Hodge filtration and operations in higher Hochschild (co)homology and applications to higher string topology

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Résumé

This paper is based on lectures given at the Vietnamese Institute for Advanced Studies in Mathematics and aims to present the theory of higher Hochschild (co)homology and its application to higher string topology. There is an emphasis on explicit combinatorial models provided by simplicial sets to describe derived structures carried or described by Higher Hochschild (co)homology functors. It contains detailed proofs of results stated in a previous note as well as some new results. One of the main result is a proof that string topology for higher spheres inherits a Hodge filtration compatible with an (homotopy) E_{n+1}-algebra structure on the chains for d-connected Poincaré duality spaces. We also prove that the E_n-centralizer of maps of commutative (dg-)algebras are equipped with a Hodge decomposition and a compatible structure of framed E_n-algebras. We also study Hodge decompositions suspensions and products by spheres, both as derived functors and combinatorially.

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1 Introduction and Overview

This paper is based on lectures given at the Vietnamese Institute for Advanced Studies in Mathematics. It aims to present both the theory of higher Hochschild (co)homology and its application to higher string topology. It contains detailed proofs of results stated in the note [Gi3] as well as some new results building on our previous work [GTZ3, Gi4] notably. One of the main new result is an application of the techniques of Higher Hochschild (co)homology to study higher string topology [1] and prove that, in addition to its already rich algebraic package, the latter inherits an additional Hodge filtration (compatible with the rest of the structure). We also prove that the E_n-centralizer of maps of commutative (dg-)algebras are equipped with a Hodge decomposition and a compatible structure of framed E_n-algebras [2] and study Hodge decompositions suspensions and products by spheres generalizing the ones of [P] and dual results of [TW], see below for more details on these results.

1. also called Brane topology [CV, GTZ3]
2. also called n-disk-algebras or unoriented E_n-algebras [AF, Gi4]

This various results are also a pretext to illustrate the techniques of higher order Hochschild Homology in the case of commutative differential graded algebras, both using its derived (in an ∞-categorical sense) interpretation and functoriality and emphasizing on and using its nice *combinatorial* structure and how to use it. The emphasis on this latter point is another benefit of this paper compared to most of the literature we know [3] and a good way to get a feeling on the behavior and benefits of higher Hochschild (co)homology, in, we hope, a gentle way.

Higher Hochschild (co)homology was first emphasized by Pirashvili in [P] in order to understand the Hodge decomposition of Hochschild homology and how to generalize it. *Higher Hochschild (co)homology is* in fact a joint *invariant of both topological spaces* (or their homotopy combinatorial avatar : simplicial sets) *and commutative differential graded algebras* (CDGA for short). As the name suggests, it is a generalization for commutative (dg-)algebras of the standard Hochschild homology of dg-associative algebras. It is also a special case [GTZ2, AF] of factorization homology [4] [BD, Lu3, AF] which get extra-functoriality and is one of the easiest one to compute and manipulate [5].

Standard Hochschild (co)homology is the (co)homology theory controlling deformations into associative algebras (or dg-categories) [G, L2, Lu3]. Besides algebra, it has tremendous applications in geometry, mathematical physics and algebraic topology see for instance [K, KS1, KS2, BNT, KS, Ca, CaTu, Ka, KKL, We2, CJ, CV, Ch, FTV, C1, C2, Tr, TZ, ArTu, Ho, RZ, Gi4] which have triggered the search for higher generalization and applications. For instance, by [F, Lu3, GTZ3, GY], Higher Hochschild cohomology over n-spheres S^n controls deformations of cdgas into E_n-algebras, generalizing the aforementioned case $n = 1$ of (possibly homotopy) associative algebras.

The Hochschild homology groups of an *associative* algebra A with value in a A-bimodule M are defined as

$$HH_n(A, M) \cong H_n\big(A \otimes^{\mathbb{L}}_{A\otimes A^{op}} A\big) \cong Tor_n^{A\otimes A^{op}}(A, M).$$

while Hochschild cohomology groups are defined as

$$HH^n(A, M) \cong H^n\big(\mathbb{R}Hom_{A\otimes A^{op}}(A, M)\big) \cong Ext^n_{A\otimes A^{op}}(A, M),$$

these definitions giving right away the correct derived definition of this functors. There is a *standard* chain complex $CH^{std}_\bullet(A, M)$ (resp. $CH^\bullet_{std}(A, M)$) that computes Hochschild homology (resp. cohomology) [G, L1]. One can extend these definitions to sheaves, differential graded algebras and algebras of smooth functions. Especially important for geometrical applications of Hochschild theory is the Hochschild-Kostant-Rosenberg (HKR for short) Theorem asserting that, if

3. though for instance [TW, GTZ] are also giving many details on it

4. also called chiral homology, an homology theory for n-dimensional framed manifold and E_n-algebras

5. in fact, this paper (and the concomitant lectures) were partially thought as an introduction to ideas and features of factorization homology in a special case of independent interest but which does not require as much higher homotopical background as the general theory

$A = C^\infty(M)$, then $HH_n(A, A) \cong \Omega^n_{dR}(M)$ and $HH^n(A, A) \cong \Gamma(M, \wedge^n TM)$. Further, there is another differential $B : CH^{std}_\bullet(A, A) \to CH^{std}_{\bullet+1}(A, A)$, the Connes operator, which induces the de Rham operator in the above isomorphism and allows to define cyclic (co)homologies (in his various forms). A consequence of this fact led to Non Commutative geometry where one replace forms and vector fields by Hochschild (co)homology of an operator algebra or a dg-category, de Rham cohomology by cyclic homology and so on. More generally, in algebraic geometry, Hochschild homology (for schemes, derived schemes, dg-categories and stacks) are useful models for forms and vector fields as well as many operations [Ca, PTVV].

It was long understood that Hochschild homology was related to loop spaces and that the B operator shall be think as a circle action. This idea was made into a theorem by proving that *Hochschild homology* is (functions on) a *derived loop space* and that the circle action is equivalent to the data of B (or de Rham forms under HKR equivalence), for instance see [TV2, TV3]. And in fact, Higher Hochschild homology gives a theory which describes more general derived mapping spaces (those with sources an homotopy type of topological space) [GTZ2].

Hochschild (co)homology of *commutative* (dg-)algebras has an important additional virtue: the (co)homology and (co)chains (with value in a *symmetric* bimodule) inherits a Hodge filtration (and decomposition in characteristic zero) given by a γ-ring structure (or equivalently *Adams* operations), first noticed by Gerstenhaber-Schack [GS] and Loday [L1]. These operations have been highly studied in the literature and this structure is fundamental in many geometric applications. In particular, for sheaves, it gives the usual Hodge decomposition of complex algebraic schemes [We2]. In fact, the Hodge decomposition is the correct analogue on Hochschild chains of the weight filtration on forms. In particular, it gives the *correct* (derived) weight structure on the mixed complex given by the standard Hochschild complex which allows to interpret derived *closed forms* in symplectic and Poisson derived geometry [PTVV, CPTVV].

In algebraic topology, Hochschild homology has been used intensively as a *model for* the forms (or cochains) on (*free* or based depending on the module coefficient) *loop spaces* to explicitly compute it, but also as a powerful device to study their structures [CJ, W]; these ideas already goes back to the early eighties [Ch, VB, Go]. For instance Hochschild (co)homology is an *algebraic model for string topology* [CV, Tr, TZ], that is the rich algebraic structure possessed by (chains on) the free loop spaces that was discovered by Chas-Sullivan [CS] and became a major topic in algebraic topology ever since. Indeed, the Hochschild cohomology of cochains algebras $C^*(X)$ is isomorphic [CJ, FTV] to the chains on the free loop space if X is a simply connected manifold:

$$H_*(LX) \cong HH^*(C^*(X), C_*(X)) \cong HH^*(C^*(X), C^*(X))[d]. \tag{1.1}$$

The isomorphism (1.1) is an isomorphism of Gerstenhaber [6] algebras (that is analogue of Poisson algebras with a cohomological degree -1 Lie bracket).

6. also called $Pois_2$-algebras in this paper

When X is a triangulated oriented Poincaré duality space, applying Sullivan's techniques, Tradler and Zeinalian proved that the Hochschild cohomology $HH^\bullet(C^*(X), C^*(X))$ is a Batalin-Vilkoviski (BV for short)-algebra (whose underlying Gerstenhaber algebra is the usual one) [TZ]. The intrinsic reason for the existence of this BV-structure is that Poincaré duality is a up to homotopy version of a Frobenius structure and that for Frobenius algebras, the Gerstenhaber structure in Hochschild cohomology is always BV [Tr, W]. Note that the cochains algebra can always be made into a *homotopy commutative* algebra and in fact into a CDGA in characteristic zero so that the restriction to cdgas for higher Hochschild is irrelevant with respect to the study of algebraic models for (co)chains on mapping spaces. This was a point of view developed for instance in [GTZ3, Gi3] and that we are explaining and pushing forward in some direction in the present paper.

Higher Hochschild (co)homology is modeled over spaces in the same way the usual Hochschild (co)homology is modeled on circles (as we have been alluding to earlier). More precisely, it is a rule which associate to any space X, commutative (dg-)algebra A and A-module M, homology groups $HH_X(A, M)$ and in fact chain complexes $CH_X(A, M)$ *functorial in every argument*, such that for $X = S^1$, one recovers the usual Hochschild homology (see Section 3.3 for more details). The *functoriality with respect to spaces* is a key feature which *allows* us *to derive algebraic operations* on the higher Hochschild chain complexes from maps of topological spaces. For instance Adams operations studied in Section 4 and higher operations studied in Section 5.

These higher Hochschild constructions pass to the associated ∞-categories and and are in fact constructions of (higher) *derived functors* of these derived *∞-categories*. Being indeed associated to homotopy types of spaces, one can naturally use simplicial sets as a model for spaces to define them (and this is indeed how they were originally defined). This allows to give nice combinatorial (co)chains models of higher Hochschild (co)homology, each simplicial model giving a different resolution fo the same homotopy type. We take advantage of them in several places in this paper. This combinatorial structure and the functoriality has also been fruitfully used in [TW] to study linear representations of $Out(F_n)$, where F_n is the free group on generators, via its action on wedges of circles.

To sum-up, the *philosophy* is that (higher) Hochschild (co)homology should be thought of as some kind of functions on a "mapping space" from X to some "derived space" and the *gain* is algebraic structures/operators induced by maps of spaces as well as algebraic models for mapping spaces and new invariants for spaces and algebras. Simplicial sets models gives in turn nice combinatorial and simpler complexes to compute these invariants. Using the relationship of higher Hochschild with invariants of mapping spaces allows in turn to transfer this rich structure to the latter ones as we will show in § 7.2 (also see [GTZ, GTZ3]).

Let us now describe the content of the paper. The first section precises our notations and recall a few facts from homotopical algebra and our conventions for ∞-categories.

In Section 3.1, we define higher Hochschild (co)chain complexes as right and left Γ-modules (as in [P, GTZ, Gi3]), that is as (co)simplicial objects, and gives several examples, while in § 3.3 we explain how Hochschild homology gives rise to a (derived) ∞-functor $(X, A) \mapsto \mathbf{CH}_X(A)$ from the ∞-categories of spaces **Top** and the ∞-category **CDGA** of cdgas with value in **CDGA**. We also spell out generalization of this for module coefficients and cohomology (for pointed spaces), specifying many various compatibilities, modules structures carried by this functors, some of them were only implicit in the literature. We give the axiomatic characterization of Higher Hochschild chains, see Theorem 3.24 which is a multiplicative analogue of the standard Eilenberg-Steenrod axioms at the chain level.

In Section 4, we apply the functoriality with respect to continuous maps to define the Hodge decomposition (in characteristic zero) or filtrations of higher Hochschild (co)homology of cdgas with values in bimodules over products of spheres by a space or iterated suspensions. We first define the operations in geometric terms, using the canonical degree k-maps of a sphere, giving rise to the definition of the operations for the *derived functors* of Hochschild (co)homology, see Theorem 4.14 and 4.25. We then spell out combinatorial models of them for the standard models of higher Hochschild models of spheres in § 4.4. We use these models to refine our results on the Hodge decomposition, see Theorem 4.17. One of the properties of higher Hochschild homology is a Fubini (or exponential law) result stating that higher Hochschild homology $\mathbf{CH}_{X\times Y}(A)$ over a product space is (equivalent to) higher Hochschild of Higher Hochschild, namely $\mathbf{CH}_X(\mathbf{CH}_Y(A)$. In § 4.6, we give explicit small combinatorial model for expressing this equivalence and computing the Hochschild (co)chains of product spaces as well as suspensions. We then use them and our results for the standard models of spheres to get again nice combinatorial description of the γ-ring structure on suspensions and products. We call these models the Eilenberg-Zilber models.

In Section 5, we study higher operations possessed by Hochschild cohomology over spheres, which generalize the standard cup-product. Our results here refine the centralizer construction of [Lu3, GTZ3] in the case of CDGAs (and does not hold for arbitrary E_n-algebras). Indeed, we proved that given a CDGA map $f : A \to B$, the Hochschild homology $\mathbf{CH}_{S^d}(A, B)$, which is the (E_d-)centralizer of f, has a structure of *framed*[7]-E_d-algebra[8] (Theorem 5.11) and we prove it is further compatible with respect to the Hodge decomposition (Corollary 5.25). We also give derived construction of this structure (mainly following ideas of [GTZ3]) as well as a nice combinatorial model (which was only briefly alluded to in [Gi3]), see Theorem 5.21.

In Section 6, we give an higher version of the Hochschild-Kostant-Rosenberg Theorem (HKR for short) for Hochschild (co)homology of formal spaces X. This results gives a powerful way of computing the Hochschild (co)homology functors in terms of the coalgebra structure of $H_*(X)$ and semi-free resolution of the algebra. We also prove that the HKR equivalence preserves the Hodge decomposition see Corollary 6.8 and Theorem 6.3.

In Section 7, we explain how to use Higher Hochschild homology as a model for

7. or unoriented in the terminology of [Gi4], or that of a d-disk algebra in the one of [AF]
8. said otherwise it has an additional action of the orthogonal group $O(d)$ for which the structure maps are equivariant

mapping spaces in algebraic topology. More precisely, we recall from [GTZ] how to generalize the classical Chen iterated integrals from loop and path spaces to all mapping spaces and then show how to apply the results of the previous sections to study higher string topology operations. The main result is Theorem 7.7 which asserts the existence of a chain level E_{d+1}-structure (or homotopy $Pois_{d+1}$-algebra structure) on the chains of the free sphere space $\mathrm{Map}(S^d, X)$ of a d-connected closed manifold, which is multiplicative with respect to the Hodge filtration induced by the power maps of S^d.

The techniques we use relies on CDGAs and as such only applies in topology in characteristic zero. Most of our results can be generalized over other coefficients or even $\mathbb{Z}$ (and some have been in [GTZ3]) but at the price of working with E_∞-algebras for which we can not use HKR theorem anymore (hence loosing an important computational tool) nor the very nice combinatorial model we have. This make the construction of Higher Hochschild very dependent of higher homotopical techniques and possibly less intuitive.

2 Notations, Conventions and a few standard facts

We fix a ground *field* k of *characteristic* 0. We will also use the following notations and conventions

— If (C, d_C) is a cochain complex, $C[i]$ is the cochain complex such that $C[i]^n := C^{n+i}$ with differential $(-1)^i d_C$. We will mainly work with cochain complexes and adopt the convention that a chain complex is a cochain complex with opposite grading when we need to compare gradings.

— An ∞-category will be a complete Segal space. Any model category gives rise to an ∞-category.

— We write $\mathsf{k}\text{-}\mathsf{Mod}^{dg}$ for the category of cochain complexes and k-**Mod** for its associated ∞-category. We will use the abbreviation dg for differential graded. We will use the words *(co)homology* for an object of these ∞-categories (in other words a complex thought up to quasi-isomorphism) and use the words *(co)homology groups* for the actual groups computed by taking the quotient of the (co)cycles by (co)boundaries (for instance see Definition 3.19).

— sSet **and** Top: sSet is the (model) category of simplicial sets, that is functors from $\Delta^{op} \to Set$ where Δ is the simplex category of finite sets $n_+ := \{0, \dots, n\}$ with order preserving maps. We also have the (model) category of topological spaces Top. These two categories are Quillen equivalent: $|-| : \mathbf{sSet} \underset{\sim}{\overset{\sim}{\leftrightarrows}} \mathbf{Top} : \Delta_\bullet(-)$. Here $\Delta_\bullet : \mathsf{Top} \to \mathsf{sSet}$ the singular set functor defined by $\Delta_n(X) = \mathrm{Map}_{\mathsf{Top}}(\Delta^n, X)$, where Δ^n is the standard n-dimensional simplex, and $|Y_\bullet|$ the geometric realization. Their associated ∞-categories, respectively denoted **sSet** and **Top** are thus equivalent.
These four (∞)-categories are *symmetric monoidal* with respect to disjoint union.

