algebra structure on X.

- We denote $\underline{haut}(X)$ the derived prestack group of homotopy automorphisms of the underlying complex X taken in the model category of chain complexes[8]. It is defined by

  $$dgArt^{aug}_{\mathbb{K}} \ni A \mapsto haut_{Mod_A}(X \otimes A),$$

  where $haut_{Mod_A}$ is the simplicial monoid of homotopy automorphisms in the category of A-modules.

[8] Precisely we consider the projective model structure

- We define $\underline{haut}_{P_\infty}(X,\psi)$ to be the derived prestack group associated to the automorphisms of (X,ψ)[9] in the ∞-category $P_\infty - Alg[W_{qiso}^{-1}]$:

$$dgArt_{\mathbb{K}}^{aug} \ni A \mapsto \mathrm{Iso}_{P_\infty - Alg(Mod_A)[W_{qiso}^{-1}]}\big(X\otimes A, X\otimes A\big)$$

  where, for any ∞-category $\mathcal{C}$, we write $\mathrm{Iso}_{\mathcal{C}}$ for the space of maps in the underlying maximum ∞-groupoid of $\mathcal{C}$.

Note that, since X is cofibrant (like any chain complex over a field) and $(-)\otimes A$ is a left Quillen functor, the homotopy automorphisms $\underline{haut}(X)$ above are exactly the self quasi-isomorphisms of $X\otimes A$. We prove in Theorem 2.22 below that $\underline{haut}_{P_\infty}(X,\psi)$ is indeed a derived prestack group.

Let us describe more precisely this derived group: consider the presheaf of ∞-categories over $CDGA_{\mathbb{K}}^{op}$ defined by

$$\begin{array}{rcl}\underline{P_\infty - Alg} : CDGA_{\mathbb{K}} & \to & Cat_\infty \\ R & \longmapsto & P_\infty - Alg(Mod_R^{cof})[W_{qiso}^{-1}]\end{array}$$

where Cat_∞ is the ∞-category of ∞-categories. Here Mod_R^{cof} is the subcategory of cofibrant R-modules in the projective model structure. Let us take then the maximal sub-∞-groupoid of $P_\infty - Alg(Mod_R^{cof})[W_{qiso}^{-1}]$ for each R, getting an ∞-groupoid valued presheaf. Then, the based loop space at a point (X,ψ) is exactly $\underline{haut}_{P_\infty}(X,\psi)$. An explicit construction for this is given by, for any cdga R, the Dwyer-Kan simplicial loop groupoid [22] of the quasi-category $P_\infty - Alg(Mod_R)[W_{qiso}^{-1}]$. Then the Kan complex of paths from $(X\otimes R, \phi\otimes R)$ to itself in this simplicial loop groupoid is a model for $\underline{haut}_{P_\infty}(X,\psi)(R)$ (this is similar to example 2.16).

We now describe a "point-set" model for the construction of those (∞-categorical) derived groups of P_∞-algebras automorphisms. First, we introduce a related and useful construction.

The presheaf of Dwyer-Kan classification spaces. The assignment

$$A \mapsto wP_\infty - Alg(Mod_A^{cof}),$$

where the $w(-)$ stands for the subcategory of weak equivalences and cof for cofibrant A-modules, defines a weak presheaf of categories in the sense of [1, Definition I.56]. It sends a morphism $A\to B$ to the functor $-\otimes_A B$. which is symmetric monoidal, hence lifts at the level of P_∞-algebras. This is not a strict presheaf, since the composition of morphisms $A\to B\to C$ is sent to the functor $(-)\otimes_A B\otimes_B C$, which is naturally isomorphic (but not equal) to $(-)\otimes_A C$. This weak presheaf can be strictified into a presheaf of categories. Applying the nerve functor to this then defines an ∞-groupoid.

Definition 2.18. We denote

$$\underline{\mathcal{N}wP_\infty - Alg} : A \mapsto \mathcal{N}wP_\infty - Alg(Mod_A^{cof}) \tag{2.5}$$

for the *simplicial presheaf of Dwyer-Kan classification spaces* given by the above construction, that is the coherent nerve of the presheaf of categories induced by the subcategory of weak-equivalences in $P_\infty - Alg$.

[9]that is, the automorphisms or weak self-equivalences of (X,ψ) in this ∞-category

We also denote

$$\underline{\mathcal{N}wCh_{\mathbb{K}}} : A \mapsto Mod_A^{cof} \tag{2.6}$$

the simplicial presheaf of quasi-coherent modules of [89, Definition 1.3.7.1].

The loop space on $\underline{\mathcal{N}wP_\infty - Alg}$ based at a P_∞-algebra (X, ψ) is then the strictification of the weak simplicial presheaf

$$\Omega_{(X,\psi)}\underline{\mathcal{N}wP_\infty - Alg} : A \mapsto \Omega_{(X\otimes A,\psi\otimes A)}\mathcal{N}wP_\infty - Alg(Mod_A^{cof}).$$

Lemma 2.19. *The pointwise loop space defined above is pointwise equivalent to the loop space functor in the projective model category of simplicial presheaves, where we consider simplicial presheaves with values in pointed simplicial sets.*

Proof. The pointed loop space functor on the projective model category of simplicial presheaves $SPsh(\mathcal{C})$ on a model category $\mathcal{C}$ is defined on any simplicial presheaf F as the homotopy pullback $pt \times_F^h pt$. In the model category setting, a homotopy pullback is computed as the limit of a fibrant resolution of the pullback diagram in $SPsh(\mathcal{C})^I_{inj}$, where I is the small category $\{\bullet \to \bullet \leftarrow \bullet\}$ and inj means that we consider this diagram category equipped with the injective model structure. Moreover, we have a Quillen equivalence

$$SPsh(\mathcal{C})^I_{proj} \leftrightarrows SPsh(\mathcal{C})^I_{inj}$$

where $SPsh(\mathcal{C})^I_{proj}$ is the projective model category of I-diagrams and the adjunction is given by the identity functors. In particular, this implies that every fibrant resolution in $SPsh(\mathcal{C})^I_{inj}$ is a fibrant resolution in $SPsh(\mathcal{C})^I_{proj}$. In the projective model structure $SPsh(\mathcal{C})^I_{proj}$, fibrations are the same as in the projective model category of functors $Fun(\mathcal{C} \times I, sSet)_{proj}$. So a fibrant resolution in $SPsh(\mathcal{C})^I_{inj}$ is pointwise a fibrant resolution in $sSet^I_{inj}$. Moreover, limits in $SPsh(\mathcal{C})$ are determined pointwise. This implies that the pullback of a fibrant resolution of a pullback diagram in simplicial presheaves is given, pointwise, by the pullback of a fibrant resolution of a pullback diagram in simplicial sets. That is, the homotopy pullback defining the loop space functor for simplicial presheaves, when valued at a given object of $\mathcal{C}$, gives the homotopy pullback defining the loop space functor for pointed simplicial sets. □

Homotopy automorphism presheaves as loops over the presheaf of Dwyer-Kan classification spaces. In the case of an operad O, there is an easy model for $\underline{haut}_{O_\infty}$. Indeed, in that case, O_∞-algebras inherits a canonical model category structure and $\underline{haut}_{O_\infty}$ is the ∞-functor associated to a simplicial presheaf given by the simplicial monoid of homotopy automorphisms of (X, ψ) in the model category of O_∞-algebras. That is the simplicial sub-monoid of self weak equivalences in the usual homotopy mapping space $Map_{O_\infty - Alg}(X, X)$ (see for instance [52, Chapter 17]). Thus this weak simplicial presheaf is

$$A \longmapsto haut_{O_\infty}(X \otimes A, \psi \otimes A)_{Mod_A}$$

where $haut_{O_\infty}(X \otimes A, \psi \otimes A)_{Mod_A}$ is the simplicial monoid of homotopy automorphisms of $(X \otimes A, \psi \otimes A) \in O_\infty - Alg(Mod_A^{cof})$. Note that by definition, this homotopy automorphism are computed by taking a cofibrant resolution of $(X \otimes A, \psi \otimes A)$ to get a cofibrant-fibrant object (all algebras are fibrant), and then considering weak self-equivalences of it. Our simplicial presheaf is then its strictification (see [1, Section I.2.3.1]).

In the case of a general properad P, there is no model category structure anymore on the category of P_∞-algebras. However, we can still define the simplicial monoid $\underline{L^H wP_\infty - Alg}(X,\psi)$ of homotopy automorphisms in the simplicial or hammock localization (with respect to quasi-isomorphisms) of P_∞-algebras, following Dwyer-Kan [20, 21]. Note that by [21], in the case when $P_\infty - Alg$ is a model category (that is, P is an operad), we have a homotopy equivalence

$$\underline{haut}_{P_\infty}(X,\psi) \simeq \underline{L^H wP_\infty - Alg}(X,\psi)$$

(taking the model category construction for the left side of this equivalence), so the two constructions agree. In both cases, these are models of the pointed loop space $\Omega_{(X,\psi)}\underline{\mathcal{N}wP_\infty - Alg}$ on the simplicial presheaf of Dwyer-Kan classification spaces:

Lemma 2.20. *Let P_∞ be a cofibrant prop. Then $\underline{haut}_{P_\infty}(X,\psi)$ is equivalent to*

$$A \in dgArt_{\mathbb{K}}^{aug} \longmapsto \Omega_{(X\otimes A,\psi\otimes A)}\Big(\mathcal{N}wP_\infty - Alg(Mod_A^{cof})\Big).$$

Further, $\underline{haut}(X)$ is equivalent to

$$A \in dgArt_{\mathbb{K}}^{aug} \longmapsto \Omega_{X\otimes A}\Big(\mathcal{N}wMod_A^{cof}\Big).$$

Proof. This comes from the fact that, for any relative category (C,W) and any object X of C, the connected component of X in $\mathcal{N}W$ is equivalent to the classifying space $BLW(X,X)$. Therefore there is an equivalence $LW(X,X) \simeq \Omega_X\mathcal{N}W$ of simplicial monoids. Hence we can define the presheaf of homotopy automorphisms, or self-weak equivalences, $\underline{haut}_{P_\infty}(X,\psi)$ is equivalent to the following simplicial presheaf

$$\underline{haut}_{P_\infty}(X,\psi) : A \in dgArt_{\mathbb{K}}^{aug} \longmapsto \Omega_{(X\otimes A,\psi\otimes A)}\mathcal{N}wP_\infty - Alg(Mod_A^{cof}).$$

The proof for $\underline{haut}(X)$ is similar. □

Prestacks of algebras. We will now prove that what we called the derived group of automorphisms of an algebra is indeed a derived prestack group. As a first step, we need the following version of Rezk's homotopy pullback theorem [81] :

Proposition 2.21. *Let P_∞ be a cofibrant prop and X be a chain complex. The forgetful functor $P_\infty - Alg \to Ch_{\mathbb{K}}$ induces a homotopy fiber sequence*

$$\underline{P_\infty\{X\}} \to \underline{\mathcal{N}wP_\infty - Alg} \to \underline{\mathcal{N}wCh_{\mathbb{K}}}$$

of simplicial presheaves over augmented Artinian cdgas (see 2.18 for the notations).

Proof. We explain briefly how [99, Theorem 0.1] can be transposed in the context of simplicial presheaves of cdgas. The identification of the homotopy fiber of the forgetful map

$$\underline{\mathcal{N}wP_\infty - Alg} \to \underline{\mathcal{N}wCh_{\mathbb{K}}}$$

with the simplicial presheaf $\underline{P_\infty\{X\}}$ follows from the two following facts. First, we can identify it pointwise with $Map(P_\infty\otimes A, End_{X\otimes A}^{Mod_A})$, where $End_{X\otimes A}^{Mod_A}$ is the endomorphism prop of $X\otimes A$ in the category of A-modules. This comes from the extension of [99, Theorem 0.1] to A-linear P_∞-algebras, which holds true trivially by replacing chain complexes by A-modules as target category in the universal functorial constructions of [99, Section 2.2] (A-modules are equipped with exactly the same operations than chain complexes which are needed in this construction: directs sums, suspensions, twisting cochains). Second, for any morphism of cdgas $f : A \to B$, the tensor product $(-) \otimes_A B$ induces an isomorphism of simplicial sets

$$Map(P_\infty \otimes A, End_{X\otimes A}^{Mod_A}) \cong Map(P_\infty \otimes B, End_{X\otimes B}^{Mod_B})$$

fitting in a commutative square

$$\begin{array}{ccc} Map(P_\infty \otimes A, End_{X\otimes A}^{Mod_A}) & \xrightarrow{\cong} & \underline{P_\infty\{X\}}(A) \\ {\scriptstyle (-)\otimes_A B}\downarrow & & \downarrow{\scriptstyle f} \\ Map(P_\infty \otimes A, End_{X\otimes A}^{Mod_A}) & \xrightarrow{\cong} & \underline{P_\infty\{X\}}(B) \end{array}$$

(see for instance [98, Section 3]) so that we get a morphism of homotopy fiber sequences

$$\begin{array}{ccccc} \underline{P_\infty\{X\}}(A) & \longrightarrow & \underline{\mathcal{N}wP_\infty - Alg}(Mod_A) & \longrightarrow & \mathcal{N}wMod_A \\ \downarrow & & {\scriptstyle (-)\otimes_A B}\downarrow & & {\scriptstyle (-)\otimes_A B}\downarrow \\ \underline{P_\infty\{X\}}(B) & \longrightarrow & \underline{\mathcal{N}wP_\infty - Alg}(Mod_B) & \longrightarrow & \mathcal{N}wMod_B. \end{array}$$

□

Theorem 2.22. *The simplicial presheaf $\underline{haut}_{P_\infty}(X,\psi)$ is a derived prestack group in the sense of Definition 2.10*[10].

In particular $\widehat{\underline{haut}_{P_\infty}(X},\psi)_{id}$ is a derived formal group.

Proof. First, recall that $\underline{haut}_{P_\infty}(X,\psi)$ is equivalent to $\Omega_{(X,\psi)}\underline{\mathcal{N}wP_\infty - Alg}$, and that we already know it is a presheaf with values in grouplike simplicial monoids, hence a group object in simplicial presheaves. Second, we use the simplicial presheaf version of Rezk's pullback theorem [81] for algebras over properads, that is, the homotopy fiber sequence

$$\underline{P_\infty\{X\}} \to \underline{\mathcal{N}wP_\infty - Alg} \to \underline{\mathcal{N}wCh_{\mathbb{K}}}$$

of simplicial presheaves over augmented Artinian cdgas, taken over the base point X given by Proposition 2.21. This homotopy fiber sequence induces a homotopy fiber sequence

$$\Omega_{(X,\psi)}\underline{\mathcal{N}wP_\infty - Alg} \to \Omega_X\underline{\mathcal{N}wCh_{\mathbb{K}}} \to \underline{P_\infty\{X\}}$$

hence the fiber sequence

$$\underline{haut}_{P_\infty}(X,\psi) \to \underline{haut}(X) \to \underline{P_\infty\{X\}}.$$

(by Lemma 2.20). Now we combine this result with Lemma 2.13(1) to deduce that $\underline{haut}_{P_\infty}(X,\psi)$ preserves weak equivalences and is infinitesimally cohesive. For this, we just have to check that the two right-hand terms of the fiber sequence satisfy these properties and use that this homotopy fiber sequence is in particular a pointwise homotopy fiber sequence. Concerning $\underline{haut}(X)$ this is already known from see [66, Section 5.2], and concerning $\underline{P_\infty\{X\}}$ this follows from its isomorphism with the Maurer-Cartan simplicial presheaf in Theorem 1.13.

[10]that is an object of $Mon_{E_1}^{gp}(SPsh_\infty^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op}))$

In particular, $\widehat{\underline{haut}_{P_\infty}(X,\psi)}_{id}$ is a derived formal group. Note that we could have directly proved this last statement by taking the formal completion of the fiber sequence above (that is, the componentwise homotopy fiber of this diagram over each appropriate base point), and then apply Lemma 2.13(1) to $\widehat{\underline{haut}(X)}_{id}$ (which is a formal derived group) and $\underline{\widehat{P_\infty\{X\}}}_{\varphi}$ (which is a derived formal moduli problem). $\square$

Remark 2.23. The classification space $\underline{\mathcal{N}wP_\infty - Alg}$ decomposes as a coproduct of the classifying spaces of homotopy automorphisms of P_∞-algebras

$$\underline{\mathcal{N}wP_\infty - Alg} \cong \coprod_{[Y,\phi]\in\pi_0\mathcal{N}wP_\infty - Alg} B\underline{haut}_{P_\infty - Alg}(Y,\phi)$$

where $[Y,\phi]$ ranges over quasi-isomorphism classes of P_∞-algebras. Restricting the homotopy pullback of Proposition 2.21 to the connected component $B\underline{haut}(X)$ of the base space, we get a homotopy pullback

$$\begin{array}{ccc} \underline{P_\infty\{X\}} & \longrightarrow & \coprod_{[Y,\phi],Y\simeq X} B\underline{haut}_{P_\infty - Alg}(Y,\phi) \\ \downarrow & & \downarrow \\ pt & \hookrightarrow & B\underline{haut}(X) \end{array}$$

where the coproduct $\coprod_{[Y,\phi],Y\simeq X}$ ranges over P_∞-algebras so that $Y\simeq X$ as complexes. So Rezk homotopy pullback theorem and its version above tells us that $\coprod_{[Y,\phi],Y\simeq X} B\underline{haut}_{P_\infty - Alg}(Y,\phi)$ can be seen as a homotopy quotient of $\underline{P_\infty\{X\}}$ by the action of $\underline{haut}(X)$. From a deformation theoretic perspective, this means that at a "tangent level", the deformation theory of $\psi : P_\infty \to End_X$ corresponds to deformations of the P_∞-algebra (X,ψ) which preserves the differential of the underlying complex X, whereas the deformations associated to $\underline{haut}_{P_\infty - Alg}(X,\psi)$ deform the differential as well (that is, it takes into account the action of $\underline{haut}(X)$ on $\underline{P_\infty\{X\}}$). We are going to see in Section 3 how to formalize properly this idea.

Remark 2.24. Let us explain the relationship between the classifying presheaf of algebras and the derived formal group of homotopy automorphisms in the neighbourhood of the identity. Recall the construction

$$\underline{haut}_{P_\infty}(X,\psi) : A \in dgArt^{aug}_{\mathbb{K}} \longmapsto \Omega_{(X\otimes A,\psi\otimes A)}\mathcal{N}wP_\infty - Alg(Mod_A^{cof}),$$

from which we deduce

$$\widehat{\underline{haut}_{P_\infty}(X,\psi)}_{id} = \Omega_*\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}$$

where Ω_* is the loop space for pointed functors as explained in Section 2. Using the decomposition of the nerve of weak equivalences into classifying spaces of homotopy automorphisms pointed out in Remark 2.23, we see moreover that for any augmented Artinian cdga R, there is a decomposition

$$\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}(R) \cong \coprod_{[Y,\phi]|(Y,\phi)\otimes_R\mathbb{K}\simeq(X,\psi)} Bhaut_{P_\infty - Alg(Mod_R)}(Y,\phi).$$

Equivalently, $\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}(R)$ is homotopy equivalent to the maximal ∞-subgroupoid of the ∞-category $P_\infty - Alg(Mod_R)[W_{qiso}^{-1}]$ generated by R-linear P_∞-algebras (Y,ϕ) such that $(Y,\phi)\otimes_R \mathbb{K} \simeq (X,\psi)$, that is

$$\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}(R) \cong P_\infty - Alg(Mod_R)[W_{qiso}^{-1}] \times^h_{P_\infty - Alg(Ch_{\mathbb{K}})[W_{qiso}^{-1}]} \{(X,\psi)\}.$$

The space $\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}$ encapsulates the whole deformation theory of (X,ψ) in the ∞-category $P_\infty - Alg[W_{qiso}^{-1}]$ as we can think of it, that is, an R-deformation of (X,ψ) is an R-linear P_∞-algebra whose restriction modulo R is quasi-isomorphic to (X,ψ), and equivalences between R-deformations are defined by compatible R-linear quasi-isomorphisms whose restriction modulo R is homotopic to $Id_{(X,\psi)}$. This is the natural generalization, to the differential graded setting, of classical deformations of degree zero algebras. Although it is not clear that such a construction provides a derived formal moduli problem, one can however associates to it the derived formal group $\underline{haut}_{P_\infty}(X,\psi)$ via a loop space construction, and by the general formalism explained in Section 2 we have

$$\mathbb{T}_{\widehat{\underline{haut}_{P_\infty}(X,\psi)}_{id}} = Lie(L(\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}))$$

where L is the completion of $\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}$ in a formal moduli problem. Another way to state this is that in general $\underline{\widehat{\mathcal{N}wP_\infty - Alg}}_{(X,\psi)}$ is 1-proximate in the sense of [66].

Remark 2.25. In the special case of operads acting on algebras concentrated in degree 0, we can say more. Let A be a P-algebra in vector spaces whose underlying vector space is of finite dimension, then by [89, Prop.2.2.6.8], the classifying presheaf $\underline{\mathcal{N}wP - Alg}$ is actually a derived 1-geometric stack, which implies by [65, Corollary 6.5] and [67, Lemma 2.1.7] that its restriction to $dgArt_{\mathbb{K}}^{aug}$ is infinitesimally cohesive. Consequently $\underline{\widehat{\mathcal{N}wP - Alg}}_A$ is already a derived formal moduli problem in this case and

$$Lie\big(\widehat{\underline{haut}_P(A)}_{id}\big) \;\cong\; Lie(\underline{\widehat{\mathcal{N}wP - Alg}}_A)$$

by Proposition 2.15.

In the special case where P is a non-positively graded dg operad and A a non-positively graded dg algebra, this classifying presheaf is not known to be a derived geometric stack, nevertheless it is homotopy equivalent to the nerve of the tangent complex of A according to [49, Theorem 2.3.4].

This Lie algebra recovers in particular various known deformation complexes in the litterature, once one has an explicit formula to compute it, as we are going to detail in Section 5.

2.4. **The fiber sequence of deformation theories.** We now relate precisely the two moduli problems of algebraic structures, that is those governed by the mapping space $\underline{P_\infty\{X\}}$ and the homotopy automorphisms space $\underline{haut_{P_\infty - Alg}}(X,\psi)$ (Definitions 2.17 and 1.10).

