# Goldman bracket for 2-dimensional orbifolds

Grégory Ginot* Behrang Noohi†

May 13, 2017

**Abstract**

A Goldman bracket for oriented differentiable stacks of dimension 2 was defined in our previous paper, using string topology techniques. Here we specialized to the case of 2-dimensional orbifolds. We construct a stack analogue of the character variety, which has a symplectic (or Poisson) coarse moduli space and then construct a Lie algebra homomorphism from the Goldman Lie algebra to its function for orbifolds. We also prove that for orbifolds obtained as a quotient of the hyperbolic plane by a Fuchsian group, our Goldman bracket agrees with Chas-Gadgil ones and as a corollary one obtains that the Goldman Lie bracket of orbifolds encodes the geometric intersection numbers of the orbifold. Finally we construct a Chas-Sullivan type generalization of Goldman Lie algebra of unoriented curves for all oriented stacks (in particular manifolds) and extend Goldman Lie algebra homomorphism for unoriented curves to the case of surface orbifolds. Our main tool is the naturality of Chas-Sullivan construction with respect to embeddings.

# Contents



*Sorbonne Universités, UPMC Univ Paris 06, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Univ Paris Diderot, Sorbonne Paris Cité, F-75005, Paris, France

†School of Mathematical Sciences, Queen Mary, Univ. of London, Mile End Road, London E1 4NS, United Kingdom

# Introduction

Goldman Lie algebra [Go1] is a beautiful and intriguing structure found in the eighties on linear combinations of (free homotopy classes of) curves in an oriented surface. It combines the usual composition of loops with the same base point with the intersection product inside the surface. This Lie algebra relates deeply the topology of surface and combinatorics of curves on it with representation theory, precisely the symplectic structure of character varieties $Hom(\pi_1(\Sigma), G)/G$ (which also describes the moduli space of flat connections on $G$-bundles over $\Sigma$).

Later, in their seminal work on string topology, Chas-Sullivan [ChSu] generalized the Goldman bracket (for oriented curves) to all oriented manifolds and in all homological degrees. Precisely, they proved that the $S^1$-equivariant homology $H_*^{S^1}(LM)$ of the free loop space $LM = \mathrm{Map}(S^1, M)$ of an oriented manifold has a Lie bracket of degree $2-\dim(M)$ which, in degree 0 for $M = \Sigma$ a surface, is precisely the Goldman bracket. This result is obtained by mixing the Batalin-Vilkovisky algebra structure of $H_*(LM)[\dim(M)]$ discovered by Chas-Sullivan with the Gysin sequence in equivariant homology. An interpretation of the even part of this Lie algebra for even dimensional manifolds in terms of the symplectic structure of moduli space of flat connections was studied in [AZ].

In [GiNo], we generalized Chas-Sullivan Lie algebra to the case of arbitrary oriented (differentiable) stacks, building on our work [BGNX] on operations for (co)homology of topological stacks and string topology for stacks. In particular, for a stack of dimension 2, the degree 0 equivariant homology $H_0^{S^1}(L\mathfrak{X})$ is a Lie algebra which we call *Goldman Lie algebra of* $\mathfrak{X}$. The primary goal of this paper is to study the Goldman algebra for dimension 2 orbifolds as well as its relation with character varieties. Our strategy is to take advantage of the particularly simple form of surface orbifolds as well as the functoriality of Goldman Lie algebra of a stack with respect to open embeddings that we proved in [GiNo] (and is recalled in Proposition 2.2). A nice feature of surface orbifolds is that one can study this bracket in terms of geometry of the coarse moduli space and simple group theoretic argument. Further, the bracket can be studied in terms of immersed curves on the moduli space away of the orbifold locus which provides a rather geometric understanding of the structure.

At the same time as our preprint [GiNo], Chas-Gadgil [ChGa] have also defined a generalization of Goldman bracket for specific orbifolds obtained as quotient $[G\backslash\mathbb{H}]$ of the hyperbolic plane by a Fuchsian group $G$ (in particular these orbifolds are reductive). Their definition is very different from ours and defined purely in group theoretic terms taking somehow advantage of the simple connexity of $\mathbb{H}$. We proved in Section 2 (see Theorem 3.2 for a more precise statement) that our bracket is isomorphic with Chas-Gadgil ones.

**Theorem 0.1** *The Chas-Gadgil Lie algebra of* $[G\backslash\mathbb{H}]$ *is canonically isomorphic to the Lie algebra on* $H_0^{S^1}(L[G\backslash\mathbb{H}])$ *given by the Chas-Sullivan type construction from [GiNo].*

Chas-Gadgil construction was in fact motivated by the study of intersection of immersed curves in a surface, for which Goldman bracket is a very useful tool. In fact, Goldman already showed in his original paper [Go2] that, for homotopy classes of loops, one of which having a simple representative, vanishing of the bracket was equivalent to the curves admitting disjoint representatives. This is one instance of the fact that the Goldman bracket does reflect very accurately the geometric intersection of free loops, that is the minimal number of intersections between representatives. In their recent preprint, Chas-Gadgil [ChGa] proved that the geometric intersection of curves is indeed given by the number of terms (with multiplicities) in the Goldman bracket (asymptotically). Using Chas-Gadgil result, the previous theorem and the fact that there is an epimorphism of Lie algebras from the Goldman Lie algebra of the complement $\mathfrak{U}$ of the orbifold locus onto the one of $\mathfrak{X}$, we proved that the main result of [ChGa] also holds for all (non-necessarily reduced) oriented 2-dimensional orbifolds.

**Theorem 0.2** *The geometric intersection number of two curves* $\alpha$ *and* $\beta$ *in* $\mathfrak{X}$ *is equal to the number of terms (counted with multiplicities) of the Goldman bracket* $\dfrac{1}{pq}[\alpha^p, \beta^q]$ *for all but finitely many* $q \neq pc$.

A similar statement holds for self-intersection of a curve (Theorem 4.5).

One of the main motivation behind Goldman discovery of the bracket on free homotopy classes of curves was the study of the Poisson bracket of character variety. For an oriented surface $\Sigma$ and algebraic group $G$, the *character variety* $\chi_{\Sigma,G}$ is the quotient space of $Hom(\pi_1(\Sigma), G)$ by the adjoint action of $G$ and can be identified with the coarse moduli space of flat connections on a $G$-principal bundle $P$ over $\Sigma$. The (smooth part) of character varieties carries an important symplectic (or Poisson if the surface is not closed) structure (see [Go1, Au, BiGu] for instance).

One of the main result of Goldman [Go1] is that this symplectic structure in the case of classical matrix Lie groups can be studied in terms of the Goldman Lie algebra of curves. More precisely, Goldman proved that if $G = GL_n(\mathbb{K})$ (seen as a subgroup of $GL_{n\cdot\dim(\mathbb{K})}(\mathbb{R})$ in the standard way), the the Trace

$$Tr : GL_n(\mathbb{K}) \subset GL_{n\cdot\dim(\mathbb{K})}(\mathbb{R}) \to \mathbb{R}$$

induces a *Lie algebra homomorphism*

$$Tr_*([\alpha]) = \big(\varphi \mapsto Tr\big(\varphi(\alpha)\big)\big) \tag{0.1}$$

from the Goldman Lie algebra to the (Poisson algebra of) functions $\mathcal{O}_{\chi_{\Sigma,G}}$ on the character variety.

We generalize this result for orbifolds as follows. First, we define in Section 5 a smooth stack $\mathfrak{Ch}_{\mathfrak{Y},G}$ of flat connection on principal $G$-bundles over an arbitrary differentiable stack $\mathfrak{Y}$. Then we define the character variety $\chi_{\mathfrak{Y},G}$ of the stack $\mathfrak{Y}$ to be the coarse moduli space of $\mathfrak{Ch}_{\mathfrak{Y},G}$. We prove that for *an orbifold*, this stack inherits a symplectic (or Poisson) structure. The restriction to orbifolds is crucial for the following reason: orbifolds are differentiable stacks $\mathfrak{X}$ whose tangent and cotangent complexes are actually *vector bundles* over $\mathfrak{X}$. In particular, forms on on orbifold can be described simply in terms of invariant one forms on an atlas of the orbifold. This allows to essentially define symplectic orbifold as a kind of equivariant notion of the usual definition see [LeMa]. Studying symplectic structures for general differentiable stack[1] seems to require to work in a broader (and much more complicated) context of derived geometry differentiable stacks. With that input, we prove that Goldman trace maps is well-defined for orbifolds as well and we prove

**Theorem 0.3** *Let $\mathfrak{Y}$ be a connected oriented effective orbifold of dimension 2 and $G = GL_n(\mathbb{K})$ (with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$). The map $Tr_* : H_0^{S^1}(L\mathfrak{Y}, \mathbb{Z}) \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is a Lie algebra homomorphism.*

The above trace map from the Goldman Lie algebra for surfaces is, however, not a Lie algebra homomorphism for other groups, for instance for orthogonal groups and in particular does not capture interesting bracket of the Poisson structure of the character variety. To accommodate the important orthogonal group case, Goldman in his seminal paper [Go2] also defines a Lie algebra structure on the linear combinations of (free homotopy classes of) *unoriented* curves (but still in an oriented surface), which is the quotient of the of $H_0(LM)$ by the $\mathbb{Z}/2\mathbb{Z}$ action reversing orientation of loops. Goldman proves that one can also define a Lie algebra structure on the free module, denoted $\mathbb{Z}\overline{\pi}$, of unoriented strings, by setting

$$\{\overline{\alpha}, \overline{\beta}\} := \sum_{p \in \alpha \pitchfork \beta} \mathrm{sign(p)}\big(\overline{\alpha \bullet_p \beta} - \overline{\alpha \bullet_p \beta^{-1}}\big)$$

[1]not to be mistaken with symplectic groupoids

where $p$ runs through (transverse) intersection points; the sign is the one of the intersection of the curves, and $\alpha \bullet_p \beta$ means the composition of the loops $\alpha$ and $\beta$ based at $p$.

Goldman then proved that the above trace map induces on the quotient a well-defined Lie algebras homomorphism from $\mathbb{Z}\overline{\pi}$ to the functions on the character variety associated to an orthogonal or spin group.

From the point of view of equivariant homology, one can see the underlying module $\mathbb{Z}\overline{\pi}$ as the $O(2)$-equivariant homology $H_0^{O(2)}(LM)$ with respect to the $O(2)$-action on the loops. This raises the question of defining a Chas-Sullivan type generalization of the unoriented Goldman Lie algebras for all oriented manifolds, orbifolds or even general differentiable stacks.

We prove (Theorem 6.1) that the **BV**-algebra structure on the homology free loop space, combined with transfer homomorphisms we studied in [GiNo], provides a Lie algebra structure on $H_*^{O(2)}(LM)[2-\dim(M)]$.

This result applies to all oriented stacks, and in particular to oriented manifolds, for which it is new as far as we know. Though we are primarily interested in orbifolds, we prove the results for arbitrary stacks since the proof is not harder in this generality, relying on the machinery of operations in (co)homology which was established in this general context in [BGNX].

For dimension 2-stacks we thus get a Lie algebra structure on $H_0^{O(2)}(\mathrm{L}\mathfrak{X})$ the free module generated by unoriented strings. In homological degree 0, we have further an identification of $H_0^{O(2)}(LM)$ with the image $p_*(H_0^{S^1}(LM))$ along the natural projection. We prove that the Lie bracket on unoriented strings can be refined on all $p_*(H_*^{S^1}(\mathrm{L}\mathfrak{X})$ yielding another Lie algebra $(p_*(H_0^{S^1}(LM), \widetilde{\{-,-\}})$:

**Theorem 0.4** *Let $\mathfrak{X}$ be an oriented manifold, orbifold or differentiable stack For $x, y \in p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X},\mathbb{Z})\big) \subset H_*^{O(2)}(\mathrm{L}\mathfrak{X},\mathbb{Z})$ the formula*

$$\widetilde{\{x,y\}} := p_*\big(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\}\big),$$

*where $p_*^{-1}(y)$ is any pre-image of $y$ by $p_*$, is well-defined and makes the (sub)space $p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X},\mathbb{Z})\big)[2-d])$ a Lie algebra.*

For a 2-dimensional oriented stack, we thus get a Lie algebra structure in homological degree 0, which we call the *Goldman Lie algebra of unoriented strings.*

We relate (see Proposition 6.9) the various Lie algebra structures on $H_*^{S^1}(\mathrm{L}\mathfrak{X})$ and $H_*^{O(2)}(\mathrm{L}\mathfrak{X})$ and show that the Goldman bracket of unoriented strings indeed refine the unoriented Chas Sullivan bracket $\{-,-\}_{O(2)}$. We then extend the Goldman homomorphim for orthogonal group to surface orbifolds.

**Theorem 0.5** *Let $\mathfrak{Y}$ be a connected oriented effective orbifold of dimension 2 and $G = O_n(\mathbb{K})$, $O_{p,q}$, $U_{p,q}$, $Sp_{p,q}$, $Sp_n(\mathbb{R})$ or $Sp_n(\mathbb{C})$. The linear map*

$$Tr_* : \Big(H_0^{O(2)}(L\mathfrak{Y},\mathbb{Z}), \widetilde{\{-,-\}}\Big) \longrightarrow \mathcal{O}_{\chi_{\mathfrak{Y},G}}$$

*is a Lie algebra homomorphism.*

*Plan of the paper:* We recall the (string topology type) definition of the Goldman algebra for stacks along with its functoriality properties in Section 1, as well as some basic facts and definitions for stacks. We prove in Section 2 that Chas-Gadgil Lie bracket is a special case of the Goldman bracket for general orbifolds. In section 3, we use the functoriality property of the Goldman bracket and Chas-Gadgil main result to study the geometric intersection of curves on an orbifold. In section 4, we define the character variety of a stack, its symplectic structure and extend Goldman homomorphism to orbifolds. In Section 5, we define two main Lie algebras structures on unoriented strings for all arbitrary stacks (even though we are motivated by the orbifold case only). We then extend Goldman homomorphism for orthogonal groups to orbifolds.

**Notations.** For stacks and group actions we will mostly follow the notation of [GiNo]. Throughout the paper we will consider (co)homology with coefficients in $k$, a commutative unital ring. If $G$ is a group and $g \in G$, we denote $g^h = hgh^{-1}$ its conjugate by $h \in G$. If $G$, $H$ are Lie groups, we denote $\mathrm{Hom}(G, H)$ the space of group homomorphisms form $G$ to $H$.

# 1 Background on stacks

## 1.1 Generalities

For a quick review of the result on stack in the style that will be used in this paper we refer the reader to the prequel [GiNo], Sectons 2 and 3. For the convenience of the reader we very briefly recall some of the main definitions and facts on stacks.

By a *topological stack* $\mathfrak{X}$ we mean a stack, over the Grothendieck site $\mathsf{CGTop}$ of compactly generated topological spaces, that is of the form $\mathfrak{X} \cong [R\backslash X]$ for some topological groupoid $[R \rightrightarrows X]$. In the case of a differentiable (or Lie) groupoid, we say that $\mathfrak{X}$ is a *differentiable stack*. In particular if a topological (resp. Lie) group $G$ acts continuously (resp. smoothly) on a space (resp. manifold) $X$, then we have the topological (resp. differentiable) stack $[G\backslash M]$ which is the quotient stack associated to the transformation groupoid $X \times G \rightrightarrows X$ where the source and target maps are the projection and the action; the groupoid structure is induced by the group structure of $G$. The classifying stack of a topological group $G$ is defined to be $\mathfrak{B}G := [G\backslash \bullet]$ where $\bullet$ is a point.

When the groupoid $[R \rightrightarrows X]$ is étale (i.e., the source and target maps are local homeomorphisms) and the diagonal $R \to X \times X$ is proper, the stack $\mathfrak{X}$ is called an *orbifold* (or *Deligne-Mumford*). Every (differentiable) orbifold is locally of the form $[G\backslash \mathbb{R}^n]$, where $G$ is a finite group acting (smoothly) on $\mathbb{R}^n$; reciprocically one can define orbifolds purely in terms of charts of that type (see [ALR, Mo]).

Every stack $\mathfrak{X}$ (on $\mathsf{CGTop}$) has an associated *coarse moduli space*, denoted $\mathfrak{X}_{\mathrm{coarse}}$, together with a map $\mathfrak{X} \to \mathfrak{X}_{\mathrm{coarse}}$ which is universal among maps from $\mathfrak{X}$ to topological spaces. When $\mathfrak{X} \cong [G\backslash X]$ is a quotient stack of a space by a group action, then the coarse moduli space is simply the orbit space $G\backslash X$ given by the ordinary quotient. More generally the coarse moduli space of a topological stack isomorphic to the quotient $[R\backslash X]$ of a topological groupoid is the quotient space of $X$ by the equivalence relation defined by $R$.

There are various different ways to define *(co)homology of a topological stack*. One can use simplicial resolutions as in [Beh1], classifying space as in [No2], or singular chains as in [CoNo]. A *classifying space* $X \to \mathfrak{X}$ is a (necessarily representable) map $X \to \mathfrak{X}$ from a topological space $X$ which has the property that its base extension to any topological space $T \to \mathfrak{X}$ is a weak equivalence $X_T \to T$ of topological spaces. The weak homotopy type of $\mathfrak{X}$ (in particular, its homotopy, homology, and cohomology groups) are defined to be those of $X$. It is shown in [No2] that the weak homotopy type is well-defined up to a unique isomorphism and is functorial in $\mathfrak{X}$. Further, classical (co)homological operations extend to topological stacks [BGNX], in particular to orbifolds. The notion of *orientation* of a differentiable stack is a generalization of the definition of orientation defined in terms of orientation (homology) class of the normal bundle of the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, see [BGNX]. For an orbifold as we mainly use here, this definition boils down to asking that the transition maps between the local charts $[G\backslash\mathbb{R}^n]$ are orientation preserving (Section 4.1).