- There are also *pointed versions* sSet_* and Top_* as well $\mathbf{sSet}_*$ and $\mathbf{Top}_*$ of the above (∞-)categories.
- **The category $\boldsymbol{\Gamma}$:** is the category of finite sets while Γ_* is the category of finite pointed sets.
- **CDGAs**: the model category CDGA of commutative differential graded algebras (CDGA for short) yields the ∞-category **CDGA** It has a (∞-)monoidal structure induced by tensor products of CDGAs. The mapping space between cdgas A and B is the simplicial set $\mathrm{Map}_{\mathbf{CDGA}}(A,B) \cong \mathrm{Hom}_{\mathsf{CDGA}}(A, B\otimes \Omega^*_{PL}(\Delta^n))$ where $\Omega^*_{PL}(\Delta^n))$ are the polynomial forms on the standard simplex [S1].
- **Modules over CDGAs**: there are model categories A-Mod and A-CDGA of (differential graded) modules and (differential graded) commutative algebras over a CDGA A from which we get ∞-categories A-Mod_∞ and A-CDGA_∞. The base change functor (for a map $f\colon A\to B$) lifts to an ∞-functor $f_*\colon B\text{-}\mathsf{Mod}_\infty \to A\text{-}\mathsf{Mod}_\infty$ (see [TV1]). The tensor products of A-modules gives rise a to symmetric monoidal functor on A-Mod_∞ that we denote $-\overset{\mathbb{L}}{\underset{A}{\otimes}}-$ (since it is a lift of the derived tensor product). Similarly, we denote $\mathbb{R}Hom_A(-,-)$ the internal hom of A-Mod_∞.
- We will write $\mathsf{Mod}^{\mathsf{CDGA}}$ and $\mathsf{Mod}_{\mathsf{CDGA}}$ for the categories consisting of pairs given by an algebra and a module (with respective maps explained in § 3.3) and $\mathbf{Mod}^{\mathsf{CDGA}}$ and $\mathbf{Mod}_{\mathsf{CDGA}}$ for their associated ∞-categories. We denote $\iota_* : \mathsf{Mod}_{\mathsf{CDGA}} \to \mathsf{CDGA}$ and $\iota^* : \mathsf{Mod}^{\mathsf{CDGA}} \to \mathsf{CDGA}^{op}$ the canonical functors sending a pair to the underlying algebra.
- **E_n-algebras:** there are similarly ∞-categories E_n-**Alg** of E_n-algebras where E_n is an (∞)-operad equivalent to the little n-dimensional cubes operad. We denote $\mathbf{Mod}_A^{E_k}$ the symmetric monoidal ∞-category of (E_k-)modules over an E_k-algebra A (see [Lu3, F]). Recall that, for instance, $\mathbf{Mod}_A^{E_k}$ is equivalent to the category of A-bimodules, while, if A is a CDGA, $\mathbf{Mod}_A^{E_\infty}$ is equivalent to the (∞-)category of left A-modules.
- We denote $Pois_n$ the operad controlling n-Poisson algebras (that is CDGA endowed with a cohomological degree $1-n$ Lie structure whose bracket is a graded derivation in each variable). We recall that for $n>1$, this is the homology of (any) E_n-operad (see [Fr4, Co]) and that the latter are formal under the same assumption.
- Since we are over a characteristic zero field, any operad $\mathcal{O}$ gives rise to a model category of algebras and hence to an associated ∞-category [Hi, Fr3].
- **γ-rings:** We will denote $(\gamma,0)$-Ring the category of γ-rings with trivial multiplication, see 4.1 as well as $(\gamma,0)$-**Ring** for its associated ∞-category.

Remark 2.1 We will consider various "derived categories" of algebras, modules or chain complexes, which can be described as ∞-categories. We will often use boldface typography for ∞-categories, their objects as well as functors to distinguish them from their strict analogues from which they are an enrichment of the ordinary derived category. For most of our applications, the reader does not need much about ∞-category, besides the fact that they are enrichment of homotopy categories, that is categories where one inverts weak-equivalences, for which the morphisms are topological spaces (or simplicial sets) which can be computed by cofibrant-fibrant resolutions when the ∞-category come from a model category. Also, in that context, homotopy (co)limits can be expressed with universal properties in the ∞-category world (in other words as ∞-(co)limits).

In that paper, following [R, Lu2], by an *∞-category* we mean a *complete Segal space* (though none of our results actually depends on the choice of a specific model; in particular quasi-categories [Lu1] will be equally fine). The ∞-categories we are mostly interested in will arise from *Dwyer-Kan localizations* from model categories; as alluded to, they should be thought of as nice enhancement of *derived categories* (in particular weak-equivalences have been inverted (in a non-naive way). Let us recall briefly how to get an ∞-category out of a model category; this will be our prominent source of examples. There is a simplicial structure, denoted $\mathcal{S}e\mathcal{S}p$ on the category of simplicial spaces such that a fibrant object in the $\mathcal{S}e\mathcal{S}p$ is precisely a Segal space. Rezk has shown that the category of simplicial spaces has another simplicial closed model structure, denoted $\mathcal{CS}e\mathcal{S}p$, whose fibrant objects are precisely complete Segal spaces [R, Theorem 7.2]. Let $\mathbb{R}:\mathcal{S}e\mathcal{S}p\to\mathcal{S}e\mathcal{S}p$ be a fibrant replacement functor. Let $\widehat{\cdot}: SeSp\to\mathcal{CS}e\mathcal{S}p$, $X_\bullet\to\widehat{X_\bullet}$, be the completion functor that assigns to a Segal space an equivalent complete Segal space. The composition $X_\bullet\mapsto\widehat{\mathbb{R}(X_\bullet)}$ gives a fibrant replacement functor $L_{\mathcal{CS}e\mathcal{S}p}$ from simplicial spaces to complete Segal spaces. Now, a standard idea to go from a model category to a simplicial space is to use Dwyer-Kan localization. Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be its subcategory of weak-equivalences. We denote $L^H(\mathcal{M},\mathcal{W})$ its *hammock localization*. It is a *simplicial category* such that the category $\pi_0(L^H(\mathcal{M},\mathcal{W}))$ is the *homotopy category* of $\mathcal{M}$. Any weak equivalence has (weak) inverse in $L^H(\mathcal{M},\mathcal{W})$.

Thus, a model category $\mathcal{M}$ gives rise functorially to the simplicial category $L^H(\mathcal{M},\mathcal{W})$ hence a simplicial space $N_\bullet(L^H(\mathcal{M},\mathcal{W}))$ by taking its nerve. Composing with the complete Segal Space replacement functor we get a functor $\mathcal{M}\to L_\infty(\mathcal{M}):=L_{\mathcal{CS}e\mathcal{S}p}(N_\bullet(L^H(\mathcal{M},\mathcal{W})))$ from model categories to ∞-categories.

3 Higher Hochschild (co)homology

3.1 Γ-modules and Hochschild (co)chain complexes over spaces

Let Γ be the category of finite sets and Γ_* be the category of finite *pointed* sets. We will write k_+ for the set $\{0,1,\dots,k\}$ with 0 as base point; this base point will often just be denoted by $+$. The collection of k_+ is a skeleton for Γ_*. A left Γ-module is a functor $\Gamma\to\mathrm{Vect}$ and right Γ-module is a functor $\Gamma^{\mathrm{op}}\to\mathrm{Vect}$. There are similar definitions for left and right Γ_*-modules; the latter will also simply be called pointed left or right Γ-modules. The category Mod-Γ_* of right Γ_*-modules is abelian with enough projectives and injectives and the same is true for the categories of left modules Γ_*-Mod as well as Mod-Γ, Γ-Mod. Details can be found in [P]. The significance of Γ-modules in Hochschild (co)homology was first understood by Loday [L1] who initiated the following constructions. Let A be a commutative unital algebra and M a symmetric A-bimodule. The left Γ_*-module $\mathcal{L}(A,M)$ is defined on an object I_+ with base point $+$ by

$$\mathcal{L}(A,M)(I_+) = M \otimes \bigotimes_{i \in I_+ \setminus \{+\}} A$$

and on a a map $I_+ \xrightarrow{\phi} J_+$ by, for any $(a_i \in A)_{i \in I_+ \setminus \{+\}}$ and $m_+ \in M$, the formula

$$\mathcal{L}(A,M)(\phi)\Big(m_+ \otimes \bigotimes_{i \in I_+ \setminus \{+\}} a_i\Big) = n_+ \otimes \bigotimes_{j \in J_+ \setminus \{+\}} b_j \tag{3.1}$$

where $b_j = \prod_{i \in \phi^{-1}(j)} a_i$. Here the empty product is set to be the unit 1 of A and for $j = +$, the product is givn by the module structure.

Similarly [Gi3], there is a right Γ_*-module $\mathcal{H}(A,M)$ defined on I_+ by

$$\mathcal{H}(A,M)(k_+) = Hom_k(A^{\otimes I_+ \setminus \{+\}}, M).$$

For a map $I_+ \xrightarrow{\phi} J_+$ and $f \in Hom_k(A^{\otimes J_+ \setminus \{+\}}, M)$, the linear map $\mathcal{H}(A,M)(\phi)(f) \in Hom_k(A^{\otimes I_+ \setminus \{+\}}, M)$ is given, for any $(a_i \in A)_{i \in I_+ \setminus \{+\}}$, by

$$\mathcal{H}(A,M)(\phi)(f)\Big(\bigotimes_{i \in I_+ \setminus \{+\}} a_i\Big) = b_+ \cdot f\Big(\bigotimes_{j \in J_+ \setminus \{+\}} b_j\Big) \tag{3.2}$$

where $b_j = \prod_{i \in \phi^{-1}(j)} a_i$. Again the empty product is set to be the unit 1 of A and, for $j = +$, the product denotes the module structure.

The above constructions extend naturally to the differential graded context; in that case one shall add the sign ϵ in equation (3.2) given by the usual Koszul sign rule, that is one adds the sign $(-1)^{|x| \cdot |y|}$ whenever x moves across y. This is simply the sign carried out by the standard symmetric monoidal structure of graded modules.

Given a cocommutative coalgebra C and a C-comodule N, Pirashvili [P] defined a right Γ_*-module ${}^{co}\mathcal{L}(C,N)$ given on objects by ${}^{co}\mathcal{L}(C,N)(I_+) = N \otimes C^{\otimes I_+ \setminus \{+\}}$. The action on arrows is as for $\mathcal{H}(A,M)$ replacing multiplications by comultiplications. Again both constructions naturally extends to differential graded algebras and coalgebras.

Example 3.1 Let $L_\bullet$ be a simplicial set. Then its homology is a cocommutative coalgebra and ${}^{co}\mathcal{L}(H_*(L), H_*(L))$ is a graded right Γ_*-module. In particular its degree q part yields the right Γ_*-module ${}^{co}\mathcal{L}_q(H_*(L), H_*(L))$.

Example 3.2 (non-pointed extensions) When $M = A$, formula (3.1) makes sense for all sets maps and no longer only pointed ones. Hence the functor $\mathcal{L}(A,A)$ extends *canonically into a left* Γ*-module*. Similarly, for $M = (A)^\vee$ the linear dual of the (dg-)module M, the functor $\mathcal{H}(A,(A)^\vee)$ extends canonically into a right Γ-module.

A right Γ_*-module R can be extended to a functor $\mathrm{Set}_*^{\mathrm{op}} \to \mathrm{Vect}$, where Set_* is the category of pointed sets, by taking limits :

$$\mathrm{Set}_* \ni Y \mapsto R(Y) := \lim_{\Gamma \ni X \to Y} R(X).$$

Thus, given any pointed simplicial set $Y_\bullet$ and right Γ-module R one gets a pointed cosimplicial vector space $R(Y_\bullet)$. Similarly, we extend left Γ_*-modules by colimits

$$Y \to L(Y), \quad \mathrm{Set}_* \ni Y \mapsto L(Y) := \operatorname*{colim}_{\Gamma \ni X \to Y} L(X)$$

as well as the non-pointed versions.

Replacing (graded) vector spaces by (co)chain complexes with their standard symmetric monoidal structure, one obtains left and right, pointed or not, differential graded Γ-modules. In particular formula (3.2), (3.1) and example 3.2 extend to differential graded algebras and modules over them in a canonical way [9] giving rise to left and right Γ and Γ_* dg-modules. *In these notes, we will almost always consider dg-Γ or Γ_*-modules.* In that case we obtain, for any (pointed) simplicial set $X_\bullet$ and left (pointed) Γ-dg-module (L, d) the simplicial (pointed) Γ-dg-module $L(X_\bullet)$. The Dold-Kan realization functor [We] thus produces a bicomplex whose total complex is denoted

$$L^{dg}(X_\bullet) = \mathrm{Tot}(L(X_\bullet), d, \partial). \tag{3.3}$$

where $\partial : L(X_i) \to L(X_{i-1})$ is the simplicial differential $\sum_{k=0}^{i}(-1)^k (d_k)_*$ where $d_k : X_\bullet \to X_{\bullet-1}$ are the face operators. In other words, writing $L(I)_p$ the degree p component of the chain complex $(L(I), d)$ for any $I \in \Gamma$, in (homological) degree n, one has $L^{dg}(X_\bullet)_n = \bigoplus_{p+q=n} L(X_q)_p$ with total differential $D = (-1)^q d + \partial$. Similarly, if (R, d) is a right (pointed) Γ-dg-module, we have a cosimplicial (pointed) Γ-dg-module $R(X_\bullet)$ and its totalization thus produces the bicomplex

$$R^{dg}(X_\bullet) = \mathrm{Tot}(R(X_\bullet), d, \partial^*). \tag{3.4}$$

where $\partial^* : R(X_i) \to R(X_{i+1})$ is the cosimplicial differential $\sum_{k=0}^{i}(-1)^k (d_k)^*$ where $d_k : X_\bullet \to X_{\bullet-1}$ are the face operators. In other words, writing $R(I)_p$ the degree p component of the chain complex $(L(I), d)$ for any $I \in \Gamma$, in (homological) degree n, one has $L^{dg}(X_\bullet)_n = \bigoplus_{p-q=n} L(X_q)_p$ with total differential $D = (-1)^q d + \partial$. We have considered the case of chain complexes, but one deal in the same way with cochain complexes.

Further, since the (co)chain complexes above came from (co)simplicial dg-modules, they are quasi-isomorphic to their normalized complexes, that is the subcomplexes obtained by taking the kernel of degeneracies (in the cosimplicial case) or the quotient by the image of the degeneracies (in the simplicial case). Henceforth, we will often tacitly assume that the notations L^{dg} and R^{dg} stand for normalized (co)chain complexes when there is no harm in doing so. We refer to [L2, We] for details on these standard constructions.

9. that is the dg-structure on $M \otimes \bigotimes_{i \in I_+ \setminus \{+\}} A$ is the tensor product of the underlying dg-k-modules.

A virtue of the constructions of $\mathcal{L}^{dg}(-,-)$ and $\mathcal{H}^{dg}(-,-)$ is that they are invariant under quasi-isomorphisms. To see this, also note that associated to any simplicial set we have the right Γ-module $\widetilde{C}_*(X_\bullet) : I \mapsto C_*(X_\bullet^I)$ where C_* is the singular chain complex. If $X_\bullet$ is pointed, then this gives a right Γ_*-module.

Proposition 3.3 *Let $R \in Mod\text{-}\Gamma_*$, $L \in \Gamma_*\text{-}Mod$ and $X_\bullet$ be a pointed simplicial set.*

1. *There is an natural equivalence*

$$L^{dg}(X_\bullet) \;\cong\; \widetilde{C}_*(X_\bullet) \overset{\mathbb{L}}{\underset{\Gamma_*}{\otimes}} L$$

   in the derived ∞-category of complexes over k.

2. *There is an natural equivalence*

$$R^{dg}(X_\bullet) \;\cong\; \mathbb{R}Hom_{Mod\text{-}\Gamma_*}\big(\widetilde{C}_*(X_\bullet),\, R\big)$$

   in the derived ∞-category of complexes over k.

3. *In particular there are spectral sequences:*

$$E_1^{p,q} = \mathrm{Ext}^p_{Mod\text{-}\Gamma}({}^{\mathrm{co}}\mathcal{L}_q(H_*(X_\bullet), H_*(X_\bullet))\,,\, R) \Longrightarrow H^{p+q}R^{dg}(X_\bullet),$$

$$E^1_{p,q} = \mathrm{Tor}_{\Gamma,p}({}^{\mathrm{co}}\mathcal{L}_q(H_*(X_\bullet), H_*(X_\bullet))\,,\, L) \Longrightarrow H^{p+q}L^{dg}(X_\bullet).$$

4. *The same holds for unpointed Γ-modules.*

PROOF. Note that $\widetilde{C}_*(X_\bullet)$ is the functor defined by $I \mapsto k\big[Hom_{Set}(I, X_\bullet)\big]$ which is cofibrant by [P]. As in [GTZ2, Proposition 4], we deduce an equivalence of simplicial modules

$$\widetilde{C}_*(X_\bullet) \overset{\mathbb{L}}{\underset{\Gamma_*}{\otimes}} L \cong k\big[Hom_{Set}(I, X_\bullet)\big] \underset{\Gamma_*}{\otimes} L \cong \operatorname*{colim}_{\Gamma \ni I \to X_\bullet} L(I)$$

from which the first equivalence follows. The second equivalence is dual to this one and the spectral sequences are the associated Grothendieck spectral sequences as in Theorem 2.4 in [P]. $\square$

In particular if $\alpha : X_\bullet \to Y_\bullet$ is a map of pointed simplicial sets, by functoriality it induces a map of cosimplicial vector spaces $R(Y_\bullet) \to R(X_\bullet)$ which is an isomorphism in cohomology when $\alpha_* : H_*(X_\bullet) \to H_*(Y_\bullet)$ is an isomorphism.

From the Quillen equivalence $|-| : \mathsf{sSet} \underset{\sim}{\overset{\sim}{\leftrightarrows}} \mathsf{Top} : \Delta_\bullet(-)$ between the model categories of simplicial sets and topological spaces and its pointed analogue, we deduce from Proposition 3.3

Corollary 3.4 *If $R \in Mod\text{-}\Gamma_*$ and $L \in \Gamma_*\text{-}Mod$, the functors $X_\bullet \mapsto R^{dg}(X_\bullet)$ and $X_\bullet \mapsto L^{dg}(X_\bullet)$ induces ∞-functors*

$$\mathbf{R} : \mathbf{Top}_*{}^{op} \to \mathsf{k}\text{-}\mathbf{Mod}, \qquad \mathbf{L} : \mathbf{Top}_* \to \mathsf{k}\text{-}\mathbf{Mod}.$$

Further, if R, L extends respectively to Mod-Γ and Γ-Mod, then the functors $\mathbf{R}$ and $\mathbf{L}$ extends to functors of ∞-categories

$$\mathbf{R} : \mathbf{Top}^{op} \to \mathsf{k}\text{-}\mathbf{Mod}, \qquad \mathbf{L} : \mathbf{Top} \to \mathsf{k}\text{-}\mathbf{Mod}.$$

Example 3.5 By definition, for any space X, an explicit (co)chain complex representing $\mathbf{R}(X)$ is given by $R^{dg}(X_\bullet)$ for any simplicial set $X_\bullet$ whose geometric realization is (weakly homotopy equivalent to) X. In particular, using the simplicial set functor $X \mapsto \Delta_\bullet(X)$, we get a strict functor $\mathsf{Top}_* \to \mathsf{k}\text{-}\mathsf{Mod}^{dg}$ given by $X \mapsto R^{dg}(\Delta_\bullet(X))$ representing $\mathbf{R}$. We will simply write $R(X)$ for this functor.

Of course, we will use the same construction and notation for left Γ_*-modules and the non-pointed versions.