Theorem 2.26. *There is a homotopy fiber sequence of derived prestack groups*

$$\Omega_\psi \underline{P_\infty\{X\}} \to \underline{haut_{P_\infty - Alg}}(X,\psi) \to \underline{haut}(X),$$

hence a homotopy fiber sequence of derived formal groups

$$\Omega_{\psi}\widehat{\underline{P_{\infty}\{X\}}} \rightarrow \widehat{\underline{haut_{P_{\infty}-Alg}}}(X,\psi)_{id} \rightarrow \widehat{\underline{haut}(X)}_{id},$$

and equivalently of their associated L_{∞}*-algebras*

$$g^{\psi}_{P,X} \rightarrow Lie(\underline{haut_{P_{\infty}-Alg}}(X,\psi)) \rightarrow Lie(\underline{haut}(X)).$$

Proof. Recall (see (2.1)) that the pointed loop space functor is defined on any simplicial presheaf F as the homotopy pullback $pt \times^{h}_{F} pt$. It thus commutes with homotopy fibers, and in particular the loop space ∞-functor commutes with fibers in the ∞-category of simplicial presheaves. In the fiber sequence of Proposition 2.21, we choose ψ as the base point on the left, (X,ψ) in the middle, and X on the right. Since fibers in the ∞-category of presheaves valued in simplicial monoids are determined in the underlying ∞-category of simplicial presheaves, applying the pointed loop space ∞-functor with respect to these base points, we deduce a fiber sequence of derived prestack groups

$$\Omega_{\psi}\underline{P_{\infty}\{X\}} \rightarrow \underline{haut_{P_{\infty}-Alg}}(X,\psi) \rightarrow \underline{haut}(X)$$

(using that $\Omega \circ B \simeq Id$). Hence, we get a fiber sequence of derived formal groups

$$\Omega_{\psi}\widehat{\underline{P_{\infty}\{X\}}} \rightarrow \widehat{\underline{haut_{P_{\infty}-Alg}}}(X,\psi)_{id} \rightarrow \widehat{\underline{haut}(X)}_{id}$$

(using that $\Omega\widehat{F_x} \simeq \widehat{\Omega_x F}$ for an infinitesimally cohesive ∞-functor F and $x \in F(\mathbb{K})$, and in particular that $\widehat{G_e} \simeq \Omega\widehat{BG}$ for a derived prestack group G). The corresponding fiber sequence of Lie algebras aasociated to this formal derived problems identifies with the desired one

$$g^{\psi}_{P,X} \rightarrow Lie(\underline{haut_{P_{\infty}-Alg}}(X,\psi)) \rightarrow Lie(\underline{haut}(X))$$

by Lemma 2.20 and using equivalence (2.4) □

2.5. **Equivalent deformation theories for equivalent (pre)stacks of algebras.** In derived algebraic geometry, an equivalence between two derived Artin stacks F and G induces a weak equivalence between the tangent complex over a given point of F and the tangent complex over its image in G [89]. We now prove similar statement about the tangent Lie algebras of our formal moduli problems of algebraic structures.

Recall the presheaf of categories given by the ∞-functor

$$\begin{array}{rcl} \underline{P_{\infty}-Alg} : CDGA_{\mathbb{K}} & \rightarrow & Cat_{\infty} \\ R & \longmapsto & P_{\infty}-Alg(Mod_R)[W^{-1}_{qiso}] \end{array}$$

where Cat_{∞} is the ∞-category of ∞-categories.

The idea is that [99, Theorem 0.1] implies that the formal moduli problem $\underline{P_{\infty}\{X\}}^{\psi}$ is "tangent" over (X,ψ) to the Dwyer-Kan classification space of the ∞-category of P_{∞}-algebras (see 2.3).

More precisely, recall that $F : \underline{P_{\infty}-Alg} \xrightarrow{\sim} \underline{Q_{\infty}-Alg}$ being an an equivalence of presheaves of ∞-categories means that, for every augmented Artinian cdga A,

$$F(A) : P_{\infty}-Alg(Mod_A)[W^{-1}_{qiso}] \xrightarrow{\sim} Q_{\infty}-Alg(Mod_A)[W^{-1}_{qiso}].$$

is an equivalence of ∞-categories. Relying on our previous results, we prove:

Theorem 2.27. *Let F be an equivalence of presheaves of ∞-categories*

$$F : \underline{P_\infty - Alg} \xrightarrow{\sim} \underline{Q_\infty - Alg}.$$

Then F induces an equivalence of fiber sequences of derived formal moduli problems

$$\begin{array}{ccccc}
\underline{P_\infty\{X\}}^{\psi} & \longrightarrow & B_{fmp}\widehat{\underline{haut}_{P_\infty-Alg}(X,\psi)}_{Id_{(X,\psi)}} & \longrightarrow & B_{fmp}\widehat{\underline{haut}(X)}_{Id_X} \\
\downarrow \sim & & \downarrow \sim & & \downarrow = \\
\underline{Q_\infty\{F(X)\}}^{F(\psi)} & \longrightarrow & B_{fmp}\widehat{\underline{haut}_{Q_\infty-Alg}(F(X,\psi))}_{Id_{(X,\psi)}} & \longrightarrow & B_{fmp}\widehat{\underline{haut}(X)}_{Id_X}
\end{array}$$

where $F(\psi)$ is the Q_∞-algebra structure on the image of (X,ψ) under F (and B_{fmp} is given by 2.3). Equivalently, F induces an equivalence of fiber sequences of the associated L_∞-algebras

$$\begin{array}{ccccc}
g^{\psi}_{P,X} & \longrightarrow & Lie(\underline{haut}_{P_\infty-Alg}(X,\psi)) & \longrightarrow & Lie(\underline{haut}(X)) \; . \\
\downarrow \sim & & \downarrow \sim & & \downarrow = \\
g^{F(\psi)}_{Q,F(X)} & \longrightarrow & Lie(\underline{haut}_{Q_\infty-Alg}(F(X,\psi))) & \longrightarrow & Lie(\underline{haut}(X))
\end{array}$$

Proof. Let $F : \underline{P_\infty - Alg} \to \underline{Q_\infty - Alg}$ be an equivalence of presheaves of ∞-categories. We have a commutative triangle

$$\begin{array}{ccc}
\underline{P_\infty - Alg} & \xrightarrow{F} & \underline{Q_\infty - Alg} \; . \\
& {\scriptstyle U\circ F}\searrow \quad \swarrow{\scriptstyle U} & \\
& \underline{Ch_{\mathbb{K}}} &
\end{array}$$

Applying the loop space functor (2.1) at the appropriate base points we get the commutative triangle

$$\begin{array}{ccc}
\Omega_{(X,\psi)}\underline{P_\infty - Alg} & \xrightarrow{\sim} & \Omega_{F(X,\psi)}\underline{Q_\infty - Alg} \; . \\
& \searrow \quad \swarrow & \\
& \Omega_X \underline{Ch_{\mathbb{K}}} &
\end{array}$$

But a based loop space at a point of an ∞-category is the homotopy automorphims grouplike monoid of this point, so that this triangle is actually the triangle of derived prestack groups

$$\begin{array}{ccc}
\underline{haut}_{P_\infty-Alg}(X,\psi) & \xrightarrow{\sim} & \underline{haut}_{Q_\infty-Alg}(F(X,\psi)) \; . \\
& \searrow \quad \swarrow & \\
& \underline{haut}(X) &
\end{array}$$

By Theorem 2.26, we get the equivalence of homotopy fiber sequences of derived prestack groups

$$\begin{array}{ccc} \Omega_{\psi}\underline{P_\infty\{X\}} & \xrightarrow{\sim} & \Omega_{F(\psi)}\underline{Q_\infty\{F(X)\}} \\ \downarrow & & \downarrow \\ \underline{haut}_{P_\infty-Alg}(X,\psi) & \xrightarrow{\sim} & \underline{haut}_{Q_\infty-Alg}(F(X,\psi)) \\ & \searrow \quad \swarrow & \\ & \underline{haut}(X) & \end{array} .$$

hence an equivalence of homotopy fiber sequences of the corresponding derived formal groups obtained by completion at the appropriate base points. This equivalence of fiber sequences gives an equivalence of fiber sequences of the corresponding Lie algebras by the Lurie-Pridham equivalence theorem. □

Remark 2.28. There is also a "strict" version of this theorem. Let us consider a morphism of weak presheaves of relative categories, that is, given for each cdga A by a morphism of relative categories

$$F(A) : (P_\infty - Alg(Mod_A), W_{qiso}) \to (Q_\infty - Alg(Mod_A), W_{qiso}).$$

Let us suppose that F induces an equivalence of presheaves of classification spaces

$$\mathcal{N}wF : \underline{\mathcal{N}wP_\infty - Alg} \xrightarrow{\sim} \underline{\mathcal{N}wQ_\infty - Alg}.$$

By [96, Section 3.3], this means that F induces an equivalence of weak presheaves of ∞-categories as in Theorem 2.27. Then, we can mimick the proof of Theorem 2.27 as follows: we replace the presheaves of ∞-categories by these presheaves of classification spaces, t ake based loop spaces which gives back the homotopy automorphisms as well, and apply Theorem 2.26.

3. The tangent Lie algebra of homotopy automorphisms

The goal of this section is to make Theorem 2.26 more precise, by proving that $Lie(\underline{haut}_{P_\infty-Alg}(X,\psi))$ is a semi-direct product, in a homotopical sense, of the two extremal terms of the fiber sequence, and that the later term is nothing but $End(X) = Hom_{Ch_{\mathbb{K}}}(X,X)$ equipped with the commutator of the composition product. This is actually the transposition, at the Lie algebra level, of a homotopy action of $haut(X)$ on $P_\infty\{X\}$ that we mentionned in Remark 2.23. In a few words, the tangent Lie algebra of homotopy automorphisms takes into account the action of the automorphisms of the complex X on the Maurer-Cartan elements of $g^{\psi}_{P,X}$, that is, on the space of P_∞-algebra structures on X. In Section 4, we will further provide an explicit description in properadic terms of that homotopy Lie algebras.

3.1. Homotopy representations of L_∞-algebras and a relevant application.

Recall (1.3) that the structure of a L_∞-algebra g is encoded by a (cohomological degree -1) coderivation Q_g of square zero on $Sym^{\bullet\geqslant 1}(g[1])$. Dualizing this coderivation induces an augmented cdga structure on

$$C^*_{CE}(g) := Hom(Sym^{\bullet}(g[1]), k)$$

which is called the *Chevalley-Eilenberg cochain algebra* of g; we denote ε the augmentation. For any graded module N, $Hom(Sym^\bullet(g[1]), N)$ inherits similarly a structure of graded $C^*_{CE}(g)$-module.

Definition 3.1. Let (g, Q_g) be a L_∞-algebra and $(M, d_M) \in Ch_{\mathbb{K}}$.

A homotopy representation of an L_∞-algebra g on M is a *derivation* D of *square zero* and (cohomological) degree 1 on $C^*_{CE}(g, M) := Hom_{dg}(Sym^\bullet(g[1]), M)$ such that $M \stackrel{D}{\to} C_{CE}(g, M) \stackrel{\varepsilon}{\to} M$ is equal to the inner differential d_M of the complex M. The fact that D is a derivation means precisely that it satisfies the following Leibniz relation: for $f \in C^n_{CE}(g)$, $\Phi \in C^*_{CE}(g, M)$, one has

$$D(f \cdot \Phi) = Q_g(f) \cdot \Phi + (-1)^n f \cdot D(\Phi). \tag{3.1}$$

Example 3.2. A particular case of homotopy representation is the standard notion of representation, given by a (dg-) Lie algebra morphism $g \to End(M)$ and the standard Chevalley-Eilenberg cochain complexes. This generalizes easily to any L_∞-algebra g and L_∞-morphism $g \to End_{Ch_{\mathbb{K}}}(M, M)$.

Definition 3.1 is equivalent to the data of a L_∞-algebra structure on $g \oplus M$, that is a coderivation of square zero on $Sym^{\bullet \geqslant 1}((g \oplus M)[1])$ that vanishes on the coideal spanned by $Sym^{\bullet \geqslant 2}(M)$ and whose restriction to $Sym^{\bullet \geqslant 1}(g[1])$ and M are respectively Q_g and the inner differential of M (followed by the canonical inclusions of these complexes in $Sym^{\bullet \geqslant 1}((g \oplus M)[1])$). In other words it is a square zero extension by M of the L_∞-algebra structure of g.

Example 3.3 (semi-direct product)**.** If h is a dg-Lie algebra, any dg-Lie algebra homomorphism $g \to Der(h)$ induces an action of g onto $C^{CE}_*(h)$ as the coderivation extending the g-action on h. Similarly, if h and g are L_∞-algebras and given an L_∞-algebra morphism $\varphi : g \to Der(h)$, we obtain a homotopy representation of g on $C^{CE}_*(h)$. The coalgebra structure of $C^{CE}_*(h)$ then yields respectively a cocommutative dg-coalgebra and a cdga structure on

$$CE_*(g, h) := C^{CE}_*\big(g, C^{CE}_*(h)\big), \quad CE^*(g, h) := C^*_{CE}\big(g, C^*_{CE}(h)\big). \tag{3.2}$$

The augmentations yield a cofiber sequence of cdgas

$$C^*_{CE}(g) \to CE^*(g, h) \to C^*_{CE}(h),$$

which is dual to a fiber sequence of dg-cocommutative coalgebras

$$C^{CE}_*(h) \to CE_*(g, h) \to C^{CE}_*(g),$$

which is equivalent to a fiber sequence of L_∞-algebras

$$h \to g \ltimes_f h \to g$$

forming a split extension of g by h. The *semi-direct product* $g \ltimes_f h$ is the direct sum $g \oplus h$ equipped with the L_∞-algebra structure coming from the differential on the coalgebra $CE_*(g, h) = Sym\big(g[1] \oplus h[1]\big)$.

Example 3.4. In particular the adjoint action $ad : g \to Der(g)$ of a L_∞-algebra g on itself yields the semi-direct product $g \rtimes_{ad} g$.

Semi-direct product of L_∞-algebras will appear in the construction of the diagram of equivalence of fiber sequences below

$$\begin{array}{ccccc}
g^{\varphi}_{P,X} & \xrightarrow{\sim} & g^{\varphi}_{P,X} & \xrightarrow{\sim} & g^{\varphi}_{P,X} \\
\downarrow & & \downarrow & & \downarrow \\
g^{\varphi^+}_{P^+,X} & \xrightarrow{\sim} & g^{\varphi}_{P,X} \ltimes_f End(X) & \xrightarrow{\sim} & Lie(\underline{haut}_{P_\infty}(X,\varphi)) \\
\downarrow & & \downarrow & & \downarrow \\
End(X) & \xrightarrow{\sim} & End(X) & \xrightarrow{\sim} & End(X)
\end{array}$$

where the homotopy semi-direct product in the middle is defined in the next section 3.2. It turns out that this semi-direct product is the tangent incarnation of the non trivial action of $\underline{haut}(X)$ on the moduli space $P_\infty\{X\}$ at a topological level. Taking this action into account in the deformation theory of (X,φ) means, on the one hand deforming the P_∞-algebra structure with compatible deformations of the differential (equivalence of the middle fiber sequence with the left one), on the other hand deforming (X,φ) *in the* ∞*-category of* P_∞*-algebras* (equivalence of the middle fiber sequence with the right one).

The plan is as follows. First, we construct two equivalences of fiber sequences of derived groups fitting in the diagram

$$\begin{array}{ccccc}
\Omega_{\varphi}\underline{P_\infty\{X\}} & \xrightarrow{\sim} & \Omega_{\varphi}\underline{P_\infty\{X\}} & \xrightarrow{\sim} & \Omega_{\varphi}\underline{P_\infty\{X\}} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{\varphi^+}\underline{P^+_\infty\{X\}} & \xrightarrow{\sim} & \Omega_{[\varphi]}(\underline{P_\infty\{X\}}//\underline{haut}(X)) & \xrightarrow{\sim} & \underline{haut}_{P_\infty}(X,\varphi)) \\
\downarrow & & \downarrow & & \downarrow \\
\underline{haut}(X) & \xrightarrow{\sim} & \underline{haut}(X) & \xrightarrow{\sim} & \underline{haut}(X)
\end{array} ,$$

where $\underline{P_\infty\{X\}}//\underline{haut}(X)$ is the appropriate homotopy quotient in the ∞-category of infinitesimally cohesive ∞-presheaves over $(dgArt^{aug}_{\mathbb{K}})^{op}$. Secondly, the desired fiber sequence of L_∞-algebras is induced by this diagram (taking, as usual in this paper, completions at identities to get equivalences of fiber sequences of derived formal groups). Finally, we identify the homotopy quotient $Lie(\Omega_{[\varphi]}(\underline{P_\infty\{X\}}//\underline{haut}(X)))$ with $g^{\varphi}_{P,X} \ltimes_f End(X)$.

3.2. **∞-actions in infinitesimally cohesive presheaves.** In this section, the ambient ∞-category is $SPsh^{infcoh}_{\infty}((dgArt^{aug}_{\mathbb{K}})^{op})$ and derived prestack groups are precisely the group objects (see 2.1) in it. This is a particular case of infinitesimally cohesive ∞-topos, where the theory of principal ∞-bundles developped in [77, 78] fully applies. In this setting, the general notion of ∞-action of a group object G in an ∞-category on another object X provides a homotopy quotient[11] $X//G$. This homotopy quotient comes naturally equipped with a homotopy fiber sequence

$$X \to X//G \to BG$$

[11] which is the same as the quotient in ∞-stack

with fiber X associated to the universal G-principal ∞-bundle $\bullet \to BG$. The map $X//G \to BG$ is the classifying morphism of the action of G on X. This is an analogue of the usual quotient stack by a group stack action of [45].

Remark 3.5. When G is presented by a simplicial presheaf in grouplike simplicial monoids (which is a model for derived prestack groups), the homotopy quotient $X//G$ is computed by the geometric realization of the simplicial action groupoid

$$\cdots G \times G \times X \overset{\to}{\underset{\to}{\to}} G \times X \rightrightarrows X.$$

(see for example [54] in the case of group actions in simplicial presheaves).

Let G be a group object in $SPsh_{\infty}^{infcoh}((dgArt_{\mathbb{K}}^{aug})^{op})$, i.e., a derived prestack group.

Proposition 3.6. *For any X equipped with an ∞-action of a derived prestack group object G, there is a fiber sequence of homotopy Lie algebras*

$$Lie(\Omega_* X) \to Lie(\Omega_*(X//G)) \to Lie(G)$$

Proof. Since the loop space is an homotopy pullback, the fiber sequence

$$X \to X//G \to BG$$

yields a fiber sequence of derived groups

$$\Omega_* X \to \Omega_*(X//G) \to G$$

hence the desired fiber sequence of tangent Lie algebras by 2.14. □

We still consider an object X with an ∞-action of a derived prestack group G. Recall the completion of a derived prestack group 2.11.

Lemma 3.7. *Assume that there exists a section of the (induced) projection map $\Omega_*\widehat{(X//G)}_x \overset{\pi}{\to} \widehat{G}_1$, that is a derived formal group morphism $s : \widehat{G}_1 \to \Omega_*\widehat{(X//G)}_x$ such that $\pi \circ s$ is equivalent to the identity. Then there is an equivalence of L_∞ algebras*

$$Lie\big(\Omega_*(X//G)\big) \;\cong\; Lie(\Omega_* X) \rtimes Lie(G)$$

and the fiber sequence of proposition 3.6 identifies with the semi-direct product one.

Proof. The Lie algebra functor (2.14) depends only on the associated formal group at the base point. Therefore it gives a L_∞-morphim $Lie(G) \overset{Lie(s)}{\longrightarrow} Lie\big(\Omega_*(X//G)\big)$. Composing with the adjoint action of the latter, we obtain a morphism $ad \circ Lie(s) : Lie(G) \to Der\Big(Lie\big(\Omega_*(X//G)\big)\Big)$. Since this is a map of Lie algebras, and s is a section of π, the induced action of $Lie(G)$ on $Lie\big(\Omega_*(X//G)\big)$ restricts to $ker(\pi) \cong Lie(\Omega_* X)$. Therefore we get an induced L_∞-algebra map $Lie(G) \to Der\big(Lie(\Omega_* X)\big)$ which defines the homotopy Lie algebra semi-direct product (3.3). It follows that the morphism

$$Lie(\Omega_* X) \rtimes Lie(G) \ni (x, y) \overset{\tau}{\mapsto} x + s(y) \in Lie\big(\Omega_*(X//G)\big)$$

is a L_∞-algebra map and that we have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} Lie(\Omega_* X) & \longrightarrow & Lie(\Omega_* X) \rtimes Lie(G) & \overset{Lie(\pi)}{\longrightarrow} & Lie(G) \\ \| & & \downarrow \tau & & \| \\ Lie(\Omega_* X) & \longrightarrow & Lie\big(\Omega_*(X//G)\big) & \overset{Lie(\pi)}{\longrightarrow} & Lie(G) \end{array}$$

of L_∞-algebras. The equivalence now follows from the 2 out 3 property. □

3.3. **The Lie algebra of homotopy automorphisms as a semi-direct product.** The goal of this section is to prove (and makes sense of) the formula below:

$$Lie(\underline{haut}_{P_\infty-Alg}(X,\psi)) \simeq Lie(\Omega_{[\varphi]}(\underline{P_\infty\{X\}}//\underline{haut}(X))) = g^{\psi}_{P,X} \rtimes^h End(X).$$

To do so, we will interpret $Lie(\underline{haut}_{P_\infty-Alg}(X,\psi))$ as the tangent Lie algebra of a homotopy quotient of $\underline{P_\infty\{X\}}$ by the ∞-action of $\underline{haut}(X)$.

An explicit model of the homotopy quotient is given by a homotopy version of the well known Borel construction, suitably adapted for simplicial presheaves over cdgas. In [54], the Borel construction is given by the classifying space of the translation groupoid associated to the action of a sheaf of groups G on a sheaf X, that is $EG\times_G X$. We adapt this construction to the case of an ∞-action.