For a compact topological space $Y$, the mapping stack $\mathrm{Map}(Y, \mathfrak{X})$, which can in fact be defined for arbitrary stacks $Y$ and $\mathfrak{X}$, turns out to be a topological stack. This is one of the main results of [No1], where it is also shown that $\mathrm{Map}(Y, \mathfrak{X})$ has the same weak homotopy type as the usual mapping space $\mathrm{Map}(Y, X)$, where $X \to \mathfrak{X}$ is a classifying space for $\mathfrak{X}$. (In fact, the weak equivalence is induced by the natural map $\mathrm{Map}(Y, X) \to \mathrm{Map}(Y, \mathfrak{X})$.) The case we are interested in is $Y = S^1$. In this case, we denote $\mathrm{Map}(S^1, \mathfrak{X})$ by $\mathrm{L}\mathfrak{X}$ and call it the *loop stack* of $\mathfrak{X}$. By functoriality, there is a natural $S^1$-action on $\mathrm{L}\mathfrak{X}$ (see Section 1.2 below).

## 1.2 Group actions on stacks

In [GiNo], we studied (weak) actions of a group on topological (or geometric) stacks following the work of [Ro]. We proved that if $G$ is a topological (resp., Lie) group acting on a topological (resp., differentiable) *stack* $\mathfrak{X}$, then, there is a topological (resp., differentiable) stack $[G\backslash\mathfrak{X}]$. An important class of examples of $G$-stacks is obtained as follows. If $Y$ is a compact space with a $G$-action, and $\mathfrak{X}$ a topological stack, then the mapping stack $\mathrm{Map}(Y, \mathfrak{X})$ inherits a canonical $G$-action.

Let $\mathfrak{Y}$ be a (topological) stack endowed with an action of the Lie group $O(2)$. Then it inherits a $S^1 = SO(2)$-action and we have a topological stack $[S^1\backslash\mathfrak{Y}]$.

We record the following lemma for future reference.

**Lemma 1.1** *Let $\mathfrak{Y}$ be a (topological) stack endowed with an action of the Lie group $O(2)$. The group $\mathbb{Z}/2\mathbb{Z} \cong S^1\backslash O(2)$ acts canonically on $[S^1\backslash \mathrm{L}\mathfrak{X}]$ and there is an natural isomorphism of (topological) stacks*

$$[\mathbb{Z}/2\mathbb{Z}\backslash[S^1\backslash \mathrm{L}\mathfrak{X}]] \cong [O(2)\backslash \mathrm{L}\mathfrak{X}]$$

*where $O(2)$ acts on $L\mathfrak{X}$ via its action on $S^1 \subset \mathbb{R}^2$*

Proof. it follows from the exact sequence of groups $S^1 \to O(2) \to \mathbb{Z}/2\mathbb{Z}$ which exhibits $O(2)$ as a semi-direct product. $\square$

**Example 1.2** The example we are interested in is the free loop stack $\mathrm{L}\mathfrak{X} = \mathrm{Map}(S^1, \mathfrak{X})$ of a topological stack $\mathfrak{X}$. It is endowed with an $O(2)$ action induced by the action of $O(2)$ on $S^1 = \{z \in \mathbb{C}, |z| = 1\}$, see [GiNo] for details (especially on the induced $S^1$-action on $\mathrm{L}\mathfrak{X}$). The resulting action of $\mathbb{Z}/2\mathbb{Z}$ on $S^1$ is given by $z \mapsto \overline{z}$. On the free loop space, it simply maps a loop to the same loop but going backward.

We recall that if $\mathfrak{X} \cong [G\backslash X]$ is the quotient stack of a topological group action on a space, then the (co)homology of $\mathfrak{X}$ is canonically equivalent to the $G$-equivariant (co)homology of $X$. If the group acts on a stack, then we can use the homotopy type of the topological quotient stack to define similarly equivariant (co)homology.

**Definition 1.3 ([GiNo])** Let $G$ be a group acting on a stack $\mathfrak{X}$. The $G$-equivariant homology $HH_*^G(\mathfrak{X})$ (resp. cohomology $HH_*^G(\mathfrak{X})$) of $\mathfrak{X}$ is defined to be the homology (resp. cohomology) of the quotient stack $[G\backslash\mathfrak{X}]$ :

$$H_*^G(\mathfrak{X}) := H_*([G\backslash\mathfrak{X}]), \qquad H^*_G(\mathfrak{X}) := H^*([G\backslash\mathfrak{X}]).$$

# 2 The generalized Goldman Lie algebra of an oriented differentiable stack

In this section we quickly recall some basic results about the generalized Goldman bracket for a differentiable stack $\mathfrak{X}$. We are mainly interested in the case where $\mathfrak{X}$ is a 2-dimensional orbifold.

In [BGNX, GiNo], we proved that, if $\mathfrak{X}$ is a differentiable (or Hurewicz) oriented stack (for instance an oriented orbifold),

- the shifted homology $H_*(\mathrm{L}\mathfrak{X})[1 - \dim(\mathfrak{X})]$ has a canonical structure of Batalin-Vilkovisky algebra (**BV**-algebra for short) generalizing Chas-Sullivan [ChSu] algebra for oriented manifolds;
- the shifted $S^1$-equivariant homology $H_*^{S^1}(\mathrm{L}\mathfrak{X})[2 - \dim(\mathfrak{X})]$ has a canonical structure of graded Lie algebra generalizing Chas-Sullivan [ChSu] Lie algebra structure for oriented manifolds.

**Definition 2.1** The *Goldman bracket* of an oriented 2-dimensional stack $\mathfrak{X}$ is the restriction to $H_0^{S^1}(\mathrm{L}\mathfrak{X})$ of the (degree 0) Lie bracket on $H_*^{S^1}(\mathrm{L}\mathfrak{X})$.

If $\mathfrak{X} = \Sigma$ is an ordinary surface, the induced structure on $H_0^{S^1}(\mathrm{L}\Sigma) \cong k[\pi_0(\mathrm{Map}(S^1, \Sigma))]$ is isomorphic to Goldman Lie algebra of free loops. In Section 4 we will see an explicit description of the Goldman bracket for general 2-dimensional orbifolds.

The Goldman bracket (and Chas-Sullivan loop product) are both natural with respect to open embeddings of stacks.

**Proposition 2.2 (Proposition 10.3 and Lemma 11.3 in [GiNo])** *Let $\mathfrak{X}$ be an oriented Hurewicz stack of dimension $d$, and $\mathfrak{U} \subseteq \mathfrak{X}$ be an open substack. Then, $\mathfrak{U}$ inherits a natural orientation from $\mathfrak{X}$, and the induced map $H_{*+d}(L\mathfrak{U}) \to H_{*+d}(\mathrm{L}\mathfrak{X})$ is a morphism of $BV$-algebras. Similarly, the induced map $H_*^{S^1}(L\mathfrak{U})[2-d] \to H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]$ is a morphism of graded Lie algebras.*

*Let $\mathfrak{X}$ be a 2-dimensional orbifold (not necessarily reduced), and let $\mathfrak{U} \subseteq \mathfrak{X}$ be the complement a finite set of points. The natural map $H_0^{S^1}(\mathrm{L}\mathfrak{U}) \to H_0^{S^1}(\mathrm{L}\mathfrak{X})$ is a surjective map of Lie algebras.*

# 3 Relation with Chas-Gadgil bracket

An interesting class of examples of (global quotient) 2-dimensional orbifolds is given by the quotient orbifolds $[G\backslash\mathbb{H}]$ of the hyperbolic plane by a Fuchsian group. This class of examples has been studied by Chas and Gadgil [ChGa] at the same time as the first version of our work. Their approach and definition of the Goldman bracket for this class of orbifolds uses in an explicit way the combinatorics of Fuchsian groups. Below we briefly recall their definition.

## 3.1 Recall on Fuchsian groups

By a Fuchsian group we mean a subgroup $G$ of the group $\mathrm{PSL}(2, \mathbb{R})$ of isometries of the upper half-plane $\mathbb{H}$. We will assume that all our Fuchsian groups are finitely generated. In this case, $[G\backslash\mathbb{H}]$ is an analytic orbifold with finitely many orbifold points, cusps and removed discs. The orbifold points (respectively, the cusps, the removed discs) correspond to conjugacy classes of maximal elliptic (respectively, parabolic, boundary-hyperbolic) subgroups of $G$. The finiteness of these is proved in [Bea, Corollary 10.3.3, Theorem 10.3.7].

All such conjugacy classes are cyclic. In the elliptic case they can be identified (up to conjugation) with the finite cyclic group generated by the loop around the corresponding orbifold point. In the parabolic (respectively, boundary-hyperbolic) case, they correspond, up to conjugation, to the infinite cyclic group generated by the loop around the cusp (respectively, the removed disc).

## 3.2 Chas-Gadgil Lie bracket

The underlying abelian group of the Chas-Gadgil construction of the Goldman algebras of loops on $[G\backslash\mathbb{H}]$ is the free abelian group spanned by the set $\mathrm{Conj}(G)$, where $\mathrm{Conj}(G)$ denotes the of conjugacy classes of elements in $G$. The latte can be canonically identified with the set of free homotopy classes of loops on $[G\backslash\mathbb{H}]$.

Let $\langle g\rangle$ denote the cyclic subgroup generated by the element $g \in G$.

**Definition 3.1** For hyperbolic elements in $g, h \in G$, let

$$C(g,h) \subseteq \langle g\rangle\backslash G/\langle h\rangle$$

be the set of those double cosets $\langle g\rangle b\langle h\rangle$ such that $A_g \cap tA_h \neq \emptyset$ for some (hence all) $t \in \langle g\rangle b\langle h\rangle$. Here, $A_g$ stand for the *axis* of the hyperbolic element $g$, the unique geodesic semi-circle joining the two fix points of $g$. If either of $g$ or $h$ is not hyperbolic we define $C(g,h)$ to be the empty set.

For conjugacy classes $\lfloor g\rfloor$, $\lfloor h\rfloor$, Chas and Gadgil define a bracket on $\mathbb{Z}[\mathrm{Conj}(G)]$ by the formula,

$$\{\lfloor g\rfloor, \lfloor h\rfloor\} = \sum_{b\in C(g,h)} \iota(g,h^b)\lfloor gh^b\rfloor.$$

The sign $\iota(g,h^b)$ in the above formula is $+1$ if the axes $A_g$ and $tA_h$ cross positively (with respect to the orientation of $\mathbb{H}$) and $-1$ otherwise. Chas-Gadgil [ChGa] proved that this bracket is a well-defined Lie bracket on $\mathbb{Z}[\mathrm{Conj}(G)]$.

Note that this bracket is also well-defined over any commutative unital ring $k$ instead of $\mathbb{Z}$. In what follows, we fix such a $k$, and take all homology groups to be with coefficients in $k$.

The main result of this section is that this Lie bracket coincides with the Goldman bracket of 2-dimensional stacks defined in [GiNo].

**Theorem 3.2** *Let $[G\backslash\mathbb{H}]$ be the orbifold quotient of the hyperbolic plane by a Fuchsian group $G$, and let $\mathrm{Conj}(G)$ be the set of conjugacy classes of $G$. There is a natural isomorphism of Lie algebras*

$$\left(H_0^{S^1}(\mathrm{L}[G\backslash\mathbb{H}]), [-,-]\right) \cong \left(k[\mathrm{Conj}(G)], \{-,-\}\right),$$

*where the Lie algebra on the left is the one defined in [GiNo] and the one on the right is the Chas-Gadgil Lie algebra.*

First we establish an isomorphism between the underlying $k$-modules.

**Lemma 3.3** *The set of free homotopy classes of loops on $[G\backslash\mathbb{H}]$ are in a natural bijection with $\mathrm{Conj}(G)$. We have a natural isomorphism*

$$H_0^{S^1}(\mathrm{L}[G\backslash\mathbb{H}]) \cong k[\mathrm{Conj}(G)].$$

Proof. This is an immediate consequence of [GiNo, Lemma 11.1], because $\pi_1[G\backslash\mathbb{H}] \cong G$, and the isomorphism is well defined up to conjugation. $\square$

For future reference, we elaborate the above isomorphisms. Since $[G\backslash\mathbb{H}]$ is a global quotient of a manifold by a discrete group, by [BGNX, Proposition 5.9] we have isomorphisms

$$\mathrm{L}[G\backslash\mathbb{H}] \cong \coprod_{g\in G}[\mathcal{P}_g/G] \cong \coprod_{\lfloor g\rfloor\in \mathrm{Conj}(G)}[\mathcal{P}_g/\mathrm{Stab}(g)],$$

where $\mathcal{P}_g := \{f : [0,1] \to \mathbb{H},\ f(1) = gf(0)\}$ is equipped with the compact-open topology and $G$ acts pointwise on the paths. Since $\mathbb{H}$ is contractible, we have

$$H_0^{S^1}(L[G\backslash\mathbb{H}]) \overset{\cong}{\leftarrow} H_0(L[G\backslash\mathbb{H}]) \cong k[\mathrm{Conj}(G)].$$

The latter isomorphism identifies a conjugacy class $x$ in $G$ with the homology class of any path $f : [0,1] \to \mathbb{H}$ such that $f(0) = gf(1)$, for any $g$ with $x = \lfloor g\rfloor$.

**Lemma 3.4** *Let $\alpha = \lfloor g\rfloor \in \mathrm{Conj}(G) \cong H_0^{S^1}(L[G\backslash\mathbb{H}])$ be the class of a parabolic or elliptic element $g \in G$. Then, for all $x \in H_0^{S^1}(L[G\backslash\mathbb{H}])$, one has $[\alpha, x] = 0$.*

Proof. In the elliptic case, the loop $g$ is a ghost loop, i.e., it is in the image of of the inertia group of the corresponding orbifold point. Thus, its free homotopy class can be made as small as possible in any neighborhood of the given orbifold point. So, we can arrange that it never meets $x$. Hence, $[\alpha, x] = 0$.

We can argue similarly in the parabolic case. In this case, the loop $g$ corresponds to a vertex at infinity of the a fundamental domain for $G$ in $\mathbb{H}$, thus it can be identified with a loop wrapping around the corresponding cusp in the compactification of (the underlying Riemann surface of) $[G\backslash\mathbb{H}]$. Such a loop can always be made small enough to avoid $x$. $\square$

The above lemma also holds in the Goldman algebra $\big(k[\mathrm{Conj}(G)], \{-,-\}\big)$ by definition of the Lie bracket of Chas-Gadgil.

Next we consider the case of hyperbolic elements.

**Lemma 3.5** *Let $\alpha = \lfloor g\rfloor, \beta = \lfloor h\rfloor$ be conjugacy classes of hyperbolic elements in $G$. Then the Goldman bracket in $H_0^{S^1}(L[G\backslash\mathbb{H}])$ is given by the formula*

$$[\alpha, \beta] = \sum_{b\in C(g,h)} \iota(g, h^b)\lfloor gh^b\rfloor.$$

*Here $C(g,h)$ is still the subset of the double coset $\langle g\rangle b\langle h\rangle$ of those double cosets $\langle g\rangle b\langle h\rangle \in \langle g\rangle\backslash G/\langle h\rangle$ such that $A_g \cap tA_h \neq \emptyset$ for $t \in \langle g\rangle b\langle h\rangle$. Further the sign $\iota(g, h^b)$ is still $+1$ if the axis $A_g$ and $tA_h$ crosses positively (with respect to the orientation of $\mathbb{H}$) and $-1$ if not.*

Proof. The class $\alpha$ and $\beta$ are respectively represented by any geodesic path $\tau_g : [0,1] \to \mathbb{H}$, $\tau_h : [0,1] \to \mathbb{H}$ such that $\tau_g(1) = g.\tau_g(0)$, $\tau_h(1) = h.\tau_h(0)$ which induces well defined loops $\Omega g$, $\Omega h : S^1 \to [G\backslash\mathbb{H}]$ on the orbifold. By [ChGa, Proposition 2.5], we can find such shortest geodesic paths $\tau_g$, $\tau_h$, such that $\Omega g$ and $\Omega y$ have precisely $C(g,h)$ intersections, transverse and at double points only.

By [GiNo, Lemma 11.2], the Goldman bracket $[\alpha, \beta]$ is the image of the Goldman bracket of $\Omega g, \Omega h]$ computed in the complement $\Sigma_G$ of the orbifold locus of $[G\backslash\mathbb{H}]$. We are thus left to counting it for the standard Goldman bracket on a surface [Go2]. In particular, it is given by the sum

$$[\Omega g, \Omega h] := \sum_{P \in \Omega g \cup \Omega h} \epsilon(P) \lfloor \Omega g \bullet_P \Omega h \rfloor$$

over all intersection points $P$ of $\Omega g$ and $\Omega y$, where $\Omega g \bullet_P \Omega h$ is the loop starting at $P$, following $\Omega g$ back to $P$ and then following $\Omega h$. The sign $\epsilon(P)$ is $+1$ if the tangent vectors of $\Omega g$ and $\Omega h$ at $P$ forms a direct basis of the oriented surface $\Sigma_G$, and $-1$ otherwise.