By functoriality, the counit $X_\bullet \to \Delta_\bullet(|X_\bullet|)$ of the adjunction yields canonical (co)chain complexes maps

$$R(X) \to R^{dg}(X_\bullet), \qquad L^{dg}(X_\bullet) \to L(X) \tag{3.5}$$

which allows to compare effectively constructions done on different simplicial models.

3.2 Combinatorial Higher Hochschild (co)chains

We will now study in depth the Γ_*-modules $\mathcal{L}(A,M)$ and $\mathcal{H}(A,M)$. Contrary to an arbitrary left Γ-modules, these functors will inherit more structures coming from the algebra structure on A and give rise to what is called Higher Hochschild (co)chains Let $(A = \bigoplus_{i\in\mathbb{Z}} A^i, d, \mu)$ be a CDGA and M be a differential graded symmetric bimodule. Let us first consider the unpointed case. As seen above, we have $\mathcal{L}(A,A)(I) = A^{\otimes I}$. Since the tensor products of CDGAs is a CDGA, $\mathcal{L}(A,A)(I)$ inherits a cdga structure and further the maps (3.1) are maps of CDGAs as well [10]. Hence, $I \mapsto \mathcal{L}(A,A)(I)$ is a functor from from sets to differential graded commutative algebras.

Now, if $Y_\bullet$ is a simplicial set, we also get the simplicial CDGA $\mathcal{L}(A,A)(Y_\bullet)$. Applying the Dold-Kan construction [11], $\mathcal{L}(A,A)^{dg}(Y_\bullet)$ is canonically a CDGA whose product is induced by the *shuffle product* which is defined (in simplicial degree p, q) as the composition

$$\begin{aligned} sh : \mathcal{L}(A,A)^{dg}(Y_p) \otimes \mathcal{L}(A,A)^{dg}(Y_q) &\xrightarrow{sh^\times} \mathcal{L}(A,A)^{dg}(Y_{p+q}) \otimes \mathcal{L}(A,A)^{dg}(Y_{p+q}) \\ &\cong \mathcal{L}(A\otimes A, A\otimes A)^{dg}(Y_{p+q}) \xrightarrow{\mu_*} \mathcal{L}(A,A)^{dg}(Y_{p+q}) \end{aligned} \tag{3.6}$$

where $\mu : A\otimes A \to A$ denotes the multiplication in A (which is a map of algebra since A is commutative) and, denoting s_i the degeneracies of the simplicial structure in $\mathcal{L}(A,A)^{dg}(Y_\bullet)$,

$$sh^\times(v\otimes w) = \sum_{(\mu,\nu)} sgn(\mu,\nu)(s_{\nu_q}\dots s_{\nu_1}(v) \otimes s_{\mu_p}\dots s_{\mu_1}(w)), \tag{3.7}$$

10. here the commutativity is crucial
11. Recall that we can consider normalized chains and cochains when applying Dold-Kan

where (μ,ν) denotes a (p,q)-shuffle, i.e. a permutation of $\{0,\dots,p+q-1\}$ mapping $0\le j\le p-1$ to μ_{j+1} and $p\le j\le p+q-1$ to ν_{j-p+1}, such that $\mu_1<\dots<\mu_p$ and $\nu_1<\dots<\nu_q$.

The differential $D:\mathcal{L}(A,A)^{dg}(Y_\bullet)\to\mathcal{L}(A,A)^{dg}(Y_\bullet)[1]$ is given as follows. The tensor products of chain complexes $A^{\otimes Y_i}$ has an internal differential which we abusively denote as d since it is induced by the inner differential $d:A\to A[1]$. Then, the differential on $\mathcal{L}(A,A)^{dg}(Y_\bullet)$ is given by the formula:

$$D\big(\bigotimes_{i\in Y_i}a_i\big) \quad := \quad (-1)^i d\big(\bigotimes_{i\in Y_i}a_i\big)+\sum_{r=0}^{i}(-1)^r(d_r)_*\big(\bigotimes_{i\in Y_i}a_i\big), \tag{3.8}$$

where the $(d_r)_*:\mathcal{L}(A,A)^{dg}(Y_i)\to\mathcal{L}(A,A)^{dg}(Y_{i-1})$ are induced by the corresponding faces $d_r:Y_i\to Y_{i-1}$ of the simplicial set $Y_\bullet$. From now on, we will denote by $CH_{Y_\bullet}(A)$ the CDGA $\mathcal{L}(A,A)^{dg}(Y_\bullet)$. More precisely, following the notations of [P, Gi3, GTZ].

Definition 3.6 Let $Y_\bullet$ be a simplicial set. The *Hochschild chains over $Y_\bullet$ of A* is the commutative differential graded algebra $(CH_{Y_\bullet}(A):=\mathcal{L}(A,A)^{dg}(Y_\bullet),D,sh)$.

The rule $(Y_\bullet,A)\mapsto(CH_{Y_\bullet}(A),D,sh)$ is thus a bifunctor from the ordinary discrete categories of simplicial sets and CDGAs to the ordinary discrete category of CDGAs.

Taking the normalized chains in the definition of $\mathcal{L}(A,A)^{dg}(Y_\bullet)$ yields the *normalized* Hochschild chain complex $\underline{CH}_{Y_\bullet}(A)$ which is also a functorial CDGA. It is standard that *the canonical map $CH_{Y_\bullet}(A)\to\underline{CH}_{Y_\bullet}(A)$ is an equivalence* (for instance see [L2, We]). In practice, we will usually not have to worry about taking normalized or not Hochschild chains and not always make the distinction between them if there is no harm in taking either one.

Remark 3.7 By construction, $CH_{Y_\bullet}(A)=\mathcal{L}(A,A)^{dg}(Y_\bullet)$ is naturally a *bigraded* object, with in one side a grading coming from the internal grading of A and in the other one a grading coming from the simplicial degree of $Y_\bullet$: an element in $CH_{Y_n}(A)$ is of simplicial degree n. We will write $|x|^{int}$ for the internal grading of an element and $|x|_{simp}$ for the simplicial degree.

The grading on $CH_{Y_\bullet}(A)$ is just the total grading of this bigrading, that is the sum of the two gradings, where the simplicial grading n is (as usual) viewed as an homological grading (and thus contributes to $-n$ for the cohomological grading).

Further note that the differential D splits into two differentials of bidegree $(0,1)$ and $(1,0)$; the first one being induced by the first term in (3.8) and the second one of simplicial degree -1 given by the alternating sum of the face maps (that is the second term in (3.8)).

Moreover, the bigrading is preserved both by CDGAs homomorphisms and maps of simplicial sets.

Hence, the *Hochschild chains over $Y_\bullet$ of A is canonically, in a bifunctorial way, the total complex of a bicomplex.* As such, we have standard spectral sequences to compute it as well.

Example 3.8 (The point) The point has a trivial simplicial model given by the constant simplicial set $pt_n = \{pt\}$. Hence

$$(CH_{pt_\bullet}(A), D) := A \overset{0}{\leftarrow} A \overset{id}{\leftarrow} A \overset{0}{\leftarrow} A \overset{id}{\leftarrow} A \cdots$$

which is a retract deformation of A (as a CDGA): the canonical algebra map $A \hookrightarrow CH_{pt_\bullet}(A)$, which maps A identically on its component of simplicial degree 0, is a quasi-isomorphism of CDGAs as well as is the projection on the simplicial degree 0 part $CH_{pt_\bullet}(A) \twoheadrightarrow \mathcal{L}(A,A)(pt_0) \cong A$. Note that the normalized chain complex $\underline{CH}_{pt_\bullet}(A)$ is *isomorphic* to A as a CDGA. The projection above identifies with the quotient from the unnormalized to the normalized chains.

Now, if $Y_\bullet$ is a *pointed* simplicial set, the canonical CDGA map $A \overset{\sim}{\hookrightarrow} CH_{pt_\bullet}(A) \to CH_{Y_\bullet}(A)$ makes $CH_{Y_\bullet}(A)$ a A-CDGA [12]. Then, if M is an A-module, from (3.1), we obtain an natural isomorphism of k-modules:

$$M \underset{A}{\otimes} CH_{Y_\bullet}(A) \cong \mathcal{L}(A,M)^{dg}(Y_\bullet) \tag{3.9}$$

and similarly, from (3.2), we have an natural isomorphism of k-modules:

$$Hom_A(CH_{Y_\bullet}(A), M) \cong \mathcal{R}(A,M)^{dg}(Y_\bullet) \tag{3.10}$$

Definition 3.9 Let $Y_\bullet$ be a *pointed* simplicial set, A a CDGA and M an A-module (viewed as a symmetric bimodule).

— The *Hochschild chains* of A with value in M over $Y_\bullet$ is the A-module

$$CH_{X_\bullet}(A,M) := M \underset{A}{\otimes} CH_{X_\bullet}(A).$$

It is a covariant functor with respect to maps of CDGAs, A-modules and pointed simplicial sets.

— The *Hochschild cochains* of A with value in M over $Y_\bullet$ is the A-module

$$CH^{X_\bullet}(A,M) := Hom_A(CH_{X_\bullet}(A), M).$$

It is a contravariant functor with respect to maps of CDGAs and pointed simplicial sets, but is covariant with respect to maps of modules.

Both functors are also naturally bigraded just as is $CH_{X_\bullet}(A)$ in Remark 3.7.

Of course, one can again takes normalized versions of these (co)chain complexes which are canonically quasi-isomorphic (as modules) to their unnormalized counterpart.

A special kind of modules will be provided by A-algebras. Indeed, if B is a (non-necessarily unital) CDGA over A, then it is in particular a symmetric A-bimodule. In that case, it follows as above that $\mathcal{L}(A,B)(Y_\bullet)$ is a simplicial CDGA for any simplicial set $Y_\bullet$ and applying the Dold-Kan construction, $\mathcal{L}(A,B)^{dg}(Y_\bullet)$ endowed with the shuffle product (3.6) is a CDGA as well. From the isomorphism (3.10), we obtain

12. that is a commutative algebra object in the symmetric monoidal category of differential graded A-modules

Lemma 3.10 *If B is a (non-necessarily unital) CDGA over A, then $CH_{X_\bullet}(A,B)$ is naturally (in A, B and $X_\bullet$) a CDGA.*

For any simplicial set $X_\bullet$, we also have the canonical (and actually unique) map of simplicial set $X_\bullet \to pt_\bullet$ which by functoriality and example 3.8 gives a CDGA map $CH_{X_\bullet}(A) \to A$. In particular, for a pointed simplicial set $X_\bullet$, $CH_{X_\bullet}(A)$ is canonically an A-augmented CDGA. Hence, Definition 3.9 implies that $CH^{X_\bullet}(A,M)$ and $CH_{X_\bullet}(A,M)$ inherits an action of $CH_{X_\bullet}(A)$ induced by its canonical action on itself, lifting [13] their A-module structure.

Lemma 3.11 *Let $X_\bullet$ be a pointed simplicial set. There are natural isomorphisms*

$$CH^{X_\bullet}(A,M) \cong \mathbb{R}Hom_A(CH_{X_\bullet}(A), M),$$

$$CH_{X_\bullet}(A,M) \cong M \overset{\mathbb{L}}{\underset{A}{\otimes}} CH_{X_\bullet}(A)$$

in the derived category of A-modules.

Proof. By definition $CH_{X_\bullet}(A) \cong A \otimes CH_{X_\bullet \setminus \{*\}}(A)$ as a A-CDGA, where the A-module structure is given by multiplication in the first tensor. It is thus a semi-free A-cdga. Hence $Hom_A(CH_{X_\bullet}(A), M)$ and $M \otimes_A CH_{X_\bullet}(A)$ computes the derived homomorphisms and tensor products in the model category of A-modules. $\square$

Lemma 3.12 *The natural weak equivalences of A-modules given by Lemma 3.11 lifts to weak equivalences in the derived ∞-category of $CH_{X_\bullet}(A)$-modules.*

Proof. Since the canonical maps $A \to CH_{X_\bullet}(A)$ and $CH_{X_\bullet}(A) \to A$ are maps of semi-free A-cdgas, any cofibrant resolution of $CH_{X_\bullet}(A)$ as a $CH_{X_\bullet}(A)$-module is also a cofibrant resolution as a A-module. Thus the canonical lift given by the left hand sides in Lemma 3.11 induces the desired derived $CH_{X_\bullet}(A)$-modules structures. $\square$

Example 3.13 (ground field) Let M be any k-module. Then, by definition $CH_{X_\bullet}(k,M) \cong M$, and

$$(CH_{X_\bullet}(k,M) := M \overset{0}{\leftarrow} M \overset{id}{\leftarrow} M \overset{0}{\leftarrow} M \overset{id}{\leftarrow} M \cdots$$

which is a retract deformation of M, with section obtained by mapping M identically on its component of simplicial degree 0. Note that M concentrated in (simplicial) degree 0 is precisely the normalized cochain complex associated to $CH_{X_\bullet}(k,M)$. This retract is compatible with the one given by example 3.8.

13. meaning that the A-module structure is induced from the $CH_{X_\bullet}(A)$ one through the algebra map $A \to CH_{X_\bullet}(A)$

Namely, for any pointed simplicial set $pt_\bullet \to X_\bullet$, cdga A and A-module, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, hook, "\simeq"] \arrow[dr, "\simeq"'] & CH_{X_\bullet}(k,M) \arrow[r, "(k\to A)_*"] & CH_{X_\bullet}(A,M) \\ & CH_{pt_\bullet}(A,M) \arrow[ur, "(pt_\bullet\to X_\bullet)_*"'] & \end{tikzcd} .$$

We now go over the combinatorics of several crucial examples in details.

Example 3.14 (The interval) A (pointed) simplicial model for the interval $I = [0,1]$ is given by $I_n = \{0_+, 1\cdots, n+1\}$, hence in simplicial degree n, $CH_{I_n}(A,M) = M\otimes A^{\otimes n+1}$ and the simplicial face maps are

$$d_i(a_0\otimes\cdots a_{n+1}) = a_0\otimes\cdots\otimes(a_i a_{i+1})\otimes\cdots\otimes a_{n+1}.$$

Clearly $CH_{I_\bullet}(A,M) = Bar(M,A,A)$ is the standard (two sided) Bar construction which is quasi-isomorphic to M. Similarly, the normalized chains are the standard two-sided reduced Bar construction.

Example 3.15 (The circle) The circle $S^1 \cong I/(0\sim 1)$ has (by Example 3.8) a simplicial model $S^1_\bullet$ which is the quotient $S^1_n = I_n/(0\sim n+1)\cong\{0,\dots,n\}$. One computes that the face maps $d_i : S^1_n \to S^1_{n-1}$, for $0\le i\le n-1$ are given by $d_i(j)$ is equal to j or $j-1$ depending on $j=0,\dots,i$ or $j=i+1,\dots,n$ and $d_n(j)$ is equal to j or 0 depending on $j=0,\dots,n-1$ or $j=n$. For $i=0,\dots,n$, the degeneracies $s_i(j)$ is equal to j or $j+1$ depending on $j=0,\dots,i$ or $j=i+1,\dots,n$. This is the standard simplicial model of S^1 cf. [L2, § 6.4.2]. Thus, $CH_{S^1_\bullet}(A,M) = \bigoplus_{n\ge 0} M\otimes A^{\otimes n}$ and the differential agrees with the *usual one on the Hochschild chain complex* $C_\bullet(A,M)$ of A with values in a (symmetric) bimodule M (see [L2]). **In particular,** $CH_{S^1_\bullet}(A)$ **is the usual Hochschild chain complex of** A; this is the motivation behind the terminology.

We will see in details the derived and functorial properties of the Hochschild cochains in the next section 3.3.

Example 3.16 (The 2-dimensional sphere) The sphere S^2 has a simplicial model $S^2_\bullet = I^2_\bullet/\partial I^2_\bullet$ i.e. $S^2_n = \{(0,0)\}\coprod\{1\cdots n\}^2$, where we take $(0,0)$ as the base point (if needed). Here the face and degeneracies maps are somehow the quotient of diagonal ones as taken from Example 3.14. Hence, the ith differential is given by $d_i^{S^2_\bullet}(p,q) = (0,0)$ in the case that $d_i(p)=0$ or $d_i(q)=0$ (where d_i is the i^{th}-face map of $S^1_\bullet$), or setting otherwise $d_i(p,q) = (d_i(p), d_i(q))$. For $i\le n-1$, we obtain $d_i(a_{(0,0)}\otimes\cdots\otimes a_{(k,k)})$ is equal to:

$$\begin{array}{lccc}
a_{(0,0)}\otimes & & & \\
 & \otimes(a_{(i-1,i)}a_{(i-1,i+1)})\otimes & \cdots & \otimes a_{(i-1,n)} \\
 & \otimes(a_{(i,i)}a_{(i,i+1)}a_{(i+1,i)}a_{(i+1,i+1)})\otimes & \cdots & \otimes(a_{(i,n)}a_{(i+1,n)}) \\
 & \otimes(a_{(i+2,i)}a_{(i+2,i+1)})\otimes & \cdots & \otimes a_{(i+2,n)} \\
 & \vdots & & \vdots \\
 & \otimes(a_{(n,i)}a_{(n,i+1)})\otimes & \cdots & \otimes a_{(n,n)}
\end{array}$$

Example 3.17 (higher spheres) We will especially focus on spheres S^d with $d>1$. First, similarly to S^2, we have the *standard model* $S^d_\bullet := (I_\bullet)^d/\partial(I_\bullet)^d \cong S^1_\bullet\wedge\cdots\wedge S^1_\bullet$ (d-factors) for the sphere S^d. Hence $S^d_n \cong \{\underline{0}\}\coprod\{1\cdots n\}^d$ and the face operators are similar to those of Example 3.16 (except that, instead of a matrix, we have a dimension d-lattice) and face maps are obtained by simultaneously multiplying each i^{th}-hyperplane with $(i+1)^{\text{th}}$-hyperplane in each dimension. The last face d_n is obtained by multiplying all tensors of all n^{th}-hyperplanes with a_0. We get this way the **standard Hochschild cochain complex** $C^{S^d_\bullet}(A,M)$ which is the cochain complex associated to $\mathcal{H}(A,M)(S^d_\bullet)$.