Let $\mathcal{P}$ a cofibrant prop, and $\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P}$ the bisimplicial set defined by $(\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P})_{m,n} = (\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n]\otimes P})_m$, where the w denotes the subcategory of morphisms which are weak equivalences in $\mathcal{E}$. We get a diagram

$$\begin{array}{ccccccc}
P\{X\} & \longrightarrow & diag\mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-]\otimes P} & \xrightarrow{\sim} & diag\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-]\otimes P} & \xleftarrow{\sim} & \mathcal{N}w(\mathcal{E}^{cf})^{P}\ , \\
\downarrow & & \downarrow & & & & \downarrow \\
pt & \longrightarrow & \mathcal{N}(fw\mathcal{E}^{cf}) & & \xrightarrow{\qquad\sim\qquad} & & \mathcal{N}(w\mathcal{E}^{cf})
\end{array}$$

where the fw denotes the subcategory of morphisms which are acyclic fibrations in $\mathcal{E}$. The crucial point here is that the left-hand commutative square of this diagram is a homotopy pullback, implying that we have a homotopy pullback of simplicial sets (see [98, Theorem 0.1])

$$\begin{array}{ccc}
P\{X\} & \longrightarrow & \mathcal{N}(wCh_{\mathbb{K}}^{P}) \\
\downarrow & & \downarrow \\
\{X\} & \longrightarrow & \mathcal{N}wCh_{\mathbb{K}}.
\end{array}$$

Therefore $P_\infty\{X\}$ can be identified with the homotopy fiber

$$\begin{array}{ccc}
P_\infty\{X\} & \longrightarrow & diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^\bullet}\quad . \\
\downarrow & & \downarrow \\
\{X\} & \longrightarrow & \mathcal{N}fwCh_{\mathbb{K}} \sim \mathcal{N}wCh_{\mathbb{K}}.
\end{array}$$

The main goal of this section is to prove the following result which identifies the homotopy quotient of $_\infty\{X\}$ by automophisms of the underlying cochain complex:

Theorem 3.8. *Let X be a cochain complex and $\phi : P_\infty \to End_X$ be a prop morphism. There exists a commutative square*

$$\begin{array}{ccc}
Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)} P_\infty\{X\} & \xrightarrow{\sim} & diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^\bullet} \\
\pi\downarrow & & \downarrow \\
Bhaut_{\mathbb{K}}(X) & \xrightarrow{\sim} & \mathcal{N}wCh_{\mathbb{K}}|_X
\end{array}$$

where π is a Kan fibration obtained by simplicial Borel construction and the horizontal maps are weak equivalences of simplicial sets, inducing an equivalence

$$\Omega_{[\varphi]}\left(Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)}P_\infty\{X\}\right)\xrightarrow{\sim}\Omega_{(X,\varphi)}\left(diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^\bullet}\right)\simeq\Omega_{(X,\varphi)}\mathcal{N}wCh_{\mathbb{K}}^{P_\infty}$$

of derived prestack groups.

For this aim, we need to define the action of $haut_{\mathbb{K}}(X)$ on $P_\infty\{X\}$. First, let us recall some homotopical properties of props under base change of cdgas:

Lemma 3.9. (1) *Any cdga A induces a Quillen adjunction*

$$(-)\otimes A : Prop(Ch_{\mathbb{K}})\rightleftarrows Prop(Mod_A) : U$$

between the model category of $\mathbb{K}$-linear dg props and the model category of A-linear dg props, where $(-)\otimes A$ is the aritywise tensor product by A and U is the forgetful functor.

(2) *Any morphism of cdgas $u : A\to B$ induces a Quillen adjunction*

$$u_* := (-)\otimes_A B : Prop(Mod_A)\rightleftarrows Prop(Mod_B) : u^*$$

between the model category of A-linear dg props and the model category of B-linear dg props, where u^* is the classical restriction functor sending any B-module to the same underlying complex with the A-module structure induced by u and the tensor product defining u_* is the aritywise tensor product of a prop by a cdga.

(3) *For any complex X and for any chain morphism $f : X\to Y$, the functor u_* induces well defined prop morphisms*

$$End_{X\otimes A}^{Mod_A}\to End_{X\otimes B}^{Mod_B}$$

$$End_{f\otimes A}^{Mod_A}\to End_{f\otimes B}^{Mod_B}$$

where $End_{(-)}^{Mod_A}$ and $End_{(-)}^{Mod_B}$ denotes respectively the endomorphism prop in the category of dg A-modules and in the category of dg B-modules.

(4) *Given a cofibrant dg prop in $\mathbb{K}$-modules P_∞ and a complex X, the functor u_* induces also a well defined simplicial map of* Kan complexes

$$P_\infty\otimes A\{X\otimes A\}\to P_\infty\otimes B\{X\otimes B\}$$

where the left hand side and right hand side mapping spaces are taken respectively in the model category of A-linear dg props and in the model category of B-linear dg props.

Proof. Recall that Mod_A is a cofibrantly generated symmetric monoidal model category whose fibrations and weak equivalences are those induced by the forgetful functor to complexes (so degreewise surjections and quasi-isomorphisms) and tensor product is defined for any pair of A-modules M and N by the coequalizer

$$A\otimes M\otimes N\rightrightarrows M\otimes N\to M\otimes_A N$$

whose pair of arrows are defined by the A-module structures of M and N.

Claim (1) is [98, Lemma 3.4].

Let $u : A\to B$ be a morphism of cdgas. It induces an adjunction

$$u_* := (-)\otimes_A B : Mod_A\rightleftarrows Mod_B : u^*$$

in which the right adjoint u^* preserves obviously fibrations and weak equivalences, so that it forms actually a Quillen adjunction. The left adjoint is a strong symmetric monoidal functor via the natural isomorphisms

$$M \otimes_A N \otimes_A B \cong (M \otimes_A B) \otimes_B (N \otimes_A B)$$

for $M, N \in Mod_B$ and the right adjoint is a lax monoidal functor via the natural maps $u^*(M) \otimes_A u^*(N) \to u^*(M \otimes_B N)$. An adjunction between a strong symmetric monoidal left adjoint and a lax monoidal right adjoint suffices to lift this adjunction at the level of props

$$u_* := (-) \otimes_A B : Prop(Mod_A) \rightleftarrows Prop(Mod_B) : u^*.$$

Fibrations and weak equivalences of dg props are defined by the forgetful functor from props to $\mathbb{N}\times\mathbb{N}$-indexed collections of complexes, so these are respectively aritywise surjections and aritywise quasi-isomorphisms. The right adjoint preserves such morphisms, so the adjunction above forms actually a Quillen adjunction finishing claim (2).

The first morphism of claim (3) follows directly from the fact that u_* is strong symmetric monoidal. The second one is induced by the first one, considering that the endomorphism prop of a morphism $f : X \to Y$ is given by the pullback

$$\begin{array}{ccc} End_{f\otimes A}^{Mod_A} & \longrightarrow & End_{Y\otimes A}^{Mod_A} \\ \downarrow & & \downarrow {\scriptstyle (f\otimes A)^*} \\ End_{X\otimes A}^{Mod_A} & \underset{(f\otimes A)_*}{\longrightarrow} & Hom_{XY}^{Mod_A} \end{array}$$

where $Hom_{XY}^{Mod_A}$ is the sequence of complexes $\{Hom_{Mod_A}((X \otimes A)^{\otimes_A m}, (Y \otimes A)^{\otimes_A n})\}_{m,n\in\mathbb{N}}$ and the morphisms $(f \otimes A)^*$ and $(f \otimes A)_*$ are defined in each arity (m, n) respectively by postcomposing with $(f \otimes A)^{\otimes n}$ and precomposing with $(f \otimes A)^{\otimes m}$. The morphism

$$End_{f\otimes A} \to End_{f\otimes B}$$

then follows by applying the pullback functor to the morphism of diagrams

$$\begin{array}{ccccc} End_{X\otimes A}^{Mod_A} & \longleftarrow & Hom_{XY}^{Mod_A} & \longrightarrow & End_{Y\otimes A}^{Mod_A} \\ \downarrow & & \downarrow & & \downarrow \\ End_{X\otimes B}^{Mod_B} & \longleftarrow & Hom_{XY}^{Mod_B} & \longrightarrow & End_{Y\otimes B}^{Mod_B} \end{array}$$

induced by u_*.

To prove (4), we first need to observe that u_* is compatible with the simplicial structure of the mapping spaces, thus defines a simplicial functor between the corresponding simplicially enriched categories. For this, we recall from the general definition of simplicial mapping spaces in model categories combined with [98, Proposition 2.5] that in the model category of A-linear dg props, the simplicial mapping space between two props P and Q is given by

$$Map_{Prop(Mod_A)}(P, Q) = Mor_{Prop(Mod_A)}(P, Q \otimes_A (A \otimes \Omega_\bullet))$$

with a simplicial structure induced by the simplicial cdga $A \otimes \Omega_\bullet$. We apply this to the particular case $P = P_\infty \otimes A$ and $Q = End_{X\otimes A}^{Mod_A}$. The functor u^* induces, for every n, an application

$$\begin{aligned} & Mor_{Prop(Mod_A)}(P_\infty \otimes A, End_{X\otimes A}^{Mod_A} \otimes_A (A \otimes \Omega_\bullet)) \\ \to \quad & Mor_{Prop(Mod_B)}((P_\infty \otimes A) \otimes_A B, (End_{X\otimes A}^{Mod_A} \otimes_A (A \otimes \Omega_\bullet)) \otimes_A B) \\ = \quad & Mor_{Prop(Mod_B)}(P_\infty \otimes B, (End_{X\otimes A}^{Mod_A} \otimes_A B) \otimes_B (A \otimes \Omega_\bullet)) \otimes_A B) \\ = \quad & Mor_{Prop(Mod_B)}(P_\infty \otimes B, (End_{X\otimes B}^{Mod_B} \otimes_B (\Omega_\bullet \otimes B)) \end{aligned}$$

which is compatible with the simplicial structures of $A \otimes \Omega_\bullet$ and $B \otimes \Omega_\bullet$, hence defines a simplicial map

$$P_\infty \otimes A\{X \otimes A\} \to P_\infty \otimes B\{X \otimes B\}.$$

To conclude, for these mapping spaces to be Kan complexes it is sufficient to prove that the source is cofibrant and the target is fibrant. The source of each mapping space is cofibrant thanks to (1) (since P_∞ is cofibrant in $\mathbb{K}$-linear dg props). Dg A-modules are all fibrant because fibrations of A-modules are defined by the forgetful functor and every complex over a field is fibrant. Moreover, fibrant A-linear dg props are the aritywise fibrant ones, so every A-linear dg prop is fibrant. This concludes the proof. □

These compatibilities allow us to get the functoriality, up to homotopy, of the ∞-action we are going to consider below.

First, let us point out that we have a natural equivalence $haut(X) \simeq fhaut(X)$, where $fhaut(X)$ is the simplicial submonoid of $haut(X)$ whose vertices are the self acyclic fibrations $X \overset{\sim}{\twoheadrightarrow} X$. This equivalence is simply given by the functorial factorization properties of the underlying model category (which replace functorially any weak equivalence by a weakly equivalent acyclic fibration), implying that every self-weak equivalence of X is in the connected component of a self acyclic fibration of X (recall that every complex over a field is both fibrant and cofibrant, so X is already a fibrant-cofibrant object).

Second, we can transfer P_∞-algebra structures along acyclic fibrations, hence a map

$$\begin{array}{rcl} haut(X) \times P_\infty\{X\} & \to & P_\infty\{X\} \\ (f, \psi) & \longmapsto & Rf_*\psi \end{array}$$

as follows. We associate to f its equivalent acyclic fibration Rf. Let End_{Rf} be the dg prop associated to the morphism Rf. It is defined by the coreflexive equalizer

$$End_{Rf}(n,m) = Eq\Big(Hom(X^{\otimes n}, X^{\otimes m}) \times Hom(Y^{\otimes n}, Y^{\otimes m}) \rightrightarrows Hom(X^{\otimes n}, Y^{\otimes m}) \Big)$$

where the maps are given by either postcomposition or precomposition by Rf. Note that End_{Rf} has a natural prop structure and two canonical prop maps Rf_*, Rf^* to End_X and End_Y. We get a lifting

$$\begin{array}{ccccc} 0 & \longrightarrow & End_{Rf} & \xrightarrow{Rf_*} & End_X \\ \downarrow & \nearrow & \downarrow{\scriptstyle Rf^*} & & \\ P_\infty & \xrightarrow[\psi]{} & End_X & & \end{array} \tag{3.3}$$

since, by [33, Lemma 7.2], the right vertical morphism is an acyclic fibration in the model category of dg properads, and P_∞ is cofibrant.

With a slight abuse of notation *we note* $Rf_*\psi$ *the composite*

$$P_\infty \longrightarrow End_{Rf} \xrightarrow{Rf_*} End_X. \tag{3.4}$$

Moreover, given two homotopy automorphisms f and g, the lifts obtained by $R(g\circ f)_*\psi = (Rg\circ Rf)_*\psi$ and $Rg_*(Rf_*\psi)$ are homotopy equivalent by contractibility of the space of lifts in the commutative square above, giving the compatibility up to homotopy with the composition in $haut(X)$.

Lemma 3.10. *The map* (3.4) *induces an* ∞*-action of the derived prestack group* $\underline{haut}(X)$ *on* $\underline{P_\infty\{X\}}$.

Proof. We have to explain why such lifts gives an action at the simplicial level, then why this action is functorial in order to induce the desired action at the level of simplicial presheaves. For this, first recall that we use the fibrant replacement functor $f \mapsto Rf$ to only consider acyclic fibration. We use the base change properties of Lemma 3.9. Indeed, the lift $P_\infty \longrightarrow End_{Rf}$ induces for any Artinian cdga A a lift $P_\infty\otimes A \longrightarrow End_{Rf}^{Mod_A}$ such that for any morphism of cdgas $u : A \to B$, we have a commutative diagram

$$\begin{array}{ccc} P_\infty\otimes A & \longrightarrow & End_{Rf}^{Mod_A} \\ \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle u_*} \\ P_\infty\otimes B & \longrightarrow & End_{Rf}^{Mod_B} \end{array} \tag{3.5}$$

Note that for any cdga A, the category Mod_A is a cofibrantly generated symmetric monoidal model category satisfying the limit monoid axioms [33, Section 6.6], so that [33, Lemma 7.2] still applies to A-linear P_∞-structures in Mod_A^{cof}. Further, the extension $Rf : X\otimes A \to X\otimes A$ is an acyclic fibration between cofibrant-fibrant objects of Mod_A. And by the assertion (1) of 3.9, the endomorphism prop maps $End_{Rf}^{Mod_A} \xrightarrow{Rf^*} End_{X\otimes A}^{Mod_A}$ are still a cofibrations (all modules are already fibrant).

Now, for any $n\in\mathbb{N}$, we work in the model category $Prop(Mod_{\Omega_n})$ and apply this Lemma to the particular case $A=\Omega_n$. Recall that:

- the n-simplices of $fhaut(X)$ are determined by self acyclic fibrations of $X\otimes\Omega_n$ in Ω_n-modules;
- the set of n-simplices of $P_\infty\{X\}$ satisfies the natural isomorphism

$$P_\infty\{X\}_n = Mor_{Prop}(P_\infty, End_X\otimes\Omega_n) \cong Mor_{Prop(Mod_{\Omega_n})}(P_\infty\otimes\Omega_n, End_{X\otimes\Omega_n}^{Mod_{\Omega_n}})$$

identifying n-simplices with Ω_n-linear P_∞-structures on $X\otimes\Omega_n$,

so building our action at the level of n-simplices amounts to make the set of self acyclic fibrations of $X\otimes\Omega_n$ (as an Ω_n-module) act on $Mor_{Prop(Mod_{\Omega_n})}(P_\infty\otimes\Omega_n, End_{X\otimes\Omega_n}^{Mod_{\Omega_n}})$ by the same formula as in the case $n=0$ above.

We are now interested by the compatibility of our lifts with base changes along morphisms of cdgas. The construction above, written in the particular case $A=\Omega_n$, works exactly the same for any cdga A. Let $A\to B$ be a any morphism of cdgas. Let $f_A : X\otimes A\to X\otimes A$ an acyclic fibration in the model category of A-modules. Under the base change functor $(-)\otimes_A B$ and the fibrant replacement functor, this gives a self acyclic fibration $R(f_A\otimes_A B)$ of $X\otimes B$ in the model category of B-modules and the diagram (3.5) above ensures the compatibility of those lifts with the base change functor (lifted to the model categories of props). Further, by contractibilty of the space of lifts in the diagram (3.3), we get equivalences $R(g\circ f)_*\psi = (Rg\circ Rf)_*\psi$ and $Rg_*(Rf_*\psi)$. The compatibility of theses lifts with base changes along morphisms of cdgas explains both :

- faces and degeneracies of the simplicial structures under consideration are given by base changes along morphisms of cdgas defined by the cosimplicial structure of $\Omega_\bullet$;
- the naturality of the action with respect to morphisms of cdgas is its compatibility with the corresponding base change functors.

Thus, we get an ∞-action of $\underline{haut}(X)$ on $\underline{P_\infty\{X\}}$ in simplicial presheaves over artinian cdgas. □

With this ∞-action we start the proof of Proposition 3.8.

Proof of Propoposition 3.8. First we construct the commutative diagram

$$\begin{array}{ccc} Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)} P_\infty\{X\} & \longrightarrow & diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^\bullet}|_X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ Bhaut_{\mathbb{K}}(X) & \longrightarrow & \mathcal{N}wCh_{\mathbb{K}}|_X \end{array}$$

in the following way:

$$\begin{array}{ccc} ((f_k,...,f_0),\varphi : P_\infty\otimes\Delta^k\rightarrow End_X) & \longrightarrow & ((X\phi)\overset{\sim}{\rightarrow}(X,f_k.\varphi)...\overset{\sim}{\rightarrow}(X,(f_k\circ...\circ f_1).\varphi)) \\ \downarrow{\scriptstyle \pi=p^*_{haut_{\mathbb{K}}(X)}} & & \downarrow{\scriptstyle forget} \\ (f_{k-1},...,f_0) & \longrightarrow & (X\overset{f_{k-1}}{\rightarrow}...\overset{f_0}{\rightarrow}X) \end{array}$$

where the left vertical map is the projection associated to the Borel construction and the right vertical map forgets the $P_\infty\otimes\Delta^k$-algebra structure. The top horizontal map transfers the $P_\infty\otimes\Delta^k$-algebra structure on X along the sequence of quasi-isomorphisms given by $f_k,...,f_0$ and the bottom horizontal map is just an inclusion. It is clear by definition of faces and degeneracies in the simplicial structures involved that these four maps are simplicial.

It remains to prove that the two horizontal maps are weak equivalences. For the bottom arrow, it follows from the work of Dwyer-Kan [21] which identifies the connected components of the classification space of a model category with the classifying complexes of homotopy automorphisms $Bhaut(X)$.

For the top arrow, we have a morphism of homotopy fibers over X

$$\begin{array}{ccccc} P_\infty\{X\} & \longrightarrow & Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)} P_\infty\{X\} & \longrightarrow & Bhaut_{\mathbb{K}}(X) \\ \downarrow{\scriptstyle =} & & \downarrow & & \downarrow{\scriptstyle \sim} \\ P_\infty\{X\} & \longrightarrow & diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^\bullet} & \longrightarrow & \mathcal{N}wCh_{\mathbb{K}}|_X \end{array}$$

inducing another morphism of homotopy fibers

$$\begin{array}{ccccc} \Omega_{[\varphi]}\left(Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)}P_{\infty}\{X\}\right) & \longrightarrow & haut_{\mathbb{K}}(X) & \longrightarrow & P_{\infty}\{X\} \\ \downarrow & & \downarrow\sim & & \downarrow = \\ \Omega_{(X,\varphi)}\left(diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^{\bullet}}\right) & \longrightarrow & \Omega_X\mathcal{N}wCh_{\mathbb{K}}|_X & \longrightarrow & P_{\infty}\{X\} \end{array}$$

taken over the base point φ. □

Proposition 3.11. *For any chain complex X, the tangent Lie algebra $Lie(\underline{haut}(X))$ of $\underline{haut}(X)$ is equivalent to $End(X)=Hom_{Ch_{\mathbb{K}}}(X,X)$ equipped with the commutator of the composition product as Lie bracket.*

Proof. This follows from Lurie-Pridham correspondence applied to the formal moduli problem $B\underline{haut}(X)$. Recall (Definition 2.14) that $Lie(\underline{haut}(X))=Lie(\widehat{\underline{haut}(X)}_{id})$ Further, $\widehat{\underline{haut}(X)}_{id}$ is the based loop space (i.e. the automorphsims) of $ObjDefo_X: dgArt_{\mathbb{K}}^{aug}\rightarrow sSet$, the deformation object functor of X of [66, Section 5.2]. The latter is a 1-proximate formal moduli problem in the sense of [66, Section 5.1] with associated formal moduli problem denoted $L(ObjDefo_X)$. By [8, Lemma 2.11], there is an equivalence of formal moduli problems

$$\widehat{\underline{haut}(X)}_{id}\cong\Omega\big(ObjDefo_X\big)\xrightarrow{\cong}\Omega\big(L(ObjDefo_X)\big). \tag{3.6}$$

Using this equivalence (3.6) with Proposition 2.15 then shows that

$$Lie(\underline{haut}(X))\cong\mathfrak{L}_{L(ObjDefo_X)}.$$

By [66, Theorem 5.2.8, Theorem 3.3.1], the Lie algebra associated to $L(ObjDefo_X)$ is precisely $Hom_{Ch_{\mathbb{K}}}(X,X)$ with its (dg-)Lie algebra structure. □

We deduce:

Proposition 3.12. *There is an equivalence of homotopy fiber sequences of Lie algebras*

$$\begin{array}{ccccc} g_{P,X}^{\varphi} & \longrightarrow & Lie(\underline{haut}_{P_{\infty}-Alg}(X,\psi)) & \longrightarrow & End(X) \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ g_{P,X}^{\varphi} & \longrightarrow & Lie(\Omega_{[\varphi]}(\underline{P_{\infty}\{X\}}//\underline{haut}(X))) & \longrightarrow & End(X) \end{array}$$

Proof. By Proposition 3.8, we have a morphism of homotopy fiber sequences

$$\begin{array}{ccccc} P_{\infty}\{X\} & \longrightarrow & Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)}P_{\infty}\{X\} & \longrightarrow & Bhaut_{\mathbb{K}}(X)\,. \\ \downarrow = & & \downarrow & & \downarrow\sim \\ P_{\infty}\{X\} & \longrightarrow & diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^{\bullet}} & \longrightarrow & \mathcal{N}wCh_{\mathbb{K}}|_X \end{array}$$

Applying the based loop functor we get an equivalence of homotopy fiber sequences of derived groups, since we also have, by Proposition 3.8, that the map

$$\Omega_{[\varphi]}\left(Ehaut_{\mathbb{K}}(X)\times_{haut_{\mathbb{K}}(X)}P_{\infty}\{X\}\right)\xrightarrow{\sim}\Omega_{(X,\varphi)}\left(diag\mathcal{N}fwCh_{\mathbb{K}}^{P\otimes\Delta^{\bullet}}\right)$$
$$\simeq\Omega_{(X,\varphi)}\mathcal{N}wCh_{\mathbb{K}}^{P_{\infty}}$$

is an equivalence. Taking the formal completions at the appropriate base points and applying the *Lie*-algebra ∞-functor combined with Proposition 3.11, we obtain the desired equivalence of homotopy fiber sequences of Lie algebras. □

As we have already seen, the forgetfull functor mapping P_∞-algebras to their underlying complexes induces a morphism of the homotopy automorphisms derived prestack groups of both categories.

Lemma 3.13. *Let (X,ψ) be a P_∞-algebra. The forgetful derived formal group morphism*
$$\widehat{\underline{haut}_{P_\infty-Alg}}(X,\Psi)_{id} \longrightarrow \widehat{\underline{haut}}(X)_{id}$$
has a section in derived formal groups.