Following [S] and [ChGa], we see that the intersections of $\Omega g$ and $\Omega h$ are labelled by the set of translates of the geodesic path $t.\tau_h$ such that $A_g \cap tA_h \neq \emptyset$. This shows that the composition $\Omega g \bullet_P \Omega h$ is precisely the loop on $[G\backslash\mathbb{H}]$, corresponding to the loop obtained as the class of $ght ht^{-1}$. It is immediate that the sign of th crossing of $A_g$ and $tA_h$ is preciesly $\iota(g, h^t)$. $\square$

Proof of Theorem 3.2. By Lemma 3.3, we are left to check that the bracket $[\alpha, \beta]$ coincides with $\{\alpha, \beta\}$ on representatives of conjugacy classes of $G$. This is the case if either $\alpha$ or $\beta$ is non-hyperbolic by Lemma 3.4. We are left to the case of hyperbolic elements which is the content of Lemma 3.5. $\square$

# 4 Geometric intersection numbers of loops on 2-dimensional orbifolds

## 4.1 2-dimensional orbifolds

In this section, we assume that $\mathfrak{X}$ is a 2-dimensional differentiable orbifold of finite type with finitely many orbifold points. This means that, locally, $\mathfrak{X}$ is isomorphic to a quotient stack $[H\backslash\mathbb{R}^2]$, where $H$ is a finite group acting on $\mathbb{R}^2$ by rotations via a cyclic quotient of $H$. When this cyclic quotient is non-trivial, the image of the origin in $[H\backslash\mathbb{R}^2]$ (or $\mathfrak{X}$) is called an *orbifold point.* We require that there are finitely many such orbifold points in $\mathfrak{X}$. By $\mathfrak{X}$ being of *finite type* we mean that the underlying surface of $\mathfrak{X}$, by which we mean the coarse moduli space of $\mathfrak{X}$, is of finite type (i.e., can be compactified by adding finitely many points, or equivalently, has finitely generated fundamental group). We say that $\mathfrak{X}$ is *reduced* if away from the orbifold locus it is isomorphic to a surface.

We say that $\mathfrak{X}$ is *oriented* if the transition maps between orbifold charts are orientation preserving. Equivalently, the underlying surface of $\mathfrak{X}$ is oriented.

**Lemma 4.1** *Let $\mathfrak{X}$ be a 2-dimensional differentiable orbifold, and let $X$ be its coarse moduli space. Suppose that $X$ is endowed with an analytic structure. Then, there is a unique analytic structure on $X$ making the moduli map $\mathfrak{X} \to X$ analytic.*

PROOF. By uniqueness, it is enough to prove the statement locally, so we may assume that $X = \mathbb{D}$ is the unit disc, and $\mathfrak{X} = [H\backslash U]$, where $U$ is diffeomorphic to $\mathbb{R}^2$ and $H = \mathbb{Z}/n\mathbb{Z}$ acts by rotations around the origin. We thus have a smooth branched covering $q\colon U \to \mathbb{D}$ which is an $\mathbb{Z}/n\mathbb{Z}$-cover away from the origin. We need to show that there is a unique analytic structure on $U$ making $p$ analytic.

There is a $\mathbb{Z}/n\mathbb{Z}$-equivariant diffeomorphism $f\colon U \to \mathbb{D}$ relative to $X$, as in the commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "q", "\mathbb{Z}/n\mathbb{Z}"'] & & \mathbb{D} \arrow[dl, "\mathbb{Z}/n\mathbb{Z}"] \\ & X = \mathbb{D} & \end{tikzcd}$$

and that this diffeomorphism is unique up to the $\mathbb{Z}/n\mathbb{Z}$-rotation action on $\mathbb{D}$. The analytic structure on $U$ that makes $f$ analytic is the sought after analytic structure that makes $q$ analytic, and this is clearly unique. $\square$

By the above lemma, any oriented 2-dimensional differentiable orbifold $\mathfrak{X}$ of finite type with finitely many orbifold points can be endowed with an analytic structure, hence is isomorphic to a global stack $[G\backslash\mathbb{X}]$, where $\mathbb{X}$ is $\mathbb{H}$, $\mathbb{C}$ or a weighted projective line $\mathbb{P}(m,n)$; see [BeNo]. In the former case, which is the case we are mainly interested in, we say that $\mathfrak{X}$ is *hyperbolic.* In this case, $G$ is a finitely generated Fuchsian group.

**Remark 4.2** In fact, by choosing the analytic structure near the cusps (i.e., the puncture points of the underlying surface of $\mathfrak{X}$) to be isomorphic to the punctured disc $\mathbb{D}\backslash\{0\}$ (as opposed to an annulus), we may further assume that $G$ is a Fuchsian group of the first kind and has finite covolume. But we will not need this extra assumption in this paper.

A general 2-dimensional orbifold $\mathfrak{X}$ as above can be realized as an $H$-gerbe over a reduced orbifold $\mathfrak{X}_{\text{red}}$, for some finite group $H$; see [BeNo, Proposition 4.6]. The map $\mathfrak{X} \to \mathfrak{X}_{\text{red}}$ is a weak Serre fibration by [No3, Proposition 4.6]. Therefore, we have an exact sequence

$$H \to \pi_1\mathfrak{X} \to \pi_1\mathfrak{X}_{\text{red}} \to 1. \tag{4.1}$$

This sequence is exact on the left when $\mathfrak{X}$ is hyperbolic (because in this case $\mathfrak{X}_{\text{red}}$ is also hyperbolic, so $\pi_2\mathfrak{X}_{\text{red}}$ is trivial).

**Lemma 4.3** *Let $\mathfrak{X}$ be a 2-dimensional orbifold which is an $H$-gerbe over a reduced orbifold $\mathfrak{X}_{\text{red}}$. Then, the following are equivalent:*

*1) $\mathfrak{X}$ is of finite type and has finitely many orbifold points;*

*2) $\mathfrak{X}_{\mathrm{red}}$ is of finite type and has finitely many orbifold points;*

*3) $\pi_1\mathfrak{X}_{\mathrm{red}}$ is finitely generated;*

*4) $\pi_1\mathfrak{X}$ is finitely generated.*

PROOF. The equivalence of (1) and (2) is clear because $\mathfrak{X}$ and $\mathfrak{X}_{\mathrm{red}}$ have the same underlying surface and the same set of orbifold points. The fact that (2) implies (3) follows from van Kampen, and (3) and (4) are equivalent because of the exact sequence (4.1). To prove that (3) implies (2), note that, by adding a few extra punctures, we may assume that $\mathfrak{X}_{\mathrm{red}}$ is hyperbolic, hence of the form $[G\backslash\mathbb{H}]$ for some finitely generated Fuchsian group $G$. The claim now follows from the fact that $G$ has finitely many maximal elliptic and parabolic conjugacy classes; see Section 3.1. $\square$

## 4.2 Intersection of free loops on oriented 2-dimensional orbifolds

We outline a geometric way of defining intersections for free homotopy classes of loops on a general oriented 2-dimensional orbifold.

**Definition 4.4** Let $\mathfrak{U} \subset \mathfrak{X}$ be the complement of the orbifold locus of a reduced orbifold $\mathfrak{X}$.

- Let $\alpha, \beta$ be free homotopy classes of loops on $\mathfrak{X}$ (that is, element of $H_0(L\mathfrak{X})$). The geometric intersection number of $\alpha$ and $\beta$ is the minimum number of transverse intersections of loops representing $\alpha$ and $\beta$ and lying in $\mathfrak{U}$.

- The self-intersection number of $\alpha$ is the minimum number of transverse intersections of a loop representing $\alpha$ and lying in $\mathfrak{U}$.

For a general non-reduced orbifold, we first write it as an $H$-gerbe over a reduced orbifold $\mathfrak{X}_{\mathrm{red}}$. To define the intersection of $\alpha$ and $\beta$ in $\mathfrak{U}$, we simply count each intersection point $x$ of $\alpha_{\mathrm{red}}$ and $\beta_{\mathrm{red}}$ in $\mathfrak{U}_{\mathrm{red}}$ with multiplicity $|H|$ (since $\{x\} \times_{\mathfrak{B}G} \{x\}$ is a disjoint union of $|H|$-many points). Self-intersection of loops is defined similarly.

The above definition makes sense because, by [GiNo, Lemma 11.3], any loop $\alpha$ in $\mathfrak{X}$ can be lifted to $\mathfrak{U}$.

For reduced orbifolds of the type $[G\backslash\mathbb{H}]$, Chas and Gadgil have proved that this intersection number is computed stably by the Goldman bracket of powers of $\alpha$ and $\beta$. We now explain how to extend their results to all oriented reduced orbifolds of dimension 2.

**Theorem 4.5** *Let $\mathfrak{X}$ be a 2-dimensional orbifold of finite type and $\alpha, \beta \in H_0^{S^1}(\mathrm{L}\mathfrak{X})$ be free homotopy classes of loops on $\mathfrak{X}$.*

1. *The geometric intersection number of $\alpha$ and $\beta$ is equal to the number of non-zero terms (counted with multiplicities) of the Goldman bracket $[\alpha^p, \beta^q]$ divided by $pq$, for all but finitely many $q \neq pc$ (where $c$ is the ratio of the translation length of any two hyperbolic elements in $\mathfrak{U}$ presenting $\alpha$ and $\beta$ or $0$ if either $\alpha$ or $\beta$ can be presented by non-hyperbolic elements).*

2. *If $\alpha$ is simple (that is, not equal to $\beta^n$ for some $\beta$), then the self-intersection number of $\alpha$ is equal to the number of non-zero terms (counted with multiplicities) of the Goldman bracket $[\alpha^p, \alpha^q]$ divided by $pq$, for all but finitely many $q \neq p$.*

PROOF. We may assume that $\mathfrak{X}$ is connected. The proof of the two statement are similar, so we only prove the first one.

First, we reduce the claim to the case of a reduced orbifold. An arbitrary connected 2-dimensional orbifold $\mathfrak{X}$ of finite type with finitely many orbifold points can be written as an $H$-gerbe $\pi\colon \mathfrak{X} \to \mathfrak{X}_{\mathrm{red}}$ over a reduced orbifold $\mathfrak{X}_{\mathrm{red}}$, where $H$ is a finite group; see Section 4.1. The exact sequence (4.1) gives us the short exact

$$1 \to H/K \to \pi_1\mathfrak{X} \xrightarrow{\pi_*} \pi_1\mathfrak{X}_{\mathrm{red}} \to 1,$$

where $K$ is the kernel of the map $H \cong \pi_1(\mathfrak{B}H) \to \pi_1(\mathfrak{X})$ induced by $\pi$ (whose fibers are isomorphic to the stack $\mathfrak{B}H$). There is a map

$$\tau\colon k[\mathrm{Conj}(\pi_1(\mathfrak{X}_{\mathrm{red}}))] \to k[\mathrm{Conj}(\pi_1(\mathfrak{X}))]$$

relating the Goldman brackets in $\mathfrak{X}_{\mathrm{red}}$ and in $\mathfrak{X}$ which is defined as follows. It sends the conjugacy class $\lfloor\gamma\rfloor \in \mathrm{Conj}(\pi_1(\mathfrak{X}_{\mathrm{red}}))$ to

$$\tau(\lfloor\gamma\rfloor) := |K| \sum \lfloor\pi_*^{-1}(\gamma)\rfloor,$$

where $|K|$ is the cardinality of $K$. By [GiNo, Proposition 11.12], we have

$$\{\alpha, \beta\}_{\mathfrak{X}} = \tau\big(\{\pi_*(\alpha), \pi_*(\beta)\}_{\mathfrak{X}_{\mathrm{red}}}\big),$$

where the right hand side is the image by $\tau$ of the Goldman bracket on $\mathfrak{X}_{\mathrm{red}}$. In particular, the Goldman bracket $\{\alpha, \beta\}_{\mathfrak{X}}$ contains exactly $|H|$-many times the number of (non-zero) terms of the Goldman bracket $\{\pi_*(\alpha), \pi_*(\beta)\}_{\mathfrak{X}_{\mathrm{red}}}$. By definition of the intersection of loops in an orbifold (Definition 4.4), dividing both sides of the equality claimed in (i) by $|H|$ reduces the claim to the case of the reduced orbifold $\mathfrak{X}_{\mathrm{red}}$. So, without loss of generality, we may assume that $\mathfrak{X}$ is reduced.

For $\mathfrak{X}$ reduced, the complement $\mathfrak{U}$ of the orbifold locus is a finite type Riemann surface. By removing a few more points from $\mathfrak{U}$, if necessary, we may assume that $\mathfrak{U}$ is hyperbolic, i.e., is of the form $[G\backslash\mathbb{H}]$ for a finitely generated Fuchsian group acting freely on $\mathbb{H}$. By Theorem 3.2, the result is now an immediate consequence of the main theorem of Chas and Gadgil [ChGa] (where $c$ is the ratio of the translation length of the hyperbolic elements presenting $\alpha$ and $\beta$). $\square$

# 5 Stacky symplectic character variety and Goldman homomorphism for orbifolds

In this section, we generalize Goldman's Lie algebra map (0.1) to orbifolds.

## 5.1 Symplectic orbifolds

We recall the definition of a symplectic structure on an orbifold due to Lerman and Malkin [LeMa]. If the orbifold $\mathfrak{X}$ is presented by a Lie groupoid $X_1 \rightrightarrows X_0$ (with source and target maps denoted $s$ and $t$, respectively), then the (differential graded) algebra $\Omega^*(\mathfrak{X})$ of differential forms on $\mathfrak{X}$ is isomorphic to

$$\Omega^*(X_0)^{X_1} := \{\omega \in \Omega^*(X_0) \,|\, s^*(\omega) = t^*(\omega)\},$$

the subalgebra of equivariant forms on $X_0$. Let $A \to TX_0$ be the Lie algebroid of $X_1 \rightrightarrows X_0$. Then, the Lie algebra of vector fields on $\mathfrak{X}$ is isomorphic to the Lie algebra $\Gamma(TX_0/A)^{X_1}$ of equivariant global sections of the bundle $TX_0/A$ over $X_0$. For a 2-forms $\omega$, its contraction along vector fields gives rise to a canonical map $\rho_\omega : \Gamma(T\mathfrak{X}) \to \Omega^1(\mathfrak{X})$. The 2-form $\omega$ is *non-degenerate* if $\rho_\omega$ is an isomorphism. A *symplectic structure* on $\mathfrak{X}$ is defined to be a closed non-degenerate 2-form $\omega \in \Omega^2(\mathfrak{X})$.

## 5.2 The character stack

First we begin with a general observation. Let $\pi$ and $G$ be sheaves of group over a site. We write $\mathrm{Mor}(\pi, G)$ for the sheaf of homomorphisms from $\pi$ to $G$. We define $\mathfrak{B}i(G, \pi)$ to be the stack that assigns to each object $T$ the groupoid of $(G, \pi)$-bimodule whose left $G$-module structure is principal. That is, an object in $\mathfrak{B}i(G, \pi)(T)$ is pair $(P, \mu)$, where $P \to T$ is a left $G$-torsor and $\mu\colon P \times \pi \to P$ is a $G$-equivariant right action of $\pi$ on $P$ relative to $T$.

**Proposition 5.1** *We have natural equivalences of stacks*

$$\mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G) \cong \mathfrak{B}i(G, \pi) \cong [G \backslash \mathrm{Mor}(\pi, G)],$$

*were $G$ acts on $\mathrm{Mor}(\pi, G)$ by left conjugation on the target.*

Proof. To prove the equivalence $\mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G) \cong \mathfrak{B}i(G, \pi)$ we exhibit a natural equivalence between the groupoid of $T$-points,

$$\mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G)(T) \cong \mathfrak{B}i(G, \pi)(T),$$

for every object $T$ in the site.

First let us work out the left hand side. By definition of the mapping stack, we have

$$\mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G)(T) \cong \mathrm{Mor}_{Stacks}(T \times \mathfrak{B}\pi, \mathfrak{B}G),$$

which in turn is equivalent to the groupoid of maps from the pre-stack $T \times [\pi \backslash \bullet]$ to $\mathfrak{B}G$; here $[\pi \backslash \bullet]$ is $\pi$ viewed as a one-object groupoid over the site (whose stackification then becomes $\mathfrak{B}\pi$).

The latter groupoid is equivalent the the groupoid of pairs $(P, \mu)$, where $P \to T$ is a left $G$-torsor and $\mu \colon P \times \pi \to P$ is a $G$-equivariant right action of $\pi$ on $P$ relative to $T$. In other words, $P$ is a $(G, \pi)$-bimodule whose left $G$-module structure is principal. This proves the desired equivalence.

To prove the equivalence $\mathfrak{B}i(G, \pi) \cong [G \backslash \mathrm{Mor}(\pi, G)]$, we first describe the groupoid $[G \backslash \mathrm{Mor}(\pi, G)](T)$. By the torsor description of the quotient stack, this is equivalent to the groupoid of pairs $(P, f)$, where $P \to T$ is a left $G$-torsor and $f \colon P \to \mathrm{Mor}(\pi, G)$ is $G$-equivariant. By abuse of notation, we denote the adjoint family $P \times \pi \to G$ of homomorphisms also by $f$.

The equivalence $\mathfrak{B}i(G, \pi)(T) \to [G \backslash \mathrm{Mor}(\pi, G)](T)$ is defined by sending $(P, \mu)$ to $(P, f)$, where $f$ is defined as the composition

$$P \times \pi \xrightarrow{(\mu, \mathrm{pr}_1)} P \times_T P \cong G \times P \xrightarrow{\mathrm{pr}_1} G.$$

Symbolically, for $x \in P$ and $\alpha \in \pi$, $f_x(\alpha) \in G$ is defined by the equation $f_x(\alpha)x = x\alpha$. To verify that for every $x \in P$ the map $f_x \colon \pi \to G$ is a group homomorphism one uses the fact that, for every $g \in G$ and $\alpha \in \pi$, $f_{gx}(\alpha) = g f_x(\alpha) g^{-1}$, which is a consequence of the fact that the actions of $G$ and $\pi$ on $P$ commute.