We also have the *small model* $S^d_{sm_\bullet}$ which is the simplicial set with exactly two *non-degenerate* simplex, one in degree 0 and one in degree d. Then, $S^d_{sm\,n} = pt$ for $n<d$ and $S^d_{sm\,n} \cong \{0_+\}\coprod\{1,\dots,\binom{n}{d}\}$ for $n\geq d$. Using this model, it is straightforward to check the following computation of the first homology groups of $CH_{S^d}(A)$ for a commutative algebra A:

$$H_n(CH_{S^d}(A)) \cong H_n(CH_{S^d_{sm_\bullet}}(A)) \begin{cases} = A & \text{if } n=0 \\ = 0 & \text{if } 0<n<d \\ = \Omega^1_A & \text{if } n=d \end{cases}$$

where Ω^1_A is the A-module of Kähler differentials (see [L2, We]).

Of course, we also have the **singular complexes** $CH_{\Delta_\bullet(S^d)}(A,M)$, $CH^{\Delta_\bullet(S^d)}(A,M)$ which are the cochain complexes associated to the *fibrant* simplicial set which in dimension n is the set of maps $\Delta^n\to S^d$. By definition, those are the strict functors associated to $\mathcal{L}(A,M)$ and $\mathcal{H}(A,M)$ as in Example 3.5.

When working with this combinatorial definition of the Hochschild chain complex, it is often useful to think of it by writing the tensor products on a model given by the geometric realization of the simplicial sets, putting the tensors $a_{x_i}\in A^{\otimes X_i}$, for each x_i in X_i at the position on $|X_\bullet|$ given by the point $\eta(\underline{t},x_i)$ where $\eta:\Delta^i\times X_i\to|X_\bullet|$ is the canonical projection. This helps seeing the tensors and studying the differential and functorial structure. The sphere example above illustrates this and we refer to [GTZ] for many more examples of this.

3.3 Derived Hochschild (co)chains

By Corollary 3.4, the definitions of Hochschild (co)chains (Definitions 3.6 and 3.9) passes to (∞-)derived categories. These lifts do retain their additional algebraic structures as we will see.

To encode the package of the many various functoriality, we will introduce some notations.

Let us denote $\mathsf{Mod}_{\mathsf{CDGA}}$ the category of pairs (A, M) consisting of a CDGA A and a A-module M whose morphisms from (A, M) to (B, N) are pairs $(A \xrightarrow{f} B, M \xrightarrow{\phi} N)$ where $f : A \to B$ is a CDGA map and $\phi : M \to f_*(N)$ is an A-module homomorphism. Note that we have an obvious *covariant* functor $\iota_* : \mathsf{Mod}_{\mathsf{CDGA}} \to \mathsf{CDGA}$ sending (A, M) to A, as well as, for any CDGA A, an equivalence of categories $\{A\} \times_{\mathsf{CDGA}} \mathsf{Mod}_{\mathsf{CDGA}} \to A - \mathrm{Mod}$ induced by (A, M) goes to M. Further, since an algebra A is canonically a module over itself, we have the faithful functor $\mathsf{CDGA} \hookrightarrow \mathsf{Mod}_{\mathsf{CDGA}}$ given by $A \mapsto (A, A)$.

Let us also denote $\mathsf{Mod}^{\mathsf{CDGA}}$ the category of pairs (A, M) consisting of a CDGA A and a A-module M whose morphisms from (A, M) to (B, N) are now pairs $(B \xrightarrow{f} A, M \xrightarrow{\phi} N)$ where $f : B \to A$ is a CDGA map and $\phi : f_*(M) \to N$ is an A-module homomorphism. Here we denote as it is standard by $f_*(N)$ the canonical A-module structure on N induced by the map f. We have now an obvious functor $\iota^* : \mathsf{Mod}_{\mathsf{CDGA}} \to \mathsf{CDGA}^{op}$ sending (A, M) to A, as well as, for any CDGA A, an equivalence of categories $\{A\} \times_{\mathsf{CDGA}^{op}} \mathsf{Mod}_{\mathsf{CDGA}} \to A - \mathrm{Mod}$ induced by (A, M) goes to M. Further we also have a faithful functor $\mathsf{CDGA}^{op} \hookrightarrow \mathsf{Mod}^{\mathsf{CDGA}}$ given by $A \mapsto (A, A^\vee)$ where the dual $A^\vee$ is endowed with its canonical A-module structure.

In particular, functors out of $\mathsf{Mod}^{\mathsf{CDGA}}$ yields naturally contravariant functors with respect to maps of CDGAS but covariant functors with respect to modules maps, while functors out of $\mathsf{Mod}_{\mathsf{CDGA}}$ yields natural covariant functors on both variables. The choice of CDGA has an upper and lower notation respectively is designed to suggest this co(ntra)variance properties with respect to maps of CDGAs. All the above categories have standard simplicial enrichment of their morphims (given by tensoring by polynomial forms $\Omega^*_{Pl}(\Delta^\bullet)$ on simplices at the target) and the above functors preserve the enrichment.

Finally, we denote $\mathbf{Mod}^{\mathsf{CDGA}}$, $\mathbf{Mod}_{\mathsf{CDGA}}$ the ∞-categories corresponding to these categories (see[Lu3, F, Fr4, GTZ3] for details on ∞-categories of modules). The above described functors passes to these ∞-categories; for instance we still have $\iota_* : \mathbf{Mod}_{\mathsf{CDGA}} \to \mathbf{CDGA}$ and $\iota^* : \mathbf{Mod}^{\mathsf{CDGA}} \to \mathbf{CDGA}^{op}$.

Proposition 3.18 ([GTZ2, GTZ3]) *The Hochschild chains functor $(X_\bullet, A) \mapsto CH_{X_\bullet}(A)$ (Definition 3.6) lifts*[14] *as a functor of ∞-categories*

$$\mathbf{CH}_{(-)}(-) : \mathbf{Top} \times \mathbf{CDGA} \to \mathbf{CDGA}, \quad (X, A) \mapsto \mathbf{CH}_X(A).$$

The Hochschild chains $(X_\bullet, A, M) \mapsto CH_{X_\bullet}(A, M)$ and cochains $CH^{X_\bullet}(A, M)$ (Definition 3.9) lifts respectively as functors of ∞-categories

$$\mathbf{CH}_{(-)}(-,-) : \mathbf{Top}_* \times \mathbf{Mod}_{\mathsf{CDGA}} \to \mathbf{Mod}_{\mathsf{CDGA}}, \quad (X, M) \mapsto \mathbf{CH}_X(\iota_*(M), M)$$

14. said otherwise, the combinatorial definition of Hochschild chains has an natural derived enhancement

fitting into a commutative diagram

$$\begin{tikzcd}
\mathbf{Top}_* \times \mathbf{Mod}_{\mathsf{CDGA}} \arrow[r, "\mathbf{CH}_{(-)}(-,-)"] \arrow[d, "i\times\iota_*"'] & \mathbf{Mod}_{\mathsf{CDGA}} \arrow[d, "\iota_*"] \\
\mathbf{Top} \times \mathsf{CDGA} \arrow[r, "\mathbf{CH}_{(-)}(-)"'] & \mathsf{CDGA}
\end{tikzcd}$$

and

$$\mathbf{CH}^{(-)}(-,-) : {\mathbf{Top}_*}^{op} \times \mathbf{Mod}^{\mathsf{CDGA}} \to \mathbf{Mod}^{\mathsf{CDGA}}, \quad (X, M) \mapsto \mathbf{CH}^X(\iota^*(M), M)$$

fitting into a commutative diagram

$$\begin{tikzcd}
\mathbf{Top}_* \times \mathbf{Mod}_{\mathsf{CDGA}} \arrow[r, "\mathbf{CH}_{(-)}(-,-)"] \arrow[d, "i\times\iota^*"'] & \mathbf{Mod}_{\mathsf{CDGA}} \arrow[d, "\iota^*"] \\
\mathbf{Top} \times \mathsf{CDGA}^{op} \arrow[r, "\mathbf{CH}_{(-)}(-)"'] & \mathsf{CDGA}^{op}.
\end{tikzcd}$$

Definition 3.19 (Higher Hochschild (co)homology and Higher Hochschild (co)homology groups) We will refer to $\mathbf{CH}_X(A)$ as the (higher) *Hochschild homology*[15] of A over the space X and $\mathbf{CH}^X(A, M)$ as (higher) *Hochschild cohomology.*

The higher Hochschild *(co)homology groups* are defined as their (co)homology groups[16]; they will be denoted by $HH_X(A)_* := H_*(\mathbf{CH}_X(A))$, $HH_X(A, M)_* := H_*(\mathbf{CH}_X(A, M))$ and $HH^X(A, M)^* := H^*(\mathbf{CH}^X(A, M))$.

Proof. The first statement is in [GTZ2] and the other ones in [GTZ3]. Since we will use explicit construction henceforth of this functor using the combinatorial models of Section 3.2, we sketch the proof of the last part of the proof. The fact that the Hochschild cochains functor lifts to ∞-functors is Corollary 3.4 and the commutativity of the diagrams is provided by Lemma 3.12. To check the claimed functoriality, we may assume that A is a cofibrant CDGA and M a cofibrant B-module. A CDGA map $A \to B$ gives a A-cdga structure to B and we can thus assume (taking a resolution if necessary) that B is a cofibrant A-cdga. Finally, we can also assume N is cofibrant as a B-module. Then a pair $(A \xrightarrow{f} B, M \xrightarrow{\phi} N)$ yields first the CDGA map $f_* : CH_{X_\bullet}(A) \to CH_{X_\bullet}(B)$ which is a morphism of cofibrant $CH_{X_\bullet}(A)$-cdga. Then the chain complex morphism

$$CH_{X_\bullet}(A, M) \cong M \otimes_A CH_{X_\bullet}(A) \xrightarrow{\phi\otimes_A f_*} N \otimes_A CH_{X_\bullet}(B) \to N \otimes_B CH_{X_\bullet}(B) \cong CH_{X_\bullet}(B, N)$$

15. using the now standard terminology of calling homology an object of an ∞-derived category of complexes, not to be mistaken with their associated homology groups

16. which are actually k-modules

is a morphism of $CH_{X_\bullet}(A)$-modules. This provides the desired structure for the first functor. The structure for the second functor is obtained similarly. Given $\psi : f_*(N) \to M$, we get a chain complex morphism

$$CH^{X_\bullet}(B,N) \cong Hom_B(CH_{X_\bullet}(B),N) \xrightarrow{f^*} Hom_A(CH_{X_\bullet}(A),N) \\ \xrightarrow{\psi_*} Hom_A(CH_{X_\bullet}(A),M) \cong CH^{X_\bullet}(A,M)$$

which is a map of $CH_{X_\bullet}(B)$-modules by construction. $\square$

Remark 3.20 (Hochschild chains over singular set are universal representatives) By definition, the Hochschild homology algebra $\mathbf{CH}_X(A)$ is represented by the CDGA $CH_{X_\bullet}(A)$ for any simplicial set $X_\bullet$ whose geometric realization is weakly equivalent to X (this is what means the fact that $\mathbf{CH}_{(-)}(-)$ lifts $CH_{(-)}(-)$). A similar result holds for $\mathbf{CH}^X(A,M)$ and $\mathbf{CH}_X(A,M)$.

In particular, we have always a canonical choice of representative for any space X. That is the Hochschild chains over the singular set $\Delta_\bullet(X)$:

$$CH_X(A) := CH_{\Delta_\bullet(X)}(A) \simeq \mathbf{CH}_X(A).$$

Further, for any simplicial set model $X_\bullet$ of $X \cong |X_\bullet|$, the canonical map $X_\bullet \to \Delta_\bullet(|X_\bullet|)$ (adjoint to $\eta : \Delta^\bullet \times X_\bullet \to |X_\bullet|$) gives a canonical CDGA quasi-isomorphisms

$$CH_{X_\bullet}(A) \xrightarrow{\simeq} CH_{\Delta_\bullet(|X_\bullet|)}(A).$$

The later allows in practice to compare effectively constructions and computations done with various models of a space. Again, we have similar statements for (co)chains with coefficients in the categories $\mathsf{Mod}^{\mathsf{CDGA}}$ and $\mathsf{Mod}_{\mathsf{CDGA}}$.

Remark 3.21 (skeletal filtration) By Remark 3.7 and the previous one, any choice of a simplicial set model $X_\bullet$ of X gives rise to a bigraded chain complex representing $\mathbf{CH}_X(A)$ (or other variant), and such chain complex came with a canonical quasi-isomorphism $\eta_* : CH_{X_\bullet}(A) \to CH_{\Delta_\bullet(X)}(A)$ which preserves the bigrading, since maps of simplicial sets preserves this bigrading. Consequently, the derived functors $\mathbf{CH}_{(-)}(-,-)$ and $\mathbf{CH}^{(-)}(-,-)$ are canonically filtered by a weight induced by the simplicial degree. We call this filtration the **skeletal filtration**.

Remark 3.22 Let us consider the case where the objects (A,M) in $\mathsf{Mod}_{\mathsf{CDGA}}$ are such that M is in fact a CDGA over A. That is, we consider the category $\mathrm{CAlg}(\mathsf{Mod}_{\mathsf{CDGA}})$ of commutative algebra objects in the symmetric monoidal[17] category $\mathsf{Mod}_{\mathsf{CDGA}}$ and its associated ∞-category $\mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}})$. By Lemma 3.10, we have that for any $(A,B) \in \mathrm{CAlg}(\mathsf{Mod}_{\mathsf{CDGA}})$, the Hochschild chain $CH_{X_\bullet}(A,B)$ is a CDGA over $CH_{X_\bullet}(A)$ and the proof of Proposition 3.18 shows that

17. for its standard monoidal structure given by tensoring over A

Proposition 3.23 *The Hochschild chains $(X_\bullet, A, B) \mapsto CH_{X_\bullet}(A,B)$ lifts to a functor of ∞-categories*

$$\begin{aligned}\mathbf{CH}_{(-)}(-,-) : \mathbf{Top}_* \times \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}) &\to \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}),\\ (X,B) &\mapsto \mathbf{CH}_X(\iota_*(B),B)\end{aligned}$$

fitting into a commutative diagram

$$\begin{tikzcd}
\mathbf{Top}_* \times \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}) \arrow[r, "\mathbf{CH}_{(-)}(-,-)"] \arrow[d, "i\times\iota_*"'] & \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}) \arrow[d, "\iota_*"] \\
\mathbf{Top} \times \mathsf{CDGA} \arrow[r, "\mathbf{CH}_{(-)}(-)"'] & \mathsf{CDGA}.
\end{tikzcd}$$

Putting together Theorem 4.2.2 and Theorem 4.3.1 in our paper [GTZ2], we get

Theorem 3.24 *The Hochschild Homology functor $\mathbf{CH}_{(-)}(-) : \mathbf{Top} \times \mathbf{CDGA} \to$ $\mathbf{CDGA}$ is the unique ∞-functor satisfying the following axioms*

1. **value on a point:** *there is a natural equivalence $CH^\bullet_{pt}(A) \cong A$ in* **CDGA**.
2. **(infinite) monoidal:** *the canonical maps*
$$\bigotimes_{i\in I} \mathbf{CH}_{X_i}(A) \longrightarrow \mathbf{CH}_{\coprod_{i\in I} X_i}(A)$$
are natural equivalences (in **CDGA***).*
3. **homotopy glueing/pushout:** *CH sends homotopy pushout in Top to homotopy pushout in CDGA. More precisely, given maps $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ in* **Top**, *and $W \cong X \bigcup^h_Z Y$ a homotopy pushout, there is a natural equivalence*
$$\mathbf{CH}_W(A) \cong \mathbf{CH}_X(A) \overset{\mathbb{L}}{\underset{\mathbf{CH}_Z(A)}{\otimes}} \mathbf{CH}_Y(A).$$

Furthermore, we have natural equivalences

$$\mathbf{CH}_{X\times Y}(A) \xrightarrow{\sim} \mathbf{CH}_X\left(\mathbf{CH}_Y(A)\right))$$

in **CDGA** *and an natural weak-equivalence*

$$Map_{Top}\big(|X_\bullet|, Map_{\mathsf{CDGA}}(A,B)\big) \cong Map_{\mathsf{CDGA}}(CH_{X_\bullet}(A), B) \tag{3.11}$$

between the mapping spaces of the associated model categories of spaces and CDGAs.

Note that axiom 2 is trivially (the map being an isomorphism) satisfied at the level of Hochschild chains for any family of simplicial sets $(X_{\bullet,i})_{i\in I}$. The pushout axiom is also easily made explicit on a given simplicial set model. Indeed, if $f : Z_\bullet \to X_\bullet$

is an inclusion of simplicial sets (e.g. a cofibration of simplicial sets), then the canonical map

$$CH_{X_\bullet}(A) \underset{CH_{Z_\bullet}(A)}{\otimes} CH_{Y_\bullet}(A) \longrightarrow CH_{W_\bullet}(A) \tag{3.12}$$

is a quasi-isomorphism and a model for the equivalence

$$\mathbf{CH}_{|W_\bullet|}(A) \cong \mathbf{CH}_{|X_\bullet|}(A) \overset{\mathbb{L}}{\underset{\mathbf{CH}_{|Z_\bullet|}(A)}{\otimes}} \mathbf{CH}_{|Y_\bullet|}(A).$$

The gluing axiom also extends to Hochschild homology with value in a A-module in a straightforward way, see [GTZ3, Gi4] for details.

Let us mention another derived interpretation of Hochschild cohomology functor. The canonical operad map $\mathcal{C}_n \to \mathit{Com}$ where $\mathcal{C}_n$ is the n-cube operad (or any model for E_n-algebras) induces a canonical functor $\mathbf{Mod}^{\mathsf{CDGA}} \to \mathbf{Mod}_{E_n}$ from cdgas and modules over it to E_n-algebras and E_n-modules [18]. Given an E_n-algebra A, we get the sub-∞-category $\mathbf{Mod}^A_{E_n}$ of E_n-A-modules and, we have the derived functor $\mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(-,-) : {\mathbf{Mod}^A_{E_n}}^{op} \times \mathbf{Mod}^A_{E_n} \to \mathsf{k}\text{-}\mathsf{Mod}^{dg}$.

Proposition 3.25 ([GTZ3]) *One has an natural equivalence* $\mathbf{CH}^{S^n}(A, M) \cong \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A, M)$.