Proof. By Lemma 2.20, since being group-like is a property, it is sufficient to construct a E_1-monoid morphism

$$\Omega_{X\otimes R}\Big(\mathcal{N}wCh_R\Big) \longrightarrow \Omega_{(X\otimes R,\psi\otimes R)}\Big(\mathcal{N}wP_\infty - Alg(Mod_R^{cof})\Big) \tag{3.7}$$

of simplicial presheaves which is a section of the forgetful morphism. Then, by applying the homotopy fiber functor over the identities we get a section of the forgetful derived formal group morphism. Since we are taking loop spaces at $X\otimes R$ and $(X\otimes R,\psi\otimes R)$, we can restrict the considered spaces respectively to the connected component $(\mathcal{N}wCh_R)_{X\otimes R}$ of $X\otimes R$ and the connected component $\mathcal{N}wP_\infty - Alg(Mod_R^{cof})_{(X\otimes R,\psi\otimes R)}$ of $(X\otimes R,\psi\otimes R)$. We search for a pointed map
$$(\mathcal{N}wCh_R)_{X\otimes R} \to \mathcal{N}wP_\infty - Alg(Mod_R^{cof})_{(X\otimes R,\psi\otimes R)}$$
(where the base points are respectively $X\otimes R$ and $(X\otimes R,\psi\otimes R)$ whose composite with the forgetful map is the identity, so that applying the pointed loop space functor gives us the map 3.7.

For simplicity, we will use the description of props as strict small symmetric monoidal dg categories and morphisms of props as symmetric monoidal dg functors. Let us note $Fun_{dg}^{\otimes}(-,-)$ for the category of symmetric monoidal R-dg functors between dg categories with symmetric monoidal natural dg transformations. The category of $P_\infty\otimes R$-algebras whose underlying complex is $X\otimes R$ can then alternately be described as $Fun_{dg}^{\otimes}(P_\infty\otimes R, End_{X\otimes R})$: one checks that such a natural transformation is defined by a collection of maps $\{\tau(n):(X\otimes R)^{\otimes n}\to(X\otimes R)^{\otimes n}\}_{n\in\mathbb{N}}$ and that, by strict monoidality and by its compatibility with the functors, it is uniquely determined by the data of the morphism $\tau(1):X\otimes R\to X\otimes R$ compatible with the P_∞-algebra structures. The prop morphism $\psi\otimes R$ then defines a functor, and the precomposition by $\psi\otimes R$ induces a functor of R-dg categories
$$(\psi\otimes R)^* : Fun_{dg}^{\otimes}(End_{X\otimes R}, End_{X\otimes R}) \to Fun_{dg}^{\otimes}(P_\infty\otimes R, End_{X\otimes R})$$
(precomposition of a natural transformation $\tau:F\to G$ by a functor H gives still a natural transformation $\tau\circ H : F\circ H\to G\circ H$), hence the simplicial map
$$\mathcal{N}wFun_{dg}^{\otimes}(End_{X\otimes R}, End_{X\otimes R}) \to \mathcal{N}wFun_{dg}^{\otimes}(P_\infty\otimes R, End_{X\otimes R}) \hookrightarrow \mathcal{N}wP_\infty - Alg(Mod_R^{cof}).$$
This map sends $Id_{End_{X\otimes R}}$ to $(X\otimes R,\psi\otimes R)$, so it sends the connected component of $Id_{End_{X\otimes R}}$ in the simplicial nerve into $\mathcal{N}wP_\infty - Alg(Mod_R^{cof})_{(X\otimes R,\psi\otimes R)}$. The connected component of $Id_{End_{X\otimes R}}$ is $Bhaut(X)$. Indeed, by strict monoidality, a

natural weak self-equivalence of the identity functor $Id_{End_{X\otimes R}}$ is uniquely determined by a self weak equivalence of $X\otimes R$ and vice-versa. Finally, the composite with the forgetful map is obviously $Id_{Bhaut(X)}$ because $(\psi\otimes R)^*$ does not change nor the underlying complex $X\otimes R$ neither its chain self-weak equivalences. □

Remark 3.14. Another way to build a section is to consider the composite map

$$\Omega_{id}\underline{End_X\{X\}}\to\Omega_\psi\underline{P_\infty\{X\}}\to\underline{haut_{P_\infty}(X,\psi)}$$

where the first map is the looping of the precomposition by ψ between the simplicial mapping spaces and the second map is the looping of the map induced by Rezk's homotopy pullback theorem. Then, one notices that $\underline{End_X\{X\}}$ is, for each cdga R, the nerve $\mathcal{N}wFun_{dg}^{\otimes}(End_{X\otimes R},End_{X\otimes R})$ considered in the proof above, so $\Omega_{id}\underline{End_X\{X\}}$ is $\Omega_{id}\underline{\mathcal{N}wFun_{dg}^{\otimes}(End_X,End_X)}=\Omega_{id}B\underline{haut(X)}=\underline{haut(X)}$.

Now we can state properly our result. Recall that $g_{P,X}^{\psi}$ is the L_∞-algebra encoding the formal moduli problem $\underline{P_\infty\{X\}^{\psi}}$.

Theorem 3.15. *There is an equivalence of L_∞-algebras*

$$Lie(\underline{haut_{P_\infty-Alg}}(X,\psi))\ \cong\ g_{P,X}^{\psi}\rtimes^h End(X).$$

Proof. Using the $haut_{\mathbb{K}}(X)$ action (Lemma 3.10) and proposition 3.11, we can identify the fiber sequence of Theorem 2.26 with

$$g_{P,X}^{\varphi}\longrightarrow Lie(\underline{haut}_{P_\infty-Alg}(X,\psi))\longrightarrow End(X).$$

The result is then a direct consequence of Proposition 3.12 once we identify the (homotopy) Lie algebra $Lie(\Omega_{[\varphi]}(\underline{P_\infty\{X\}}//\underline{haut}(X)))$ with the semi-direct product $g_{P,X}^{\psi}\rtimes^h End(X)$. By Lemma 3.7, lemma 2.20 and the commutativity of the right square of the diagram of Proposition 3.8, we only need to find a section of the formal group morphism associated to the derived prestack group map defined, on any artinian R, by

$$\underline{haut}_{P_\infty-Alg}(X,\psi)(R)\cong\Omega_{(X\otimes R,\varphi)}\mathcal{N}wCh_R^{P_\infty}$$
$$\longrightarrow\Omega_X(\mathcal{N}wCh_R|_X)\cong\underline{haut}(X)(R).$$

This is given by Lemma 3.13. □

4. An explicit model via the operad of differentials

We now provide an explicit formula to express the Lie algebra structure of the homotopy automorphisms of a P_∞-algebra, which is crucial to consider deformation complexes of algebraic structures which also encode compatible deformations of the differential. For this, we use a construction originally due to Merkulov [75], which gives a conceptual explanation of how one can express the deformation theory inside $P_\infty-Alg$ as deformations of a P_∞-algebra structure in the (pro)peradic sense plus compatible deformations of the differential, and formalizes properly Remark 2.23.

Precisely, Theorem 3.15 express the Lie algebra structure of the homotopy automorphisms of a P_∞-algebra as a semi-direct product involving the standard operadic deformation complex of P_∞-algebras. In the next two subsections we actually express the semi-direct product explicitly as an L_∞-algebra $g_{P^+,X}^{\psi^+}$, obtained as a Maurer-Cartan twisting of a convolution L_∞-algebra involving a "plus construction" for properads. The use of the + construction to deform usual deformation complex of morphism of properads also appear as a crucial part in Merkulov-Willwacher study of quantization functors [76].

4.1. **The operad of differentials.** We start by recalling the following definition of Merkulov [75].

Definition 4.1. Let P be any dg properad with presentation $P = \mathcal{F}(E)/(R)$ and differential δ. We define P^+ to be the dg-properad with presentation $\mathcal{F}(E^+)/(R)$ and differential δ^+ where the Σ-biobject E^+ is defined by

$$E^+(1,1) = E(1,1) \oplus \mathbb{K}[1] \text{ and } E^+(m,n) = E(m,n).$$

In other word we add to E a generating operation u of degree -1, with one input and one output. The differential δ^+ is modified so that its restriction to E is still δ and further

$$\delta^+(u) = u \otimes u \in E(1,1) \otimes E(1,1).$$

The role of the generator u is thus to twist of a complex X when we consider a P^+-algebra structure on X. The following is proved in [75] (and also follows from the argument of 4.4).

Lemma 4.2. *The construction $P \mapsto (P)^+$ is an endofunctor $(-)^+ : Prop \to Prop$ of the category of dg-properads.*

Further, properad morphisms $\varphi^+ : P^+ \to End_{(X,d)}$ for a given complex X with differential d corresponds to properad morphisms $P \to End_{(X,d-\varphi^+(u))}$ for X equipped with the twisted differential $d - \varphi^+(u)$.

In particular, if X is a graded vector space then P^+-algebra structures on X equip X simultaneously with a P-algebra structure and a compatible differential.

Let us reinterpret this construction by defining the following operad:

Definition 4.3. The operad of differentials Di is the quasi-free operad $Di = (\mathcal{F}(E), \partial)$, where $E(1) = \mathbb{K}\delta$ with δ a generator of degree -1, $E(n) = 0$ for $n \neq 1$ and $\partial(\delta) = \delta \circ \delta$ is the operadic composition $\circ : Di(1) \otimes Di(1) \to Di(1)$.

We will do an abuse of notation and still note Di the properad freely generated by this operad.

Lemma 4.4. *Let (V, d_V) be a complex.*

(1) A Di-algebra structure $\phi : Di \to End_V$ on V is a twisted complex $(V, d_V - \delta_V)$ where δ_V is the image of the operadic generator δ under ϕ.

(2) A morphism of Di-algebras $f : (V, d_V - \delta_V) \to (W, d_W - \delta_W)$ is a chain morphism $f : (V, d_V) \to (W, d_W)$ which satifies moreover $f \circ (d_V - \delta_V) = (d_W - \delta_W) \circ f$ (it is a morphism of twisted complexes).

Proof. (1) The morphism ϕ is entirely determined by the image of the generator δ. Since

$$Di(1) \to Hom(V, V)$$

is a morphism of complexes, its compatibility with the differentials reads

$$\phi(\partial(\delta)) = d_V \circ \delta_V + \delta_V \circ d_V$$

which gives the equation of twisting cochains

$$\delta_V^2 = d_V \circ \delta_V + \delta_V \circ d_V,$$

hence

$$(d_V - \delta_V)^2 = d_V^2 + \delta_V^2 - d_V \circ \delta_V - \delta_V \circ d_V = 0.$$

(2) A Di-algebra structure on V is given by a morphism $Di(V) \to V$, and a Di-algebra morphism $f : V \to W$ is a chain morphism fitting in the commutative square

$$\begin{array}{ccc} Di(V) & \xrightarrow{Di(f)} & Di(W) \\ \downarrow & & \downarrow \\ V & \xrightarrow[f]{} & W \end{array} .$$

Since a Di-algebra structure is determined by the image of the generator δ via $Di(1) \otimes V \to V$, this amounts to the commutativity of the square

$$\begin{array}{ccc} Di(1) \otimes V & \xrightarrow{Di(1)\otimes f} & Di(1) \otimes W \\ \downarrow & & \downarrow \\ V & \xrightarrow[f]{} & W \end{array} ,$$

which is exactly saying that f is a morphism of twisted complexes. □

Remark 4.5. Let us note that Di is a non-positively graded quasi-free operad, but not a *cofibrant* operad. Indeed, to be a retract of a relative cell complex in the model category of dg operads, it needs a filtration Di lacks of (and the same holds true for the corresponding dg properad). By [73, Corollary 40], quasi-free properads $(\mathcal{F}(E), \partial)$ endowed with a Sullivan filtration are cofibrant, and any properad admits a resolution by such. A Sullivan filtration (inspired by the Sullivan filtrations of rational homotopy theory) is an exhaustive increasing filtration $(E_i)_{i \geq 0}$ such that $E_0 = \{0\}$, the maps $E_i \to E_{i+1}$ are split dg-monomorphisms of Σ-modules and $\partial(E_i) \subset \mathcal{F}(E_{i-1})$. In the case of the properad generated by the operad Di, the Σ-module of generators is reduced to $E(1,1) = \mathbb{K}\delta$, so there is no other possibility of filtration than the trivial one given by $E_0 = \{0\}$ and $E_1 = E$, which is not a Sullivan filtration since $\partial(E) \neq \{0\}$. Neither the operad nor the properad Di are cofibrant, they are only Σ-cofibrant.

The only effect of the plus construction on the cohomology of a properad P is to add a new generator of arity $(1,1)$ to H^*P whose square is zero. That will imply the next lemma.

Lemma 4.6. *If $\varphi : P \to Q$ is a quasi-isomorphism of properads, then the induced map $\varphi^+ : P^+ \to Q^+$ is a quasi-isomorphism.*

In other words,the endofunctor $(-)^+ : Prop \to Prop$ preserves weak equivalences of (dg-)properads.

Proof. Let $\varphi : P \xrightarrow{\sim} Q$ be a quasi-isomorphism of dg props whose collections of generators are respectively E_P and E_Q, such that $E_P^+(1,1) = E(1,1) \oplus \mathbb{K}u_P$ and $E_Q^+(1,1) = E(1,1) \oplus \mathbb{K}u_Q$. Then $H^*(\varphi^+)$ sends $[u_P]$ to $[u_Q]$ (where $[-]$ denotes the cohomology class) and coincides with $H^*(\varphi)$ on the other generators. The only relations satisfied by $[u_P]$ and $[u_Q]$ are that they are both of square zero so $H^*(\varphi^+)$ is still a prop isomorphism, hence φ^+ is a quasi-isomorphism. □

Lemma 4.7. *The (pr)operad Di is contractible in the sense that the initial morphism $I \to Di$ and the projection $Di \to I$ are quasi-isomorphisms.*

Proof. To define our contracting homotopy, let us analyze a bit more the structure of Di. We have $Di(n) = 0$ for $\neq 1$ and

$$Di(1) = \mathbb{K} \oplus \bigoplus_{n \geq 1} \mathbb{K}\delta^{\circ n}$$

where $\circ$ is the operadic composition and δ is of degree -1 (so δ^n is of degree $-n$). The differential ∂ is the extension to $\mathcal{F}(E)$ of the map $E \to \mathcal{F}(E), \delta \mapsto \delta \circ \delta$. For sign reasons, we have $\partial(\delta \circ \delta) = 0$. By recursion, we deduce that for any natural integer n, we have $\partial(\delta^{\circ 2n}) = 0$ and $\partial(\delta^{\circ 2n+1}) = \delta^{\circ 2n+2}$. Elements of odd degrees are not cycles and elements of even degrees are boundaries except for the identity operation in degree zero, which defines the only non trivial class in homology, so the homology of Di reduces to I.

An equivalent way to state this is that there is a chain homotopy between $Id_{Di(1,1)}$ and the composite $Di(1,1) \to I(1,1) \to Di(1,1)$ (the projection determined by sending δ to 0 followed by the inclusion). This chain homotopy $h : Di(1,1)_{-n} \to Di(1,1)_{-n-1}$ (where n is a natural integer and the subscript is the homological degree) is defined by $h(\delta^{\circ n}) = \delta^{n+1}$. All complexes over $\mathbb{K}$ are fibrant and cofibrant, so chain homotopies are equivalent to homotopies in the sense of model category theory. Given that the converse composite, the inclusion followed by the projection, equals the identity, this means that these form a homotopy equivalence. All chain complexes here being fibrant and cofibrant, a homotopy equivalence is a quasi-isomorphism. Moreover, $Di(m,n) = I(m,n) = 0$ for $(m,n) \neq (1,1)$, so the initial morphism and the projection are indeed homotopy inverse quasi-isomorphisms of properads. □

This operad Di is a model for the moduli problem associated to derived homotopy self-equivalences $\underline{haut}(X)$. Indeed, the operadic moduli space $\underline{Di\{X\}}$ of a Di-algebra X controls the homotopy automorphism of the underlying complex:

Proposition 4.8. *There is an isomorphism of dg Lie algebras*

$$g_{Di,X}^{triv} \simeq Lie(\underline{haut}(X)).$$

Proof. The operad Di is a quasi-free resolution of I in the sense of Theorem 1.13. It is of the form $Di = (\mathcal{F}(s^{-1}C), \partial)$, where C is a cooperad generated by a single generator u of degree 0 with a coproduct determined by $\Delta_C(u) = u \otimes u$. Moreover, the trivial Di-algebra structure $triv$ sends u to 0, so the Lie bracket on $g_{Di,X}^{triv} = Hom_\Sigma(\overline{C}, End_X)$ is just the convolution Lie bracket obtained by taking the graded commutator of the convolution product. At the level of complexes, we have

$$\begin{aligned} g_{Di,X}^{triv} &= Hom_\Sigma(\overline{C}, End_X) \\ &= Hom_{Ch_{\mathbb{K}}}(\mathbb{K}u, Hom(X,X)) \\ &\cong End(X). \end{aligned}$$

It remains to compare the Lie structures. Since both Lie brackets are graded commutators of associative products, we just have to compare these products. The product on $End(X)$ is the composition of homomorphisms. The product on $g^{triv}_{Di,X}$ is the convolution product, obtained on two elements $f, g : \mathbb{K}u \to End(X)$ by applying first the infinitesimal cooperadic coproduct $\Delta_{(1)}$ to u, then replacing the vertices by $f(u)$ and $g(u)$, and finally composing these maps in $End(X)$. Under the identification of $Hom_{Ch_{\mathbb{K}}}(\mathbb{K}u, End(X))$ with $End(X)$, this gives exactly the composition product on $End(X)$ so the two structures agree. □

Remark 4.9. At the level of derived formal groups, this means that we have an equivalence $\Omega_{triv}\widehat{\underline{Di\{X\}}} \simeq \widehat{\underline{haut(X)}}_{id}$ (by taking the loops on the corresponding derived formal moduli problems).

4.2. **Computing the tangent Lie algebra of homotopy automorphisms.** In this section, we relate $Lie\big(\underline{haut_{P_\infty}}(X,\psi)\big)$ with the plus construction.

Lemma 4.10. *Let P be a properad. There is a commutative square of properads*

$$\begin{array}{ccc} Di & \longrightarrow & P^+_\infty \\ \downarrow & & \downarrow \\ I & \longrightarrow & P_\infty \end{array} \tag{4.1}$$

where $Di \to I$ and $P^+_\infty \to P_\infty$ are the forgetful maps (sending the generator of Di to 0), the upper horizontal map is the inclusion and the lower horizontal map is the initial morphism. The commutative square (4.1) is a pushout.

Proof. We compare first P^+_∞ and $P_\infty \vee Di$, where $\vee$ stands for the coproduct of properads (see [73, Appendix A.3] for its definition). Since the free properad functor $\mathcal{F}$ is a left adjoint, it preserves coproducts and thus comes with natural isomorphisms $\mathcal{F}(M \oplus N) \cong \mathcal{F}(M) \vee \mathcal{F}(N)$. If we take the coproduct $P_\infty \vee Q_\infty$ of two quasi-free properads $P_\infty = (\mathcal{F}(M), \partial_P)$ and $Q_\infty = (\mathcal{F}(M), \partial_Q)$, then via the previous isomorphism we can define a differential on $\mathcal{F}(M \oplus N)$ by taking the derivation associated to

$$\partial_P|_M \oplus \partial_Q|_N : M \oplus N \to \mathcal{F}(M) \oplus \mathcal{F}(N) \hookrightarrow \mathcal{F}(M \oplus N)$$

by universal property of derivations and the fact that this morphism satisfies the twisting cochain equation. In the case where $Q = Di$, it turns out that the free properad underlying P^+_∞ is $\mathcal{F}(M \oplus \mathbb{K}d)$ and the differential on P^+_∞ (see [75]) coincides with the one above, yielding a properad isomorphism

$$P^+_\infty \cong P_\infty \vee Di.$$

Remind that the coproduct is defined as the following pushout diagram over the initial object

$$\begin{array}{ccc} I & \longrightarrow & Di \\ \downarrow & & \downarrow \\ P_\infty & \longrightarrow & P_\infty \vee Di \end{array}$$

and let us consider the following pushout diagram

$$\begin{array}{ccc} Di & \longrightarrow & I \\ \downarrow & & \downarrow \\ P_\infty \vee Di & \longrightarrow & \widetilde{P_\infty}. \end{array}$$

We already know that $P_\infty \vee Di \cong P_\infty^+$, so to conclude the proof of this Lemma we have to show that $\widetilde{P_\infty} = P_\infty$. For this, let us remark that concatenating these two pushout diagrams

$$\begin{array}{ccccc} I & \longrightarrow & Di & \longrightarrow & I \\ \downarrow & & \downarrow & & \downarrow \\ P_\infty & \longrightarrow & P_\infty \vee Di & \longrightarrow & \widetilde{P_\infty} \end{array}$$

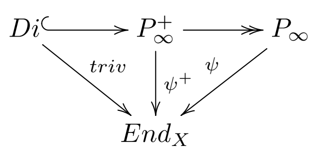
gives a new pushout diagram

$$\begin{array}{ccc} I & \longrightarrow & I \\ \downarrow & & \downarrow \\ P_\infty & \longrightarrow & \widetilde{P_\infty} \end{array}$$

where the upper horizontal map is Id_I, so that the pushout is indeed P_∞. □

Remark 4.11. The properads Di and P_∞^+ are not cofibrant. Moreover, the model category of properads is not left proper, actually even not left proper relatively to the model category of Σ-bimodules, so the square above is not a homotopy pushout of properads.

Remark 4.12. Let us note that, for any P_∞-algebra (X, ψ), we have a commutative diagram of properad morphisms



where the maps relating Di, P_∞ and P_∞^+ are the ones defined in the Lemma (4.10).

Lemma 4.13. *For any cofibration of properads $P \to Q$ and any properad R, the precomposition map of simplicial mapping spaces $Map_{Prop}(Q, R) \to Map_{Prop}(P, R)$ is a fibration in the Kan-Quillen model structure of $sSet$.*

Proof. This follows essentially the argument line of [10, Proposition 5.3], given that our choice of functorial simplicial resolution in the model category of properads is defined by the aritywise tensor product with Sullivan's algebra of polynomial forms on the standard simplices as in [10].

For this reason, we do not reproduce here the full proof of [10, Proposition 5.3] but rather points out the modifications needed to adapt it to the properadic context. First, recall that the category of dg properads is tensored over $CGDA_{\mathbb{K}}$. This external tensor product $\otimes_e$ is defined for any properad P and any cdga A by $(P \otimes_e A)(m, n) = P(m, n) \otimes A$, with the composition product on $P \otimes_e A$ defined by the one of P on the factor P and the product of A on the factor A. In particular, as explained in Definition 1.8, the external tensor product by the simplicial Sullivan cdga of standard simplices $(-) \otimes_e \Omega^\bullet$ forms a functorial simplicial resolution in the model category of dg properads. The simplicial mapping space of dg properads is then defined by

$$Map_{Prop}(P, Q) = Mor_{Prop}(P, Q \otimes_e \Omega^{bullet}) \cong Mor_{Prop(Mod_{\Omega^\bullet})}(P \otimes_e \Omega^\bullet, Q \otimes_e \Omega^\bullet)$$

analogously to the case of cdgas considered in [10, Section 5].