One can run this construction backwards to construct $(P, \mu)$ from $(P, f)$, thereby producing the inverse equivalence. $\square$

**Corollary 5.2** *Suppose $\pi$ is a finitely generated discrete group. Then,* $\mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G)$ *is a topological (respectively, differentiable, analytic, algebraic, etc.) stack whenever $G$ is a topological (respectively, differentiable, analytic, algebraic, etc.) group.*

Proof. This is because $\mathrm{Mor}(\pi, G)$ is closed inside the product $G^S$, where $S$ is a set of generators for $\pi$. $\square$

**Remark 5.3** In the above corollary the finite generation of $\pi$ is not necessary in the topological setting. Also, in the differentiable setting, the statement remains valid without this assumption if we allow Fréchet manifolds. The statement is also valid in the algebraic setting if we assume $G$ is affine.

Now, let $\mathfrak{Y}$ be a connected topological stack and write $\pi := \pi_1(\mathfrak{Y}, y_0)$ for its fundamental group (taken at some base point $y_0$ whose choice is not relevant here). Let $G$ be a topological (or differentiable, analytic, algebraic, etc.) group.

We denote by $\mathfrak{B}^{\mathrm{fl}}G$ the stack over $\mathsf{Diff}$ of flat principal $G$-bundles, whose fiber over a manifold $U$ is the groupoid $\mathfrak{B}^{\mathrm{fl}}G_{|U}$ of principal $G$-bundles over $U$ together with a flat connection on the bundle.

We define the *character stack* of $\mathfrak{Y}$ to be the mapping stack (over the site where $G$ lives)

$$\mathfrak{Ch}_{\mathfrak{Y},G} := \mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G).$$

Note that, if we write $\Pi$ for the fundamental groupoid of $\mathfrak{Y}$, then the choice of the base point $y_0$ yields a canonical stack morphism

$$\mathrm{Map}(\mathfrak{B}\Pi, \mathfrak{B}G) \to \mathrm{Map}(\mathfrak{B}\pi, \mathfrak{B}G)$$

which is an equivalence since $\mathfrak{Y}$ is connected. This provides a base point free version of the character stack.

**Proposition 5.4** *Let $\mathfrak{Y}$ be a connected topological stack with fundamental group $\pi$. Let $G$ be a topological group. We have a natural equivalence of stacks*

$$\mathfrak{Ch}_{\mathfrak{Y},G} \cong [G \backslash \mathrm{Mor}(\pi, G)],$$

*were $G$ acts on $\mathrm{Mor}(\pi, G)$ by left conjugation on the target. Furthermore, if $G$ is a Fréchet (respectively, affine algebraic) group, then the character stack $\mathfrak{Ch}_{\mathfrak{Y},G}$ is a Fréchet (respectively, algebraic) stack. If $\pi$ is finitely generated and $G$ is a Lie (respectively, complex, algebraic) group, then the character stack $\mathfrak{Ch}_{\mathfrak{Y},G}$ is a differentiable (respectively, analytic, algebraic) stack. Furthermore, one has a natural isomorphism $\mathfrak{Ch}_{\mathfrak{Y},G} \cong \mathrm{Map}(\mathfrak{Y}, \mathfrak{B}^{fl}G)$ of stacks over $\mathsf{Diff}$.*

PROOF. Follows from Proposition 5.1 and Corollary 5.2. Also see Remark 5.3. □

**Remark 5.5** When $\mathfrak{Y}$ has trivial homotopy groups in degree $n \geq 2$, then the character stack $\mathfrak{Ch}_{\mathfrak{Y},G}$ is also isomorphic to the mapping stack $\mathrm{Map}(\mathfrak{Y}, B^{con}G)$ where $B^{con}G$ is the stack of bundles equipped with connections.

**Example 5.6** Take $\mathfrak{Y}$ to be the circle $S^1$. Then one finds that the mapping stack $L\mathfrak{B}^{\mathrm{fl}}G = Map(S^1, \mathfrak{B}^{\mathrm{fl}}G)$ is isomorphic to $[\mathrm{Mor}(\mathbb{Z}, G)/G] = [G\backslash G]$ where $G$ acts on itself by conjugation, that is the inertia stack of $\mathfrak{B}G$.

For the rest of this section we will assume that $\mathfrak{Y}$ is a connected 2-dimensional orbifold of finite type, and $G$ is a reductive Lie group with a $G$-invariant non-degenerate pairing $K : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$.

**Definition 5.7** We define the character variety of $\mathfrak{Y}$ to be the coarse moduli space $\chi_{\mathfrak{Y},G} := (\mathfrak{Ch}_{\mathfrak{Y},G})_{\mathrm{coarse}}$ of the character stack.

In other words, $\chi_{\mathfrak{Y},G}$ is the quotient space $G \backslash \mathrm{Mor}(\pi, G)$. It has a well defined structure of a real-valued ringed space (see [Ka]) (when $G$ is finitely generated) and can also be seen as real analytification of an algebraic variety as in [Go1, Go2]. We have a morphism of topological stacks $\mathfrak{Ch}_{\mathfrak{Y},G} \to \chi_{\mathfrak{Y},G}$. Functions on $\chi_{\mathfrak{Y},G}$ are simply $G$-equivariant functions on the smooth ringed space[2] $\mathrm{Mor}(\pi, G)$. We are now interested in vector fields and forms on the character variety.

[2] which can be seen as a manifold or an algebraic variety

**Lemma 5.8** *Let $\phi : \pi \to G$ be a group homomorphism.*

1. *The sections of $T\chi_{\mathfrak{Y},G}$ at $\phi$ are naturally isomorphic to $H^1(\mathfrak{Y}, \mathfrak{g}_\phi)$, where $\mathfrak{g}_\phi$ is the Lie algebra $\mathfrak{g}$ viewed as a local coefficient system on $\mathfrak{Y}$ via $\phi$.*

2. *If we further assume that $\mathfrak{Y}$ is of* finite type *(in other words that $\pi$ is of finite presentation), then the fiber of $\Omega^1(\chi_{\mathfrak{Y},G})$ at $\phi$ is isomorphic to $H_1(\mathfrak{Y}, \mathfrak{g}_\phi)$.*

Proof. By proposition 5.4, we have a presentation of $\chi_{\mathfrak{Y},G}$ by the (transformation) Lie groupoid $\mathrm{Mor}(\pi, G) \times G \rightrightarrows \mathrm{Mor}(\pi, G)$ associated to the conjugation action of $G$ on itself. By [Ka, Theorem 3], we have a natural isomorphism $T_\phi \mathrm{Mor}(\pi, G) \cong Z^1(\pi, \mathfrak{g}_\phi)$, where the latter stands for the space of the 1-cycles $\pi \to \mathfrak{g}$ for the action of $\pi$ on $\mathfrak{g}$ via $\phi$. Similarly, the tangent space at $\phi$ to the $G$-orbit of $\phi$ is isomorphic to the spaces of 1-coboundaries $B^1(\pi, \mathfrak{g}_\phi)$. Hence, $T_\phi \mathrm{Mor}(\pi, G) \times G \cong Z^1(\pi, \mathfrak{g}_\phi) \oplus \mathfrak{g}$ with $A_\phi \cong \mathfrak{g}$ and thus we have a natural isomorphism

$$\Gamma(TX_0/A)_\phi^{X_1} \cong H^1(\pi, \mathfrak{g}_\phi). \tag{5.1}$$

Now let $q : Y \to \mathfrak{Y}$ be a classifying space for $\mathfrak{Y}$. Then $q : \pi_1(Y) \to \pi$ is an isomorphism and thus $q$ induces natural isomorphisms

$$H^1(\pi, \mathfrak{g}_\phi) \xrightarrow{\simeq} H^1(\pi_1(Y), \mathfrak{g}_\phi) \cong H^1(Y, \mathfrak{g}_\phi) \xleftarrow{\simeq} H^1(\mathfrak{Y}, \mathfrak{g}_\phi)$$

where the middle isomorphism is induced by Hurewicz and universal coefficient theorem. This proves (1). When $\mathfrak{Y}$ is of finite type, $H^1(\mathfrak{Y}, \mathfrak{g}_\phi)$ is finite dimensional and the cap product $H^1(\mathfrak{Y}, \mathfrak{g}_\phi) \otimes H_1(\mathfrak{Y}, \mathfrak{g}_\phi{}') \to H_0(\mathfrak{Y}, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi{}') \to H_0(\mathfrak{Y}) \cong \mathbb{R}$ exhibits $H_1(\mathfrak{Y}, \mathfrak{g}_\phi{}')$ as the linear dual of $H^1(\pi, \mathfrak{g}_\phi)$. Then (2) follows from the isomorphism $\mathfrak{g}_\phi{}' \to \mathfrak{g}_\phi$ induced by the non-degenerate pairing $K : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. $\square$

To define symplectic a structure on the character stack, we assume further that $\mathfrak{Y}$ is compact. In this case, we have a fundamental class $[\mathfrak{Y}]$ and thus a natural isomorphism $H^2(\mathfrak{Y}) \cong \mathbb{R}$ induced by capping with $[\mathfrak{Y}]$. Then, for any $\phi : \pi \to G$, we can define a linear map $\omega_\phi$ as the composition

$$\omega_\phi : H^1(\mathfrak{Y}, \mathfrak{g}_\phi)^{\otimes 2} \xrightarrow{\cup} H^2(\mathfrak{Y}, \mathfrak{g}_\phi^{\otimes 2}) \xrightarrow{K_*} H^2(\mathfrak{Y}, \mathbb{R}) \cong \mathbb{R}, \tag{5.2}$$

where $K_*$ is well defined since $K$ is $G$-invariant. The collection $(\omega_\phi)_{\phi \in \mathrm{Mor}(\pi,G)}$ defines an anti-symmetric bilinear form on $T\chi_{\mathfrak{Y},G}$, hence a 2-form on $\chi_{\mathfrak{Y},G}$, denoted $\omega$.

Let us now define a bracket on the functions of $\chi_{\mathfrak{Y},G}$ which will prove to be a Poisson bracket. Here we assume again that $\mathfrak{Y}$ is of finite type. We first define the 2-vector field $\widetilde{\omega} \in \Lambda^2 T\chi_{\mathfrak{Y},G}$ as the collection, for all $\phi : \pi \to G$,

$$H_1(\mathfrak{Y}, \mathfrak{g}_\phi)^{\otimes 2} \xrightarrow{\Cap} H_0(\mathfrak{Y}, \mathfrak{g}_\phi^{\otimes 2}) \xrightarrow{K_*} H_0(\mathfrak{Y}, \mathbb{R}) \cong \mathbb{R}$$

where $\Cap$ is the intersection product in homology. This defines a 2-vector field by Lemma 5.8. Now, for functions $f, g$ on $\chi_{\mathfrak{Y},G}$, we define a pairing

$$[f, g] := \widetilde{\omega}(df, dg). \tag{5.3}$$

Note that this formula makes sense as soon as $\mathfrak{Y}$ is an oriented orbifold, not necessarily a compact one.

**Theorem 5.9** *Let $\mathfrak{Y}$ be an oriented orbifold of dimension 2, which is of finite type.*

1. *If $\mathfrak{Y}$ is compact, the 2-form $\omega \in \Omega^2(\chi_{\mathfrak{Y},G})$ is closed and non-degenerate. Hence it makes the character variety $\chi_{\mathfrak{Y},G}$ a symplectic ringed space. Furthermore, the Poisson bracket on functions is given by the bracket (5.3).*
2. *If $\mathfrak{Y}$ is a punctured compact[3] orbifold and is further effective, then the above bracket (5.3) also makes its character variety a Poisson algebra.*

Note that in this paper we will only need part 2 when $\mathfrak{Y}$ is punctured Riemann surface in which case it is standard [Au, BiGu].

Proof. We first assume $\mathfrak{Y}$ is compact. Then the closedness of the 2-form $\omega$ follows from the identification $H^1(\mathfrak{Y}, \mathfrak{g}_\phi) \cong H^1(\pi, \mathfrak{g}_\phi)$ (see (5.1)) and [Ka, Theorem 4] (where we take $\varphi$ to be induced by the cap product along the fundamental class of $\mathfrak{Y}$). We prove it is non-degenerate as follows. Let $\delta_\phi : H^1(\mathfrak{Y}, \mathfrak{g}_\phi) \to \big(H^1(\mathfrak{Y}, \mathfrak{g}_\phi)\big)'$ be the map induced by the pairing $\omega_\phi$. Since $\mathfrak{Y}$ is compact, as in Lemma 5.8.2, we have an identification $\big(H^1(\mathfrak{Y}, \mathfrak{g}_\phi)\big)' \cong H_1(\mathfrak{Y}, {\mathfrak{g}_\phi}')$ under which $\delta_\phi$ becomes

$$H^1(\mathfrak{Y}, \mathfrak{g}_\phi) \xrightarrow{-\cap[\mathfrak{Y}]} H_1(\mathfrak{Y}, \mathfrak{g}_\phi) \xrightarrow{\delta_K} H_1(\mathfrak{Y}, {\mathfrak{g}_\phi}')$$

where the first map is Poincaré duality isomorphism and the second one $\delta_K$ is induced by the pairing $K : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. The pairing being non-degenerate, the last map is an isomorphism as well.

The same argument shows that if $H_f$ is the Hamiltonian vector field associated to a function $f$, then $df = \delta_K(H_f \cap [\mathfrak{Y}])$ is its Poincaré dual composed with the identification of $\mathfrak{g}$ and $\mathfrak{g}'$ induced by $K$. In particular, since the intersection product is the Poincaré dual to the cup product, we obtain that the Poisson bracket on functions is given by (5.3).

We now assume $\mathfrak{S}$ is effective and punctured. Since it is also of finite type, it means that $\mathfrak{S}$ is isomorphic to a compact Riemann surface with boundary and a finite sets of orbifold points (not located on the boundary). We wish to prove that $\mathcal{O}_{\chi_{\mathfrak{S},G}}$ is a Poisson algebra and we let $S_1 \rightrightarrows S_0$ be an étale presentation of $\mathfrak{S}$. It is enough to prove the result on a connected component of $\chi_{\mathfrak{S},G}$, which we do now in the case of the component of the identity; the other cases being similar. By Proposition 5.4, we are left to prove the result in the case of the

[3] in other words is isomorphic to the complement of a discrete set of non-overlapping disks in a compact orbifold

coarse moduli space of the global quotient stack of the space $\mathcal{F} \subset \Omega^1(S_0, \mathfrak{g})^{Y_1}$ of flat connections on the trivial $G$-bundle on $\mathfrak{S}$ by the group $\mathcal{G} = Map(S_0, G)^{S_1}$ of $S_1$-equivariant maps. We refer to [Beh2] for details on flat connections on stacks and [LeMa] for the special case of forms on orbifolds. Let also $\mathcal{G}_\partial$ be the normal subgroup of $\mathcal{G}$ of those maps which are the identity at the punctures. There is a closed 2-form on $\mathcal{F}$ defined at any flat connection $\alpha \in \Omega^1(S_0, \mathfrak{g})^{Y_1}$ as

$$\widetilde{\omega}_\alpha(x, y) = K_*(x \wedge y) \cap [\mathfrak{S}]$$

where $[\mathfrak{S}]$ is the fundamental class of $\mathfrak{S}$; one notes that the pairing actually do not depend on $\omega$, in particular it is closed. Since $K$ is non-degenerate, the pairing induced by this 2-form is non-degenerate as well. Hence, $\widetilde{\omega}$ is a symplectic structure on the vector space $\Omega^1(S_0, \mathfrak{g})^{Y_1}$. The standard argument of [Au, §2] (or [BiGu]) shows that the curvature can be identified with the moment map of the subgroup $\mathcal{G}_\partial$. In particular, the quotient $\Omega^1_{flat}(S_0, \mathfrak{g})^{Y_1}/\mathcal{G}_\partial$ is symplectic. Further, the $\mathcal{G}$ action on this quotient is symplectic and thus makes the quotient $(\mathcal{F}/\mathcal{G}_\partial)/\mathcal{G} \cong (\mathcal{F}/\mathcal{G})$ a Poisson space (see [Au, §2], [BiGu]).

Now, recall that the identification of $[\mathcal{F}/\mathcal{G}]$ with the character stack $\mathfrak{Ch}_{\mathfrak{S},G}$ is induced by the holonomy map which sends a flat connection $\alpha \in \mathcal{F}$ and a loop $\gamma \in \pi_1(\mathfrak{S})$ to its holonomy.

Under this isomorphism, the same computation as above in the point 1) using Poincaré duality for orbifolds with boundary shows that the Poisson bracket is induced by formula (5.3). $\square$

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a map of connected topological stacks, then we have an induced map $f_* : \pi_1(\mathfrak{X}) \to \pi_1(\mathfrak{Y})$ inducing a topological stack morphisms $f^* : \mathfrak{Ch}_{\mathfrak{Y},G} \to \mathfrak{Ch}_{\mathfrak{X},G}$ and $f^* : \chi_{\mathfrak{Y},G} \to \chi_{\mathfrak{X},G}$. Passing to the function, we finally get the map $(f^*)^* : \mathcal{O}_{\chi_{\mathfrak{X},G}} \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$.