4 Hodge filtration and λ-operations on Hochschild (co)homology over spheres and suspensions

In this section we will define and study the Hodge filtration on higher Hochschild (co)homology, both from the derived point of view (which will gives geometric and universal characterization) as well as on combinatorial models (which allows computations and will allow to strengthen some results as well). The Hodge filtration is induced by a structure of γ-ring, a classic notion going back to the genesis of K-theory, but will be a very specific kind of such : it will be a γ-ring with trivial multiplication (not to be mistaken with the multiplicative structure of Hochschild homology itself !). We start with the basic definitions and recollections of this structure.

Disclaimer: in this section 4, unless otherwise stated, to simplify the combinatorial identities, we assume all the Hochschild (co)chains complexes to be *normalized* without changing the notations.

4.1 γ-rings and lambda operations

In this section we recall a few results and notations about the classical theory of γ-rings. We refer to [H], [Kr] for detailed treatment. Traditionally, γ-rings are additional structures on a ring provided by families of self-maps λ^k, γ^k and ψ^k (the

18. sometimes called E_n-bimodules since for $n = 1$, they correspond to homotopy bimodules over an homotopy associative algebra

Adams operations), for $k \geq 0$, which are related by universal formulas allowing to deduce all the operations from the first one and the ring structure.

In this paper we are interested in a very special kind of such γ-ring, namely the case where we see the ring is equipped with the *zero multiplication* which are the ones relevant for the algebraic Hochschild complex see [L1]. We thus only give the definition in that case.

Definition 4.1 Let R be a $\mathbb{Z}$-module. A structure of *γ-ring with zero multiplication* on R is a family of *linear* maps $(\lambda^k : R \to R)_{k\geq 0}$ such that

1. $\lambda^0 = 0$ and $\lambda^1 = id$,
2. $\lambda^k \circ \lambda^\ell = \lambda^{k\ell}$ for any $k, l \geq 0$.

Remark 4.2 For the sign for the operations λ^k, we follow the"more geometric" sign convention from [MCa, L2]. the reader shall be careful that in [L1, H, Kr], they consider the same operations but multiplied by the sign $(-1)^{k-1}$.

Since the multiplication is set to be null, following this convention, the usual Adams operations associated to the γ-ring structure are given by $\psi^k = k\lambda^k$.

Remark 4.3 Classically, in [AT], one considers *unital* ring when defining a γ-ring. Definition 4.1 is simply a translation of the classical definition of a (special) γ-ring on the unital ring $k \oplus R$ (where R is an ideal with 0-multiplication). The corresponding operations $\overline{\lambda}^k$ on $\mathbb{Z} \oplus R$, with the sign conventions of [H, AT], are (necessarily) given by $\overline{\lambda}^k(n, x) = \big(\binom{n}{k}, (-1)^{k-1}\lambda^k(x)\big)$.

In addition to the Adams operations, we can also define the maps γ^k, defined, for $x \in R$, by

$$\gamma^k(x) = (-1)^{k-1}\sum_{i=0}^{n-1}\binom{k-1}{i}(-1)^{k-i-1}\lambda^{k-i}.$$

This formula is a specialization in our null ring structure case of a general formula valid for all γ-rings; in fact one can see that γ^k is just (the component in R of) $\overline{\lambda}^k(x + k - 1)$ with the notation of remark 4.3.

An important feature of γ-rings is that they carry a natural filtration. The filtration of a γ-ring R, is the filtration $F_n^\gamma R$ ($n \geq 0$) defined by

$$F_p^\gamma X = \langle \gamma^{p_1}(x_1)...\gamma^{p_s}(x_s)\,;\, x_1, .., x_s \in R \text{ and } p_1 + ... + p_s \geq p\rangle. \tag{4.1}$$

The notation $\langle y_1...y_p\rangle$ stands for the abelian group generated by monomials $y_1...y_p$. This filtration has the property that, for $n \geq 1$, if $x \in F_n^\gamma R$, then

$$\lambda^k(x) \;\equiv\; k^{n-1}x \text{ modulo } F_{n+1}^\gamma R.$$

Of course, in our case of interest of null ring structure, the filtration has a simpler form since we only need to consider a single monomial. Note, for instance, that $F_0^\gamma = F_1^\gamma = R$ so the first step of the filtration is not relevant. This motivates to change the grading by 1, which will recover the traditional Hochschild grading and is compatible with the Hodge filtration of de Rham complexes via Hochschild Kostant Rosenberg Theorem.

We define the ***Hodge filtration*** to be the filtration $(F^{\gamma}_{n+1}R)_{n\geq 0}$. When R is a module over a ring of characteristic zero, and is further convergent, it is well-known (for instance see [H, Kr], that the filtration always splits yielding the ***Hodge decomposition*** of R:

$$R \cong \prod_{n\geq 0} R^{(n)}, \quad \text{where } R^{(n)} := F^{\gamma}_{n+1}R/F^{\gamma}_{n+2}R.$$

We thus have in that case that $\lambda^k(x) = k^n x$ for all x in $R^{(n)}$.

Example 4.4 If $(R, (\lambda^k)_{k\geq 0})$ is a λ-ring with zero multiplication, then, for any $d > 0$, the family $(\widetilde{\lambda}^k)_{k\geq 0}$ defined by $\widetilde{\lambda}^k := \lambda^{k^d}$ also satisfy Definition 4.1; hence is a different λ-ring with zero multiplication structure on R. However, the Hodge filtration and decomposition associated to this new structures are essentially the same. Indeed, if $x \in F^{\gamma}_{n+1}R$, one has that

$$\widetilde{\lambda}^k(x) - k^{dn}x \; \in F^{\gamma}_{n+1}R.$$

On the other hand, the filtration $(F^{\widetilde{\gamma}}_n)_{n\geq 0}$ associated to the operations $(\widetilde{\lambda}^k)_{k\geq 0}$ must satisfy, for any $k \geq 1$, if $x \in F^{\gamma}_{m+1}R$, then $\widetilde{\lambda}^k(x) - k^m x \in F^{\widetilde{\gamma}}_{n+1}R$. In characteristic zero, this implies immediately for all n that $F^{\widetilde{\gamma}}_{dn+1}R = \cdots = F^{\widetilde{\gamma}}_{dn+d}R$ and in particular the Hodge decomposition associated to the new operations $(\widetilde{\lambda}^k)_{k\geq 0}$ satisfy $\widetilde{R}^{(n)} = \{0\}$ if d is not a divisor of n and $\widetilde{R}^{(dj)} = R^{(j)}$. We will see a natural example of such in Section 4, see Theorem 4.17.

We will also be interested in the case where the underlying module R of a γ-ring zero multiplication also admits another (non-trivial) product structure and that the structure on R is compatible with the product:

Definition 4.5 A γ-ring with zero multiplication $(R, (\lambda^k)_{k\geq 0})$ which is also a (dg-)commutative k-algebra is said to be a *multiplicative γ-ring with zero multiplication* if the maps $\lambda^k : R \to R$ are maps of k-algebras (with respect to the non-trivial multiplication).

In other words, a multiplicative γ-ring with zero multiplication is a γ-ring with zero multiplication in the symmetric monoidal category of (dg-)commutative algebras. A standard example is the Hochschild chain complex of a commutative algebra [L2].

Example 4.6 Let A be a cdga and W be a dg-A-module. Then $Sym_A(W)$ inherits a *multiplicative* dg-γ-ring with zero multiplication structure by setting $\lambda^k(w) = k.w$ for any $w \in W$ and extending (in the unique way) as an algebra map. In that case, the weight n piece of the Hodge decomposition is precisely $Sym^{(n)}_A(W)$. This is for instance the case of the standard Hodge decomposition $\Omega^*(A) \cong Sym_A(\Omega^1_A)$ for a smooth algebra.

An interesting particular case of this construction is given as follows. Let $Sym(V)$ be a polynomial algebra on a graded k-module V. Assume given a differential d on $Sym(V)$ such that $d(V) \subset Sym^{\geq 1}(V)$. In other words, $(Sym(V), d)$ is a semi-free cdga. Then for any integer p, one can define a multiplicative γ-ring with zero multiplication structure on

$$Sym(V \oplus V[p]) \cong Sym_{Sym(V)}(Sym(V) \otimes V[p]),$$

$$Sym(V \oplus V^\vee[p]) \cong Sym_{Sym(V)}(Sym(V) \otimes V^\vee[p])$$

by setting $A = Sym(V)$ and $W = Sym(V) \otimes V[p]$ (or $W = Sym(V) \otimes V^\vee[p]$). Extending the differential on $Sym(V)$ to a differential on $Sym(V \oplus V[p])$ as the unique differential satisfying $d(v[p]) = (-1)^p s(d(v)$ where s is the unique differential whose restriction to V is the shift $V \to V[p]$. We obtain a multiplicative dg-γ-ring with zero multiplication where V is on weight 0 and $V[p]$ (respectively $V^\vee[p]$) in weight 1 with respect to the Hodge decomposition.

We will write $(\gamma, 0) - \mathsf{CDGA}$ and $(\gamma, 0) - \mathbf{CDGA}$ for the category of multiplicative γ-rings with zero multiplication and its associated ∞-category, that is the (∞) categories of γ-rings with trivial multiplication in the symmetric monoidal categories CDGA and $\mathbf{CDGA}$ respectively.

Remark 4.7 The notion of (dg-)commutative multiplicative γ-ring structure with trivial multiplication can be extended to any symmetric monoidal (∞-)category to define $(\gamma, 0) - \mathcal{C}$ for any such category $\mathcal{C}$.

4.2 Edgewise subdivision and simplicial approach to λ-operations

We will define functorial λ-operations on Hochschild (co)homology over spheres using topological operations on spheres representing the degree k-maps $S^d \to S^d$. However, at the (co)chain level, theses operations will be defined only on the singular models which is a very large complex. As we will see, we can find concrete combinatorial description of them on small complexes representatives of Higher Hochschild (co)chains. For this, we will follow an idea due to McCarthy relying on the edgewise subdivision functor [BHM] and the construction of natural system leading to power maps. The result will be Definition 4.11 below.

The (k-th) Edgewise subdivision is the functor $\mathrm{sd}_k : \Delta \to \Delta$ (where $k \geq 1$) which is defined on objects by

$$\mathrm{sd}_k(n-1)_+ = (kn-1)_+$$

and, if $f : (n-1)_+ \to (m-1)_+$ is non-decreasing,

$$\mathrm{sd}_k(f)(in + j) = im + f(j).$$

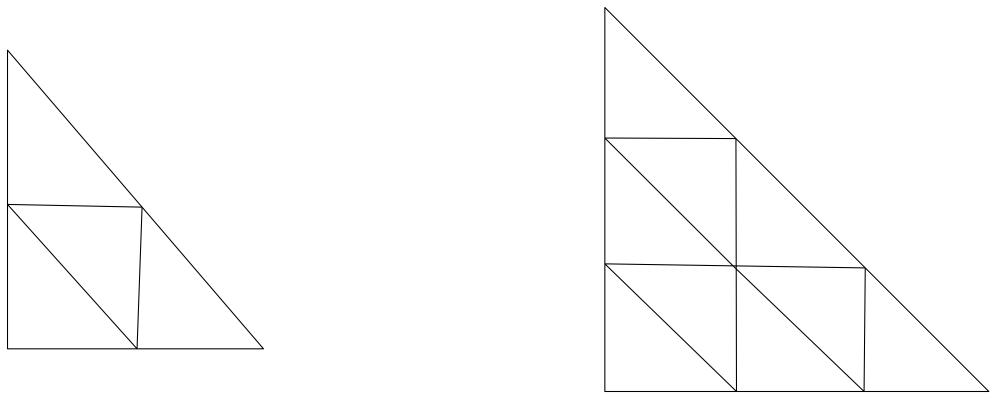

Figure 1: The first edgewise subdivisions $\mathrm{sd}_2\Delta^2$ and $\mathrm{sd}_3\Delta^2$ of the 2-simplex .

It thus gives rise to functors $\mathrm{sd}_k : \mathsf{sSet} \to \mathsf{sSet}$ on simplicial sets by precomposition by sd_k:

$$X_\bullet \mapsto \mathrm{sd}_k(X_\bullet) := \Delta^{op} \xrightarrow{\mathrm{sd}} \Delta^{op} \xrightarrow{X_\bullet} \mathsf{Set}.$$

The edgewise subdivision is a subdivision like the barycentric subdivision; in particular, one has homeomorphisms $|\mathrm{sd}_r(X)_\bullet| \underset{\cong}{\xrightarrow{D_r}} |X_\bullet|$ induced by

$$\begin{array}{ccc} X_{rn-1} \times \Delta_+^{n-1} & \xrightarrow{D_r} & X_{rn-1} \times \Delta_+^{rn-1} \\ (x,u) & \mapsto & (x, \frac{u}{r} \oplus \ldots \oplus \frac{u}{r}) \quad (r \text{ factors}) \end{array} \tag{4.2}$$

($cf.$ [BHM] Lemma 1.1). Hence for any $R \in \text{Mod-}\Gamma_*$ and pointed simplicial set $X_\bullet$, one has $|R(X_\bullet)| \cong |R(\mathrm{sd}_k(X)_\bullet)|$. There is an explicit quasi-isomorphism $\mathcal{D}_k^\bullet : R^{dg}(\mathrm{sd}_k(X)_\bullet) \to R^{dg}(X_\bullet)$ due to McCarthy [MCa] representing this equivalence [19]. This is constructed as follows. For positive integers k, n, set

$$\mathcal{S}_{n,k} := \{(\sigma,\tau) \in \Sigma_n \times Hom_\Delta\big((n-1)_+, k_+\big) \,/\, \sigma(i) > \sigma(i+1) \Rightarrow \tau(i-1) < \tau(i)\}.$$

Each element $\mu = (\sigma, \tau)$ corresponds to a non-degenerate n-simplex of the k-fold subdivision of the standard n-simplex ([MCa]), as is concretely realized by the following rule $\mu \mapsto \delta_\mu \in Hom_\Delta\big((kn+k-1)_+, n_+\big)$ where

$$\delta_\mu(i) := \begin{cases} 0 & \text{if} \quad i \le \tau(0)(n+1) + \sigma(1) - 1 \\ j & \text{if} \quad \tau(j-1)(n+1) + \sigma(j) \le i \le \tau(j)(n+1) + \sigma(j+1) - 1 \end{cases} \tag{4.3}$$

where $1 \le j \le n$ in the second line. It is easy to check that δ_μ is a well-defined map in the simplicial category Δ (because τ is non-decreasing).

When $Y_\bullet$ is a simplicial set and $R \in Mod{-}\Gamma$, then $R(Y_\bullet)$ is a cosimplicial module and we have $\mathrm{sd}_k R(Y_\bullet) = R(\mathrm{sd}_k(Y)_\bullet)$ hence $R^{dg}(\mathrm{sd}_k(Y)_\bullet) \cong Tot(\mathrm{sd}_k R(Y_\bullet))$ where Tot is the totalization (or Dold-Kan) chain complex associated to a cosimplicial module.

19. and thus realizing $\mathbf{R}(D_k^{-1})$

Definition 4.8 Let $\mathcal{R}_\bullet$ be a cosimplicial module and $\mathcal{L}_\bullet$ be a simplicial module. We define $\mathcal{D}_k^\bullet : \mathrm{sd}_k(\mathcal{R}_\bullet) \to \mathcal{R}_\bullet$ by the formula

$$\mathcal{D}_k^\bullet := \sum_{\mu=(\sigma,\tau)\in \mathcal{S}_{n,k}} (-1)^\tau \, (\delta_\mu^*)^* \tag{4.4}$$

and similarly $\mathcal{D}_{k,\bullet} : \mathcal{L}_\bullet \to \mathrm{sd}_k(\mathcal{L}_\bullet)$ by the formula

$$\mathcal{D}_{k,\bullet} := \sum_{\mu=(\sigma,\tau)\in \mathcal{S}_{n,k}} (-1)^\tau \, \delta_\mu^*. \tag{4.5}$$

In particular, formula (4.5) can be applied to get a map $R^{dg}(\mathrm{sd}_k(X)_\bullet) \to R^{dg}(X_\bullet)$ and formula (4.4) can be applied to to get a map $L^{dg}(X_\bullet) \to L^{dg}(\mathrm{sd}_k(X)_\bullet)$.

Lemma 4.9 ([MCa], Corollary 3.7) *The maps $\mathcal{D}_k^\bullet$ and $\mathcal{D}_{k,\bullet}$ are natural quasi-isomorphisms which induces the inverse ${D_r^{-1}}_*$ of the homeomorphism D_r in (co)homology.*

To define the λ-operations on $Tot(\mathcal{L}_\bullet)$, we need a way to compose $\mathcal{D}_k$ with maps sending back $\mathrm{sd}_k(\mathcal{L}_\bullet)$ to $\mathcal{L}_\bullet$. This is possible when $\mathcal{L}$ is in fact a left Γ-module. First note:

Remark 4.10 (The underlying (co)simplicial structure of Γ-modules) Any (pointed) left Γ-module L has an underlying (pointed) simplicial module structure $L_\bullet$ given by $L_n := L(n_+)$ and, for any non-decreasing map $f \in Hom_\Delta(n_+, m_+)$, by $L(\widetilde{f}) : L_m \to L_n$ where $\widetilde{f} : m_+ \to n_+$ is the map defined by

$$\widetilde{f}(i) = \begin{cases} j & \text{if } f(j-1) < i \le f(j) \\ 0 & \text{if there is no such } j \text{ as above.} \end{cases}$$

Similarly, any right Γ-module R has an underlying cosimplicial module given by $R^n := R(n_+)$ and $R(\widetilde{f}) : R^n \to R^m$ and, if the Γ-module was pointed, so is he induced cosimplicial module.

We will use the following natural system of maps to define λ-operations. We set $\widetilde{\varphi}_n^k : (kn-1)_+ \to (n-1)_+$ to be the pointed finite set maps defined by

$$\widetilde{\varphi}_n^k(in+j) = j. \tag{4.6}$$

By functoriality, if R is a right Γ_*-module we get cosimplicial maps

$$\widetilde{\varphi}^{k*}(R) : R^\bullet \to \mathrm{sd}_k(R^\bullet)$$

and, for $L \in \Gamma_* - Mod$, simplicial maps

$$\widetilde{\varphi}_*^k(L) : \mathrm{sd}_k(L_\bullet) \to L_\bullet.$$

Definition 4.11 For a R a right Γ_*-module, we denote $\varphi^k : R^\bullet \to R^\bullet$ the composition $\varphi^k = \mathcal{D}_k^\bullet \circ \widetilde{\varphi}^{k*}$.