Then we adapt the proof of [10, Lemma 5.2] to dg properads, and for this, we just have to check that for any properad P, the external tensor product $P \otimes_e (-)$ preserves equalizers and finite products of cdgas. The forgetful functor from properads to Σ-bimodules creates all limits, and limits in Σ-bimodules are created termwise, so limits of properads are created termwise as well. Moreover, limits of cdgas are created in chain complexes, in particular equalizers and finite products. So the argument reduces to check that the tensor product of chain complexes preserves equalizers and finite products, which is true because the tensor product by a flat module preserves finite limits, and all modules over a field are flat.

Finally, let us focus on the proof of [10, Proposition 5.3]. The existence of the desired lifting reduces to the "Mayer-Vietoris" argument used in loc. cit. Indeed, pullbacks of properads are created termwise, and acyclic fibrations in the model category of properads are termwise surjective quasi-isomorphisms. So, the maps $A(u)$ and $p(m,n)$ (following the notations of [10, Proposition 5.3]), where $p(m,n)$ is the map in arity (m,n) given by the properad fibration p, are surjective quasi-isomorphisms, and we just have to check that the map $(A(u)\otimes id, id \otimes p(m,n))$ is a surjective quasi-isomorphim. ☐

Lemma 4.14. *Let X be a chain complex. The diagram (4.1) induces an homotopy pullback*

$$\begin{array}{ccc} \underline{P_\infty\{X\}} & \longrightarrow & \underline{P_\infty^+\{X\}} \\ \downarrow & & \downarrow \\ \underline{I\{X\}} & \longrightarrow & \underline{Di\{X\}}. \end{array}$$

of simplicial presheaves.

Proof. The commutative square induced by (4.1) is a strict pullback, simply because the splitting of P_∞^+ as a coproduct $P_\infty \wedge Di$ induces a splitting of $\underline{P_\infty^+\{X\}}$ as $\underline{P_\infty\{X\}} \times \underline{Di\{X\}}$. Indeed, the universal property of the coproduct induces a canonical isomorphism in each simplicial dimension

$$Mor_{Prop}(P_\infty^+, End_X \otimes \Omega^\bullet) \cong Mor_{Prop}(P_\infty, End_X \otimes \Omega^\bullet) \times Mor_{Prop}(Di, End_X \otimes \Omega^\bullet)$$

which is compatible with the simplicial structure, because the latter is determined only by the simplicial resolution of the target. Note that pullbacks of simplicial sets are given by pullbacks of sets in each simplicial dimension, so actually we only need the isomorphisms of sets above: it proves that we have a pullback of sets in each simplicial dimension, hence a pullback of simplicial sets. Then, we conclude by the fact that limits of simplicial presheaves are determined pointwise in simplicial sets.

By Lemma 4.13, the map $\underline{P_\infty^+\{X\}} \rightarrow \underline{Di\{X\}}$ is a fibration, and the Kan-Quillen model structure on $sSet$ is right proper, so this pullback is a homotopy pullback. ☐

Remark 4.15. The splitting of $\underline{P_\infty^+\{X\}}$ used in the proof is not a homotopy product (these are not Kan complexes), so it does not commute with homotopy fiber functors. Consequently, it does not induces a splitting of the corresponding Lie algebra as a direct sum. Note also that $\underline{P_\infty^+\{X\}}$ is fibrant, but not a fibrant resolution of $\underline{P_\infty\{X\}}$.

Proposition 4.16. *The maps of simplicial presheaves of lemma 4.14 induces a fiber sequence of L_∞-algebras*

$$g^{\psi}_{P,X} \to g^{\psi^+}_{P^+,X} \to g^{triv}_{Di,X}.$$

Proof. Since I is the initial properad, the simplicial presheaf $\underline{I\{X\}}$ is nothing but the constant presheaf sending everything to the point. Therefore, the homotopy pullback of lemma 4.14 is a homotopy fiber sequence of simplicial sets which can be pointed by the sequence of base points

$$\psi \mapsto \psi^+ \mapsto triv.$$

Taking the corresponding based loops and using proposition 2.15, we deduce a fiber sequence of derived groups whose corresponding homotopy fiber sequence of L_∞-algebras is the one of the proposition. □

Remark 4.17. An alternate way to get this fiber sequence, starting from Lemma 4.10, is to observe that this pushout induces a pullback of convolution L_∞-algebras

$$\begin{array}{ccc} g_{P,X} & \longrightarrow & g_{P^+,X} \\ \downarrow & & \downarrow \\ g_{I,X} & \longrightarrow & g_{Di,X} \end{array}$$

and that $g_{I,X} = 0$, so that this a fiber sequence of L_∞-algebras. Along this fiber sequence, the Maurer-Cartan element ψ of $g_{P,X}$ is sent to ψ^+, which is in turn sent to *triv*. Twisting our L_∞-algebras by these Maurer-Cartan elements produces a new fiber sequence

$$g^{\psi}_{P,X} \to g^{\psi^+}_{P^+,X} \to g^{triv}_{Di,X}.$$

Moreover, the second arrow is a surjection, hence a fibration in the model category of L_∞-algebras, and all objects are fibrant, so this fiber sequence is a homotopy fiber sequence.

To conclude, we compare this fiber sequence with the fiber sequence

$$g^{\psi}_{P,X} \to Lie(\underline{haut_{P_\infty - Alg}}(X,\psi)) \to Lie(\underline{haut}(X)).$$

of Theorem 2.26 to obtain:

Theorem 4.18. *There is a quasi-isomorphism of L_∞-algebras*

$$g^{\psi^+}_{P^+,X} \simeq g^{\varphi}_{P,X} \ltimes_f End(X) \simeq Lie(\underline{haut}_{P_\infty}(X,\psi)).$$

Proof. We already have by Proposition 3.8 an equivalence of homotopy fiber sequences

$$\begin{array}{ccccc} \underline{P_\infty\{X\}} & \longrightarrow & \underline{\mathcal{N}wP_\infty - Alg}|_X & \longrightarrow & B\underline{haut}(X) \ . \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \underline{P_\infty\{X\}} & \longrightarrow & \underline{P_\infty\{X\}}//\underline{haut}(X) & \longrightarrow & B\underline{haut}(X). \end{array}$$

To conclude the proof, we have to compare the lower fiber sequence with the fiber sequence

$$\underline{P_\infty\{X\}} \to \underline{P^+_\infty\{X\}} \to \underline{Di\{X\}}.$$

Precisely, we apply twice the décalage construction to get two fiber sequences

$$\Omega_{[\varphi]}\underline{P_\infty\{X\}}//\underline{haut}(X) \to \underline{haut}(X) \to \underline{P_\infty\{X\}} \tag{4.2}$$

and

$$\Omega_{\varphi^+}\underline{P^+_\infty\{X\}} \to \Omega_{triv}\underline{Di\{X\}} \to \underline{P_\infty\{X\}}. \tag{4.3}$$

We do not expect to get directly an equivalence at this level, however, our strategy is to define a commutative square

$$\begin{array}{ccc} \Omega_{triv}\underline{Di\{X\}} & \longrightarrow & \underline{P_\infty\{X\}} \\ \downarrow & & \downarrow \\ \underline{haut}(X) & \longrightarrow & \underline{P_\infty\{X\}} \end{array} \tag{4.4}$$

hence inducing a morphism between the corresponding homotopy fiber sequences, so that the two vertical arrows induce equivalences of L_∞-algebras at the tangent level, after completion of the derived groups at the appropriate base points.

The right hand vertical arrow of (4.4) is just the identity morphism. Recall that, by [32, Theorem 5.2.1], if O_∞ is the cobar construction on a Σ-cofibrant dg cooperad, then the homotopies between two morphisms $\varphi, \psi : O_\infty \to End_X$ are in bijection with ∞-quasi-isotopies in $O_\infty - Alg$ between the corresponding O_∞-algebras, that is, ∞-quasi-isomorphisms whose first level lies in the connected component of the identity in $haut(X)$.

In particular, a loop in $\Omega_\varphi O_\infty\{X\}$, that is, a self-homotopy of φ, induces a self-∞-isotopy of (X, φ). In the particular case where the operad is augmented and φ is the trivial O_∞-algebra structure on X (that is, it factorizes through the augmentation $O_\infty \to I$), then such a self-∞-isotopy is just a self quasi-isomorphism in the connected component of the identity. Consequently, there is a natural map

$$\Omega_{triv}\underline{Di\{X\}} \to \Omega_{triv}\underline{\mathcal{B}^c\mathcal{B}Di\{X\}} \to \underline{haut}(X)$$

where $\mathcal{B}^c\mathcal{B}Di$ is the bar-cobar resolution of Di, the first arrow is induced by the operad morphism $\mathcal{B}^c\mathcal{B}Di \to Di$ and the second arrow is the one explained above which takes loops on the trivial algebra structure to self-∞-isotopies of the trivial algebra structure, that is, to self-quasi-isomorphisms. This natural map makes the commutative square (4.4) above commutes, and becomes an isomorphism when restricting the target to the connected component of id_X. This means that, even though this map is not an equivalence, taking the tangent Lie algebras of the derived formal groups obtained after completions at the appropriate base points (trivial loop on the left, id_X on the right) leads to a quasi-isomorphism of Lie algebras

$$g^{triv}_{Di,X} \xrightarrow{\simeq} Lie(\widehat{\underline{haut}(X)}_{id}) = End(X).$$

Now, this commutative square (4.4) induces a morphism between the homotopy fibers given by (4.2) and (4.3)

$$\Omega_{\varphi^+}\underline{P^+_\infty\{X\}} \to \Omega_{[\varphi]}\underline{P_\infty\{X\}}//\underline{haut}(X).$$

Since this square becomes a square of quasi-isomorphisms of Lie algebras at the tangent the level, the induced morphism between the Lie algebras of the fibers is a quasi-isomorphim as well

$$g^{\varphi^+}_{P^+,X} = Lie(\Omega_{\varphi^+}\widehat{\underline{P^+_\infty\{X\}}}) \xrightarrow{\simeq} Lie(\Omega_{[\varphi]}\underline{P_\infty\{X\}}\widehat{//}\underline{haut}(X)).$$

Therefore, Proposition 3.12 gives us the equivalence of the later Lie algebra with $Lie(\underline{haut}_{P_\infty}(X,\varphi))$. Moreover $Lie(\Omega_{[\varphi]}\widehat{\underline{P_\infty\{X\}}//\underline{haut}}(X))$ is canonically equivalent to the homotopy semi-direct product $g^{\varphi}_{P,X} \ltimes_f End(X)$ by Theorem 3.15. ☐

Remark 4.19. Although it is interesting to see the role of the Borel construction here, there is an alternate proof of Theorem 4.18 which makes no use of it. Let us sketch it; for this, we compare the fiber sequences

$$\Omega_{[\varphi]}\underline{P_\infty\{X\}}//\underline{haut}(X) \to \underline{haut}(X) \to \underline{P_\infty\{X\}}$$

and

$$\underline{haut}_{P_\infty}(X,\varphi) \to \underline{haut}(X) \to \underline{P_\infty\{X\}}$$

by checking that actually, in both cases we are considering the fibers of the same map from $\underline{haut}(X)$ to $\underline{P_\infty\{X\}}$, which is the map sending a homotopy automorphism to its action on φ. Hence an equivalence of Lie algebras

$$Lie(\underline{haut}_{P_\infty}(X,\varphi)) \simeq Lie(\Omega_{[\varphi]}\underline{P_\infty\{X\}}//\underline{haut}(X)).$$

Then, the argument line of the proof above provides the equivalence

$$Lie(\Omega_{[\varphi]}\underline{P_\infty\{X\}}//\underline{haut}(X)) \simeq g^{\varphi^+}_{P^+,X}.$$

Theorem 4.18 shows that the + construction is crucial to study deformation of dg-algebras and not just deformations of algebraic structures on a fixed complex.

Example 4.20 (Strict associative algebras)**.** Let (A,ψ) be a (necessarily strict) associative algebra concentrated in degree 0. Then by proposition 3.11, one has $\underline{haut}(A) \cong Hom(A,A)$ the Lie algebra (concentrated in degree 0) of endomorphisms of the underlying vector space of A.

It is a standard computation ([74, 75]) that the Lie algebra $g^{\psi}_{Ass} = \mathfrak{L}_{\underline{Ass_\infty\{A\}}^{\psi}}$ is isomorphic to the subcomplex $C^{\bullet\geq 2}(A,A)[1] = \bigoplus_{n\geq 2} Hom(A^{\otimes n},A)$ of the shifted Hochschild cochain complex $C^{\bullet}(A,A)[1]$ with Lie bracket given by the restriction of the standard Gerstenhaber complex. Using these equivalences, the action given by lemma 3.10 of $Hom(A,A)$ on $C^{\bullet\geq 2}(A,A)[1]$ is given by the usual action of $Hom(A,A) = C^1(A,A)[1]$ on the Hochschild complex given by the Lie bracket. Therefore, we have that

$$Lie(\underline{haut}_{Ass_\infty}(A,m)) \cong C^{\bullet\geq 1}(A,A)[1]) \cong Hom(A^{\otimes>1},A)[1] \rtimes Hom(A,A).$$

The latter can be deduced by an immediate computation mimicking [75] from the operad Ass^+_∞ obtained by the usual Koszul resolution of Ass, using theorem 4.18.

In particular, the moduli space $\underline{Ass_\infty\{A\}}^{\psi}(\mathbb{K}[[t]])$ controls the algebra structures on $A[[t]]$ whose reduction modulo t is the given one.

However, the moduli space $\underline{haut}_{Ass_\infty}(A,\psi)(\mathbb{K}[[t]])$ controls the algebra structures on $A[[t]]$ whose reduction modulo t is the given one, up to isomorphism of algebras which are the identity modulo t.

In other words, the set of connected components of such deformations in the first case is the set of all possible deformations, while the connected component of the derived prestack group $\underline{haut}_{Ass_\infty}(A,\psi)$ are the set of all possible deformations modulo the standard gauge equivalences.

Example 4.20 can be generalized to algebras concentrated in degree 0 over other operads, see Section 6.1.

Note that this result extends for dg-algebras (and A_∞-algebras as well), see 5.1.

If P_∞ is semi-free, there is a nice explicit description of $Lie(\underline{haut}_{P_\infty}(X,\psi))$.

Corollary 4.21. *Let $P_\infty = (\mathcal{F}(s^{-1}\overline{C}),\partial) \xrightarrow{\sim} P$ be a cofibrant quasi-free resolution of an operad P where C is a cooperad and (X,ψ) be a P_∞-algebra. One has an equivalence of Lie algebras*

$$Lie(\underline{haut}_{P_\infty}(X,\psi)) \cong Hom_\Sigma(\overline{C}\oplus I, Q) \cong Coder(\overline{C}(X[1]))^{D_\Psi} \rtimes End(X,X)$$

where the last term is the L_∞-algebra of coderivations of the cofree coalgebra on $X[1]$ twisted by the Maurer Cartan element D_Ψ (the coderivation of square zero corresponding to the P_∞-algebra structure Ψ) and the action of $End(X,X)$ is given by the composition of coderivations of $C(X[1])$.

Proof. The isomorphism of Lie algebras given in [63, Proposition 10.1.17] induces an isomorphism of Lie algebras

$$g_{P,X} \cong Coder(\overline{C}(X[1])).$$

Then, the P_∞-algebra structure ψ is a Maurer-Cartan element in $g^{\psi}_{P,X}$, whose image under the Lie algebra isomorphism above gives a Maurer-Cartan element D_ψ in $Coder(\overline{C}(X[1]))$, that is, a degree 1 coderivation of square zero. Twisting this isomorphism by those Maurer-Cartan elements gives an isomorphism

$$g^{\psi}_{P,X} \cong Coder(\overline{C}(X[1]))^{D_\psi}.$$

Therefore the equivalence between the r.h.s and l.h.s in the theorem follows from Theorem 4.18. The tangent action of $End(X)$ on $g^{\psi}_{P,X}$ induced by the action of $\underline{haut}_{P_\infty}(X,\psi)$ on $\underline{P_\infty\{X\}}$ (lifting, for any $f \in haut_{P_\infty}(X,\psi)$, the P_∞-structures along $End_f \to End_X$ by Lemma 3.10) gives under this isomorphism an action of $End(X)$ on $Coder(\overline{C}(X[1]))^{D_\psi}$ defined by the composition of coderivations. The equivalence between the middle term and the r.h.s of the equivalences now follows from proposition 4.8 and the fact that the dg-operad Di is of the form $Di = (\mathcal{F}(\mathbb{K}\delta),\partial) \cong (\mathcal{F}(s^{-1}I),\partial)$(Definition 4.3). □

Example 4.22. Let A be a dg-associative algebra. We can consider the truncated and full Hochschild complexes of A, respectively $CH^{\bullet>0}(A)$ and $CH_*(A)$ as in 4.20. The *full version* controls the deformation theory of the category of A-modules which is in general another kind of formal moduli problem. Equivalently, the full Hochschild complex controls the deformations of A as a *curved A_∞-algebra* [79].

However, in the case where A is concentrated in degree zero, we observe that, first, deformations of A are deformations as a strict Poisson algebra or as a strict associative algebra, and second, curved and uncurved deformations are equivalent. Consequently, the space of Maurer Cartan elements are the same for the truncated and the untruncated versions of ¨ Hochschild complexes.

This observation is crucial in the study of formality theorems for Poisson algebras and deformation quantization of Poisson structures on manifolds [59, 85]. Let us fix $A = \mathcal{C}^\infty(\mathbb{R}^d)$ the algebra of smooth functions on $\mathbb{R}^d$, and consider two complexes. First, the full Hochschild complex $CH^*(A,A)$, second, the complex of polyvector fields $T_{poly}(A) = \left(\bigoplus_{k\geq 0}\bigwedge^k Der(A)[-k]\right)[1]$ (recall that vector fields are derivations of the ring of smooth functions). The complex of polyvector fields also forms a (shifted) Lie algebra with a Lie structure induced by the bracket of vector fields. The classical Hochschild-Kostant-Rosenberg theorem (HKR for short) states that the cohomology of $CH^*(A,A)$ is precisely $T_{poly}(A)$. However, the HKR quasi-isomorphism is not compatible with their respective Lie algebra structures. In [59], Kontsevich proved that the HKR quasi-isomorphism lifts to an L_∞-quasi-isomorphism

$$T_{poly}(A)[1] \stackrel{\sim}{\to} CH^*(A,A)[1]$$

by building an explicit formality morphism. An alternative proof of the formality theorem is due to Tamarkin [85] and provides a formality quasi-isomorphism of homotopy Gerstenhaber algebras (that is E_2-algebras). Here $T_{poly}(A)$ is actually the deformation complex of the trivial Poisson algebra structure. In general, the full Hochschild complex $CH^*(A,A)$ controls deformations of A as a curved algebra, but since A is in degree zero, the space of Maurer Cartan elements obtained from the full Hochschild complex is the same as the one from the truncated Hochschild complex. This is important, because the formality theorem holds for the full complex but not for the truncated one. This formality theorem implies the equivalence of the associated formal moduli problems. Then, applying these moduli problems to the ring of formal power series $\mathbb{K}[[t]]$, one gets that the the set of isomorphism classes Poisson algebra structures on $A[[\hbar]]$ without constant term is in bijection with gauge equivalence classes of $*_\hbar$-products (that is, associative formal deformations of the product of A).

There aer similar phenomenon for other categories of algebraic structures such as (shifted) Poisson ones. See 5 and 6.

5. Examples

5.1. **Deformations of E_n-algebras.** We now generalize example 4.20 to the homotopy setting and to higher algebras, that is E_n-algebras.The latter are higher generalizations of homotopy associative algebras and form a hierarchy of "more and more" commutative and homotopy associative structures, interpolating between homotopy associative or A_∞-algebras (that is E_1-algebras) and E_∞-algebras.

Algebras governed by E_n-operads and their deformation theory play a prominent role in a variety of topics such as the study of iterated loop spaces, Goodwillie-Weiss calculus for embedding spaces, deformation quantization of Poisson manifolds and Lie bialgebras, factorization homology and derived symplectic/Poisson geometry [58, 59, 66, 69, 12, 29, 28, 36, 44, 48, 55, 60, 71, 79, 85, 92].

To define E_n-algebras, one first note that the configuration spaces of (rectilinear embeddings of) n-disks into a bigger n-disk gather into a topological operad D_n, called the little n-disks operad. An E_n-operad (in chain complexes) is a dg-operad quasi-isomorphic to the singular chains $C_*(D_n)$ of the little n-disks operad. There is an ∞-functor from E_n-algebras to L_∞-algebras whose composition with the forgetful functor to chain complexes is the shift $X \mapsto X[1-n]$.

Given an ordinary associative algebra A, its endomorphisms $Hom_{biMod_A}(A,A)$ in the category $biMod_A$ of A-bimodules is isomorphic to the center $Z(A)$ of A. Deriving this hom object gives the Hochschild cochain complex $C^*(A,A) \cong \mathbb{R}Hom_{biMod_A}(A,A)$ of A, and the associated Hochschild cohomology $HH^*(A,A)$ of A satisfies $HH^0(A,A) = Z(A)$. For higher structures, one has the following definition (see [28, 69, 44]).

Definition 5.1. The (full) Hochschild complex of an E_n-algebra A, computing its higher Hochschild cohomology, is the derived hom $C^*_{E_n}(A,A) = \mathbb{R}Hom^{E_n}_A(A,A)$ in the category of (operadic) A-modules[12] over E_n.

[12] note that the operadic E_1-module are precisely the bimodules

The Deligne conjecture endows the Hochschild cochain complex with an E_{n+1}-algebra structure [44, Theorem 6.28] or [28, 69]. Associated to an E_n-algebra A, one also has its cotangent complex L_A, which classifies square-zero extensions of A [28, 69].

Definition 5.2 ([28])**.** The tangent complex T_A of an E_n-algebra A is the dual $T_A := Hom_A^{E_n}(L_A, A) \cong \mathbb{R}Der(A, A)$.

The latter isomorphism gives a L_∞-structure to T_A and Francis [28, 69] has proved that $T_A[-n]$ has a canonical structure of E_{n+1}-algebra (lifting the L_∞-structure). He further proved that there is a fiber sequence

$$T_A[-n] \to CH^*_{E_n}(A, A) \to A$$

where the first map is a map of E_{n+1}-algebras.