**Proposition 5.10** *Let $i : \mathfrak{S} \hookrightarrow \mathfrak{Y}$ be an open embedding of oriented connected 2-dimensional orbifolds. The induced map $(i^*)^* : \mathcal{O}_{\chi_{\mathfrak{S},G}} \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is a map of Poisson algebras.*

Proof. The open embedding $i : \mathfrak{S} \hookrightarrow \mathfrak{Y}$ preserves the intersection product by [GiNo, §10]. Hence, we have a commutative diagram

$$\begin{tikzcd}
H_1(\mathfrak{S}, \mathfrak{g}_{\phi\circ i})^{\otimes 2} \arrow[r, "\Cap"] \arrow[d, "i_*"'] & H_0(\mathfrak{S}, \mathfrak{g}_{\phi\circ i}^{\otimes 2}) \arrow[r, "K_*"] \arrow[d, "i_*"'] & H_0(\mathfrak{S}, \mathbb{R}) \arrow[d, "i_*"'] \arrow[dr, "\cong"] & \\
H_1(\mathfrak{Y}, \mathfrak{g}_{\phi})^{\otimes 2} \arrow[r, "\Cap"] & H_0(\mathfrak{Y}, \mathfrak{g}_{\phi}^{\otimes 2}) \arrow[r, "K_*"] & H_0(\mathfrak{Y}, \mathbb{R}) \arrow[r, "\cong"] & \mathbb{R}
\end{tikzcd}$$

so that the induced map $i^* : \chi_{\mathfrak{Y},G} \cong G\backslash \mathrm{Mor}(\pi, G) \to G\backslash \mathrm{Mor}(\pi_1(\mathfrak{S}), G) \cong \chi_{\mathfrak{S},G}$ induces a map of Poisson algebras on the functions by Theorem 5.9. $\square$

## 5.3 Goldman Lie algebras homomorphim for orbifolds

Now let us define a map from the Goldman Lie algebra to the functions on the character stack as follows. Let $Tr : G \to \mathbb{R}$ be the trace map given by the composition of an embedding of $G$ in $GL_n(\mathbb{R})$ followed by the usual trace function on real valued matrices. Since it is a $G$-invariant map, it defines a function on the stack $\mathrm{Map}(S^1, BG^{con}) \cong [G\backslash G]$ (Example 5.6). By composition, we also have a differentiable stack morphism $\mathrm{Map}(S^1, \mathfrak{Y}) \times \mathrm{Map}(\mathfrak{Y}, BG^{con}) \to \mathrm{Map}(S^1, BG^{con})$ hence, passing to functions, a map

$$\pi_0(\mathrm{Map}(S^1, \mathfrak{Y}) \times \mathcal{O}_{[G\backslash G]}) \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}.$$

Choosing the Trace $Tr$ as the function, we thus get a map

$$Tr_* : \pi_0(\mathrm{Map}(S^1, \mathfrak{Y})) \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$$

Precisely, for $[\gamma] \in H_0^{S^1}(L\mathfrak{Y}) = \pi_0(\mathrm{Map}(S^1, \mathfrak{Y})$ the class associated to a loop $\gamma : S^1 \to \mathfrak{Y}$, we have that

$$Tr_*(\gamma) : \phi \mapsto Tr(\phi(\gamma)) \tag{5.4}$$

where $\phi \in \mathrm{Mor}(\pi, G)$.

**Remark 5.11** The map $Tr_*$ we just defined depends on the embedding $G \subset GLn(\mathbb{R})$. We can also replace the Trace $Tr$ by any $G$-invariant function $f : G \to \mathbb{R}$ (in particular such a $f$ is canonically a function on $\mathrm{Map}(S^1, BG^{con})$).

The Trace map is natural:

**Lemma 5.12** *Let $i : \mathfrak{X} \to \mathfrak{Y}$ be a map of 2-dimensional oriented orbifolds. The following square*

$$\begin{tikzcd} H_0^{S^1}(L\mathfrak{X}) \arrow[r, "Tr_*"] \arrow[d, "i_*"'] & \mathcal{O}_{\chi_{\mathfrak{X},G}} \arrow[d, "(i^*)^*"] \\ H_0^{S^1}(L\mathfrak{Y}) \arrow[r, "Tr_*"] & \mathcal{O}_{\chi_{\mathfrak{Y},G}}. \end{tikzcd} \tag{5.5}$$

*is commutative.*

PROOF. For $\gamma \in \pi_1(\mathfrak{X})$ and $\phi : \pi_1(\mathfrak{Y}) \to G$, we have

$$\begin{aligned} (i^*)^* \circ Tr_*(\gamma)(\phi) = Tr_*(\gamma)(i^*(\phi)) &= Tr_*(\gamma)(\phi \circ i) \\ &= Tr_*(\phi(i \circ \gamma)) \\ &= Tr_*(i_*(\gamma))(\phi). \end{aligned}$$

□

Following Goldman, for $G = GL_m(\mathbb{K})$, (with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$), we *choose* the standard embedding[4] of $G$ in $GL_n(\mathbb{R})$ to define the Trace $Tr$[5] and accordingly we take for $K$ the trace form associated to this standard representation[6].

The following result is an orbifold generalization of Goldman standard result for closed surfaces.

**Theorem 5.13** *Let $\mathfrak{Y}$ be a connected oriented finite type orbifold of dimension 2 and $G = GL_n(\mathbb{K})$ (with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$). The map $Tr_* : H_0^{S^1}(L\mathfrak{Y}) \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is a Lie algebra homomorphism.*

In the above theorem we take $k = \mathbb{Z}$ for the coefficients of the homology group.

Proof. Let $\mathfrak{S}$ be the complement of finitely many points in $\mathfrak{Y}$ so that we have epimorphisms $i_* : \pi_1(\mathfrak{S}) \to \pi$ and $H_0^{S^1}(L\mathfrak{S}) \to H_0^{S^1}(L\mathfrak{Y})$. For any $\phi : \pi \to G$, By Hurewicz theorem we also have a linear surjection $i_* : H_1(\mathfrak{S}, \mathfrak{g}_{\phi\circ i}) \to H_1(\mathfrak{Y}, \mathfrak{g}_\phi)$. By Lemma 5.12 and Proposition 5.10 we have a commutative diagram

$$\begin{tikzcd} H_0^{S^1}(L\mathfrak{S}) \arrow[r, "Tr_*"] \arrow[d, "i_*"'] & \mathcal{O}_{\chi_{\mathfrak{S},G}} \arrow[d, "(i^*)^*"] \\ H_0^{S^1}(L\mathfrak{Y}) \arrow[r, "Tr_*"] & \mathcal{O}_{\chi_{\mathfrak{Y},G}}. \end{tikzcd}$$

whose vertical arrows are Lie algebras homomorphims and futher, the left one is surjective. We want to show that the lower arrow is a map of Lie algebras It is thus enough to prove that the top horizontal arrow is a Lie algebra homomorphism to finish the proof.

Let us first do the easier case where $\mathfrak{Y}$ is further *reduced*. Then $\mathfrak{S}$, being the complement of the orbifold locus of $\mathfrak{Y}$, is an ordinary punctured Riemann surface. Goldman classical result [Go2, Theorem 3.5] (or more accurately [BiGu, Theorem 3.2]) applies to $\mathfrak{S}$ to show that $Tr_* : H_0^{S^1}(L\mathfrak{S}) \to \mathcal{O}\chi_{\mathfrak{S},G}$ is a Lie algebra homomorphism. Indeed, to compute the bracket $[Tr_*(\gamma), Tr_*(\beta)]$ we can use the Poisson bracket formula (5.3) which amounts to compute the Poincaré dual of the Hamiltonian vector field associated to $Tr_*(\gamma)$. By [Go2, Proposition 3.7], this is given by the class $\lfloor \gamma \otimes V(Tr_*)(\phi(\gamma)) \rfloor$ where $V(Tr) : G \to \mathfrak{g}$ is the variation of $Tr$. Thus the top horizontal arrow of (5.5) is a morphism of Lie algebras which concludes the proof in the reduced case.

Now if $\mathfrak{X}$ is non-reduced, then it is an $H$-gerbe over a reduced one $\mathfrak{X}_{red}$ (see 4.1). Taking $\mathfrak{S}$ to be the complement of finitely many points in $\mathfrak{X}$, we can assume $\mathfrak{S}$ to be a neutral $H$-gerbe over an hyperbolic surface $\mathfrak{S}_{\mathrm{red}} \subset \mathfrak{X}_{red}$. We let $\pi : \mathfrak{S} \to \mathfrak{S}_{red}$ denote the gerbe structure map.

---

[4]as the subgroup of $GL_{\dim(m)\mathbb{K})}(\mathbb{R})$ which centralizes the $\mathbb{K}$-structure on $\mathbb{K}^m$

[5]for instance, the map $Tr : GL_n(\mathbb{C}) \to \mathbb{R}$ is 2 times the real part of the usual trace $GL_n\mathbb{C}) \to \mathbb{C}$.

[6]for instance, $K : GL_n(\mathbb{C}) \to \mathbb{R}$ is 2 times the real part of the usual trace $GL_n\mathbb{C}) \to \mathbb{C}$.

As we have seen above, we only need to prove that $H_0^{S^1}(L\mathfrak{S}) \xrightarrow{Tr_*} \mathcal{O}_{\chi_{\mathfrak{S},G}}$ is a Lie algebra homomorphim to conclude the proof of the theorem.

By [GiNo, Section 11.3], we have that

$$\{x,y\}_{\mathfrak{S}} = \sum \pi_*^{-1}\big(\{\pi_*(x),\pi_*(y)\}_{\mathfrak{S}_{red}}\big) \tag{5.6}$$

in $H_0^{S^1}(\mathrm{L}\mathfrak{S})$. We now study the Poisson bracket on functions of $\chi_{\mathfrak{S},G}$. We refer to [BGNX] (in particular Sections 9 and 16) for details on Gysin maps and intersection product. The intersection pairing of an oriented orbifold $\mathfrak{X}$ $H_*(\mathfrak{X},\mathfrak{g})\otimes H_*(\mathfrak{X},\mathfrak{g}) \to H_*(\mathfrak{X}\times\mathfrak{X},\mathfrak{g}\otimes\mathfrak{g})) \overset{\mathsf{diag}^!}{\to} H_*(\mathfrak{X},\mathfrak{g}\otimes\mathfrak{g})[\dim(\mathfrak{X})]$ is the Gysin map associated to the diagonal $\mathsf{diag}:\mathfrak{X}\to\mathfrak{X}\times\mathfrak{X}$. Note that the diagonal map for $\mathfrak{S}$ factors as

$$\mathfrak{S} \longrightarrow \mathfrak{S}\times_{\mathfrak{S}_{red}}\mathfrak{S} \longrightarrow \mathfrak{S}\times\mathfrak{S}$$

where the first map is a principal $H$-bundle map. Hence the Gysin map $\mathsf{diag}^!$ in that case is the composition

$$H_*(\mathfrak{S}\times\mathfrak{S},\mathfrak{g}\otimes\mathfrak{g})) \xrightarrow{\pi^*\mathsf{diag}^!} H_{*-2}(\mathfrak{S}\times_{\mathfrak{S}_{red}}\mathfrak{S},\mathfrak{g}\otimes\mathfrak{g})) \xrightarrow{\tau_H} H_{*-2}(\mathfrak{X},\mathfrak{g}\otimes\mathfrak{g})$$

of the Gysin map associated to the pullback $\mathfrak{S}\times_{\mathfrak{S}_{red}}\mathfrak{S} \longrightarrow \mathfrak{S}\times\mathfrak{S}$ and the usual transfer for covering spaces by [BGNX, Lemma 9.4, formula 9.3.1, and 9.2.1]. Further, by naturality of Gysin morphisms ([BGNX, 9.2.2]), we obtain a commutative diagram

$$\begin{tikzcd}
H_1(\mathfrak{S},\mathfrak{g}_{\phi\circ\pi})^{\otimes 2} \arrow[r, "\pi^*\mathsf{diag}^!"] \arrow[d, "(\pi_*)^{\otimes 2}"'] & H_0(\mathfrak{S}\times_{\mathfrak{S}_{red}}\mathfrak{S},\mathfrak{g}_{\phi\circ\pi}^{\otimes 2}) \arrow[d, "\pi\times_{\mathfrak{S}_{red}}\pi_*", "\cong"'] \\
H_1(\mathfrak{S}_{red},\mathfrak{g}_{\phi})^{\otimes 2} \arrow[r, "\Cap"] & H_0(\mathfrak{S}_{red},\mathfrak{g}_{\phi}^{\otimes 2})
\end{tikzcd}$$

where the right vertical map is an isomorphism since it is induced in degree 0 by a $H\times H$-gerbe map. Hence, the intersection product for $\mathfrak{S}$ is given, for $x,y\in H_1(\mathfrak{S})$, by the formula

$$(x\Cap y) = \tau\circ\big(\pi\times_{\mathfrak{S}_{red}}\pi\big)_0^{-1}\big(\pi_*(x)\Cap\pi_*(y)\big). \tag{5.7}$$

The form $K:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ is given by the real part of the trace map, more precisely by the formula $\lambda\mathsf{Re}(Tr(XY))$ where $\lambda=1$ if $\mathbb{K}=\mathbb{R}$ and $\lambda=2$ if $\mathbb{K}=\mathbb{C},\mathbb{H}$. To compute the bracket (5.3) of $Tr_*(\lfloor x\rfloor)$ and $Tr_*(\lfloor y\rfloor)$, we apply the above formula to $dTr_*(\lfloor x\rfloor)_\varphi, dTr_*(\lfloor y\rfloor)_\varphi \in H_1(\mathfrak{X},\mathfrak{g}_\phi)$. Using Goldman classical computations [Go1, Proposition 3.7, 1.5 and 1.7], we obtain, for any $\varphi:\pi_1(\mathfrak{X})\to G$, $\lfloor x\rfloor,\lfloor y\rfloor\in H_0^{S^1}(\mathrm{L}\mathfrak{S})$,

$$[Tr_*(\lfloor x\rfloor),Tr_*(\lfloor y\rfloor)]_\varphi \quad = \quad \sum_{h\in H}\lambda\mathsf{Re}\Big(Tr\big((\pi_*(\lfloor x\rfloor)\otimes\varphi(x))\Cap(\pi_*(\lfloor y\rfloor))\otimes\varphi(hy)\big)\Big)$$

where we identify $\varphi(z) \in G$ with its image $\mathfrak{g}$ under the natural inclusion $G = GL_n(\mathbb{K}) \hookrightarrow M_n(\mathbb{K}) = \mathfrak{gl}_n(\mathbb{K})$. We can choose $x$ and $y$ to be a (linear combination) of curves such that (each of) the curves (in the linear combination) $\pi(x)$, $\pi(y)$ lies in the complement of the orbifold locus of the reduced orbifold $\mathfrak{S}_{red}$ and have transversal intersections with only double points. We denote $\pi(x)\#\pi(y)$ the set of transversal intersections and we denote as usual $\mathrm{sgn}(p)$ the sign of this intersection, that is $+1$ if the orientation given by the tangent vectors of $\pi_(x)$ and $\pi(y)$ agrees with the one of $\mathfrak{S}_{red}$ and $-1$ otherwise. We then get

$$\big[Tr_*(\lfloor x\rfloor), Tr_*(\lfloor y\rfloor)\big]_\varphi \quad = \sum_{\substack{h \in H,\\ p\in \pi(x)\#\pi(y)}} \mathrm{sgn}(p)\lambda\mathsf{Re}\big(Tr\big(\varphi(x_p \cdot hy_p)\big)\big)\big) \tag{5.8}$$

where $z_p$ means the representative of $z \in H_0^{S^1}(\mathrm{L}\mathfrak{S})$ in $\pi_1(\mathfrak{S}, \pi^{-1}(p))$ of the loop $z : S^1 \to \mathfrak{S}$ starting at a chosen lift $\pi^{-1}(p)$; by invariance under the conjugation action, the choice of the lift does not matter.

We also have by [GiNo, Proposition 11.12] and using that the curves $\pi(x)$, $\pi_(y)$ lies in an ordinary surface, that

$$\begin{aligned}
Tr_*\big(\{\lfloor x\rfloor, \lfloor y\rfloor\}_{\mathfrak{S}}\big)(\varphi) &= \lambda\mathsf{Re}\big(Tr\big(\sum \varphi\big(\pi_*^{-1}\{\pi_*(\lfloor x\rfloor), \pi_*(\lfloor y\rfloor)\}_{\mathfrak{S}_{red}}\big)\big)\\
&= \lambda\mathsf{Re}\big(Tr\big(\sum \varphi\big(\pi_*^{-1} \sum_{p\in\pi(x)\#\pi(y)} \mathrm{sgn}(p)(\pi(x)_p\cdot\pi(y)_p)\big)\big)\big)\\
&= \sum_{\substack{h \in H,\\ p\in \pi(x)\#\pi(y)}} \mathrm{sgn}(p)\lambda\mathsf{Re}\big(Tr\big(\varphi(x_p \cdot hy_p)\big)\big)\big)
\end{aligned}$$

which, by identity (5.8), shows that $H_0^{S^1}(L\mathfrak{S}) \xrightarrow{Tr_*} \mathcal{O}_{\chi_{\mathfrak{S},G}}$ is a Lie algebra homomorphism. There is nothing left to prove. $\square$

**Remark 5.14** Theorem 5.13 is a generalization of Goldman standard result for orbifolds but only use the character variety and not the full stack structure. Let $\mathcal{O}_{\mathfrak{Ch}_{\mathfrak{Y},G}}$ be the functions on the character stack. This is a (equivalence class of) commutative Hopf algebroid and in particular, for any presentation of $Z_1 \rightrightarrows Z_0$ of the stack, we get a simplicial commutative algebra defined as the nerve of the Hopf algebroid $\mathcal{O}_{Z_0} \rightrightarrows \mathcal{O}_{Z_1}$. Its cohomology is by definition the cohomology of the functions on the stack.