Similarly, for L a left Γ_*-module, we define $\varphi^k : L_\bullet \to L_\bullet$ to be the composition $\varphi^k = \widetilde{\varphi}_*^k \circ \mathcal{D}_{k,\bullet}$.

Note that $\varphi^1 = \mathrm{id}$ and further that the maps φ^k gives us *lambda operations* (on a trivial ring structure) in the sense of Definition 4.1, that is we have

Lemma 4.12 *The maps $\varphi^\bullet$ satisfy $\varphi^k \circ \varphi^l = \varphi^{kl}$.*

Proof. Let $X : \Gamma_* \to \mathsf{Set}$, this is in particular a simplicial set by remark 4.10 and then equation (4.6) defines a functor $\widetilde{\varphi}_*^k : \mathrm{sd}_k X(\bullet) \to X(\bullet)$. The lemma for a left Γ-module follows now from the commutative diagram

$$\begin{tikzcd}
{|X(\bullet)|} \arrow[r, "D_k^{-1}"] \arrow[d, "D_{kl}^{-1}"'] & {|\mathrm{sd}_k X(\bullet)|} \arrow[r, "\widetilde{\varphi}_*^k"] \arrow[dl, "D_l^{-1}"'] & {|X(\bullet)|} \arrow[d, "D_l^{-1}"] \\
{|\mathrm{sd}_{kl} X(\bullet)|} \arrow[rr, "\mathrm{sd}_l(\widetilde{\varphi}_*^k)"] \arrow[drr, bend right, "\widetilde{\varphi}_*^{kl}"'] & & {|\mathrm{sd}_l X(\bullet)|} \arrow[d, "\widetilde{\varphi}_*^l"] \\
 & & {|X(\bullet)|}
\end{tikzcd}$$

where we use $|-|$ stands for the realization of a simplicial dg-module (that is dg-extension of Dold-Kan functor). The commutativity of the lower triangle is a consequence of (4.6) and the upper one by definition of D_m. The middle trapeze commutes since $\widetilde{\varphi}_*^l$ is a natural transformation $\mathrm{sd}_l X((\bullet)) \to X(\bullet)$. The proof for a right Γ_*-module is similar. $\square$

4.3 Hodge filtration for Hochschild cochains over spheres and suspensions

We will now use the functoriality over spaces of Hochschild (co)homology to give a λ-ring (with zero multiplication), and consequently a *Hodge*-filtration, structure on the case of a spheres or any other suspension ΣX of a pointed space. For circles, this structure will recover Loday and Gerstenhaber-Schack ones.

We will also gives combinatorial explicit representations of these structures.

The key idea is the fact that the circle has canonical power maps since it is a group. Precisely, identifying S^1 with $\{z \in \mathbb{C}/|z| = 1\}$ with base point its unit 1, define

$$\lambda^k : S^1 \to S^1, \quad z \mapsto z^k. \tag{4.7}$$

Note that these power maps are *pointed*, thus, we can extend them to any space obtained as a suspension $\Sigma X = S^1 \wedge X$ by

$$\lambda^k : S^1 \wedge X \xrightarrow{\lambda^k \wedge id} S^1 \wedge X. \tag{4.8}$$

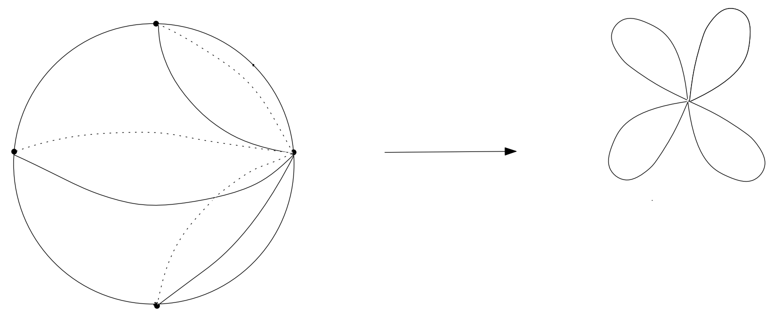

Figure 2: The pinching map $\lambda^4 : S^1 \wedge S^1 \to S^1 \wedge S^1$ obtained by collapsing to the base point the 3 circles on the sphere on the left.

This applies in particular to higher spheres $S^d = S^1 \wedge \cdots \wedge S^1$. In that case the power maps, for $k \geq 1$, factors as the composition

$$\lambda^k = S^d \stackrel{pinch^{(k)}}{\longrightarrow} S^d \vee \cdots \vee S^d \stackrel{\vee id}{\longrightarrow} S^d \tag{4.9}$$

where the first map is a pinching map collapsing k-many S^{d-1}-spheres, see picture 2.

Similarly we have operations

$$\lambda^k : S^1 \times X \stackrel{\lambda^k \times id}{\longrightarrow} S^1 \times X. \tag{4.10}$$

defined for a product of S^1 with a space.

From definition (4.7) and (4.8), we immediately obtain (in all cases)

$$\lambda^k \circ \lambda^l = \lambda^{k+l}. \tag{4.11}$$

By functoriality with respect to continuous maps, we obtain operations on the singular chains associated to the singular simplicial set of Y and in the ∞-derived category:

Definition 4.13 Let Y be any space equal to S^d or the suspension ΣX or product $S^1 \times X$ of any pointed space. For any cdga A and A-module M we define the *power operations* to be

$$\lambda^{k,\bullet} : CH^{\Delta_\bullet(Y)}(A,M) \stackrel{(\lambda^k)^*}{\longrightarrow} CH^{\Delta_\bullet(Y)}(A,M), \tag{4.12}$$

$$\lambda^k_\bullet : CH_{\Delta_\bullet(Y)}(A,M) \stackrel{(\lambda^k)_*}{\longrightarrow} CH_{\Delta_\bullet(Y)}(A,M) \tag{4.13}$$

$$\boldsymbol{\lambda}^k : \mathbf{CH}^Y(A,M) \stackrel{(\lambda^k)^*}{\longrightarrow} \mathbf{CH}^Y(A,M), \tag{4.14}$$

$$\boldsymbol{\lambda}^k : \mathbf{CH}_Y(A,M) \stackrel{(\lambda^k)_*}{\longrightarrow} \mathbf{CH}_Y(A,M). \tag{4.15}$$

By definition, the operations defined on Hochschild (co)homology are represented by the operations λ^k (4.12), (4.13) defined on the Hochschild (co)chains over the singular set $\Delta_\bullet(Y)$ and are cochain maps on the later ones.

We now give the main theorem for the Hodge filtration for Hochschild cohomology; we will cover the case of homology shortly after.

Theorem 4.14 *Let Y be S^d or the suspension ΣX, or product $S^1 \times X$ by the circle, of any pointed space X. The maps $\lambda^{k\bullet}$ defined on the singular complex $CH^{\Delta_\bullet(Y)}(A,M)$ are functorial with respect to maps of algebras and modules and satisfy the identity*

$$\lambda^{p,\bullet} \circ \lambda^{q,\bullet} = \lambda^{pq,\bullet}$$

for any $p,q \in \mathbb{Z}$ making $CH^{\Delta_\bullet(Y)}(-,-)$ a functor from $\mathsf{Mod}^{\mathsf{CDGA}}$ to γ-rings with zero multiplication in the category $\mathsf{Mod}^{\mathsf{CDGA}}$.

The maps $\boldsymbol{\lambda}^k$ are similarly functorial and satisfy $\boldsymbol{\lambda}^p \circ \boldsymbol{\lambda}^q \cong \boldsymbol{\lambda}^{pq}$ endowing $\mathbf{CH}^Y(-,-)$ with the structure of an functor from $\mathbf{Mod}^{\mathsf{CDGA}}$ to γ-rings with zero multiplication in the ∞-category $\mathbf{Mod}^{\mathsf{CDGA}}$. Moreover

1. *If k is of characteristic 0, then there is an natural splitting*

$$CH^{\Delta_\bullet(Y)}(A,M) = \prod_{j\geq 0} CH^{\Delta_\bullet(Y),(j)}(A,M)$$

where the subchain complexes $CH^{\Delta_\bullet(Y),(j)}(A,M)$ are equal to $\ker(\lambda^{k\bullet} - k^j.\mathrm{id})$ (for all $k \geq 0$).

2. *The map induced by λ^k on $\mathbf{CH}^{S^d}(A,M)$ agrees with the map*

$$\mathbf{CH}^{S^d}(A,M) \xrightarrow{\mathrm{f_k}^*} \mathbf{CH}^{S^d}(A,M)$$

for any map $f_k : S^d \to S^d$ which is of degree k [20].

3. *The above decomposition yields one on Hochschild cohomology groups: $HH^Y(A,M) = \prod_{j\geq 0} HH^{Y,(j)}(A,M)$ and we have natural equivalences*

$$HH^{Y,(j)}(A,M) = H^*\Big(\ker\big(\mathbf{CH}^Y(A,M) \overset{\boldsymbol{\lambda}^k - k^j.id}{\longrightarrow} \mathbf{CH}^Y(A,M)\big)\Big)$$
$$= \ker\Big(HH^Y(A,M) \overset{\boldsymbol{\lambda}^k - k^j.id}{\longrightarrow} HH^Y(A,M)\Big) \cong H^*(CH^{\Delta_\bullet(Y),(j)}(A,M)).$$

4. *The various Hodge filtrations preserve the skeletal filtration* [21].

Following the terminology from Section 4.1, we call the natural filtrations/decompositions given by the γ-ring structures, the Hodge filtrations or decompositions of $\mathbf{CH}^Y(A,M)$ and $CH^{\Delta_\bullet(Y)}(A,M)$ and similarly for the ones induced on the cohomology groups $HH^Y(A,M)$.

20. that is $\pi_d(f_k,*)(1) = k$
21. see remark 3.21, this is the filtration induced by the simplicial degree.

Proof. The naturality of the maps λ^k follows from Proposition 3.18 asserting the triple functoriality of Hochschild cochains and cohomology functors which, in view of identity (4.11), also gives the identities:

$$\lambda^{p\bullet} \circ \lambda^{q\bullet} = \lambda^{pq\bullet}, \qquad \boldsymbol{\lambda}^p \circ \boldsymbol{\lambda}^q \cong \boldsymbol{\lambda}^{pq}.$$

The map $\boldsymbol{\lambda}^k$ is a derived enhancement of $\lambda^{k\bullet}$ by Definition 3.9 (and Proposition 3.18). In particular we obtain that $(CH^{\Delta_\bullet(Y)}(A,M),(\lambda^{k,\bullet})_{k\geq 1})$ is a strict γ-ring with trivial multiplication (that is an object of $(\gamma,0)$-ring), representing the derived system $(\mathbf{CH}^Y(A,M),(\boldsymbol{\lambda}^k)_{k\geq 1}))$. It follows that the latter inherits a canonical structure of an object in the associated ∞-category $(\gamma,0)$**-ring** as claimed.

Note also that the factorization (4.9) implies that the map $\lambda^k : S^d \to S^d$ is precisely of degree k. Now, claim 2. follows from the fact that two maps $S^d \to S^d$ are homotopic if and only if they are of the same degree.

Assertion 3. follows from 1. and the fact that $\lambda^{k\bullet}$ on $CH^{\Delta_\bullet(Y)}(A,M)$ induces the map $\boldsymbol{\lambda}^k$ on the Hochschild cohomology $\mathbf{CH}^Y(A,M)$.

Now we prove Assertion 1. As already seen, the equality $\lambda^{p\bullet}\circ\lambda^{q\bullet} = \lambda^{pq\bullet}$ defines a γ-ring structure with zero multiplication on the cochain *complex* $CH^{\Delta_\bullet(Y)}(A,M)$. We thus obtain a complete Hodge filtration F^γ and Adams operations as in Section 4.1. Then the general theory of λ-rings ([H, Theorem 4.5] , [Kr]) implies that, when $k \supset \mathbb{Q}$, there is a decomposition into eigenspaces of the Adams operations which satisfies the claimed properties.

Assertion 4 is a consequence of the fact that at the cochain level, the maps $\lambda^{k,\bullet}$ preserves the simplicial degree (as in Remark 3.7) of the Hochschild cochain complex. $\square$

4.4 Hodge filtration on Hochschild cochains on the standard model

We have define λ-operations on the Hochschild cohomology $\mathbf{CH}^Y(-,-)$ and further found a functorial Hodge decomposition on the explicit, but very huge, chain complex given by the singular set $\Delta_\bullet(Y)$ of the space Y. We are going to define operations on the much smaller complex given by the standard model of the spheres using the power operations from Definition 4.11. What allows us to do that is the following proposition 4.15 (whose result does *not* for arbitrary models). That will allow us to slightly refine the Theorem 4.14 using combinatorial identities given by eulerian idempotents.

Proposition 4.15 *Let $S^d_\bullet$ be the standard models of the spheres (see example 3.17). The (co)simplicial modules $\mathcal{H}(A,M)(S^d_\bullet)$ and $\mathcal{L}(A,M)(S^d_\bullet)$ lift respectively to right and left Γ_*-modules.*

By lifting we mean that the (co)simplicial structure obtained from the Γ_*-modules as in remark 4.10 is the original one.

Proof. For $d=1$, this is an observation of Loday [L1]. The proof generalizes to higher spheres, by applying the same construction diagonally (here we are taking advantage of the fact that the standard models without its base point is given by d-power of the case $d=1$). Namely, for a pointed set I_+, denote $I := I_+ \setminus \{+\}$ and set

$$\mathcal{L}(A,M)(I_+) := M \otimes A^{(I^d)}$$

and $\mathcal{H}(A,M)(I_+) := Hom_k\big(A^{\otimes(I^d)}, M\big)$. For $\phi : I_+ \to J_+$, we define

$$\mathcal{L}(A,M)(\phi)\Big(m_+ \otimes \bigotimes_{(i^1,\dots,i^d)\in I^d} a_{(i^1,\dots,i^d)}\Big) = n_+ \otimes \bigotimes_{(j^1,\dots,j^d)\in J^d} b_{(j^1,\dots,j^d)}$$

where

$$b_{(j^1,\dots,j^d)} = \prod_{(i^1,\dots,i^d)\in(\phi^d)^{-1}(j^1,\dots,j^d)} a_{(i^1,\dots,i^d)}$$

and one puts 1 if the preimage $(\phi^d)^{-1}$ at one set of indexes is empty. It is immediate to check that the induced simplicial structure is the one of example 3.17 (or 3.16 or 3.15 if $d=2,1$). The construction for $\mathcal{H}(A,M)(\phi)$ is similar. $\square$

Hence, if $S^d_\bullet$ is the standard simplicial set model for the d-sphere, Proposition 4.15, identities (3.9), (3.10) and definitions 4.11 and 3.9 gives us the following operations:

$$\varphi^{k,\bullet} : CH^{S^d_\bullet}(A,M) \cong \mathcal{H}(A,M)^{dg}(S^d_\bullet) \longrightarrow \mathcal{H}(A,M)^{dg}(S^d_\bullet) \cong CH^{S^d_\bullet}(A,M) \tag{4.16}$$

$$\varphi^k_\bullet : CH_{S^d_\bullet}(A,M) \cong \mathcal{L}(A,M)^{dg}(S^d_\bullet) \longrightarrow \mathcal{L}(A,M)^{dg}(S^d_\bullet) \cong CH_{S^d_\bullet}(A,M) \tag{4.17}$$

which are (co)chains homomorphisms.

We *denote similarly* $\boldsymbol{\varphi}^k$ *the induced maps on Hochschild (co)homology*

$$\boldsymbol{\varphi}^k : \mathbf{CH}^{S^d}(A,M) \to \mathbf{CH}^{S^d}(A,M), \quad \boldsymbol{\varphi}^k : \mathbf{CH}_{S^d}(A,M) \to \mathbf{CH}_{S^d}(A,M)$$

which are well defined since the maps $\varphi^{k,\bullet}$ and $\varphi^k_\bullet$ preserves quasi-isomorphisms.

One can give a direct combinatorial definitions of the operations $\varphi^{k\bullet}$ and $\varphi^k_\bullet$ as follows. Let $\Sigma_{n,j}$ be the subset of permutations of Σ_n with $j-1$ descents. We recall ([L2]) that a *descent* for $\sigma\in\Sigma_n$ is an index i such that $\sigma(i) > \sigma(i+1)$. The Γ_*-modules structures of Proposition 4.15 satisfies that $\sigma\in\Sigma_n$ acts on $f\in C^{S^d_n}(A,M) = Hom(A^{\otimes n^d}, M))$ by

$$\sigma^*(f)(\cdots\otimes a_{i^1,\dots,i^d}\otimes\cdots) = f(\cdots\otimes a_{\sigma^{-1}(i^1),\dots\sigma^{-1}(i^d)}\otimes\cdots)$$

as well as by

$$\sigma_*\Big(m\otimes\bigotimes a_{i^1,\dots i^d}\Big) = m\otimes\bigotimes a_{\sigma^{-1}(i^1),\dots\sigma^{-1}(i^d)}$$

on Hochschild chains $C_{S^d_n}(A,M)$.

Lemma 4.16 *In simplicial degree n, the operation $\varphi^{k,n} : C^{S^d_n}(A,M) \to C^{S^d_n}(A,M)$ on cochains is equal to*

$$\varphi^{k,n} = \sum_{i=0}^{k-1} \sum_{\sigma\in\Sigma_{n,k-i}} (-1)^{\sigma} \binom{n+k}{n} \sigma^*.$$

On the chains $C_{S^d_n}(A,M)$, one has

$$\varphi^{k}_n = \sum_{i=0}^{k-1} \sum_{\sigma\in\Sigma_{n,k-i}} (-1)^{\sigma} \binom{n+k}{n} \sigma_*.$$

PROOF. This is an explicit computation using formulas (4.4), (4.5), (4.6), the definition of power maps (Definition 4.11) and Proposition 4.15. The computation, in the case $d=1$, is given in details in [MCa]; it is the same for $d>1$ using the structure of Proposition 4.15. $\square$

Theorem 4.17 *The operations $\varphi^{k,\bullet}$ defined on the singular complex $CH^{S^d_\bullet}(A,M)$ are functorial with respect to maps of algebras and modules, satisfy*

$$\varphi^{k,\bullet}\circ\varphi^{l,\bullet} = \varphi^{kl,\bullet}$$

making $CH^{S^d}(-,-)$ into a functor from $\mathsf{Mod}^{\mathsf{CDGA}}$ to γ-rings with zero multiplication in the category $\mathsf{Mod}^{\mathsf{CDGA}}$.