A corollary of our theorem 4.18 is the following operadic identification of the tangent complex T_A of an E_n-algebra (5.2):

Corollary 5.3. *The E_n-Hochschild tangent complex T_A of an E_n-algebra A is naturally weakly equivalent as an L_∞-algebra to $g^{\psi^+}_{E_n^+,A}$:*

$$T_A \simeq Lie(\underline{haut}_{E_n}(A, \psi)) \simeq g^{\psi^+}_{E_n^+,A},$$

where ψ^+ is the E_n^+-algebra structure on A trivially induced by its E_n-algebra structure $\psi : E_n \to End_A$ as above, and $\underline{haut}_{E_n}(A)$ is the derived prestack group of homotopy automorphisms of A as an E_n-algebra.

Proof. According to [28, Lemma 4.31], the homotopy Lie algebra of homotopy automorphisms $Lie(\underline{haut}_{E_n}(A, \psi))$ is equivalent to the tangent complex T_A of A. Hence Theorem 4.18 implies the corollary. □

In particular, Theorem 4.18 implies that the tangent complex T_A of an A_n-algebra splits as a semi-direct product of $End(A)$ with the operadic deformation complex of A as an E_n-algebra.

5.2. **Deformation complexes of $Pois_n$-algebras.** We now introduce Tamarkin deformation complexes of a $Pois_n$-algebra [87] and prove that these complexes *do control deformations* of (dg-)$Pois_n$-algebras.

We denote by $Pois_n$ the operad of $Pois_n$-algebras and $uPois_n$ the operad of unital $Pois_n$-algebras.

Let A be a dg $Pois_n$-algebra, with structure morphism $\psi : Pois_n \to End_A$. We denote by $CH^*_{Pois_n}(A, A)$ its $Pois_n$-Hochschild cochain complex, also referred to as its $Pois_n$-deformation complex as defined by Tamarkin [87] and Kontsevich [58]. Following Calaque-Willwacher [11], we note that this complex is given by the suspension

$$CH^*_{Pois_n}(A, A) := Hom_\Sigma(uPois_n{}^*\{n\}, End_A)[-n] \tag{5.1}$$

of the underlying chain complex of the convolution Lie algebra. Here $(-)^*$ is the linear dual and $\{n\}$ is the operadic n-iterated suspension. The inclusion of $Pois_n$ in $uPois_n$ induces a splitting (as a graded space)

$$CH^*_{Pois_n}(A, A) \cong A \oplus Hom_\Sigma(Pois_n{}^*\{n\}, End_A)[-n] \tag{5.2}$$

and also gives rise to the *truncated* deformation complex

$$CH^{(\bullet>0)}_{Pois_n}(A,A) = Hom_\Sigma({Pois_n}^*\{n\}, End_A)[-n] \tag{5.3}$$

obtained by deleting the "unit part" A, which is more relevant to deformations of $Pois_n$-algebras[13], see Lemma 5.6. Note that both complexes are naturally bigraded with respect to the internal grading of A and the "operadic" grading coming from ${uPois_n}^*$. The notation $CH^{(\bullet>0)}_{Pois_n}(A,A)$ is there to suggest that we are taking the subcomplex with positive weight with respect to the operadic grading.

The suspensions $CH^*_{Pois_n}(A,A)[n]$ and $CH^{(\bullet>0)}_{Pois_n}(A,A)[n]$ have *canonical* L_∞*-structures* since they are convolution algebras, and $CH^{(\bullet>0)}_{Pois_n}(A,A)[n]$ is canonically a sub L_∞-algebra of $CH^*_{Pois_n}(A,A)[n]$. Tamarkin [87] (see also [58, 11]) proved that *the complex* $CH^*_{Pois_n}(A,A)$ *actually inherits a (homotopy)* $Pois_{n+1}$*-algebra structure* lifting this L_∞-structure. Further, by (5.2) we have an exact sequence of cochain complexes

$$0 \longrightarrow CH^{(\bullet>0)}_{Pois_n}(A,A) \longrightarrow CH^*_{Pois_n}(A,A) \longrightarrow A \longrightarrow 0 \tag{5.4}$$

which yields after suspending the exact triangle

$$A[n-1] \stackrel{\partial_{Pois_n}[n-1]}{\longrightarrow} CH^{(\bullet>0)}_{Pois_n}(A,A)[n] \longrightarrow CH^*_{Pois_n}(A,A)[n]. \tag{5.5}$$

Remark 5.4. The map $\partial_{Pois_n} : A \subset CH^*_{Pois_n}(A,A) \to CH^{(\bullet>0)}_{Pois_n}(A,A)$ is the part of the differential in the cochain complex $CH^*_{Pois_n}(A,A) = A \oplus CH^{(\bullet>0)}_{Pois_n}(A,A)$ which comes from the operadic structure. That is $\partial_{Pois_n}(x) \in Hom(A,A)$ is the map $a \mapsto \pm[x,a]$ where the bracket is the bracket of the $Pois_n$-algebra. The Jacobi identity for the Lie algebra $A[n-1]$ implies that the sequence (5.5) is a sequence of L_∞-algebras.

Remark 5.5. The operad $Pois_n$ is denoted e_n in [11, 87] and the complex $CH^*_{Pois_n}(A,A)$ is simply denoted $def(A)$ in Tamarkin [87]. We prefer to use the notations we have introduced by analogy with (operadic) Hochschild complexes.

The next Lemma compares the L_∞-algebra structure of the truncated $Pois_n$ Hochschild complex and the one associated to the derived prestack group of homotopy automorphisms of a $Pois_n$-algebra:

Lemma 5.6. *Let A be a dg $Pois_n$-algebra with structure map $\psi : Pois_n \to End_A$. There is an equality of dg Lie algebras*

$$g^{\psi^+}_{Pois_n^+,A} = CH^{(\bullet>0)}_{Pois_n}(A,A)$$

where the right hand side is the truncated cochain complex of a $Pois_n$-algebra defined by Tamarkin as above.

Proof. According to the definition of the plus construction $(-)^+$ given in Section 4, we have

$$Pois^+_{n\infty} = \Omega(Pois^*_n\{n\})^+ = (\mathcal{F}(\overline{Pois^*_n\{n+1\}})^+, \partial^+)$$

where $Pois_{n\infty}$ is the minimal model of $Pois_n$, $(-)^*$ is the linear dual, $\{n\}$ is the operadic n-iterated suspension, Ω is the operadic cobar construction and $\overline{}$ is the

[13] as opposed to deformation of categories of modules

coaugmentation ideal of a coaugmented cooperad. Recall that the collection of generators $\overline{Pois_n^*\{n+1\}}^+$ is given by

$$\overline{Pois_n^*\{n+1\}}^+(1) = \overline{Pois_n^*\{n+1\}}(1) \oplus \mathbb{K}[1] = \overline{Pois_n^*\{n+1\}}(1) \oplus \mathbb{K}d$$

where d is a generator of degree 1 and

$$\overline{Pois_n^*\{n+1\}}^+(r) = \overline{Pois_n^*\{n+1\}}(r)$$

for $r > 1$. The restriction of the differential ∂^+ on the generators decomposes into $\partial^+ = \partial + \delta$ where ∂ is the differential of the minimal model and δ is defined by $\delta(d) = d \otimes d$ and zero when evaluated on the other generators (note that, by the Koszul sign rule and for degree reasons, we have $\delta^2(d) = 0$ so we get a differential indeed). Now let $\psi^+ : Pois_{n\infty}^+ \to End_A$ be the operad morphism induced by ψ, thus a Maurer-Cartan element of the convolution graded Lie algebra $g_{Pois_n^+,A}$. We twist this Lie algebra by ψ to get a dg Lie algebra $g_{Pois_n^+,A}^{\psi^+}$ with the same Lie bracket and whose differential is defined by

$$\pm(d_A)_* + [\psi, -]$$

where $(-)_*$ denotes the post-composition, d_A is the differential on End_A induced by the differential of A, the $\pm$ sign is defined according to the Koszul sign rule and $[-,-]$ is the convolution Lie bracket. Note here that the Koszul dual cooperad has no internal differential. We refer the reader to [63, Chapter 12] for more details about such convolution Lie algebras. Now let us point out that

$$\overline{Pois_n^*\{n+1\}}^+(1) = \overline{Pois_n^*\{n+1\}}(1) \oplus \mathbb{K}[1] = (\overline{Pois_n^*\{n\}}(1) \oplus \mathbb{K})[1],$$

which implies that

$$g_{Pois_n^+,A}^{\psi^+} = Hom_\Sigma(\overline{Pois_n^*\{n\}} \oplus I, End_A)^\psi = Conv(Pois_n^*\{n\}, End_A)$$

where $Conv(Pois_n^*\{n\}, End_A)$ is the convolution Lie algebra of [11, Section 2.2]. This is an equality of dg Lie algebras, because the convolution bracket is defined by the action of the infinitesimal cooperadic coproduct on the coaugmentation ideal, so is the same on both sides. □

Lemma 5.6 together with Theorem 4.18 implies that

Corollary 5.7. *The truncated Tamarkin deformation complex $CH_{Pois_n}^{(\bullet>0)}(A,A)$ controls deformations of A into the ∞-category of dg $Pois_n$-algebras, in other words is the tangent Lie algebra of the derived prestack group $\underline{haut}_{Pois_{n\infty}}(A)$, where $Pois_{n\infty}$ is a cofibrant resolution of $Pois_n$.*

Remark 5.8. The proof of Lemma 5.6 also shows that the deformation complex $g_{Pois_n,A}^{\psi}$ of the formal moduli problem $\underline{Pois_{n\infty}\{A\}}^\psi$ is given by the L_∞-algebra $CH_{Pois_n}^{(\bullet>1)}(A,A)[n]$, which is the kernel

$$CH_{Pois_n}^{(\bullet>1)}(A,A)[n] := \ker\left(CH_{Pois_n}^{(\bullet>0)}(A,A)[n] \twoheadrightarrow Hom(A,A)[n]\right) \tag{5.6}$$

and is thus a even further truncation of $CH_{Pois_n}^*(A,A)$. The situation is thus similar to what happens in deformation theory of associative algebras.

One can also wonder which deformation problem is controled by the full complex $CH^*_{Pois_n}(A, A)$. In view of our results and classical results on deformation theory of E_n-algebras ([57, 79, 28]), we make the following

Conjecture. Let $n \geq 2$ and let A be an n-Poisson algebra. The L_∞-algebra structure of the full shifted Poisson complex $CH^*_{Pois_n}(A)[n]$ controls the deformations of Mod_A into E_{n-1}-monoidal dg categories.

Here, when $n \leq 1$, some shift is needed on the linear enrichment of the $E_{|n-1|}$-monoidal dg-category, according to the red shift trick [92, 91].

This conjecture is deeply related to the deformation theory of shifted Poisson structures in derived algebraic geometry, in the sense of [12]. Precisely, if X is a derived Artin stack locally of finite presentation and equipped with an n-shifted Poisson structure, then its sheaf of principal parts (which controls the local deformation theory on X and whose modules describe the quasi-coherent complexes over X) forms a sheaf of mixed graded $Pois_{n+1}$-algebras. The deformation theory of the category of quasi-coherent complexes should then be controled by a full shifted Poisson complex.

5.3. **Bialgebras.** Let us conclude our series of examples with one of properadic nature. Here we are interested in associative and coassociative bialgebras, and refer the reader to Example 7.7 for a precise definition as well as the construction of the corresponding properad $Bialg$.

What we call the *Gerstenhaber-Schack complex* is the total complex of a bicomplex, defined by

$$C^*_{GS}(B, B) \cong \prod_{m,n \geq 1} Hom_{dg}(B^{\otimes m}, B^{\otimes n})[-m-n]. \tag{5.7}$$

The horizontal differential is defined, for every n, by the Hochschild differential associated to the Hochschild complex of B seen as an associative algebra with coefficients in the B-bimodule $B^{\otimes n}$. The vertical differential is defined, for every m, by the co-Hochschild differential associated to the co-Hochschild complex of B seen as a coassociative coalgebra with coefficients in the B-bicomodule $B^{\otimes m}$. The compatibility between these differentials, which gives us a well defined bicomplex, follows from the distributive law relating the product and the coproduct of the bialgebra B (see [38, 75] for details). Combining Theorem 4.18 with the computation of [75], we get:

Theorem 5.9. *The Gerstenhaber-Schack complex is quasi-isomorphic to the L_∞-algebra controlling the deformations of dg bialgebras up to quasi-isomorphisms:*

$$C^*_{GS}(B, B) \cong g^{\varphi^+}_{Bialg^+_\infty, B} \simeq Lie(\underline{haut}_{Bialg_\infty}(B))$$

.

Hence the Gerstenhaber-Schack complex is indeed the L_∞-algebra controling the derived deformation theory of dg bialgebras in a precise meaning, something new since the introduction of this complex by Gerstenhaber and Schack in their seminal paper [38]. Moreover, as emphasized by the results of [75] and [46], this complex plays a crucial role in deformation quantization.

6. Concluding remarks and perspectives

To conclude, let us give an overview of the various deformation complexes considered in the litterature and their *derived* and *underived* formal moduli problems.

6.1. **Algebras over operads in vector spaces.** Let X be a vector space and P an operad with Koszul dual C. Then the cohomological grading on the convolution Lie algebra $g_{P,X} = Hom_{\Sigma}(C, End_X)$ is entirely determined by the "weight grading" of operations in the cooperad C. In particular, in the case where X is of finite dimension n, this means that the degree 0 Lie subalgebra of $g_{P,X}$ is nothing but $gl(X)$, whose associated Lie group is the general Lie group $GL(X)$. This is the gauge group acting on the Maurer-Cartan elements of $g_{P,X}$, so that the moduli set of Maurer-Cartan elements is

$$\mathcal{MC}(g_{P,A}) = Mor(P, End_A)/GL(A).$$

The deformation complex $g_{P,X}^{\varphi}$ of a P-algebra $A = (X, \varphi)$ controls then the deformations of A *as a P-algebra*, up to *linear automorphisms of A*. If we replace C by $\overline{C}$ in the definition of $g_{P,X}$, which is what we did in the present paper, then there is no non trivial gauge group acting anymore, and the Maurer-Cartan moduli set is just

$$\mathcal{MC}(g_{P,A}) = Mor(P, End_A).$$

The corresponding *underived* formal moduli problem, or classical deformation functor, controls the deformations of A as a P-algebra in the category of vector spaces, up to isomorphisms.

In the derived setting, one replaces Artinian algebras by their dg enhancement, so that the simple description above in terms of gauge group action does not exist anymore (notice that, although V is in degree 0, we have to consider $haut(V \otimes A)$ for any *differential graded* local Artinian algebra A, and $V \otimes A$ is not in degree zero anymore). The relevant theory, described in Section 4, defines the appropriate deformation problems as loops over the homotopy quotient of the moduli space of P-algebra structures by a homotopy automorphisms group. Moreover, it turns out, as we explained in Remark 2.25, that the corresponding derived formal moduli problem is given by the formal completion $\widehat{\underline{\mathcal{N}wP - Alg}}_A$ at A of the n-geometric derived Artin stack of n-dimensional P-algebras.

To clarify the link between our *derived* construction and the *underived* deformation functor described above, let us restrict our derived moduli problem to local Artinian algebras. In this context, provided that P is an operad in vector spaces, the simplicial presheaves $\underline{P_\infty\{X\}}$ and $\underline{haut}(X, \varphi)$ are actually discrete. Indeed, given a local Artinian algebra R, each vertex of the Kan complex $P_\infty\{X \otimes m_R\}$ factors uniquely through the composite

$$P_\infty \twoheadrightarrow P \rightarrow End_{V \otimes m_R}$$

because of degree reasons, commutation with the differentials and the fact that $End_{X \otimes m_R}$ is concentrated in degree zero. Moreover, for the same reasons, a homotopy between two such maps cannot be anything else than the identity, so that finally

$$P_\infty\{X \otimes m_R\} \cong Mor_{prop}(P, End_{X \otimes m_R}) = MC(g_{P,X})$$

and the corresponding pointed functor over (X, φ) is

$$P_\infty\{X \otimes m_R\}^{varphi} \cong MC(g_{P,X}^{\varphi}).$$

Note that this is coherent with the fact that left homotopies between such maps are in bijection with ∞-isotopies of (X,φ), which boils down to $Id_{(X,\varphi)}$ when X is in degree zero. Also for degree reasons, the simplicial presheaf $\underline{haut}(X,\varphi)$ is equivalent to the discrete presheaf defined, for each local Artinian algebra R, by the strict automorphism group $Aut(X\otimes m_R)$, that is, the algebraic group of automorphisms of (X,φ). The homotopy action of $\underline{haut}(X,\varphi)$ on $\underline{P_\infty\{X\}}$ is then nothing but the gauge action described above, so that the semi-direct product $g^{\varphi^+}_{P^+,X}\simeq\simeq g^{\varphi}_{P,X}\ltimes_{hol} End(X)$ becomes the dg convolution Lie algebra considered in [63, Section 12.2.22].

6.2. **Differential graded algebras over operads.** A differential graded structure on the object X carries non trivial homotopies, and taking into account this new homotopy data that do not exist in the degree zero case involves deformations of P-algebra structures into P-algebra structures *up to homotopy*, that is P_∞-algebra structures, and therefore taking into account the non trivial homotopy type of the moduli space $P_\infty\{X\}$. Given a P_∞-algebra $A=(X,\varphi)$, there are a priori three possible variants of derived deformation problems one could look at:

(1) Deformation theory of the operad morphism $\varphi : P_\infty \to End_A$;
(2) Deformation theory of A in the ∞-category of P_∞-algebras *up to ∞-isotopies*;
(3) Deformation theory of A in the ∞-category of P_∞-algebras *up to quasi-isomorphisms.*

Problem (1) is, as we saw before, controled by the derived formal moduli problem

$$\underline{P_\infty\{X\}}(R) = hofib_\varphi(P_\infty\{X\otimes R\}\to P_\infty\{X\})$$

whose associated L_∞-algebra is $g^{\varphi}_{P,X}$ (constructed with $\overline{C}$). Problem (2) is the setting in which [63, Section 12.2.22] takes place: an R-deformation of a P-algebra A in the sense of (2) is a an R-linear P_∞-algebra $\tilde{A}\simeq A\otimes R$ with a $\mathbb{K}$-linear P_∞-algebra ∞-isomorphism $\tilde{A}\otimes_R\mathbb{K}\xrightarrow{\sim} A$, where $(-)\otimes_R\mathbb{K}$ is defined by the augmentation of R. Two deformations are equivalent if they are related by an R-linear ∞-isomorphism whose restriction modulo m_R is the identity, that is ∞-isotopies. It turns out that, in the operadic case, problems (1) and (2) are equivalent: by [32, Theorem 5.2.1] homotopies between morphisms from P_∞ to End_X are in bijection with ∞-isotopies between the corresponding P_∞-algebras, and by [63, Section 12.2.22] the later are also controled by the convolution L_∞-algebra. Here the gauge group of the deformation complex (for this moduli problem) of a P_∞-algebra A is isomorphic to the group of ∞-isotopies of A.

We spent some time in this article to deal with Problem (3), which had previously no known construction in the framework of derived deformation theory. As explained before, an R-deformation of a P-algebra A in the sense of (3) is a an R-linear P_∞-algebra $\tilde{A}\simeq A\otimes R$ with a $\mathbb{K}$-linear P_∞-algebra quasi-isomorphism $\tilde{A}\otimes_R\mathbb{K}\xrightarrow{\sim} A$, where $(-)\otimes_R\mathbb{K}$ is defined by the augmentation of R. We built a *derived formal group* $\widehat{\underline{haut}_{P_\infty}}(A)_{id}$ whose corresponding L_∞-algebra admits two equivalent descriptions

$$Lie(\widehat{\underline{haut}_{P_\infty}}(A)_{id})\simeq g^{\varphi}_{P,X}\ltimes_{hol} End(X)\simeq g^{varphi^+}_{P^+,X}$$

where the middle one exhibits this moduli problem as originating from the homotopy quotient of the space of P_∞-algebra structures on X by the homotopy action of self-quasi-isomorphisms $haut(X)$, and the right one explains how one can encode this explicitely as simultaneous compatible deformations of the P_∞-algebra structure *and* the differential of X.

Another way to compare deformation problems (2) and (3) is to recall that there are equivalences of ∞-categories

$$P_\infty - Alg[W_{qiso}^{-1}] \simeq P - Alg[W_{qiso}^{-1}] \simeq \infty - P_\infty - Alg[W_{\infty-qiso}^{-1}]$$

where the first equivalence is induced by the operadic quasi-isomorphism $P_\infty \xrightarrow{\sim} P$, and the second equivalence is induced by the strictification theorem of [63, Chapter 12], the later ∞-category being the one of P_∞-algebras with ∞-morphisms, and with ∞-quasi-isomorphisms as weak equivalences. Problem (3) concerns deformation theory in the ∞-category of P_∞-algebras *up to ∞-quasi-isomorphisms*, hence is a relaxed version of Problem (2) in this sense.

6.3. **Algebras over properads.** There is no well defined (homotopy invariant) notion of ∞-morphism of algebras over properads at present, though recent progresses have been made in [53]. So problem (2) does not make sense anymore in this more general setting. However, as we proved in the previous sections, problems (1) and (3) can be properly formalized and explicitely described by means of homotopy theory and derived algebraic geometry methods. One of the main additional difficulties when passing from operads to properads is the absence of model category structure on the corresponding kinds of algebras, which makes the situation more subtle to deal with both from the viewpoints of ∞-category theory and derived algebraic geometry.

7. Appendix: recollections on props, homotopical algebra and ∞-categories

The goal of this appendix is to briefly review several key notions and results from model categories and props that will be used in the present paper, as well as their homotopical algebra and associated ∞-categories.

7.1. **Symmetric monoidal categories over a base category.** Symmetric monoidal categories over a base category formalize how a given symmetric monoidal category can be tensored and enriched over another category, in a way compatible with the monoidal structure:

Definition 7.1. Let $\mathcal{C}$ be a symmetric monoidal category. A symmetric monoidal category over $\mathcal{C}$ is a symmetric monoidal category $(\mathcal{E}, \otimes_{\mathcal{E}}, 1_{\mathcal{E}})$ endowed with a symmetric monoidal functor $\eta : \mathcal{C} \to \mathcal{E}$, that is, an object under $\mathcal{C}$ in the 2-category of symmetric monoidal categories.

This defines on $\mathcal{E}$ an external tensor product $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ by $C \otimes X = \eta(C) \otimes_{\mathcal{E}} X$ for every $C \in \mathcal{C}$ and $X \in \mathcal{E}$. This external tensor product is equipped with the following natural unit, associativity and symmetry isomorphisms:

(1) $\forall X \in \mathcal{E}, 1_{\mathcal{C}} \otimes X \cong X$,
(2) $\forall X \in \mathcal{E}, \forall C, D \in \mathcal{C}, (C \otimes D) \otimes X \cong C \otimes (D \otimes X)$,
(3) $\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, C \otimes (X \otimes Y) \cong (C \otimes X) \otimes Y \cong X \otimes (C \otimes Y)$.