We write $H^\bullet(\mathcal{O}_{\mathfrak{Ch}_{\mathfrak{Y},G}})$ its cohomology. It is a graded commutative algebra which, in degree 0 is precisely $H^0(\mathcal{O}_{\mathfrak{Ch}_{\mathfrak{Y},G}}) \cong \mathcal{O}^G_{\mathrm{Mor}(\pi,G)}$. It will be pleasant to see whether all the algebra of functions is a Poisson algebra.

However, note that the character stack $\mathfrak{Ch}_{\mathfrak{Y},G}$ is really a stack and not an orbifold (unless $G$ is finite), so that we do not have a nice definition of symplectic structure. Further, it was proved in [AZ] that, for $\mathfrak{Y}$ a surface, the Goldman map can be lifted to a map of Lie algebras from the even degree part $(H^{S1}_{2\bullet}(L\mathfrak{Y})$

of the equivariant homology with values in the functions on a moduli space of super connections witnessing higher cohomological degree $H^i(\mathfrak{Y}, \mathfrak{g})$ and not only the degree $i = 1$ provided by Lemma 5.8. In view of the analogous situation in algebraic geometry, this suggests that differentiable stacks shall be embedded in a larger category of derived differentiable stack (not yet detailled in the literature) for which it seems reasonnable to expect that a derived character stack $\mathbb{R}\mathfrak{Ch}_{\mathfrak{Y},G}$ shall have a symplectic (or Poisson) structure.

**Question 5.15** *Is there a good notion of derived Fréchet Poisson stacks making* $\mathbb{R}\mathfrak{Ch}_{\mathfrak{Y},G}$ *Poisson in such a way that there is an natural linear morphism*

$$H^\bullet(\mathcal{O}_{[G\backslash G]}) \to H^\bullet(\mathrm{Map}_{dgLie}\left(H_\bullet^{S1}(L\mathfrak{Y}), H^\bullet(\mathcal{O}_{\mathbb{R}\mathfrak{Ch}_{\mathfrak{Y},G}})\right)$$

*(where* $\mathrm{Map}_{dgLie}$ *is the mapping space of Lie algebras homomorphisms), which in degree* $0$ *induces the above Goldman transformation* $Tr_*$ *?*

# 6 Lie algebras of unoriented strings

Theorem 5.13 also applies to other groups, for instance orthogonal or symplectic groups, provided one replaces free loops by *unoriented* free loops [Go1]. Indeed (see [Go1, Tu]) the free abelian group $\mathbb{Z}\overline{\pi}$, where $\overline{\pi}$ is set of free homotopy classes of unoriented loops on a surface $\Sigma$, has a Lie algebra structure and the trace defines a Lie algebra homomorphism with values in the functions of the character variety of orthogonal or symplectic groups (see *loc. cit.* or Theorem 6.13 below for a precise statement).

In this section we first give Chas-Sullivan type generalizations[7] of this Lie algebra structure of unoriented loops for arbitrary oriented stacks, and then to prove that there is a generalization of Goldman homomorphism for oriented orbifolds.

Note that $\mathbb{Z}\overline{\pi}$ is the degree 0 homology group $H_0^{O(2)}(L\Sigma)$ where the group $O(2)$ acts on $L(\Sigma)$ via its natural action on $S^1$. We recall that $H_i^{O(2)}(L\Sigma) \cong H_i([O(2)\backslash L\Sigma])$ where $[O(2)\backslash L\Sigma]$ is the quotient stack (see 2), which is homotopic to the stack $[Diff(S^1), \mathrm{L}\mathfrak{X}]$ of unoriented strings (see [GiNo, Remark 7.5]). From this identification and Goldman result, is natural to look for Lie algebra structures on $H_*^{O(2)} \cong H_*([O(2)\backslash \mathrm{L}\mathfrak{X}]$ for any oriented stack $\mathfrak{X}$.

## 6.1 Goldman and Chas-Sullivan Lie algebras of unoriented loops

By Lemma 1.1, we have a topological stack isomorphism $[O(2)\backslash \mathrm{L}\mathfrak{X}] \cong [\mathbb{Z}/2\mathbb{Z}\backslash[S^1\backslash \mathrm{L}\mathfrak{X}]]$. There is thus, by [GiNo, Lemma 8.1 and Definition 8.2], a transfer homomorphism

$$T^{\mathbb{Z}/2\mathbb{Z}} : H_i^{O(2)}(\mathrm{L}\mathfrak{X}) \cong H_i([O(2)\mathrm{L}\mathfrak{X}]) \to H_i([S^1\backslash \mathrm{L}\mathfrak{X}]) \cong H_i^{S^1}(\mathrm{L}\mathfrak{X}) \qquad (6.1)$$

[7]there are three of them, which are equivalent in characteristic zero but not in general

associated to the quotient

$$p : [S^1\backslash \mathrm{L}\mathfrak{X}] \longrightarrow [\mathbb{Z}/2\mathbb{Z}\backslash[S^1\backslash \mathrm{L}\mathfrak{X}]] \cong [O(2)\backslash \mathrm{L}\mathfrak{X}].$$

We also have, by [GiNo, Lemma 8.1 and Definition 8.2] again, a transfer map

$$T^{O(2)} : H_*^{O(2)}(\mathrm{L}\mathfrak{X}) \to H_{*+1}(\mathrm{L}\mathfrak{X}). \tag{6.2}$$

We denote the Chas-Sullivan product by $\star : H_*(\mathrm{L}\mathfrak{X})^{\otimes 2} \to H_{*-\dim(\mathfrak{X})}(\mathrm{L}\mathfrak{X})$ (see [ChSu] for manifolds and [BGNX] for arbitrary stacks).

We construct two Lie algebras as follows.

**Theorem 6.1** *Let $\mathfrak{X}$ be an oriented differentiable*[8] *stack of dimension $d$.*

1. *For $x, y \in H_*^{O(2)}(\mathrm{L}\mathfrak{X}) \cong H_*([O(2)\backslash \mathrm{L}\mathfrak{X}])$, the formula*

   $$\{x, y\}_{O(2)} := (-1)^{|x|} q_*\big(T^{O(2)}(x) \star T^{O(2)}(y)\big)$$

   *makes the equivariant homology $H_*^{O(2)}(\mathrm{L}\mathfrak{X})[2-d]$ into a graded Lie algebra. Here $T^{O(2)}$ is the above transgression map and $q : \mathrm{L}\mathfrak{X} \to [O(2)\backslash \mathrm{L}\mathfrak{X}]$ is the canonical projection.*

2. *Assume further that the multiplication by 2 is injective in the ground ring $k$. For $x, y \in p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big) \subset H_*^{O(2)}(\mathrm{L}\mathfrak{X})$ the formula*

   $$\widetilde{\{x, y\}} := p_*\big(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\}\big),$$

   *where $p_*^{-1}(y)$ is any pre-image of $y$ by $p_*$, is well-defined and makes the (sub)space $p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)[2-d])$ a Lie algebra.*

For our purpose, the important examples of oriented differentiable stacks are oriented smooth manifolds, oriented orbifolds and more generally quotient stacks $[G\backslash Y]$ of an oriented smooth manifold $Y$ by a smooth and orientation preserving action of a Lie group $G$. Our result thus gives a Lie algebra structure on shifted $O(2)$-equivariant homology of the free loop space of a manifold which was not explicited in the literature to our knowledge.

In homological degree 0, the map

$$p_* : H_0^{S^1}(\mathrm{L}\mathfrak{X}) \cong k[\widehat{\pi}] \longrightarrow \big(H_0^{S^1}(\mathrm{L}\mathfrak{X})\big)_{\mathbb{Z}/2\mathbb{Z}} \cong H_0^{O(2)}(\mathrm{L}\mathfrak{X}) \cong k[\overline{\pi}]$$

is surjective. Here we denote $\widetilde{\pi_1(\mathfrak{X})}$ and $\overline{\pi_1(\mathfrak{X})}$ respectively the sets of free homotopy classes of loops and unoriented free homotopy classes of loops. Since $p_*$ is surjective, the bracket $\{-,-\}$ is thus defined on all $H_0^{O(2)}(\mathrm{L}\mathfrak{X})$.

**Corollary 6.2** *Assume $\mathfrak{X}$ is of dimension 2 and the multiplication by 2 is injective in the ground ring $k$. Then $\widetilde{\{-,-\}}$ makes $H_0^{O(2)}(\mathrm{L}\mathfrak{X})$ a Lie algebra.*

[8] or more generally, an oriented Hurewicz stack, see [BGNX] for a definition.

**Definition 6.3**
- We call call $\big(H_*^{O(2)}(\mathrm{L}\mathfrak{X}), \{-,-\}_{O(2)}\big)$ the *unoriented Chas-Sullivan*[9] Lie algebra of $\mathfrak{X}$.
- When $\dim(\mathfrak{X}) = 2$, we also call $\big(H_0^{O(2)}(\mathrm{L}\mathfrak{X}), \widetilde{\{-,-\}}\big)$ the *unoriented Goldman*[10] Lie algebra of $\mathfrak{X}$.
- Finally, for general $\mathfrak{X}$, we call $\big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)[2-d], \widetilde{\{-,-\}}\big)$ the *extended unoriented Goldman* Lie algebra of $\mathfrak{X}$.

**Remark 6.4** In degree 0 and for $\mathfrak{X} = \Sigma$ an oriented surface, the unoriented Goldman Lie algebra structure on $p_*\big(H_0^{S^1}(L\Sigma)\big) \cong H_0^{O(2)}(L\Sigma)$ is the same as the one considered by Goldman on unoriented loops. This follows from Remark 6.10, identity 4. in Lemma 5.2 and Proposition 6.7 below, since, for a loop $\alpha : S^1 \to \Sigma$, $\varepsilon_*\lfloor\alpha\rfloor = \alpha^{-1}$.

**Remark 6.5** The unoriented Goldman bracket is a refinement of the (restriction in degree 0 for a dimension 2 oriented stack of the) Chas-Sullivan bracket. Indeed, it only uses half of the term involved in the Chas-Sullivan bracket, see Proposition 6.9 below.

Before proving the theorem we collect some results related to transfer maps and the $\mathbb{Z}/2\mathbb{Z}$-action. If $\mathfrak{Y}$ is a topological stack with $O(2)$-action (we care only about $\mathfrak{Y} = \mathrm{L}\mathfrak{X}$), we denote $\varepsilon : [S^1\backslash\mathfrak{Y}] \to [S^1\backslash\mathfrak{Y}]$ the non-trivial automorphism of the string stack $[S^1\backslash\mathfrak{Y}]$ induced by the $\mathbb{Z}/2\mathbb{Z}$ action on $[S^1\backslash\mathfrak{Y}]$ (that is the map induced by the action of $-1 \in \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$).

**Lemma 6.6** *Let $\mathfrak{Y}$ be a topological stack endowed with a $O(2)$-action (for instance $\mathfrak{Y} = \mathrm{L}\mathfrak{X}$). For $x, y \in H_*^{O(2)}(\mathfrak{Y})$, one has the following identities*

1. $p_*(T^{\mathbb{Z}/2\mathbb{Z}}(x)) = 2x$;
2. $T^{O(2)}(x) = T \circ T^{\mathbb{Z}/2\mathbb{Z}}(x)$ *where* $T : H_*^{S^1}(\mathfrak{Y}) \to H_{*+1}(\mathfrak{Y})$ *is the transfer map from [GiNo, Definition 8.2].*
3. $\varepsilon_* \circ T^{\mathbb{Z}/2\mathbb{Z}}(x) = T^{\mathbb{Z}/2\mathbb{Z}}(x)$; *that is the image of* $T^{\mathbb{Z}/2\mathbb{Z}}$ *lies in the* $\mathbb{Z}/2\mathbb{Z}$-*invariant subspace of* $H_*^{S^1}(\mathfrak{Y})$;
4. *For* $z \in H_*^{S^1}(\mathfrak{Y})$, *one has*
$$T^{\mathbb{Z}/2\mathbb{Z}}(p_*(z)) = z + \varepsilon_*(z).$$

PROOF. Replacing $[S^1\backslash\mathfrak{Y}]$ by a classifying space [No2] and by functoriality of transfer [GiNo, Proposition 8.3], it is enough to prove 1. and 4. for a topological space. By [BGNX, Lemma 9.4], it reduces to the usual transfer formula for finite covering and the result is now standard. Similarly identity 3. follows

[9] or Chas-Sullivan Lie algebra of unoriented strings
[10] or Goldman Lie algebra of unoriented strings

from naturality of Gysin maps [BGNX, Section 9.2] applied to the commutative diagram

$$\begin{tikzcd} [S^1\backslash\mathfrak{Y}] \arrow[r, "\varepsilon"] \arrow[d, "p"'] & [S^1\backslash\mathfrak{Y}] \arrow[d, "p"] \\ [O(2)\backslash\mathfrak{Y}] \arrow[r, "id"] & [O(2)\backslash\mathfrak{Y}]. \end{tikzcd}$$

Identitiy 2. follows from functoriality of Gysin maps [BGNX, Section 9.2] applied to the composition

$$q : \mathfrak{Y} \xrightarrow{\pi} [S^1\backslash\mathfrak{Y}] \xrightarrow{p} [\mathbb{Z}/2\mathbb{Z}\backslash[S^1\backslash\mathfrak{Y}]] \cong [O(2)\backslash\mathfrak{Y}].$$

□

We now study the action of $\mathbb{Z}/2\mathbb{Z}$ on the Goldman bracket for oriented loops.

**Proposition 6.7** *For $a, b \in H_*^{S^1}(\mathrm{L}\mathfrak{X})$, one has*

$$\{\varepsilon_*(a), \varepsilon_*(b)\} = \varepsilon_*\big(\{a, b\}\big).$$

*In particular, the invariant subspace $\Big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\Big)^{\mathbb{Z}/2\mathbb{Z}}$ is a Lie subalgebra of $\Big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d], \{-,-\}\Big)$.*

Proof. We start with the first claim. Let us denote $m : \mathrm{Map}(8, \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ the gluing of loops, that is the map induced after composition with $\mathrm{Map}(-, \mathfrak{X})$ by the canonical pinching map $S^1 \to S^1 \vee S^1 = 8$ identifying 1 and its antipodal point[11]. We recall that $\mathrm{Map}(8, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ since $\mathfrak{X}$ is Hurewicz ([BGNX, Corollary 5.3]). We let $\tau : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ be the map exchanging the two factors. We abusively denote similarly the map exchanging the two factors of the product $\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}$. The key observation is that, a contrario of the case of based loops, the multiplication of loops $m : \mathrm{Map}(8, \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ is homotopy commutative (see [ChSu] or [BGNX, proof of Proposition 10.8]), hence $m_* : H_*(\mathrm{Map}(8, \mathfrak{X})) \to H_*(\mathrm{L}\mathfrak{X})$ is equal to $m_* \circ \tau_*$. Further, since

$$\varepsilon \circ m = m \circ \tau \circ (\varepsilon \times_{\mathfrak{X}} \varepsilon) : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$$

we thus obtain after passing to homology

$$\begin{aligned} \varepsilon_* \circ \pi_* \circ m_* = \pi_* \circ \varepsilon_* \circ m_* &= \pi_* \circ m_* \circ \tau_* \circ (\varepsilon \times_{\mathfrak{X}} \varepsilon)_* \\ &= \pi_* \circ m_* \circ (\varepsilon \times_{\mathfrak{X}} \varepsilon)_* \end{aligned}$$

[11]If $\mathfrak{X}$ is a topological space it is simply the standard map composing two loops with the same base point into one loop

as maps $H_*(\mathrm{Map}(8,\mathfrak{X})) \to H_*(\mathrm{L}\mathfrak{X})$. It follows that the right square in diagram

$$\begin{tikzcd}
H_*(\mathrm{L}\mathfrak{X})^{\otimes 2} \arrow[r, "S"] \arrow[d, "(\varepsilon_*)^{\otimes 2}"'] & H_*(\mathrm{L}\mathfrak{X}\times \mathrm{L}\mathfrak{X}) \arrow[r, "\Delta^!"] \arrow[d, "(\varepsilon\times\varepsilon)_*"] & H_{*-d}(\mathrm{Map}(8,\mathfrak{X})) \arrow[r, "\pi_*\circ m_*"] \arrow[d, "(\varepsilon\times_{\mathfrak{X}}\varepsilon)_*"] & H^{S^1}_{*-d}(\mathrm{L}\mathfrak{X}) \arrow[d, "\varepsilon_*"] \\
H_*(\mathrm{L}\mathfrak{X})^{\otimes 2} \arrow[r, "S"] & H_*(\mathrm{L}\mathfrak{X}\times \mathrm{L}\mathfrak{X}) \arrow[r, "\Delta^!"] & H_{*-d}(\mathrm{Map}(8,\mathfrak{X})) \arrow[r, "\pi_*\circ m_*"] & H^{S^1}_{*-d}(\mathrm{L}\mathfrak{X})
\end{tikzcd}$$

is commutative. Here $\Delta^!$ is the Gysin map associated to the diagonal $\Delta : M \to M\times M$ and $S$ is the cross product, see [BGNX, Section 10.1]. The middle square commutes by naturality of the Gysin map [BGNX, 9.2.2] and the left square by naturality of the cross product. Since the Chas-Sullivan loop product is the composition ([BGNX])

$$H_p(\mathrm{L}\mathfrak{X})\otimes H_q(\mathrm{L}\mathfrak{X}) \stackrel{S}{\to} H_{p+q}(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}) \stackrel{\Delta^!}{\to} H_{p+q-d}(\mathrm{Map}(8,\mathfrak{X})) \stackrel{m_*}{\to} H_{p+q-d}(\mathrm{L}\mathfrak{X}),$$

we are left to prove that the diagram

$$\begin{tikzcd}
\left(H^{S^1}_{*-1}(\mathrm{L}\mathfrak{X})\right)^{\otimes 2} \arrow[r, "T\otimes T"] \arrow[d, "(\varepsilon_*)^{\otimes 2}"'] & H_*(\mathrm{L}\mathfrak{X})^{\otimes 2} \arrow[d, "(\varepsilon_*)^{\otimes 2}"] \\
\left(H^{S^1}_{*-1}(\mathrm{L}\mathfrak{X})\right)^{\otimes 2} \arrow[r, "T\otimes T"] & H_*(\mathrm{L}\mathfrak{X})^{\otimes 2}
\end{tikzcd}$$

is commutative. Since $T$ is the Gysin map associated to $\mathrm{L}\mathfrak{X} \to [S^1\backslash \mathrm{L}\mathfrak{X}]$ ([GiNo, Definition 8.2]), this result follows again by naturality of Gysin map [BGNX, 9.2.2] and the Proposition is proved. $\square$

We now compare the formula defining the bracket $\{-,-\}_{O(2)}$ with the Goldman bracket.