Similarly, the operations $\boldsymbol{\varphi}^k$ are also functorial and give to $\mathbf{CH}^Y(-,-)$ the structure of a functor from $\mathbf{Mod}^{\mathsf{CDGA}}$ to γ-rings with zero multiplication in the ∞-category $\mathbf{Mod}^{\mathsf{CDGA}}$.

Moreover

1. *After passing to Hochschild cohomology functor, one has*

$$\boldsymbol{\varphi}^k \cong \boldsymbol{\lambda}^{k^d} : \mathbf{CH}^{S^d}(A,M) \to \mathbf{CH}^{S^d}(A,M).$$

2. *If k is of characteristic 0, then there is an natural splitting*

$$CH^{S^d_\bullet}(A,M) = \prod_{j\geq 0} CH^{S^d_\bullet,(j)}(A,M)$$

   where the sub-chain complexes $CH^{S^d_\bullet,(j)}(A,M)$ are $\ker(\varphi^{k\bullet} - k^j.\mathrm{id})$ (for all $k\geq 0$).

3. *If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, there is a natural decomposition*

$$CH^{S^d_\bullet}(A,M) = \bigoplus_{0\leq n\leq p-1} CH^{S^d_\bullet,(j)}(A,M)$$

   with each $\varphi^{n\bullet}$ acting by multiplication by n^j on $CH^{S^d_\bullet,(j)}(A,M)$.

4. *The above two decompositions yields similar ones on Hochschild cohomology groups and one has $H_*\big(CH^{S^d_\bullet,(\ell)}(A,M)\big)=0$ if $\ell\neq jd$. Denoting*
$$HH^{S^d,(j)}(A,M) := H^*\big(CH^{S^d_\bullet,(jd)}(A,M)\big),$$
one has an natural splitting
$$HH^{S^d}(A,M) = \prod HH^{S^d,(j)}(A,M).$$
Further one also has an identification
$$\begin{aligned} HH^{S^d,(j)}(A,M) &= H^*\Big(\ker\big(\mathbf{CH}^{S^d}(A,M) \stackrel{\boldsymbol{\lambda}^k - k^j.id}{\longrightarrow} \mathbf{CH}^{S^d}(A,M)\big)\Big) \\ &= H^*\Big(\ker\big(CH^{S^d_\bullet}(A,M) \stackrel{\varphi^k - k^{jd}.id}{\longrightarrow} CH^{S^d_\bullet}(A,M)\big)\Big). \end{aligned}$$
5. *In both cases, one has cochains* isomorphisms
$$CH^{S^d_\bullet,(0)}(A,M) \cong pt_*(CH^{pt_\bullet}(A,M)) \cong M$$
where pt_* is the cochain map induced by the base point $pt_\bullet \to S^d_\bullet$.
In particular the Hodge filtration on the cohomology groups satisfies $HH^{S^d,(0)}(A,M)\cong M$ and all other pieces $HH^{S^d,(j)}(A,M)$ are not in weight 0 with respect to the skeletal filtration.

In particular, from 3. and 4. we get a (partial) Hodge decomposition in positive characteristic extending the filtration of Theorem 4.14. Statement 4. also ensures that the pieces $HH^{S^d,(j)}(A,M)$ are the same as the ones given by Theorem 4.14 in characteristic zero.

Remark 4.18 Since the cohomology groups of a split complex are the direct sum of the cohomology groups of the pieces, the claims 3. and 4. in the theorem implies that the cochain complexes $CH^{S^d_\bullet,(j)}(A,M)$ *are acyclic unless* $j=di$ is a multiple of d. In which case, the cochain complex $CH^{S^d_\bullet,(id)}(A,M)$ is quasi-isomorphic to $CH^{\Delta_\bullet(S^d),(i)}(A,M)$ by 1 and 4.

Remark 4.19 Statement 5. and 4. imply that, for any simplicial model of the maps $\boldsymbol{\lambda}^k$ on a simplicial model $Y_\bullet$ of S^d inducing a γ-ring structure, the natural map (induced by the base point) $M \hookrightarrow CH^{Y_\bullet,(0)}(A,M)$ is a quasi-isomorphism. In particular, the only piece of weight 0 in the Hochschild cohomology decomposition are also of weight 0 for the skeletal filtration.

Proof of Theorem 4.17

The functoriality follows from from Theorem 3.24 as in Theorem 4.14. The formula $\varphi^{k\bullet}\circ\varphi^{l\bullet}=\varphi^{kl\bullet}$ comes from Lemma 3.12 and Lemma 4.12. Hence we have a functorial (strict) γ-ring with trivial multiplication structure on $CH^{S^d_\bullet}(A,M)$ which in turn induces one on $\mathbf{CH}^{S^d}(A,M)$ given by $\boldsymbol{\varphi}^k$ by definition of those.

Claim 4. is an immediate consequence of the other ones and the fact that the decomposition in Theorem 4.14 is also given by the same eigenspaces.

Since we have a γ-ring (with trivial multiplication) on the cochain *complex* $CH^{S^d_\bullet}(A,M)$, we have a complete Hodge filtration F^γ and Adams operations as in Section 4.1 which already implies 2. To prove also 3. we use Lemma 4.16 to analyze a little bit more the $\varphi^{k\bullet}$. Indeed, Loday [L1, Proposition 2.8] has proved that the eulerian idempotents $e_n^{(i)}$ form a basis of orthogonal idempotents of $\mathbb{Q}[\Sigma_n]$ and further that in simplicial degree n one has

$$\varphi^{k,n} = ke_n^{(1)} + \cdots + k^n e_n^{(n)}.$$

This follows precisely of the formula given in Lemma 4.16 which identifies the map $\varphi^{k,n}$ with $(-1)^{k-1}(\lambda_n^k)^*$, where $\lambda_n^k \in \mathbb{Z}[\Sigma_n]$ is given by [L1, Definition 1.6], and finally statement g) in [L1, Proposition 2.8]. The eulerian idempotents commutes with the simplicial differential (see *loc. cit.*) and thus precisely induces the splitting of the cochain complex $CH^{S^d_\bullet}(A,M)$:

$$\begin{aligned} CH^{S^d_\bullet,(j)}(A,M) &= e_\bullet^{(j)}\big(CH^{S^d_\bullet}(A,M)\big) \\ &= \ker\big(\mathbf{CH}^{S^d_\bullet}(A,M) \xrightarrow{\boldsymbol{\lambda}^k - k^j.id} \mathbf{CH}^{S^d_\bullet}(A,M)\big). \end{aligned} \quad (4.18)$$

Furthermore, the combinatorial properties of the associated operations γ^k show that $CH^{S^d_\bullet,(0)}(A,M) = CH^{S^d_0,(0)}(A,M) = M$ and $CH^{S^d_{\bullet\geq 1},(0)}(A,M) = \{0\}$; the proof is exactly the same as [L1, Théorème 3.5] using again the formula given by Lemma 4.16. This proves assertion 5 of the Theorem in characteristic zero as well; the proof of 5. in positive characteristic is an immediate consequence of it and the definition of the splitting we give below.

Now if k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, we can not define the operators $e_n^{(i)}$ on the Hochschild cochains but the operators

$$\overline{e_n^{(i)}} = \sum_{m\geq 0} e_n^{(i+(p-1)m)} : CH^{S^d_\bullet}(A,M) \to CH^{S^d_\bullet}(A,M)$$

are well defined (provided $1 \leq i \leq p-1$ and $n \geq 1$). The proof is purely combinatorial and given in [GS, Section 5]. We can thus set $\overline{e^{(i)}}$ to be the map induced by the operators $\overline{e_n^{(i)}}$ when n varies. The Hochschild differential still commutes with this operator since it does with all $e^{(i)}$, and further $\overline{e_n^{(i)}} = e_n^{(i)}$ for $n \leq p-1$. This gives the desired splitting

$$CH^{S^d_\bullet}(A,M) = \bigoplus_{0\leq i\leq p-1} \overline{e_n^{(i)}}^*\Big(CH^{S^d_\bullet,(j)}(A,M)\Big).$$

It remains to prove assertion 1. The factorization (4.9) shows that

$$\boldsymbol{\lambda}^{k^d} = \mathbf{CH}^{S^d}(A,M) \xrightarrow{\vee id^*} \mathbf{CH}^{\bigvee S^d}(A,M) \xrightarrow{pinch^{(k^d)^*}} \mathbf{CH}^{S^d}(A,M). \quad (4.19)$$

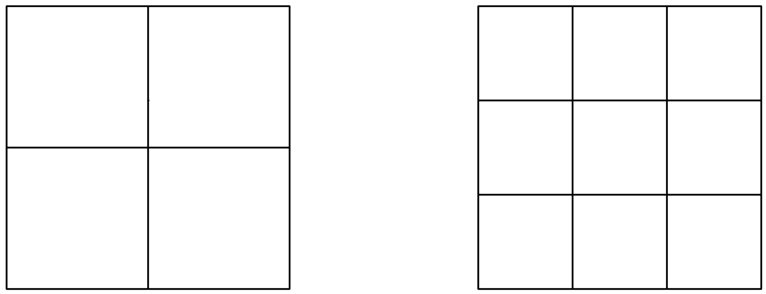

Figure 3: The $k = 2$ on the left and $k = 3$ on the right subdivisions of $I^2/\partial I^2$.

Hence we have to prove that $\varphi^{k\bullet}$ represents the same composition when passing to the derived category.

To do so, we are going to use a different model for the iterated pinching map. The sphere S^d is the quotient $I^d/\partial I^d$ of the d-cube by its boundary ($I = [0, 1]$). We can subdivide easily the cube in kd-isometric d-cube delimited by the intersections of $[0, 1]^d$ with the $d(k - 1)$ hyperplanes of equations $x_i = \frac{j}{k}$ ($i = 1 \ldots d$). See Figure 3.

Identifying the boundary of all the cubes of the subdivision (that is its codimension 1 skeleton) with the base point yields on the quotient spaces a continuous map:

$$pinch^{(k)}_{(d)} : S^d = I^d/\partial I^d \to \bigvee_{\{1,\ldots,k\}^d} S^d \tag{4.20}$$

from S^d to the wedges of kd-many copies of it.

We are now going to show that the operations ψ^k are a model for this pinching map $pinch^{(k)}_{(d)}$. We first recall that $m_+ = \{0, \ldots m\}$ is of cardinal $m + 1$ with base point 0, and that $S^d_n = (n^d)_+ = \{0\} \coprod \{1, \ldots, n\}^d$. Thus we have $\mathrm{sd}_k(S^d_\bullet)_n) = ((kn + k - 1)^d)_+$ and

$$\bigvee_{\{1\ldots k\}^d} S^d_n \;\cong\; (n^d)_+ \vee \cdots \vee (n^d)_+$$

with k^d components (which we label by elements of the set $\{1, \ldots, k\}^d$). The wedge $\bigvee S^d$ is the geometric realization of the wedge $\bigvee S^d_\bullet$ of simplicial sets.

Then we note that there is a map of simplicial set

$$p_{(k)} : \mathrm{sd}_k(S^d_\bullet) \longrightarrow \bigvee_{\{1\ldots k\}^d} S^d_\bullet$$

given, for all $0 \le i_1, \ldots, i_d \le k - 1$ and $0 \le j_1, \ldots, j_d \le n$, by

$$\begin{aligned} &p_{(k)}\big(i_1(n + 1) + j_1, \ldots, i_d(n + 1) + j_d\big) \\ &\qquad = (j_1, \ldots, j_d) \in (n^d)_+ \vee \Big(\bigvee_{\{1,\ldots,k\}^d \setminus \{(i_1,\ldots,i_d)\}} \{0\} \Big) \subset \bigvee_{\{1\ldots k\}^d} (n)^d_+ \end{aligned} \tag{4.21}$$

that is, $p_{(k)}(i_1(n+1)+j_1, \ldots, i_d(n+1)+j_d)$ is the element $(j_1, \ldots, j_d)$ viewed in the $(i_1, \ldots, i_d)^{\text{th}}$ component of the wedge $\bigvee\limits_{\{1\ldots k\}^d} S_n^d = \bigvee\limits_{\{1\ldots k\}^d} (n)_+^d$.

The following lemma identifies the realization of $p_{(k)}$ with the dk-fold pinching map.

Lemma 4.20 *The map $p_{(k)}$ is a map of simplicial sets and after taking geometric realization we have a commutative diagram*

$$\begin{tikzcd}
|S^d_\bullet| \arrow[dr, bend right, "pinch^{(k)}_{(d)}"'] & |\mathrm{sd}_k(S^d_\bullet)| \arrow[l, "D_k"'] \arrow[r, "|\widetilde{\varphi}^k_\bullet|"] \arrow[d, "|p_{(k)}|"'] & |S^d_\bullet| \, . \\
& \bigvee\limits_{\{1\ldots k\}^d} |S^d_\bullet| \arrow[ur, "\bigvee id"'] &
\end{tikzcd}$$

Proof of Lemma 4.20 The easiest and most instructive way to check the lemma is to see that, by definition of edgewise subdivision, $\mathrm{sd}_k(S^d_\bullet)$ is a decomposition of a d-cube into k^d copies of $I^d_\bullet$ (the standard model for the d-cube) whose boundary (of the big cube) has been collapsed to a point, see Figure 3. Then the map $p_{(k)}$ is the map that collapses the boundary of each of the d-cubes of the subdivision. It is thus a map obtained by quotient of simplicial sets, hence is simplicial and its realization is the quotient of the realization, which is precisely $pinch^{(k)}_{(d)}$.

Let us now put these observation in combinatorial data. This boils down to the fact that if $f : m_+ \to n_+$ is non-decreasing, then by definition of the functor sd, we have

$$\begin{aligned}
p_{(k)} \circ f^*\big(i_1(n+1)+j_1, \ldots, i_d(n+1)+j_d\big) \\
= p_{(k)}\big(i_1(m+1)+f^*(j_1), \ldots, i_d(m+1)+f^*(j_d)\big) \\
= (f^*(j_1), \ldots, f^*(j_d)) = f^*(j_1, \ldots, j_d) \\
= f^* \circ p_{(k)}\big(i_1(n+1)+j_1, \ldots, i_d(n+1)+j_d\big).
\end{aligned}$$

The composition of $\bigvee id$ with $p_{(k)}$ is just

$$p_{(k)}\big(i_1(n+1)+j_1, \ldots, i_d(n+1)+j_d\big) = (j_1, \ldots, j_d) \in (n^d)_+$$

which proves that the right triangle commutes. For the commutativity of the left triangle, recall that the standard model is obtained as iterated wedges of $S^1_\bullet$ and identifies canonically with the quotient $I^d_\bullet \partial(I^d_\bullet)$. On the realization of this model, $pinch^{(kd)}$ is obtained by taking the quotient of $[0,1]^d$ by its intersection with the $d(k-1)$ hyperplanes of equations $x_i = \frac{j}{k}$ ($i = 1 \ldots d$, $1 \le j \le k-1$. Let $t = (0 \le t_1 \le \cdots \le t_n \le 1)$ be in Δ^n; then the element $\{t\} \times \{(j_1, \ldots, j_d)\}$ becomes the point of coordinates $(t_{j_1}, \ldots, t_{j_d}) \in [0,1]^d/\partial[0,1]^d \cong |S^d_\bullet|$ in the geometric realization. The homeomorphism

$$D_k : |\mathrm{sd}_k S^d_n| = \coprod \Delta^n \times (kn+k-1)^d_+/\sim \,\to \coprod \Delta^{(kn+k-1)^d} \times (kn+k-1)^d_+/\sim$$

is given by (4.2). Hence

$$D_k\big(\{t\}\times\{(i_1(m+1)+j_1),\dots,i_d(m+1)+j_d)\}\big)=\Big(\frac{i_1}{k}+\frac{t_{j_1}}{k},\dots,\frac{i_d}{k}+\frac{t_{j_d}}{k}\Big).$$

Further for the same reasons,

$$|P_{(k)}|\big(\{t\}\times\{(i_1(m+1)+j_1),\dots,i_d(m+1)+j_d)\}\big)=(t_{j_1},\dots,t_{j_d})$$

this component lying in the sphere labeled by $(i_1,\dots,i_k)$ in the wedge. The result follows by applying the pinch map, which simply sends a point of coordinates $\frac{1}{k}\Big((i_1,\dots,i_k)+(u_{j_1},\dots u_{j_k}\Big)$ to $k(u_{j_1},\dots u_{j_k})$ in the component $(i_1,\dots,i_k)$, to the image of D_k. $\square$

Since, by definition 4.11 and (4.16), φ^k is the composition

$$CH^{S^d_\bullet}(A,M)\xrightarrow{\widetilde{\varphi}^{k*}}CH^{\mathrm{sd}_k(S^d_\bullet)}(A,M)\xrightarrow{\mathcal{D}^\bullet_k}CH^{S^d_\bullet}(A,M),$$

Lemma 4.20 and Lemma 4.9 implies that passing to the Hochschild cohomology functor $\mathbf{CH}^{S^d}(A,M)$ we get the identity

$$\varphi^k=\mathbf{CH}^{S^d}(A,M)\xrightarrow{\vee id^*}\mathbf{CH}^{\bigvee S^d}(A,M)\xrightarrow{pinch^{(k)*}_{(d)}}\mathbf{CH}^{S^d}(A,M). \tag{4.22}$$

Since $pinch^{(k)}_{(d)}$ is homotopical to $pinch^{(kd)}$, then we deduced from it and factorization (4.19) that

$$\varphi^k\ \cong\ \boldsymbol{\lambda}^{k^d}:=\mathbf{CH}^{S^d}(A,M)\longrightarrow\mathbf{CH}^{S^d}(A,M).$$

In fact, we also see that the diagram

$$\begin{tikzcd} CH^{\Delta_\bullet(S^d)}(A,M) \arrow[r, "\lambda^{k^d}"] \arrow[d, "\eta^*"'] & CH^{\Delta_\bullet(S^d)}(A,M) \arrow[d, "\eta^*"] \\ CH^{S^d_\bullet}(A,M) \arrow[r, "\varphi^k"] & CH^{S^d_\bullet}(A,M) \end{tikzcd}$$

is commuting up to an natural homotopy of cochain complexes. $\square$

Corollary 4.21 *For $d=1$, the Hodge decompositions of Hochschild cohomology groups and $CH^{S^1_\bullet}(A,M)$ are identical to Gerstenhaber-Schack ones [GS].*

Proof. It follows from equation (4.18) and the explicit definition of Gerstenhaber-Schack splitting in terms of eulerian idempotents. $\square$

Remark 4.22 From Lemma 4.20 and its proof, it is not hard to see that the realization $|\widetilde{\varphi^k}|\circ D_k^{-1}$ is a model for the continuous map

$$S^d\cong S^1\wedge\cdots\wedge S^1\xrightarrow{\lambda^k\wedge\cdots\wedge\lambda^k}S^1\wedge\cdots\wedge S^1\cong S^d$$

given by the iterated wedge of the power map λ^k on the standard circle.