The coherence constraints of these natural isomorphisms (associativity pentagons, symmetry hexagons and unit triangles which mix both internal and external tensor products) come from the symmetric monoidal structure of the functor η.

We will implicitly assume throughout the paper that all small limits and small colimits exist in $\mathcal{C}$ and $\mathcal{E}$, and that each of these categories admit an internal hom bifunctor. We suppose moreover the existence of an external hom bifunctor $Hom_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \rightarrow \mathcal{C}$ satisfying an adjunction relation

$$\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, Mor_{\mathcal{E}}(C \otimes X, Y) \cong Mor_{\mathcal{C}}(C, Hom_{\mathcal{E}}(X, Y))$$

(so $\mathcal{E}$ is naturally an enriched category over $\mathcal{C}$).

Throughout this paper we will deal with symmetric monoidal categories equipped with a model structure. We assume that the reader is familiar with the basics of model categories. We refer to to Hirschhorn [52] and Hovey [51] for a comprehensive treatment of homotopical algebra. We just recall the axioms of symmetric monoidal model categories formalizing the interplay between the tensor and the model structures (in a word, these conditions ensure that the tensor product forms a Quillen bifunctor). From the point of view of ∞-categories, if a model category is equipped with a compatible symmetric monoidal structure (that is, satisfying the conditions below), then its associated ∞-category is symmetric monoidal as well (as an ∞-category).

Definition 7.2. (1) A symmetric monoidal model category is a symmetric monoidal category $\mathcal{C}$ equipped with a model category structure such that the following axioms holds:

MM0. For any cofibrant object X of $\mathcal{C}$, the map $Q1_{\mathcal{C}} \otimes X \rightarrow 1_{\mathcal{C}} \otimes X \cong X$ induced by a cofibrant resolution $Q1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ of the unit $1_{\mathcal{C}}$ is a weak equivalence.

MM1. The pushout-product $(i_*, j_*) : A \otimes D \oplus_{A \otimes C} B \otimes C \rightarrow B \otimes D$ of cofibrations $i : A \rightarrowtail B$ and $j : C \rightarrowtail D$ is a cofibration which is also acyclic as soon as i or j is so.

(2) Suppose that $\mathcal{C}$ is a symmetric monoidal model category. A symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$ is a symmetric monoidal model category over $\mathcal{C}$ if the axiom MM1 holds for both the internal and external tensor products of $\mathcal{E}$.

Example 7.3. The usual projective model category $Ch_{\mathbb{K}}$ of unbounded chain complexes over a field $\mathbb{K}$ forms a symmetric monoidal model category.

A useful property of the pushout-product axiom MM1 is that it is equivalent to the following standard dual version:

Lemma 7.4. *(cf.* [51, Lemma 4.2.2]*) In a symmetric monoidal model category $\mathcal{C}$, the axiom MM1 is equivalent to the following one:*

MM1'. *The morphism*

$$(i^*, p_*) : Hom_{\mathcal{C}}(B, X) \rightarrow Hom_{\mathcal{C}}(A, X) \times_{Hom_{\mathcal{C}}(A,Y)} Hom_{\mathcal{C}}(B, Y)$$

induced by a cofibration $i : A \rightarrowtail B$ and a fibration $p : X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$ which is also acyclic as soon as i or p is so.

7.2. **Props, properads and their algebras.** Props generalize operads, so that algebras over props can be defined by operations with multiple outputs, contrary to operads which parametrize only operations with one single output. In particular, they are adapted to the study of bialgebra-like structures. Properads are an intermediate object between operads and props, which are close enough to operads in the sense that they are defined, like operads, as monoids in a category of symmetric sequences (contrary to props), but are sufficient to encode many interesting bialgebra-like structures. One of the key feature of *properads* is that, contrary to props, there is a good theory of bar-cobar constructions and Koszul duality for them, allowing to get explicit resolutions in deformation theory of algebraic structures. We detail some of these ideas below.

7.2.1. *Props and their algebras.* Let $\mathcal{C}$ be a symmetric monoidal category. A Σ-biobject is a double sequence $\{M(m,n) \in \mathcal{C}\}_{(m,n)\in\mathbb{N}^2}$ where each $M(m,n)$ is equipped with a right action of Σ_m and a left action of Σ_n commuting with each other. We write $\mathcal{C}^{\mathbb{S}}$ for the category of Σ-biobjects in $\mathcal{C}$.

Definition 7.5. A prop is a Σ-biobject endowed with associative horizontal composition products

$$\circ_h : P(m_1,n_1)\otimes P(m_2,n_2)\to P(m_1+m_2,n_1+n_2),$$

associative vertical composition products

$$\circ_v : P(k,n)\otimes P(m,k)\to P(m,n)$$

and units $1\to P(n,n)$ which are neutral for $\circ_v$. These products satisfy the exchange law

$$(f_1\circ_h f_2)\circ_v (g_1\circ_h g_2) = (f_1\circ_v g_1)\circ_h (f_2\circ_v g_2)$$

and are compatible with the actions of symmetric groups. The elements of $P(m,n)$ are said to be of arity (m,n).

Morphisms of props are equivariant morphisms of collections compatible with the composition products.

There is a functorial free prop construction F leading to an adjunction

$$F : \mathcal{C}^{\mathbb{S}} \rightleftarrows Prop : U$$

where U is the forgetful functor. As for operads, there is a notion of ideal in a prop, so that one can define a prop by generators and relations. This approach is particularly useful considering the definition of algebras over a prop:

Definition 7.6. (1) To any object X of $\mathcal{C}$ we can associate an endomorphism prop End_X defined by

$$End_X(m,n) = Hom_{\mathcal{C}}(X^{\otimes m}, X^{\otimes n}).$$

(2) A P-algebra is an object $X\in\mathcal{C}$ equipped with a prop morphism $P\to End_X$.

Operations of P are sent to operations on tensor powers of X, and the compatibility of a prop morphism with composition products on both sides impose the relations that such operations satisfy. This means that given a presentation of a prop P by generators and relations, the P-algebra structure on X is determined by the images of these generators and their relations. Let us give some motivating examples related to our article:

Example 7.7. A differential graded associative and coassociative bialgebra is a triple (B,μ,Δ) such that:

(i) (B,μ) is a dg associative algebra;
(ii) (B,Δ) is a dg coassociative coalgebra;
(iii) the map $\Delta : B\to B\otimes B$ is a morphism of algebras and the map $\mu : B\otimes B\to B$ is a morphism of coalgebras.

The prop *Bialg* of associative-coassocative bialgebras is generated by two degree zero operations , one generator of arity $(2,1)$ and one generator of arity $(1,2)$, which corresponds to the operations μ and Δ above wether one specifies a prop morphism $Bialg \rightarrow End_B$. It is quotiented by the ideal generated the associativity relation, the coassociativity relation, and the compatibility relation describing the condition (iii) above.

In the unitary and counitary case, one adds a generator for the unit, a generator for the counit and the necessary compatibility relations with the product and the coproduct.

Example 7.8. Lie bialgebras originate from mathematical physics, in the study of integrable systems whose gauge groups are not only Lie groups but Poisson-Lie groups, see the seminal work of Drinfeld [17].

A differential graded Lie bialgebra is a triple $(\mathfrak{g}, [,], \delta)$ such that:

(i) $(\mathfrak{g}, [,])$ is a dg Lie algebra;
(ii) $(\mathfrak{g}, \delta)$ is a dg Lie coalgebra;
(iii) the cocycle relation : the coLie cobracket of a Lie bialgebra g is a cocycle in the Chevalley-Eilenberg complex $C^*_{CE}(g, \Lambda^2 g)$, where $\Lambda^2 g$ is equipped with the structure of g-module induced by the adjoint action.

The prop *BiLie* encoding Lie bialgebras is generated by one generator of arity $(2,1)$ and one generator of arity $(1,2)$, both of degree zero, and with the signature action of Σ_2 (that is, they are antisymmetric). It is quotiented by the ideal generated by the Jacobi relation, the co-Jacobi relation, and the cocycle relation.

We can also define a *P-algebra in a symmetric monoidal category over $\mathcal{C}$*:

Definition 7.9. Let $\mathcal{E}$ be a symmetric monoidal category over $\mathcal{C}$.

(1) The endomorphism prop of $X \in \mathcal{E}$ is given by $End_X(m,n) = Hom_{\mathcal{E}}(X^{\otimes m}, X^{\otimes n})$ where $Hom_{\mathcal{E}}(-,-)$ is the external hom bifunctor of $\mathcal{E}$.

(2) Let P be a prop in $\mathcal{C}$. A P-algebra in $\mathcal{E}$ is an object $X \in \mathcal{E}$ equipped with a prop morphism $P \rightarrow End_X$.

This definition will be useful, for instance, in the case where P is a dg prop (a prop in $Ch_{\mathbb{K}}$) but algebras over P lie in a symmetric monoidal category over $Ch_{\mathbb{K}}$.

To conclude, props enjoy nice homotopical properties. Indeed, the category of Σ-biobjects $\mathcal{C}^{\mathbb{S}}$ is a diagram category over $\mathcal{C}$, so it inherits the usual projective model structure of diagrams, which can be transferred along the free-forgetful adjunction:

Theorem 7.10. *(cf.* [33, Theorem 5.5]*) The category of dg props Prop equipped with the classes of componentwise weak equivalences and componentwise fibrations forms a cofibrantly generated model category.*

7.2.2. *Properads.* Composing operations of two Σ-biobjects M and N amounts to consider 2-levelled directed graphs (with no loops) with the first level indexed by operations of M and the second level by operations of N. Vertical composition by grafting and horizontal composition by concatenation allows one to define props as before. The idea of properads is to mimick the construction of operads as monoids in Σ-objects, by restricting the vertical composition product to connected graphs. The unit for this connected composition product $\boxtimes_c$ is the Σ-biobject I given by $I(1,1) = \mathbb{K}$ and $I(m,n) = 0$ otherwise. The category of Σ-biobjects then forms a symmetric monoidal category $(Ch_{\mathbb{K}}^{\mathbb{S}}, \boxtimes_c, I)$.

Definition 7.11. A dg properad (P,μ,η) is a monoid in $(Ch_{\mathbb{K}}^{\mathbb{S}},\boxtimes_c,I)$, where μ denotes the product and η the unit. It is augmented if there exists a morphism of properads $\epsilon : P \to I$. In this case, there is a canonical isomorphism $P \cong I \oplus \overline{P}$ where $\overline{P} = ker(\epsilon)$ is called the augmentation ideal of P.

Morphisms of properads are morphisms of monoids in $(Ch_{\mathbb{K}}^{\mathbb{S}},\boxtimes_c,I)$.

Properads have also their dual notion, namely coproperads:

Definition 7.12. A dg coproperad (C,Δ,ϵ) is a comonoid in $(Ch_{\mathbb{K}}^{\mathbb{S}},\boxtimes_c,I)$.

As in the prop case, there exists a free properad functor $\mathcal{F}$ forming an adjunction

$$\mathcal{F} : Ch_{\mathbb{K}}^{\mathbb{S}} \rightleftarrows Properad : U$$

with U the forgetful functor [94]. Dually, there exists a cofree coproperad functor denoted $\mathcal{F}_c(-)$ having the same underlying Σ-biobject. This adjunction equips dg properads with a cofibrantly generated model category structure with componentwise fibrations and weak equivalences [73]. The notion of algebra over a properad is similar to algebra over a prop since the endomorphism prop restricts to an endomorphism properad. Moreover, properads also form a model category for the same reasons as props:

Theorem 7.13. *(cf.* [73, Appendix A]*) The category of dg props Prop equipped with the classes of componentwise weak equivalences and componentwise fibrations forms a cofibrantly generated model category.*

Properads are general enough to encode a wide range of bialgebra structures such as associative and coassociative bialgebras, Lie bialgebras, Poisson bialgebras, Frobenius bialgebras for instance.

7.3. **Algebras and coalgebras over operads.** Operads are used to parametrize various kind of algebraic structures consisting of operations with one single output. Fundamental examples of operads include the operad *As* encoding associative algebras, the operad *Com* of commutative algebras, the operad *Lie* of Lie algebras and the operad *Pois* of Poisson algebras. Dg operads form a model category with bar-cobar resolutions and Koszul duality [63]. An algebra X over a dg operad P can be defined in any symmetric monoidal category $\mathcal{E}$ over $Ch_{\mathbb{K}}$, alternatively as an algebra over the corresponding monad $P(-) : Ch_{\mathbb{K}} \to Ch_{\mathbb{K}}$, which forms the free P-algebra functor, or as an operad morphism $P \to End_X$ where $End_X(n) = Hom_{\mathcal{E}}(X^{\otimes n}, X)$ and $Hom_{\mathcal{E}}$ is the external hom bifunctor.

Remark 7.14. There is a free functor from operads to props, so that algebras over an operad are exactly the algebras over the corresponding prop. Hence algebras over props include algebras over operads as particular cases.

Dual to operads is the notion of cooperad, defined as a comonoid in the category of Σ-objects. A coalgebra over a cooperad is a coalgebra over the associated comonad. We can go from operads to cooperads and vice-versa by dualization. Indeed, if C is a cooperad, then the Σ-module P defined by $P(n) = C(n)^* = Hom_{\mathbb{K}}(C(n),\mathbb{K})$ form an operad. Conversely, suppose that $\mathbb{K}$ is of characteristic zero and P is an operad such that each $P(n)$ is finite dimensional. Then the $P(n)^*$ form a cooperad in the sense of [63]. We also give the definition of *coalgebras over an operad*:

Definition 7.15. (1) Let P be an operad. A P-coalgebra is a complex C equiped with linear applications $\rho_n : P(n) \otimes C \to C^{\otimes n}$ for every $n \geq 0$. These maps are Σ_n-equivariant and associative with respect to the operadic compositions.

(2) Each $p \in P(n)$ gives rise to a cooperation $p^* : C \to C^{\otimes n}$. The coalgebra C is usually said to be conilpotent if for each $c \in C$, there exists $N \in \mathbb{N}$ so that $p^*(c) = 0$ when we have $p \in P(n)$ with $n > N$.

If $\mathbb{K}$ is a field of characteristic zero and the $P(n)$ are finite dimensional, then it is equivalent to define a P-coalgebra via a family of applications $\overline{\rho}_n : C \to P(n)^* \otimes_{\Sigma_n} C^{\otimes n}$.

7.4. **Homotopy algebras.** Given a prop, properad or operad P, a homotopy P-algebra, or P-algebra up to homotopy, is an algebra for which the relations are relaxed up to a coherent system of higher homotopies. this is encoded by

Definition 7.16. A homotopy P-algebra is an algebra over a cofibrant resolution P_∞ of P.

To make this definition meaningful, one has to prove that the notion of homotopy P-algebra does not depend (up to homotopy) on a choice of resolution:

Theorem 7.17. *(cf. [98]) A weak equivalence of cofibrant dg props $P_\infty \xrightarrow{\sim} Q_\infty$ induces an equivalence of the corresponding ∞-categories of algebras*

$$P_\infty - Alg[W_{qiso}^{-1}] \xrightarrow{\sim} Q_\infty - Alg[W_{qiso}^{-1}],$$

where $P_\infty - Alg[W_{qiso}^{-1}]$ denotes the ∞-categorical localization of $P_\infty - Alg$ with respect to its subcategory of quasi-isomorphisms.

Remark 7.18. Properads have a well defined theory of bar-cobar constructions and Koszul duality [94], which allows to produce explicit cofibrant resolutions of properads. The bar-cobar resolution is a functorial cofibrant resolution but of a rather big size, whereas the resolution obtained from the Koszul dual (when P is Koszul) is not functorial but smaller and better suited for computations.

These resolutions are of the form $P_\infty = (\mathcal{F}(V), \partial)$ where ∂ is a differential obtained by summing the differential induced by the Σ-biobject V with a certain derivation. To sum up, for a (pr)operad, one can always choose a homotopy P-algebra to be an algebra over a *quasi-free resolution of* P, in which the generators give the system of higher homotopies and the relations defining a strict P-algebra become coboundaries.

7.5. **Homotopy theory of cdgas and their modules.** Before getting to the heart of the subject, let us precise that, as usual in deformation theory and (derived) algebraic geometry, the commutative differential algebras (cdga for short) that we consider here are unital. That is, we consider the category of unital commutative monoids in the symmetric monoidal model category $Ch_{\mathbb{K}}$ and note it $CDGA_{\mathbb{K}}$. Such monoids enjoy many useful homotopical properties, as they form a *homotopical algebra context* in the sense of [89, Definition 1.0.1.11]. We will not list all the properties satisfied by cdgas, but here is a non-exhaustive one that will be useful in this article:

(1) The category $CDGA_{\mathbb{K}}$ forms a cofibrantly generated model category with fibrations and weak equivalences being the degreewise surjections and quasi-isomorphisms.

(2) Given a cdga A, its category of dg A-modules Mod_A forms a cofibrantly generated symmetric monoidal model category. The model structure is, again, right induced by the forgetful functor, and the tensor product is given by $- \otimes_A -$. In particular, we have a Quillen adjunction

$$(-) \otimes A : Ch_{\mathbb{K}} \leftrightarrows Mod_A : U$$

with a strong monoidal left adjoint (hence lax monoidal right adjoint). The unit η of this adjunction is defined, for any complex X, by

$$\begin{array}{rcl} \eta(X) : X & \to & X \otimes A \\ x & \longmapsto & x \otimes 1_A \end{array}$$

where 1_A is the unit element of A (the image of $1_{\mathbb{K}}$ by the unit map of A).

(3) Base changes are compatible with the homotopy theory of modules. Precisely, a morphism of cdgas $f : A \to B$ induces a Quillen adjunction

$$f_! : Mod_A \leftrightarrows Mod_B : f^*$$

where f^* equip a B-module with the A-module structure induced by the morphism f and $f_! = (-) \otimes_A B$. Moreover, if f is a quasi-isomorphism of cdgas then this adjunction becomes a Quillen equivalence.

(4) The category of augmented cdgas $CDGA_{\mathbb{K}}^{aug}$ is the category under $\mathbb{K}$ associated to $CDGA_{\mathbb{K}}$, so it forms also a cofibrantly generated model category. Moreover, this model category is *pointed* with $\mathbb{K}$ as initial and terminal object, so that one can alternately call them *pointed* cdgas. Let us note also that augmented unital cdgas are equivalent to non-unital cdgas $CDGA_{\mathbb{K}}^{nu}$ via the Quillen equivalence

$$(-)_+ : CDGA_{\mathbb{K}}^{nu} \leftrightarrows CDGA_{\mathbb{K}}^{aug} : (-)_-$$

where $A_+ = A \oplus \mathbb{K}$ for $A \in CDGA_{\mathbb{K}}^{nu}$ and A_- is the kernel of the augmentation map of A for $A \in CDGA_{\mathbb{K}}^{aug}$.

Example 7.19. There is a simplicial cdga called the *Sullivan cdga of polynomial forms on the standard simplices*. It is given by

$$\Omega_n := Sym\left(\bigoplus_{i=0}^{n} (\mathbb{K}t_i \oplus \mathbb{K}dt_i)\right) \Big/ \left(\begin{array}{c} t_0 + \cdots + t_n = 1 \\ dt_0 + \cdots + dt_n = 0 \end{array}\right) \tag{7.1}$$

which is the algebra of piecewise linear forms on the standard simplex Δ^n, the differential being defined, on the generators, by $d(t_i) = dt_i$. The simplicial structure is induced by the cosimplicial structure of $n \mapsto \Delta^n$, see [84] for details.

This cdga and its modules will be essential when considering formal moduli problems and simplicial resolutions in the core of the paper.

For any cdga A, the category Mod_A of left dg A-modules is a (cofibrantly generated) symmetric monoidal model category tensored over chain complexes. Therefore one can define the category $P_\infty - Alg(Mod_A)$ of P_∞-algebras in Mod_A, for any cofibrant prop P_∞ as in section 7.4 and Theorem 7.17 extends to this context.

An important subcategory of augmented cdgas is the one of artinian algebras, which are the coaffine formal moduli problems.

Definition 7.20. An augmented cdga A is Artinian if

- its cohomology groups $H^n(A)$ vanish for $n > 0$ and for $n << 0$, and each of them is finite dimensional over k;
- the (commutative) ring $H^0(A)$ is artinian in the standard meaning of commutative algebra.

We denote $dgArt_{\mathbb{K}}^{aug}$ the full subcategory of $CDGA_{\mathbb{K}}^{aug}$ of Artinian cdgas.

7.6. **Relative categories versus ∞-categories.** There are many equivalent ways to model ∞-categories. Precisely, there are several Quillen equivalent models for ∞-categories we can choose to work with [7], for instance quasi-categories [68], complete Segal spaces [82], simplicial categories [6], or relative categories [3, 4]. In this paper, it will often be convenient to consider ∞-functors which are associated to "naive" functors, provided-of course-that they preserve weak equivalences. This is not necessarily posible to do that in a straightforward naive way depending on the model chosen for ∞-categories. Therefore, here, we choose to work in the homotopy theory of relative categories as developed recently by Barwick-Kan [3, 4]. This will allow us to define more easily ∞-functors starting from classical constructions, instead of going through, for instance, the cartesian fibration/opfibration formalism of [68]. For the sake of clarity, we start by recalling the main features of this theory and refer to [3, 4] for more details. Then we state some technical lemmas that will help us to go from equivalences of relative categories to equivalences of ∞-categories.

7.6.1. *∞-categories associated to relative categories or model categories.* We now recall and compare various standard ways to construct ∞-categories.

Definition 7.21. A relative category is a pair of categories $(\mathcal{C}, W_{\mathcal{C}})$ such that $W_{\mathcal{C}}$ is a subcategory of $\mathcal{C}$ containing all the objects of $\mathcal{C}$. We call $W_{\mathcal{C}}$ the category of weak equivalences of C. A relative functor between two relative categories $(\mathcal{C}, W_{\mathcal{C}})$ and $(\mathcal{D}, W_{\mathcal{D}})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(W_{\mathcal{C}}) \subset W_{\mathcal{D}}$.

We note *RelCat* the category of relative categories and relative functors. By Theorem 6.1 of [3], there is an adjunction between the category of bisimplicial sets and the category of relative categories

$$K_\xi : sSets^{\Delta^{op}} \leftrightarrows RelCat : N_\xi$$

(where K_ξ is the left adjoint and N_ξ the right adjoint) which lifts any Bousfield localization of the Reedy model structure of bisimplicial sets into a model structure on *RelCat*. In the particular case of the Bousfield localization defining the complete Segal spaces [82], one obtains a Quillen equivalent homotopy theory of the homotopy theories in *RelCat* [3]. In particular, a morphism of relative categories is a weak equivalence if and only if its image under N_ξ is a weak equivalence of complete Segal spaces. We refer the reader to Section 5.3 of [3] for the definition of the functor N_ξ. Let us just mention that it is weekly equivalent to the classifying diagram functor N defined in [82], which is a key tool to construct complete Segal spaces.