**Lemma 6.8** *For any $x, y \in H^{O(2)}_*(\mathrm{L}\mathfrak{X})$, one has*

$$\{x,y\}_{O(2)} = p_*(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\}). \tag{6.3}$$

*Further, one has*

$$T^{\mathbb{Z}/2\mathbb{Z}}\big(\{x,y\}_{O(2)}\big) = 2\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\}. \tag{6.4}$$

Proof. Let us denote by $d$ the dimension of $\mathfrak{X}$. By definition of the Lie bracket for $S^1$-equivariant loops ([GiNo, Corollary 9.6]), the bracket $\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\})$ is given (up to the sign $(-1)^{|x|}$) by the last line of the following diagram

$$\begin{tikzcd}
H^{O(2)}_{p-1}(\mathrm{L}\mathfrak{X})\otimes H^{O(2)}_{q-1}(\mathrm{L}\mathfrak{X}) \arrow[r, "(T^{O(2)})^{\otimes 2}"] \arrow[d, "\left(T^{\mathbb{Z}/2\mathbb{Z}}\right)^{\otimes 2}"'] & H_p(\mathrm{L}\mathfrak{X})\otimes H_q(\mathrm{L}\mathfrak{X}) \arrow[r, "\star"] \arrow[d, equal] & H_{p+q-d}(\mathrm{L}\mathfrak{X}) \arrow[r, "q_*"] \arrow[d, equal] & H^{O(2)}_{p+q-d}(\mathrm{L}\mathfrak{X}) \\
H^{S^1}_{p-1}(\mathrm{L}\mathfrak{X})\otimes H^{S^1}_{q-1}(\mathrm{L}\mathfrak{X}) \arrow[r, "T^{\otimes 2}"] & H_p(\mathrm{L}\mathfrak{X})\otimes H_q(\mathrm{L}\mathfrak{X}) \arrow[r, "\star"] & H_{p+q-d}(\mathrm{L}\mathfrak{X}) \arrow[r, "\pi_*"] & H^{S^1}_{p+q-d}(\mathrm{L}\mathfrak{X}). \arrow[u, "p_*"']
\end{tikzcd} \tag{6.5}$$

The commutativity of the left square in the diagram is precisely the identity 2. in Lemma 6.6 while the right square is commutative by definition of $q = p \circ \pi$. This proves identity (6.3) since the first line of the diagram is the bracket $\{-,-\}_{O(2)}$.

The second identity (6.4) follows from the first one since, by Lemma 6.6 (identity 4) and Proposition 6.7, we have that

$$\begin{aligned}\{x,y\}_{O(2)} &= T^{\mathbb{Z}/2\mathbb{Z}} \circ p_*(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\}) \\ &= \{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\} + \varepsilon_*\big(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\}\big) \\ &= \{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\} + \{\varepsilon_* \circ T^{\mathbb{Z}/2\mathbb{Z}}(x), \varepsilon_* \circ T^{\mathbb{Z}/2\mathbb{Z}}(y)\} \\ &= 2\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\}\end{aligned}$$

by invariance of the image of $T^{\mathbb{Z}/2\mathbb{Z}}$ (identity 3. in Lemma 6.6). □

Proof of Theorem 6.1.

1. The (graded) antisymmetry of the bracket $\{-,-\}_{O(2)}$ is a consequence of the fact that the Chas-Sullivan product $\star$ is graded commutative.

We are left to prove the Jacobi identity for $\{-,-\}_{O(2)}$.

$$\{x,\{y,z\}_{O(2)}\}_{O(2)} \pm \{y,\{z,x\}_{O(2)}\}_{O(2)} \pm \{z,\{x,y\}_{O(2)}\}_{O(2)} = 0.$$

The proof is similar to the one of the Goldman bracket. Indeed, since $q = p \circ \pi$, by identity 2 in Lemma 6.6 and $\mathbb{Z}/2\mathbb{Z}$-equivariance of $\pi$,

$$\begin{aligned}\{x,\{y,z\}_{O(2)}\}_{O(2)} &= (-1)^{|x|+|y|} q_*\Big(T^{O(2)}(x) \star T \circ T^{\mathbb{Z}/2\mathbb{Z}} \circ p_* \circ \pi_*\big(T^{O(2)}(y) \star T^{O(2)}(z)\big)\Big) \\ &= (-1)^{|x|+|y|} q_*\Big(T^{O(2)}(x) \star T \circ (id + \varepsilon_*) \circ \pi_*\big(T^{O(2)}(y) \star T^{O(2)}(z)\big)\Big) \\ &= (-1)^{|x|+|y|} q_*\Big(T^{O(2)}(x) \star T \circ \pi_* \circ (id + \varepsilon_*)\big(T^{O(2)}(y) \star T^{O(2)}(z)\big)\Big)\end{aligned}$$

Repeating an argument from the proof of Proposition 6.7, we have

$$\varepsilon_*\big(T^{O(2)}(y) \star T^{O(2)}(z)\big) = \varepsilon_*\big(T^{O(2)}(y)\big) \star \varepsilon_*\big(T^{O(2)}(z)\big) = T^{O(2)}(y) \star T^{O(2)}(z)$$

where the last equality follows from identity 2 and 3 in Lemma 6.6 and $\varepsilon_* \circ T = T \circ \varepsilon_*$ (by naturality of Gysin maps [GiNo, 9.2.2]). Plugging this identity in the previous one, we finally have

$$\{x,\{y,z\}_{O(2)}\}_{O(2)} = 2(-1)^{|x|+|y|} q_*\Big(T^{O(2)}(x) \star D\big(T^{O(2)}(y) \star T^{O(2)}(z)\big)\Big) \quad (6.6)$$

where $D = T \circ \pi_*$ (see [ChSu, GiNo]) is the **BV**-operator in the homology $H_*(\mathrm{L}\mathfrak{X})[-d]$. Summing up the previous identity (6.6) and its two symmetric ones arising in the Jacobi identity, we see that the Jacobi identity reduces to the **BV**-identity

$$\begin{aligned}D(a \star b \star c) - D(a \star b) \star c - (-1)^{|a|} a \star D(b \star c) - (-1)^{(|a|+1)|b|} b \star D(a \star c) + \\ + D(a) \star b \star c + (-1)^{|a|} a \star D(b) \star c + (-1)^{|a|+|b|} a \star b \star D(c) = 0\end{aligned}$$

and $q_* \circ D = p_* \circ (\pi_* \circ T) \circ \pi_* = 0$ as well as $D \circ T^{O(2)} = T \circ (\pi_* \circ T) \circ T^{\mathbb{Z}/2\mathbb{Z}} = 0$, which holds by the long exact sequence of $S^1$-equivariant homology [GiNo, Proposition 8.4].

2. We now study the bracket $\widetilde{\{-,-,\}}$ on the image $p_*(H_*^{S^1}(\mathrm{L}\mathfrak{X}))$. We start by proving the following identity:

$$T^{\mathbb{Z}/2\mathbb{Z}}\big(\widetilde{\{x,y\}}\big) = \Big\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\Big\}. \tag{6.7}$$

By definition of $\widetilde{\{-,-,\}}$, identity 4 in Lemma 6.6 and Proposition 6.7, we get

$$\begin{aligned}
T^{\mathbb{Z}/2\mathbb{Z}}\big(\widetilde{\{x,y\}}\big) &= T^{\mathbb{Z}/2\mathbb{Z}} \circ p_*\big(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\}\big) \\
&= \big\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\big\} + \varepsilon_*\big(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\}\big) \\
&= \big\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\big\} + \big\{\varepsilon_* \circ T^{\mathbb{Z}/2\mathbb{Z}}(x), \varepsilon_* \circ p_*^{-1}(y)\big\} \\
&= \big\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y) + \varepsilon_*(p_*^{-1}(y))\big\}
\end{aligned}$$

by invariance of the image of $T^{\mathbb{Z}/2\mathbb{Z}}$ (identity 3. in Lemma 6.6). Now identity (6.7) follows from

$$T^{\mathbb{Z}/2\mathbb{Z}}(y) = T^{\mathbb{Z}/2\mathbb{Z}}\big(p_*\big(p_*^{-1}(y)\big)\big) = p_*^{-1}(y) + \varepsilon_*(p_*^{-1}(y).$$

which holds by identity 4 in Lemma 6.6.

Note that the right hand side of identity (6.7) does not depend on any choice of a preimage of $y$. Since $T^{\mathbb{Z}/2\mathbb{Z}}$ is injective (by identity 1 in Lemma 6.6 and our assumption on the ground ring), it follows that the bracket $\widetilde{\{x,y\}}$ is independent of the choice of $p_*^{-1}(y)$. Thus it is well-defined. Moreover, it is thus also graded antisymmetric since $\{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}(y)\Big\}$ is so.

To prove the Jacobi identity for $\widetilde{\{-,-,\}}$, again by injectivity of $T^{\mathbb{Z}/2\mathbb{Z}}$, it is thus enough to prove that

$$T^{\mathbb{Z}/2\mathbb{Z}}\Big(\widetilde{\big\{x,\widetilde{\{y,z\}}\big\}} \pm \widetilde{\big\{y,\widetilde{\{z,x\}}\big\}} \pm \widetilde{\big\{z,\widetilde{\{x,y\}}\big\}}\Big) = 0.$$

From identity 6.7, we deduce that

$$\begin{aligned}
T^{\mathbb{Z}/2\mathbb{Z}}\Big(\widetilde{\big\{x,\widetilde{\{y,z\}}\big\}}\Big) &= \{T^{\mathbb{Z}/2\mathbb{Z}}(x), T^{\mathbb{Z}/2\mathbb{Z}}\big(\widetilde{\{y,z\}}\big)\} \\
&= \{T^{\mathbb{Z}/2\mathbb{Z}}(x), \{T^{\mathbb{Z}/2\mathbb{Z}}(y), T^{\mathbb{Z}/2\mathbb{Z}}(z)\}\}.
\end{aligned}$$

Hence the Jacobi identity for $\widetilde{\{-,-,\}}$ follows from the Jacobi identity for the Goldman bracket. □

The several Lie algebras structures on the homology unoriented strings (from the previous section), strings and free loops are related as follows.

**Proposition 6.9** *Let $\mathfrak{X}$ be an oriented differentiable (or Hurewicz) stack of dimension $d$.*

- *The transfer maps $2T^{\mathbb{Z}/2\mathbb{Z}} : H_*^{O(2)}(\mathrm{L}\mathfrak{X})[2-d] \to H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]$ and $2T^{O(2)} : H_*^{O(2)}(\mathrm{L}\mathfrak{X})[2-d] \to H_*(\mathrm{L}\mathfrak{X})[1-d]$ are graded Lie algebras homomorphisms. Here, $H_*(\mathrm{L}\mathfrak{X})[1-d]$ is the graded Lie algebra structure underlying the* **BV***-algebra structure of free loop homology (see [ChSu] for manifolds or [BGNX] for stacks).*

- *Assume that the multiplication by 2 is injective in the ground ring $k$. Then the following diagram*

$$\begin{tikzcd}
\Big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \{-,-\}_{O(2)}\Big) \arrow[r, hook] \arrow[d, hook, "\times 2"'] & \Big(H_*^{O(2)}(\mathrm{L}\mathfrak{X})[2-d], \{-,-\}_{O(2)}\Big) \arrow[d, "2T^{\mathbb{Z}/2\mathbb{Z}}"] \arrow[ddd, bend left=60, "2T^{O(2)}"] \\
\Big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \{\widetilde{-,-,}\}\Big) \arrow[r, "T^{\mathbb{Z}/2\mathbb{Z}}"] \arrow[ddr, bend right, "T^{O(2)}"] & \Big(\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big)^{\mathbb{Z}/2\mathbb{Z}}, \{-,-\}\Big) \arrow[d, hook] \\
 & \Big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d], \{-,-\}\Big) \arrow[d, "T"'] \\
 & \Big(H_*(\mathrm{L}\mathfrak{X})[1-d], \{-,-\}_{BV}\Big)
\end{tikzcd}$$

  *is a commutative diagram of* graded Lie algebras homomorphisms*.*

- *If in addition 2 is invertible in the ground ring $k$, then all the maps in the upper square of the previous diagrams are Lie algebras* isomorphisms*.*

Proof.First we note that the spaces arising in the diagram are all Lie algebras by Theorem 6.1, Proposition 6.7, identity 6.3 and [GiNo, Corollary 9.6].

Now we also note that the maps $2T^{\mathbb{Z}/2\mathbb{Z}}$ and $T^{\mathbb{Z}/2\mathbb{Z}}$ factors through the invariant $\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big)^{\mathbb{Z}/2\mathbb{Z}}$ subspace as an immediate consequence of identity 3 in Lemma 6.6.

The equality (6.4) also tells us that

$$2T^{\mathbb{Z}/2\mathbb{Z}} : \Big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \{-,-\}_{O(2)}\Big) \to \Big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d], \{-,-\}\Big)$$

is a graded Lie algebra morphism. By Lemma 6.6, $2T^{O(2)} : H_*^{O(2)}(\mathrm{L}\mathfrak{X})[2-d] \to H_*(\mathrm{L}\mathfrak{X})[1-d]$ factors as the composition $2T^{O(2)} = T \circ 2T^{\mathbb{Z}/2\mathbb{Z}}$. Since $T$ is also a graded Lie algebra map ([GiNo, Corollary 9.6 ]), the first claim is proved.

That $T^{\mathbb{Z}/2\mathbb{Z}} : \Big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \{\widetilde{-,-,}\}\Big) \longrightarrow \Big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d], \{-,-\}\Big)$ is a graded Lie algebra map is identity (6.7) that we proved earlier. To prove claim 2, we are only left to prove that $\Big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \{-,-\}_{O(2)}\Big) \xrightarrow{\times 2}$ $\Big(p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \widetilde{\{-,-,\}}\Big)$ is a graded Lie algebras morphism. Since $T^{\mathbb{Z}/2\mathbb{Z}}$ is an injective (by identity 1 in Lemma 6.6 and our assumption on the ground ring), Lie algebra map, the result follows from the first claim and commutativity of the diagram.

Further, if 2 is invertible, then $\frac{1}{2}(id+\varepsilon_*) : \big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)_{\mathbb{Z}/2\mathbb{Z}} \overset{\cong}{\to} \big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)^{\mathbb{Z}/2\mathbb{Z}}$ is an isomorphism. From the canonical isomorphisms

$$\begin{aligned}\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)_{\mathbb{Z}/2\mathbb{Z}} &\cong \big(H_*([S^1\backslash \mathrm{L}\mathfrak{X}])\big)_{\mathbb{Z}/2\mathbb{Z}} \\ &\cong (H_*([\mathbb{Z}/2\mathbb{Z}\backslash[S^1\backslash \mathrm{L}\mathfrak{X}]]) \cong H_*([O(2)\backslash \mathrm{L}\mathfrak{X}]) \cong H_*^{O(2)}(\mathrm{L}\mathfrak{X})\end{aligned}$$

and identities 1 and 4 in Lemma 6.6, we deduce that $p_* : \big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)_{\mathbb{Z}/2\mathbb{Z}} \to H_*^{O(2)}(\mathrm{L}\mathfrak{X})$ is an isomorphism and so is $T^{\mathbb{Z}/2\mathbb{Z}} : H_*^{O(2)}(\mathrm{L}\mathfrak{X}) \to \big(H_*^{S^1}(\mathrm{L}\mathfrak{X})\big)_{\mathbb{Z}/2\mathbb{Z}}$. This implies that all the maps in upper square of the diagram are bijective. □

**Remark 6.10** Proposition 6.9 implies that over a ground field of characteristic different from 2, the Lie subalgebra $\Big(\big(H_{*+2-d}^{S^1}(\mathrm{L}\mathfrak{X})\big)^{\mathbb{Z}/2\mathbb{Z}}, \{-,-\}\Big)$ is isomorphic to the Lie algebra $\Big(H_{*+2-d}^{O(2)}(\mathrm{L}\mathfrak{X}), \{-,-\}\Big)$. This is well-known for $\mathfrak{X}$ a surface in homological degreee 0 [Tu].