A simplicial category is a category enriched over simplicial sets. We denote by *SCat* the category of simplicial categories. There exists functorial cosimplicial resolutions and simplicial resolutions in any model category ([21],[52]), so model categories provide examples of (weakly) simplicially enriched categories. One recovers the morphisms of the homotopy category from a cofibrant object to a fibrant object by taking the set of connected components of the corresponding simplicial mapping space. Another more general example is the simplicial localization developed by Dwyer and Kan [19]. To any relative category Dwyer and Kan associates a simplicial category $L(\mathcal{C},W_{\mathcal{C}})$ called its simplicial localization. They developed also another simplicial localization, the hammock localization $L^H(\mathcal{C},W_{\mathcal{C}})$ [20]. By taking the sets of connected components of the mapping spaces, we get $\pi_0 L(\mathcal{C},W_{\mathcal{C}})\cong \mathcal{C}[W_{\mathcal{C}}^{-1}]$ where $\mathcal{C}[W_{\mathcal{C}}^{-1}]$ is the localization of $\mathcal{C}$ with respect to $W_{\mathcal{C}}$ (i.e. the homotopy category of $(\mathcal{C},W_{\mathcal{C}})$). The simplicial and hammock localizations are equivalent in the following sense:

Proposition 7.22. *(Dwyer-Kan [20], Proposition 2.2) Let $(\mathcal{C},W_{\mathcal{C}})$ be a relative category. There is a zigzag of Dwyer-Kan equivalences*

$$L^H(\mathcal{C},W_{\mathcal{C}})\leftarrow diag L^H(F_*\mathcal{C},F_*W_{\mathcal{C}})\rightarrow L(\mathcal{C},W_{\mathcal{C}})$$

where $F_*\mathcal{C}$ is a simplicial category called the standard resolution of $\mathcal{C}$ (see [19] Section 2.5).

Let us precise the definition of Dwyer-Kan equivalences:

Definition 7.23. Let $\mathcal{C}$ and $\mathcal{D}$ be two simplicial categories. A functor $F:\mathcal{C}\rightarrow\mathcal{D}$ is a Dwyer-Kan equivalence if it induces weak equivalences of simplicial sets $Map_{\mathcal{C}}(X,Y)\stackrel{\sim}{\rightarrow} Map_{\mathcal{D}}(FX,FY)$ for every $X,Y\in\mathcal{C}$, as well as inducing an equivalence of categories $\pi_0\mathcal{C}\stackrel{\sim}{\rightarrow}\pi_0\mathcal{D}$.

Let us compile some useful results: first, every Quillen equivalence of model categories gives rise to a Dwyer-Kan equivalence of their simplicial localizations, as well as a Dwyer-Kan equivalence of their hammock localizations (see [21] Proposition 5.4 in the case of simplicial model categories and [50] in the general case). By Theorem 1.1 of [6], there exists a model category structure on the category of (small) simplicial categories with the Dwyer-Kan equivalences as weak equivalences. Every simplicial category is Dwyer-Kan equivalent to the simplicial localization of a certain relative category (see for instance [4], Theorem 1.7) and the associated model structure is also a homotopy theory of homotopy theories. The Reedy weak equivalences between two complete Segal spaces are precisely the Dwyer-Kan equivalences between their associated homotopy theories (Theorem 7.2 of [82]).

Therefore the ∞-category associated to a relative category is thus, equivalently, the ∞-category associated to its simplicial localization or the ∞-category associated to its corresponding complete Segal space. The same construction applies to turn relative functors into ∞-functors. Moreover, it can be made functorial. For instance, given a relative category $(\mathcal{C},W_{\mathcal{C}})$, the associated quasi-category is given by the composite $N_{coh}L^H(\mathcal{C},W_{\mathcal{C}})^f$, where $L^H(-)$ is the Dwyer-Kan localization functor, $(-)^f$ is a functorial fibrant resolution in the Bergner model structure [6], and N_{coh} is the coherent nerve. In the following, given a relative category (M,W), where W is the subcategory of weak equivalences, we will denote by $M[W^{-1}]$ its ∞-categorical localization.

7.6.2. *From relative categories to homotopy automorphisms.* We collect two useful lemmas to obtain equivalences between ∞-categories of algebras, and see under which conditions they induce equivalences between the formal moduli problems controlling deformations of such algebras:

Lemma 7.24. *Let $F:(\mathcal{C},W_{\mathcal{C}})\rightleftarrows(\mathcal{D},W_{\mathcal{D}}):G$ be an adjunction of relative categories (that is, the functors F and G preserves weak equivalences) such that the* *unit and counit of this adjunction are pointwise weak equivalences. Then F induces an equivalence of ∞-categories with inverse G.*

Proof. Let us denote by $RelCat$ the category of relative categories. The objects are the relative categories and the morphisms are the relative functors, that is, the functors restricting to functors between the categories of weak equivalences. By [3, Theorem 6.1], there is an adjunction between the category of bisimplicial sets and the category of relative categories

$$K_\xi : sSets^{\Delta^{op}} \leftrightarrows RelCat : N_\xi$$

(where K_ξ is the left adjoint and N_ξ the right adjoint) which lifts any Bousfield localization of the Reedy model structure of bisimplicial sets into a model structure on $RelCat$. In the particular case of the Bousfield localization defining the model category CSS of complete Segal spaces [82, Theorem 7.2], one obtains a Quillen equivalent homotopy theory of ∞-categories in $RelCat$ [3].

As recalled in 7.6.1, a way to build the ∞-category associated to a relative category $(\mathcal{C}, W_{\mathcal{C}})$ is to take a functorial fibrant resolution $N_\xi(\mathcal{C}, W_{\mathcal{C}})^f$ of the bisimplicial set $N_\xi(\mathcal{C}, W_{\mathcal{C}})$ in CSS to get a complete Segal space. So we want to prove that $N_\xi F^f$ is a weak equivalence of CSS. For this, let us note first that the assumption on the adjunction between F and G implies that F is a strict homotopy equivalence in $RelCat$ in the sense of [3]. By [3, Proposition 7.5 (iii)], the functor N_ξ preserves homotopy equivalences, so $N_\xi F$ is a homotopy equivalence of bisimplicial sets, hence a Reedy weak equivalence. Since CSS is a Bousfield localization of the Reedy model structure on bisimplicial sets, Reedy weak equivalences are weak equivalences in CSS, then by applying the fibrant resolution functor $(-)^f$ we conclude that $N_\xi F^f$ is a weak equivalence of complete Segal spaces. □

In the formalism of Dwyer-Kan's hammock localization, an equivalence of simplicial categories $F : \mathcal{C} \to \mathcal{D}$ satisfies in particular the following property: for every two objects X and Y of $\mathcal{C}$, it induces a weak equivalence of simplicial mapping spaces

$$L^H(\mathcal{C}, W_{\mathcal{C}})(X, Y) \xrightarrow{\sim} L^H(\mathcal{D}, W_{\mathcal{D}})(F(X), F(Y)).$$

(in particular, the associated functor $Ho(F)$ at the level of homotopy categories is an equivalence). We would like this weak equivalence to restrict at the level of homotopy automorphisms:

Lemma 7.25. *Let $F : (\mathcal{C}, W_{\mathcal{C}}) \rightleftarrows (\mathcal{D}, W_{\mathcal{D}}) : G$ be an adjunction of relative categories satisfying the assumptions of Lemma 7.24. Then the restriction of F to the subcategories of weak equivalences*

$$wF : W_{\mathcal{C}} \to W_{\mathcal{D}}$$

is an equivalence of simplicial localizations (actually an equivalence of ∞-groupoids) inducing a weak equivalence of homotopy automorphisms

$$L^H W_{\mathcal{C}}(X, X) \xrightarrow{\sim} L^H W_{\mathcal{D}}(F(X), F(X)),$$

where $L^H W_{\mathcal{C}}$ is Dwyer-Kan's hammock localization of $W_{\mathcal{C}}$ with respect to itself.

Proof. This adjunction of relative categories induces, by Lemma 7.24, an equivalence of simplicial localizations between $L^H(\mathcal{C}, W_{\mathcal{C}})$ and $L^H(\mathcal{D}, W_{\mathcal{D}})$. By construction, this implies that the simplicial categories $L^H W_{\mathcal{C}}$ and $L^H W_{\mathcal{D}}$ are equivalent as well. Alternately, one could say that an equivalence of ∞-categories induce an equivalence of the associated ∞-groupoids of weak equivalences. By definition of an equivalence of simplicial categories, we get the desired equivalence between the simplicial mapping spaces of $L^H W_{\mathcal{C}}$ and their images under F in $L^H W_{\mathcal{D}}$ (that is, an equivalence of homotopy automorphisms). □

References

[1] M. Anel, *Champs de modules des catégories linéaires et abéliennes*, PhD thesis, Université Toulouse III – Paul Sabatier (2006).

[2] M. Anel, A. Joyal, *Sweedler Theory for (co)algebras and the bar-cobar constructions*, arXiv:1309.6952.

[3] C. Barwick, D. M. Kan, *Relative categories:another model for the homotopy theory of homotopy theories*, Indagationes Mathematicae 23 (2012), 42-68.

[4] C. Barwick, D. M. Kan, *A characterization of simplicial localization functors and a discussion of DK equivalences*, Indagationes Mathematicae 23 (2012), 69-79.

[5] H.-J. Baues, *The cobar construction as a Hopf algebra*, Invent. Math. **132** (1998), no. 3, 467–489.

[6] J. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. 359 (2007), 2043-2058.

[7] J. Bergner, *A survey of* $(\infty, 1)$*-categories*, in J. Baez and J. P. May, Towards Higher Categories, IMA Volumes in Mathematics and Its Applications, Springer (2010), 69-83.

[8] A. Blanc, L. Katzarkov, P. Pandit, *Generators in formal deformations of categories*, Compositio Mathematica 154 (2018), 2055-2089.

[9] G. Borot, *Lecture notes on topological recursion and geometry*, preprint arXiv:1705.09986.

[10] , A. K. Bousfield, V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, memoir of the AMS 8 (1976).

[11] D. Calaque, T. Willwacher, *Triviality of the higher Formality Theorem*, Proc. Amer. Math. Soc. 143 (2015), no. 12, 5181–5193.

[12] D. Calaque, T. Pantev, B. Toen, M. Vaquié, G. Vezzosi, *Shifted Poisson structures and deformation quantization*, J. Topol. 10 (2017), 483–584.

[13] W. Chacholski, J. Scherer, *Homotopy theory of diagrams*, Mem. Amer. Math. Soc. 736 (2002).

[14] M. Chas, D. Sullivan, *String Topology*, preprint arXiv:math/9911159.

[15] M. Chas, D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, The legacy of Niels Henrik Abel, 771–784, Springer, Berlin, 2004.

[16] K. Cieliebak, K. Fukaya, J. Latscheev, *Homological algebra related to surfaces with boundary*, preprint arXiv:1508.02741.

[17] V. G. Drinfeld, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $Gal(\overline{Q}/Q$, Algebra i Analiz 2 (1990), 149-181. English translation in Leningrad Math. J. 2 (1991), 829-860.

[18] V. G. Drinfeld, Quantum groups, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, 798–820, Amer. Math. Soc., Providence, RI.

[19] W. G. Dwyer, D. M. Kan, *Simplicial localization of categories*, J. Pure and Applied Algebra 17 (1980), 267-284.

[20] W. G. Dwyer, D. M. Kan, *Calculating simplicial localizations*, J. Pure and Applied Algebra 18 (1980), 17-35.

[21] W. G. Dwyer, D. M. Kan, *Function complexes in homotopical algebra*, Topology 19 (1980), 427-440.

[22] W. G. Dwyer, D. M. Kan, *Homotopy theory and simplicial groupoids*, Nederl. Akad. Wetensch. Indag. Math., 46, (1984), 379 – 385.

[23] B. Enriquez, P. Etingof, *On the invertibility of quantization functors*, J. Algebra 289 (2005), no. 2, 321–345.

[24] B. Enriquez, G. Halbout, *Quantization of quasi-Lie bialgebras*, J. Amer. Math. Soc. 23 (2010), 611–653.

[25] B. Enriquez, G. Halbout, *Quantization of coboundary Lie bialgebras*, Ann. of Math. (2) 171 (2010), 1267–1345.

[26] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras I*, Selecta Math. (N. S.) 2 (1996), no. 1, 1-41.

[27] P. Etingof, D. Kazdhan, *Quantization of Lie bialgebras. II, III*, Selecta Math. (N.S.) 4 (1998), no. 2, 213–231, 233–269.
[28] J. Francis, *The tangent complex and Hochschild cohomology of E_n-rings*, Compositio Mathematica 149 (2013), no. 3, 430-480.
[29] J. Francis, D. Gaitsgory, *Chiral Koszul duality*, Selecta Math. New Ser. 18 (2012), 27-87.
[30] B. Fresse, *Koszul duality of operads and homology of partition posets*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, 115–215, Contemp. Math. 346 (2004).
[31] B. Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics 1967, Springer-Verlag (2009).
[32] B. Fresse, *Operadic cobar constructions, cylinder objects and homotopy morphisms of algebras over operads*, in Alpine perspectives on algebraic topology (Arolla, 2008), Contemp. Math. 504, Amer. Math. Soc. (2009), 125-189.
[33] B. Fresse, *Props in model categories and homotopy invariance of structures*, Georgian Math. J. 17 (2010), pp. 79-160.
[34] B. Fresse, *Koszul duality of En-operads*, Selecta Math. (N.S.) 17 (2011), 363–434.
[35] B. Fresse, *Iterated bar complexes of E-infinity algebras and homology theories*, Alg. Geom. Topol. 11 (2011), pp. 747-838.
[36] B. Fresse, *Homotopy of operads and Grothendieck-Teichmuller groups: Parts 1 and 2*, Mathematical Surveys and Monographs 217, American Mathematical Society, 2017, xl+534 pages (first volume), xxxi+704 pages (second volume).
[37] B. Fresse, T. Willwacher, *The intrinsic formality of E_n-operads*, arXiv:1503.08699, to appear in the Journal of the EMS.
[38] M. Gerstenhaber, S. D. Schack, *Algebras, bialgebras, quantum groups, and algebraic deformations*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 51-92, Contemp. Math., 134, Amer. Math. Soc., Providence, RI, 1992.
[39] E. Getzler, P. Goerss, *A model category structure for differential graded coalgebras*, preprint (1999).
[40] E. Getzler, J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint arXiv:hep-th/9403055.
[41] G. Ginot, *Homologie et modèle minimal des algèbres de Gerstenhaber*, Annales Mathématiques Blaise Pascal **11** (2004) no. 1, 95-127.
[42] G. Ginot, G. Halbout, *A Formality Theorem for Poisson Manifolds*, Lett. Math. Phys. 66 (2003), 37-64.
[43] G. Ginot, T. Tradler, M. Zeinalian, *A Chen model for mapping spaces and the surface product*, Ann. Sc. de l'Éc. Norm. Sup., 4e série, t. 43 (2010), p. 811-881.
[44] G. Ginot, T. Tradler, M. Zeinalian, *Higher Hochschild cohomology of E-infinity algebras, Brane topology and centralizers of E-n algebra maps*, preprint arXiv:1205.7056.
[45] G. Ginot, B. Noohi, *Group actions on stacks and applications to equivariant string topology for stacks*, preprint arXiv:1206.5603.
[46] G. Ginot, S. Yalin, *Deformation theory of bialgebras, higher Hochschild cohomology and formality*, preprint arXiv:1606.01504 (2016).
[47] V. Hinich, *DG coalgebras as formal stacks*, J. Pure Appl. Algebra 162 (2001), 209-250.
[48] V. Hinich, *Tamarkin's proof of Kontsevich formality theorem*, Forum Math. 15 (2003), 591–614.
[49] V. Hinich, *Deformations of homotopy algebras*, Communications in Algebra 32 (2004), 473-494.
[50] V. Hinich, *Dwyer-Kan localization revisited*, Homology Homotopy Appl. 18 (2016), 27-48.
[51] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs volume 63, AMS (1999).
[52] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs volume 99, AMS (2003).
[53] E. Hoffbeck, J. Leray, B. Vallette, *Properadic homotopical calculus* , arXiv:1910.05027.
[54] J. F. Jardine, *Stacks and the homotopy theory of simplicial sheaves*, Homology Homotopy Appl. 3 (2001), 361-384.
[55] A. Kapustin, *Topological field theory, higher categories, and their applications*, in *Proceedings of the International Congress of Mathematicians. Volume III*, Hindustan Book Agency, New Delhi (2010), 2021–2043.

[56] B. Keller, *A-infinity algebras, modules and functor categories*, in Trends in representation theory of algebras and related topics, Contemp. Math. 406 (2006), 67-93.
[57] B. Keller and W. Lowen, *On Hochschild cohomology and Morita deformations*, Int. Math. Res. Not. IMRN **2009**, no. 17, 3221–3235.
[58] M. Kontsevich, *Operads and motives in deformation quantization*, Moshé Flato (1937-1998), Lett. Math. Phys. 48 (1999), no. 1, 35-72.
[59] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), no. 3, 157-216.
[60] M. Kontsevich and Y. Soibelman, Notes on A_∞-algebras, A_∞-categories and non-commutative geometry, in *Homological mirror symmetry*, 153–219, Lecture Notes in Phys., 757, Springer, Berlin.
[61] M. Kontsevich, Y. Soibelman, *Airy structures and symplectic geometry of topological recursion*, preprint arXiv:1701.09137.
[62] P. Lambrechts, and I. Volic, *Formality of the little N-disks operad*, Mem. Amer. Math. Soc. 230 (1079), 2014.
[63] J-L. Loday, B. Vallette, *Algebraic Operads*, Grundlehren der mathematischen Wissenschaften, Volume 346, Springer-Verlag (2012).
[64] J. Lurie, *Derived Algebraic Geometry VI: E_k algebras*, available at http://www.math.harvard.edu/ lurie/
[65] J. Lurie, *Derived Algebraic Geometry IX: Closed Immersions*, available at http://www.math.harvard.edu/ lurie/
[66] J. Lurie, *Derived Algebraic Geometry X: Formal Moduli Problems*, available at http://www.math.harvard.edu/ lurie/
[67] J. Lurie, *Derived Algebraic Geometry XIV: Representability Theorems*, available at http://www.math.harvard.edu/ lurie/
[68] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton University Press, Princeton, NJ, 2009.
[69] J. Lurie, *Higher Algebra*, September 2017 version, book available at http://www.math.harvard.edu/ lurie/
[70] S. MacLane, *Categorical algebra*, Bull. Amer. Math. Soc. Volume 71, Number 1 (1965), 40-106.
[71] J. P. May, *The geometry of iterated loop spaces*, Springer, Berlin, 1972.
[72] S. Merkulov, B. Vallette, *Deformation theory of representation of prop(erad)s I*, J. für die reine und angewandte Math. (Crelles Journal), Issue 634 (2009), 51-106.
[73] S. Merkulov, B. Vallette, *Deformation theory of representation of prop(erad)s II*, J. für die reine und angewandte Math. (Crelles Journal), Issue 636 (2009), 125-174.
[74] S. Merkulov, *Prop profile of Poisson geometry*, Comm. Math. Phys. 262 (2006), 117–135.
[75] S. Merkulov, *Formality theorem for quantization of Lie bialgebras*, Lett. Math. Phys. 106 (2016), no. 2, 169–195.
[76] S. Merkulov, T. Willwacher *Classification of universal formality maps for quantizations of Lie bialgebras*, arXiv:1605.01282
[77] T. Nikolaus, U. Schreiber, D. Stevenson, *Principal ∞-bundles: general theory*, J. Homotopy Relat. Struct. 10 (2015), 749-801.
[78] T. Nikolaus, U. Schreiber, D. Stevenson, *Principal ∞-bundles: presentations*, J. Homotopy Relat. Struct. 10 (2015), 565-622.
[79] Pr A. Preygel, *Thom-Sebastiani and Duality for Matrix Factorizations, and Results on the Higher Structures of the Hochschild Invariants*, Thesis (Ph.D.), M.I.T. 2012.
[80] J. P. Pridham, *Unifying derived deformation theories*, Adv. Math. 224 (2010), 772–826.
[81] C. W. Rezk, *Spaces of algebra structures and cohomology of operads*, Thesis, MIT, 1996.
[82] C. W. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. 353 (2001), 973–1007.
[83] S. Schwede, B. Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. 3 (2003), 287–334.
[84] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977), 269–331.
[85] D. Tamarkin, *Another proof of M. Kontsevich formality theorem*, preprint arXiv:math/9803025, 1998.
[86] D. Tamarkin. *Formality of chain operad of little discs*, Lett. Math. Phys. 66 (2003), 65–72.

[87] D. Tamarkin, *Deformation complex of a d-algebra is a (d+1)-algebra*, preprint arXiv:math/0010072.
[88] B. Toën, G. Vezzosi, *Homotopical Algebraic Geometry I: Topos theory*, Adv. in Math. 193 (2005), 257-372.
[89] B. Toën, G. Vezzosi, *Homotopical algebraic geometry II. Geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.
[90] B. Toën, M. Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. de l'ENS 40 (2007), 387-444.
[91] B. Toën, *Derived algebraic geometry*, EMS Surv. Math. Sci. 1 (2014), no. 2, 153–240.
[92] B. Toën, *Derived Algebraic Geometry and Deformation Quantization*, ICM lecture.
[93] B. . Toën, *Problèmes de modules formels*, Séminaire BOURBAKI 68ème année, 2015-2016, no 1111.
[94] B. Vallette, *A Koszul duality for props*, Trans. Amer. Math. Soc. 359 (2007), 4865-4943.
[95] S. Yalin, *The homotopy theory of bialgebras over pairs of operads*, J. Pure Appl. Algebra 218(2014), 973–991.
[96] S. Yalin, *Simplicial localization of homotopy algebras over a prop*, Math. Proc. Cambridge Philos. Soc. 157 (2014), no. 3, 457–468.
[97] S. Yalin, *Maurer-Cartan spaces of filtered L_∞-algebras*, J. Homotopy Relat. Struct. 11 (2016), no. 3, 375–407.
[98] S. Yalin, *Moduli stacks of algebraic structures and deformation theory*, J. Noncommut. Geom. 10 (2016), 579-661.
[99] S. Yalin, *Function spaces and classifying spaces of algebras over a prop*, Algebraic and Geometric Topology 16 (2016), 2715–2749.