If one takes $\mathbb{Z}$ as a ground ring, in the case where $\mathfrak{X} = \Sigma$ is an oriented surface, it was already noted by Goldman that we still have an isomorphism between the invariant part of the Goldman Lie algebra $H_0^{S^1}(L\Sigma)^{\mathbb{Z}/2\mathbb{Z}}$ and $\mathbb{Z}\overline{\pi} = H_0^{O(2)}(L\Sigma)$ (where $\overline{\pi}$ is the set of unoriented free homotopy classes of loops in $\Sigma$). This isomorphism is simply given by sending the class of the (unoriented) constant loop to itself and by sending the unoriented homotopy class $[x]$ of a loop $x : S^1 \to \Sigma$ onto the class $\lfloor x\rfloor + \varepsilon_*\lfloor x+\rfloor \in H_0^{S^1}(L\Sigma)$. The fact that it is an isomorphism comes from the fact that $\lfloor x\rfloor$ and $\varepsilon_*\lfloor x\rfloor$ are not freely homotopic to each other in an oriented surface (unless they are homotopically constant). That this map is a Lie algebras homomorphism follows again from Proposition 6.7 (more precisely it is the same as the proof of identity (6.7)) and the fact that the class of constant loops are in the center of the Goldman algebra.

However, this identification between $\mathbb{Z}/2\mathbb{Z}$-invariants of $H_0^{S^1}(\mathrm{L}\mathfrak{X})$ and $H_0^{O(2)}(\mathrm{L}\mathfrak{X})$ does not need to apply in general for arbitrary stacks, not even effective 2-dimensional orbifolds.

## 6.2 Goldman homomorphism for unoriented strings

We now study the unoriented version of Theorem 5.13. Recall that an embedding $G \hookrightarrow GL_m(\mathbb{R})$ followed by the usual Trace defines a $G$-invariant function on $G$ as well as the map $Tr_* : H_0^{S^1}(L\mathfrak{Y}) \cong \mathbb{Z}[\widehat{\pi_1(\mathfrak{Y})}] \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ given by identity (5.4). In general, the map $Tr_*$ is not invariant under the $\mathbb{Z}/2\mathbb{Z}$-action on

$H_0^{S^1}(L\mathfrak{Y})$ and thus does not pass to the quotient. However, this holds true for the standard embeddings of groups of automorphisms of nondegenerate bilinear forms. More precisely:

**Lemma 6.11** *Let $G = O_n(\mathbb{K})$, $O_{p,q}$, $U_{p,q}$, $Sp_{p,q}$, $Sp_n(\mathbb{R})$ or $Sp_n(\mathbb{C})$ and* choose *the trace map $Tr : G \to \mathbb{R}$ to be the one associated to their standard embedding in $GL_m(\mathbb{R})$. The map $Tr_* : H_0^{S^1}(L\mathfrak{Y}) \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is $\mathbb{Z}/2\mathbb{Z}$-invariant and thus passes to the quotient to define*

$$Tr_* : H_0^{O(2)}(L\mathfrak{Y}) \cong \Big(H_0^{S^1}(L\mathfrak{Y})\Big)_{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathcal{O}_{\chi_{\mathfrak{Y},G}}. \tag{6.8}$$

Proof. This is essentially due to Goldman. Indeed, under the assumption of the Lemma, we have that $Tr(A) = Tr(A^{-1})$ [Go1, Proof of Theorem 3.14]. Hence, for $\phi :\in \mathrm{Mor}(\widehat{\pi_1(\mathfrak{Y})}, G)$ a group morphism, and $[\gamma] \in \pi_0(\mathrm{Map}(S^1, \mathfrak{Y}))$, one has

$$\begin{aligned} Tr_*(\varepsilon([\gamma])) &= Tr\big(\phi((\gamma)^{-1})\big) \\ &= Tr\big((\phi(\gamma))^{-1}\big) \\ &= Tr\big(\phi(\gamma)\big). \end{aligned}$$

by definition of $Tr_*$ and of the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Map}(S^1, \mathfrak{Y})$. $\square$

**Remark 6.12** The mapping stack $\mathrm{Map}(S^1, \mathfrak{B}^{\mathrm{fl}}G)$ has an $\mathbb{Z}/2\mathbb{Z}$-action induced by the $O(2)$-action on $S^1$. Under the identification $\mathrm{Map}(S^1, \mathfrak{B}^{\mathrm{fl}}G) \cong [G\backslash G]$ (Example 5.6), this action is given on objects by $\varepsilon(g) = g^{-1}$. Thus the fact that a $G$-invariant map $f : G \to \mathbb{R}$ (such as $Tr$) defines a function on the quotient stack $[\mathbb{Z}/2\mathbb{Z}\backslash[G\backslash G]]$ is precisely equivalent to $f(A) = f(A^{-1})$. In view of the previous proof, this is precisely the condition needed for the map $[\gamma] \mapsto \Big(\phi \mapsto f(\phi(\gamma))\Big)$ to define a linear map $H_0^{O(2)}(L\mathfrak{Y}) \longrightarrow \mathcal{O}_{\chi_{\mathfrak{Y},G}}$.

For the above groups, the map $Tr_* : H_0^{S^1}(L\mathfrak{Y}) \to \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is not a Lie algebra map but it becomes one if one passes to the unoriented Goldman Lie algebra $H_0^{S^1}(L\mathfrak{Y})$ (with the Lie bracket $\widetilde{\{-,-\}}$) provided by Corollary 6.2.

**Theorem 6.13** *Let $\mathfrak{Y}$ be a connected oriented finite type reduced orbifold of dimension 2 and $G = O_n(\mathbb{K})$, $O_{p,q}$, $U_{p,q}$, $Sp_{p,q}$, $Sp_n(\mathbb{R})$ or $Sp_n(\mathbb{C})$. The linear map $Tr_* : \Big(H_0^{O(2)}(L\mathfrak{Y}, \mathbb{Z}), \widetilde{\{-,-\}}\Big) \longrightarrow \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is a Lie algebra homomorphism.*

In view of identity (6.4), it follows that : $2Tr_* : \Big(H_0^{O(2)}(L\mathfrak{Y}, \mathbb{Z}), \{-,-\}_{O(2)}\Big) \longrightarrow \mathcal{O}_{\chi_{\mathfrak{Y},G}}$ is also Lie algebra homomorphism.

Before proving the theorem, we prove the unoriented analogue of our result (Proposition 2.2) functoriality of Chas-Sullivan Lie algebra structure for open embeddings; it actually holds in full generality for stacks.

**Lemma 6.14** *Let $\mathfrak{X}$ be an oriented Hurewicz stack of dimension $d$, and $\mathfrak{U} \subseteq \mathfrak{X}$ an open substack. Then, $\mathfrak{U}$ inherits a natural orientation from $\mathfrak{X}$, and further*

- *the induced maps $\big(H_*^{O(2)}(L\mathfrak{U})[2-d], \{-,-\}_{O(2)}\big) \to \big(H_*^{O(2)}(\mathrm{L}\mathfrak{X})[2-d], \{-,-\}_{O(2)}\big)$, is a morphism of graded Lie algebras;*

- *if mutliplication by 2 is injective in the ground ring $k$, then the maps $\big(p_*\big(H_*^{S^1}(L\mathfrak{U})[2-d]\big), \widetilde{\{-,-\}}\big) \to p_*\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big), \widetilde{\{-,-\}}\big)$ and $\Big(\big(H_*^{S^1}(L\mathfrak{U})[2-d]\big)\big)^{\mathbb{Z}/2\mathbb{Z}}, \{-,-\}\Big) \to \Big(\big(H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]\big)\big)^{\mathbb{Z}/2\mathbb{Z}}, \{-,-\}\Big)$ are morphism of graded Lie algebras. In particular, if $\mathfrak{X}$ is of dimension 2, then the induced map $H_0^{O(2)}(L\mathfrak{U}), \widetilde{\{-,-\}}\big) \to \big(H_0^{O(2)}(\mathrm{L}\mathfrak{X}), \widetilde{\{-,-\}}\big)$ between the unoriented Goldman Lie algebras is a Lie algebra homormophism.*

Proof. By [GiNo, Proposition 10.3], $\mathfrak{U}$ is naturally oriented, $H_*(L\mathfrak{U})[1-d] \to H_*(L\mathfrak{X})[1-d]$ is a **BV**-algebra morphism and the induced map $H_*^{S^1}(\mathrm{L}\mathfrak{U})[2-d], \to H_*^{S^1}(\mathrm{L}\mathfrak{X})[2-d]$ is a morphism of graded Lie algebras (with respect to the bracket $\{-,-\}$). Since $L\mathfrak{U} \to L\mathfrak{X}$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant, the result factors through the $\mathbb{Z}/2\mathbb{Z}$-invariant subspace in homology.

Let us denote by the same letter $i_*$ the linear maps $H_*^H(L\mathfrak{U}) \to H_*^H(L\mathfrak{X})$ (where $G = \{1\}, S^1, O(2)$) induced by $L\mathfrak{U} \to \mathrm{L}\mathfrak{X}$. Applying naturality of Gysin maps [GiNo, 9.2.2] to the commutative diagram

$$\begin{tikzcd}
L\mathfrak{U} \arrow[r, "\pi"] \arrow[d] \arrow[rr, bend left, "q"] & {[S^1\backslash L\mathfrak{U}]} \arrow[r, "p"] \arrow[d] & {[O(2)\backslash L\mathfrak{U}]} \arrow[d] \\
L\mathfrak{X} \arrow[r, "\pi"] \arrow[rr, bend right, "q"'] & {[S^1\backslash L\mathfrak{X}]} \arrow[r, "p"] & {[O(2)\backslash L\mathfrak{X}]}
\end{tikzcd}$$

shows that

$$i_* \circ T^{\mathbb{Z}/2\mathbb{Z}} = T^{\mathbb{Z}/2\mathbb{Z}} \circ i_*, \qquad i_* \circ T^{O(2)} = T^{O(2)} \circ i_*. \tag{6.9}$$

Since the bracket $\widetilde{\{x,y\}}$ is defined as $p_*\big(\{T^{\mathbb{Z}/2\mathbb{Z}}(x), p_*^{-1}(y)\}\big)$, the identity $i_*(\widetilde{\{x,y\}}) = \widetilde{\{i_*(x), i_*(y)\}}$ follows from identity (6.9), $i_* \circ p_* = p_* \circ i_*$ and the fact that $i_*$ is a map of graded Lie algebras with respect to the bracket $\{-,-\}$. Similarly $i_*(\{x,y\}_{O(2)}) = \{i_*(x), i_*(y)\}_{O(2)}$ follows from identity (6.9), $i_* \circ q_* = q_* \circ i_*$ and the fact that $i_*$ is a map of algebras with respect to $\star$. $\square$

Proof of Theorem 6.13. The map $Tr_*$ is well-defined by Lemma 6.11. We now prove that it is a Lie algebra map.

Let $\mathfrak{S}$ be the complement of the orbifold locus of $\mathfrak{Y}$ so that we have epimorphisms $i : \pi_1(\mathfrak{S}) \to \pi_1(\mathfrak{Y})$, $i_* : \pi_0(\mathrm{Map}(S^1, \mathfrak{S})) \to \pi_0(\mathrm{Map}(S^1, \mathfrak{Y}))$. Moding out by the $\mathbb{Z}/2\mathbb{Z}$-action, we also get an epimorphism $i_* : \pi_0(\mathrm{Map}(S^1, \mathfrak{S}))_{\mathbb{Z}/2\mathbb{Z}} \to \pi_0(\mathrm{Map}(S^1, \mathfrak{Y}))_{\mathbb{Z}/2\mathbb{Z}}$ and consequently a linear surjection

$$i_*^{O(2)} : H_0^{O(2)}(L\mathfrak{S}) \longrightarrow H_0^{O(2)}(L\mathfrak{Y}).$$

It follows from Lemma 6.11 and diagram (5.5) that we have a commutative diagram

$$\begin{tikzcd}
H_0^{S^1}(L\mathfrak{S}) \arrow[r] \arrow[d, two heads, "i_*"'] \arrow[rr, bend left, "Tr_*"] & H_0^{O(2)}(L\mathfrak{S}) \arrow[r, "Tr_*"] \arrow[d, two heads, "i_*^{O(2)}"'] & \mathcal{O}_{\chi_{\mathfrak{S}},G} \arrow[d, "(i^*)^*"'] \\
H_0^{S^1}(L\mathfrak{Y}) \arrow[r] \arrow[rr, bend right, "Tr_*"'] & H_0^{O(2)}(L\mathfrak{Y}) \arrow[r, "Tr_*"] & \mathcal{O}_{\chi_{\mathfrak{Y}},G}.
\end{tikzcd} \tag{6.10}$$

We have seen (Proposition 5.10) that the right vertical map $(i^*)^*$ is a Lie algebras homomorphism. Note that, by definition of $Tr_*$, for any $x \in H_0^{O(2)}(L\mathfrak{S})$ and any preimage $y \in p_*^{-1}(\{x\})$, we have that $Tr_*(x) = Tr_*(p(y)) = Tr_*(y)$. Hence, by $\mathbb{Z}/2\mathbb{Z}$-invariance of $Tr_*$ (Lemma 6.11) and Lemma 6.6, we get that

$$Tr_*\big(T^{\mathbb{Z}/2\mathbb{Z}}(x)\big) = Tr_*\big(T^{\mathbb{Z}/2\mathbb{Z}}(p(y))\big) = Tr_*(y + \varepsilon_*(y)) = 2Tr_*(y) = 2Tr_*(x). \tag{6.11}$$

Since Goldman [Go1, Theorem 5.13] has proved that the map $2Tr_* : \Big(H_0^{S^1}(L\mathfrak{S})\Big)^{\mathbb{Z}/2\mathbb{Z}} \to \mathcal{O}_{\chi_{\mathfrak{S}},G}$ is a Lie algebra homomorphism, it follows from (6.11), Remark 6.4 and Proposition 6.9 that the upper left horizontal arrow in diagram (6.10) is a Lie algebra homomorphism. Since $i_*^{O(2)} : H_0^{O(2)}(L\mathfrak{S}) \longrightarrow H_0^{O(2)}(L\mathfrak{Y})$ is surjective, it is now enough to prove that it is a Lie algebra homomorphism. This is Lemma 6.14. $\square$

**Remark 6.15** As we have seen on the proof of Theorem 6.13, an immediate consequence of identity (6.11) and Remark 6.4 is that our map $Tr_* : H_0^{O(2)}(L\mathfrak{Y}) \to \mathcal{O}_{\chi_{\mathfrak{Y}},G}$ is identified with Godlman map in [Go1] under the aforementionned equivalence $H_0^{O(2)}(L\mathfrak{Y}) \xrightarrow{\simeq} \Big(H_0^{S^1}(L\mathfrak{S})\Big)^{\mathbb{Z}/2\mathbb{Z}}$.

# References


[AZ] H. Abbapsour, M. Zeinalian, *String bracket and flat connections*, Algebraic & Geometric Topology, **7** (2007) 197-231.

[ALR] A. Adem, J. Leida and Y. Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, 171, Cambridge Univ. Press, Cambridge, 2007.

[Au] M. Audin, Lectures on gauge theory and integrable systems, in *Gauge theory and symplectic geometry (Montreal, PQ, 1995)*, 1–48, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 488, Kluwer Acad. Publ., Dordrecht.

[Bea] A. F. Beardon, *The geometry of discrete groups* Graduate Texts in Mathematics Vol 91, Springer (1983).

[Beh1] K. Behrend, *Cohomology of stacks*, in Intersection theory and moduli, 249-294 (electronic) ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[Beh2] K. Behrend, *On the de Rham cohomology of differential and algebraic stacks*, Adv. Math. **198** (2005), no. 2, 583–622.

[BGNX] K. Behrend, G. Ginot, B. Noohi, P. Xu, *String topology for stacks*, Astérisque **343** (2012).

[BeNo] K. Behrend, B. Noohi, *Uniformization of Deligne-Mumford analytic curves*, J. Reine Angew. Math., **599** (2006), 111-153.

[BiGu] I. Biswas and K. Guruprasad, *Principal bundles on open surfaces and invariant functions on Lie groups*, Internat. J. Math. **4** (1993), no. 4, 535–544.

[ChSu] M. Chas, D. Sullivan, *String Topology*, arXiv:9911159 [math.GT]

[ChGa] M. Chas, S. Gadgil, *The Goldman bracket determines intersection numbers for surfaces and orbifolds*, arXiv:1209.0634

[CoNo] T. Coyne, B. Noohi, *Singular chains on topological stacks*, arXiv:1502.04995 [math.AT].

[GiNo] G. Ginot, B. Noohi, *Group actions on stacks and applications to equivariant string topology for stacks*, arXiv:1206.5603.

[Go1] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. **54** (1984), no. 2, 200–225.

[Go2] W. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. 85 (1986), no. 2, 263–302.

[Ka] Y. Karshon, *An algebraic proof for the symplectic structure of moduli space*, Proc. Amer. Math. Soc. **116** (1992), no. 3, 591–605.

[KaTo] F. W. Kamber, P. Tondeur, *On flat bundles*, Bull. Amer. Math. Soc. 72 1966 846–849.

[LeMa] E. Lerman and A. Malkin, *Hamiltonian group actions on symplectic Deligne-Mumford stacks and toric orbifolds*, Adv. Math. **229** (2012), no. 2, 984–1000.

[Mi] J. Milnor, *On the existence of a connection with curvature zero*, Comm. Math. Helv. 32 (1957), 215–223.

[Mo] I. Moerdijk, Orbifolds as groupoids: an introduction, in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, 205–222, Contemp. Math., 310, Amer. Math. Soc.

[No1] B. Noohi, *Mapping stacks of topological stacks*, J. reine angew. Math. (Crelle), **646** (2010), 117–133.

[No2] ———, *Homotopy types of topological stacks*, Adv. Math. **230** (2012), no. 4-6, 2014–2047.

[No3] ———, *Fibrations of topological stacks*, Adv. Math. 252 (2014), 612640.

[Ro] M. Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53**, Issue 1 (2005), 209-236.

[S] P. Scott, *The symmetry of intersection numbers in group theory*, Geom. Topol. **2** (1998), 11–29.

[Tu] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 6, 635–704.