4.5 Hodge filtration and λ-operations for Hochschild chains over spheres and suspensions

Let Y be a space equal to S^d, or ΣX or $S^1 \times X$. In Section 4.3, we have defined *power operations* (see (4.12) and (4.15))

$$\lambda_\bullet^k : CH_{\Delta_\bullet(Y)}(A,M) \xrightarrow{(\lambda^k)_*} CH_{\Delta_\bullet(Y)}(A,M)$$
$$\boldsymbol{\lambda}^k : \mathbf{CH}_Y(A,M) \xrightarrow{(\lambda^k)_*} \mathbf{CH}_Y(A,M).$$

on Hochschild chains (over the singular set of Y) and (the derived) Hochschild homology over Y. Further in Section 4.4, we have defined (4.17) operations

$$\varphi_\bullet^k : CH_{S^d_\bullet}(A,M) \longrightarrow CH_{S^d_\bullet}(A,M)$$

on the Hochschild chains over the standard model of the spheres inducing the operation

$$\boldsymbol{\varphi}^k : \mathbf{CH}_{S^d}(A,M) \to \mathbf{CH}_{S^d}(A,M)$$

on Hochschild homology.

When the module structure of M comes from an algebra one, since the power maps are induced by maps of spaces, we get

Proposition 4.23 *Let B be a CDGA over A. The power maps $\lambda_\bullet^k$ and $\boldsymbol{\lambda}^k$ are maps of CDGAs. Further, they make the cdga $CH_{\Delta_\bullet(Y)}(A,B)$ a multiplicative λ-ring with zero multiplication*[22].

Passing to homology, these operations provides a factorization of the functor $\mathbf{CH}_{(-)}(-) : \mathbf{Top} \times \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}) \to \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}})$ from proposition 3.23 as a functor

$$\begin{aligned}\mathbf{CH}_{(-)}(-,-) : \mathbf{Top}_* \times \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}})& \\ \longrightarrow (\gamma,0) - \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}) &\xrightarrow{forget} \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}}) \quad (4.23)\end{aligned}$$

where $(\gamma,0) - \mathbf{CAlg}(\mathbf{Mod}_{\mathsf{CDGA}})$ is the ∞-category of (non-unital) cdga's over a unital cdga endowed with an additional γ-ring with zero multiplication structure (see remark 4.7).

In particular, for $B = A$, the higher Hochschild homology functor (from Proposition 3.18) lifts canonically as a functor

$$\mathbf{CH}_{(-)}(-) : \mathbf{Top} \times \mathbf{CDGA} \longrightarrow (\gamma,0) - \mathbf{CDGA}.$$

Proof. Since the power operations as well as their identities are induced by maps of topological spaces, the first statement comes from Lemma 3.10 and Proposition 3.23. The gamma-ring structure with trivial multiplication is a consequence of identity (4.11) which yields

$$\lambda_\bullet^p \circ \lambda_\bullet^q = \lambda_\bullet^{pq}, \qquad \boldsymbol{\lambda}^p \circ \boldsymbol{\lambda}^q \cong \boldsymbol{\lambda}^{pq}.$$

22. see Definition 4.1

In particular we obtain that $(CH^{\Delta_\bullet(Y)}(A,B), sh, (\lambda^{k,\bullet})_{k\geq 1})$ is a strict multiplicative γ-ring with trivial multiplication (that is an object of $(\gamma,0)$-ring(CDGA)), representing the derived system $(\mathbf{CH}^Y(A,B), \mathbf{sh}, (\boldsymbol{\lambda}^k)_{k\geq 1}))$. It follows that the latter inherits a canonical structure of an object in $(\gamma,0)$**-ring**$(\mathbf{CDGA}) = (\gamma,0)-\mathbf{CDGA}$ as claimed. By construction, forgetting the γ-ring structures gives back the original Hochschild homology functor. The last statement follows from the first statement. $\square$

Similarly, on the small model we have

Lemma 4.24 *Let B be a CDGA over A. The maps $\varphi^k_\bullet$ (and the algebra structure of Lemma 3.10) makes $CH_{S^d_\bullet}(A,B)$ a multiplicative γ-ring with zero multiplication.*

Proof. The γ-ring structure with 0-multiplication is a consequence of Lemma 4.12. Lemma 4.16 gives an explicit combinatorial formula for the maps $\varphi^k_\bullet$ and one needs to check that these maps are multiplicative with respect to the multiplication induced by the shuffle product 3.7. This is the same combinatorial identity to check as for the usual λ-operations in [L2] (which is the case $S^1_\bullet$) or the detailed computations in [MCa, Section 5]. $\square$

We now state the properties of the Hodge filtration for Hochschild chains and homology, which are "predual" to those of Hochschild cochains and are proved exactly in the same way.

Theorem 4.25 *Let Y be S^d or the suspension ΣX, or product $S^1 \times X$ by the circle, of any (pointed in the suspension case) space X.*

1. *The power maps $\lambda^k_\bullet$ makes $CH_{\Delta_\bullet(Y)}(-,-)$ a γ-ring with zero multiplication which is a module over the multiplicative γ-ring with zero multiplication $CH_{\Delta_\bullet(Y)}(A)$, in a functorial way. Passing to the (derived) homology functor, this yields*
$$\boldsymbol{\lambda}^p \circ \boldsymbol{\lambda}^q \cong \boldsymbol{\lambda}^{pq}$$
and a factorization of higher Hochschild Homology (over Y) $\mathbf{CH}^Y(-,-)$ as a functor from $\mathbf{Mod}_{\mathsf{CDGA}}$ to $(\gamma,0)-(\mathbf{Mod}_{\mathsf{CDGA}})$[23].
2. *The operations $\varphi^k_\bullet$ satisfy*
$$\varphi^k_\bullet \circ \varphi^l_\bullet = \varphi^{kl}_\bullet$$
and makes $CH_{S^d_\bullet}(A,M)$ a γ-rings with zero multiplication which is a module over the multiplicative γ-ring with zero multiplication $CH_{S^d_\bullet}(A)$, in a functorial way[24].
3. *In Hochschild homology, one has*
$$\boldsymbol{\varphi}^k \cong \boldsymbol{\lambda}^{k^d} : \mathbf{CH}_{S^d}(A,M) \longrightarrow \mathbf{CH}_{S^d}(A,M).$$

23. See Definition 4.1, Remark 4.7
24. in other words a functor from $\mathsf{Mod}_{\mathsf{CDGA}}$ to $(\gamma,0)-(\mathsf{Mod}_{\mathsf{CDGA}})$.

4. *For $Y = S^d$, the map $\boldsymbol{\lambda}^k$ agrees with the map*

$$\mathbf{CH}_{S^d}(A, M) \xrightarrow{\mathrm{f_{k*}}} \mathbf{CH}_{S^d}(A, M)$$

   for any map $f_k : S^d \to S^d$ which is of degree k.

5. *If k is of characteristic 0, then there are natural splittings*

$$CH_{S^d_\bullet}(A, M) = \prod_{j \geq 0} CH^{(j)}_{S^d_\bullet}(A, M)$$
$$CH_{\Delta_\bullet(Y)}(A, M) = \prod_{j \geq 0} CH^{(j)}_{\Delta_\bullet(Y)}(A, M)$$

   where the sub-chain complexes $CH^{(j)}_{S^d_\bullet}(A, M)$ and $CH^{(j)}_{\Delta_\bullet(Y)}(A, M)$ are respectively $\ker(\varphi^k_\bullet - k^j.\mathrm{id})$ *and* $\ker(\lambda^k_\bullet - k^j.\mathrm{id})$ *(for all $k \geq 0$).*

6. *If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, there is a natural decomposition*

$$CH_{S^d_\bullet}(A, M) = \bigoplus_{0 \leq n \leq p-1} CH^{(j)}_{S^d_\bullet}(A, M)$$

   with each $\varphi^n_\bullet$ acting by multiplication by n^j on $CH^{(j)}_{S^d_\bullet}(A, M)$.

7. *The above decompositions yields similar ones on Hochschild homology groups: $HH_Y(A, M) = \prod_{j \geq 0} HH^{(j)}_Y(A, M)$ and we have natural equivalences*

$$\begin{aligned} HH^{(j)}_Y(A, M) &= H_*\Big(\ker\big(\mathbf{CH}_Y(A, M) \overset{\boldsymbol{\lambda}^k - k^j.id}{\longrightarrow} \mathbf{CH}_Y(A, M)\big)\Big) \\ &= \ker\Big(HH_Y(A, M) \overset{\boldsymbol{\lambda}^k - k^j.id}{\longrightarrow} HH_Y(A, M)\Big) \cong H_*(CH^{(j)}_{\Delta_\bullet(Y)}(A, M)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \cong H_*\big(CH^{(jd)}_{S^d_\bullet}(A, M)\big). \end{aligned}$$

   If $Y = S^d$, the latter group are also isomorphic to

$$H_*\Big(\ker\big(CH_{S^d_\bullet}(A, M) \overset{\varphi^k - k^{jd}.id}{\longrightarrow} CH_{S^d_\bullet}(A, M)\big)\Big).$$

   Further, $H_*\big(CH^{(\ell)}_{S^d_\bullet}(A, M)\big) = 0$ if $\ell \neq jd$.

8. *The various Hodge filtrations preserve the skeletal filtration.*

9. *One has chains* isomorphisms

$$CH^{(0)}_{S^d_\bullet}(A, M) \cong pt^*(CH_{pt_\bullet}(A, M)) \cong M$$

   where pt^* is the cochain map induced by the$S^d_\bullet \to pt_\bullet$.
   In particular the Hodge filtration on the cohomology groups satisfies $HH^{(0)}_{S^d}(A, M) \cong M$ and all other pieces $HH^{(j)}_{S^d}(A, M)$ are not in weight 0 with respect to the skeletal filtration.

PROOF. The existence of γ-ring structures with trivial multiplication follows from identity (4.11) as well as Lemma 4.12. Proposition 4.23 and Lemma 4.24 the multiplicative γ-ring structures; their proof and Proposition 3.23 then yield claims 1 and 2. The proof of the other statement is the same as those of Theorem 4.17 and Theorem 4.14. $\square$

Corollary 4.26 **i)** *The Hodge decomposition provided by Theorem 4.25 on the homology groups $HH_{S^d}(A,M)$ coincides with Pirashvili's ones in [P].*

ii) *For $d = 1$, the maps φ^m coincides with the usual Adams operations in Hochschild homology [L1].*

PROOF. Claim i) is a consequence of claim 5 in Theorem 4.25 and the main result of [P]. Claim ii) follows from the explicit combinatorial description given by Lemma 4.16 and the computations in [MCa, L1] giving an explicit description the operations in terms of descents. $\square$

4.6 Hodge filtration and the Eilenberg-Zilber model for Hochschild cochains of suspensions and products

In section 4.4, we constructed a γ-ring structure on the standard chain complex $CH^{S^d_\bullet}(-,-)$ inducing the same Hodge decomposition as the one given by the power operations $\boldsymbol{\lambda}^k$, and used it to exhibit some more structure on the Hodge filtration, for instance the positive characteristic filtration.

The construction of the operations φ^k does not extend to a product or suspension $S^1_\bullet \wedge X_\bullet$ in general because the proof of Proposition 4.15 uses the particular form of the simplicial sets $S^d_\bullet$. But we can take advantage of the product or suspension structure to define another, much smaller actually, model for the Hochschild (co)chains over $S^1_\bullet \wedge X_\bullet$ or $S^1_\bullet \times X_\bullet$. The trick is to use the "exponential rule" for the Hochschild chain cdga (see Theorem 3.24). This is realized by using iteratively the Eilenberg-Zilber, following [GTZ]. Recall from Sections 3.1 and 3.2 that sh is the shuffle product (3.6) and that $\mathcal{L}(A,A)(Y_\bullet)$ is a simplicial CDGA for any simplicial set $Y_\bullet$. Replacing A by the simplicial CDGA $\mathcal{L}(A,A)(X_\bullet)$ yields the bisimplicial CDGA

$$\mathcal{L}_{bis}(A)(X^1_\bullet, X_\bullet) := \mathcal{L}\Big(\mathcal{L}(A,A)(X_\bullet), \mathcal{L}(A,A)(X_\bullet)\Big)(Y_\bullet)$$

whose associated diagonal simplicial CDGA is $\mathcal{L}(A,A)(Y_\bullet \times X_\bullet)$. We can use the formula defining the shuffle operations $sh^\times$ (3.7) to map a bisimplicial dg-module into a dg-module in the usual way [GJ]. That is, this formula extends diagonally to give the map

$$sh^\times_{bis} : \mathcal{L}\Big(\mathcal{L}(A,A)(X_\bullet), \mathcal{L}(A,A)(X_\bullet)\Big)^{dg}(Y_\bullet) \longrightarrow \mathcal{L}(A,A)^{dg}(Y_\bullet \times X) = CH_{Y_\bullet \times X_\bullet}(A)$$

given by the formula

$$\sum_{(\mu,\nu)} sgn(\mu,\nu)\big(s^{Y_\bullet}_{\nu_q} \dots s^{Y_\bullet}_{\nu_1}\big) \circ \left(\left(s^{\mathcal{L}(A,A)(X_\bullet)}_{\mu_p} \dots s^{\mathcal{L}(A,A)(X_\bullet)}_{\mu_1}\right)^{\otimes X_p}\right). \tag{4.24}$$

Composing it with the ($Y_\bullet$-fold) shuffle $sh^\times$:

$$\mathcal{L}(A,A)^{dg}(X_\bullet)^{\otimes Y_n} \overset{(sh^\times)^{\times \# Y_n - 1}}{\longrightarrow} \mathcal{L}(A^{\otimes Y_n}, A^{\otimes Y_n})(X_\bullet)$$
$$= \mathcal{L}\Big(\mathcal{L}(A,A)(X_\bullet), \mathcal{L}(A,A)(X_\bullet)\Big)^{dg}(Y_n), \quad (4.25)$$

we get the following linear map:

$$EZ : CH_{Y_\bullet}(CH_{X_\bullet}(A)) = \bigoplus \Big(CH_{X_{p_0}}(A) \otimes \cdots \otimes CH_{X_{p_{\# Y_n}}}(A)\Big)$$
$$\overset{(sh^\times)^{\times \# Y_n - 1}}{\longrightarrow} \bigoplus \Big(CH_{X_{p_0 + \cdots + p_{\# Y_n}}}(A)\Big)^{\otimes Y_n} = \mathcal{L}\Big(\mathcal{L}(A,A)(X_\bullet), \mathcal{L}(A,A)(X_\bullet)\Big)^{dg}(Y_\bullet)$$
$$\overset{sh^\times_{bis}}{\longrightarrow} \mathcal{L}(A,A)^{dg}(Y_\bullet \times X_\bullet) = CH_{Y_\bullet \times X_\bullet}(A). \quad (4.26)$$

Proposition 4.27 *The map (4.26) $EZ : CH_{Y_\bullet}(CH_{X_\bullet}(A)) \to CH_{Y_\bullet \times X_\bullet}(A)$ is a weak-equivalence of CDGAs. It is further natural in A, $X_\bullet$, $Y_\bullet$. In particular, $CH_{Y_\bullet}(CH_{X_\bullet}(A))$ is a model for $\mathbf{CH}_{Y \times X}(A) \in \mathbf{CDGA}$.*

Proof. The second statement is an immediate consequence of the first one, which is proved in [GTZ]. Now, let $f_\bullet : X_\bullet \to X'_\bullet$, $g : Y_\bullet \to Y'_\bullet$ be simplicial sets morphisms. Since the shuffle operation (3.7) is a lax symmetric monoidal from bisimplicial dg-modules to simplicial dg-modules, we get a commutative diagram

$$\begin{tikzcd}
CH_{Y_\bullet}(CH_{X_\bullet}(A)) \arrow[r, "CH_{Y_\bullet}(f_\bullet)"] \arrow[d, "(sh^\times)^{\times \# Y_\bullet - 1}"'] & CH_{Y_\bullet}(CH_{X'_\bullet}(A)) \arrow[r, "g_\bullet"] \arrow[d, "(sh^\times)^{\times \# Y_\bullet - 1}"'] & CH_{Y'_\bullet}(CH_{X'_\bullet}(A)) \arrow[d, "(sh^\times)^{\times \# Y_\bullet - 1}"] \\
\bigoplus (CH_{X_\bullet}(A))^{\otimes Y_\bullet} \arrow[r, "f_\bullet^{\otimes Y_\bullet}"] \arrow[d, "sh^\times_{bis}"'] & \bigoplus \big(CH_{X'_\bullet}(A)\big)^{\otimes Y_\bullet} \arrow[r, "(id)^{\otimes g_\bullet}"] \arrow[d, "sh^\times_{bis}"'] & \bigoplus \big(CH_{X'_\bullet}(A)\big)^{\otimes Y'_\bullet} \arrow[d, "sh^\times_{bis}"] \\
CH_{Y_\bullet \times X_\bullet}(A) \arrow[r, "f_\bullet \times id"] \arrow[rr, bend right, "f_\bullet \times g_\bullet"'] & CH_{Y_\bullet \times X_\bullet}(A) \arrow[r, "id \times g_\bullet"] & CH_{Y_\bullet \times X_\bullet}(A)
\end{tikzcd}$$

which shows the functoriality of EZ with respect to pairs of maps of simplicial sets. The functoriality with respect to maps of cdgas is an immediate consequence of the formula defining the shuffle operations. $\square$

Not also that by functoriality, $CH_{X_\bullet}(A,M)$ and $CH^{X_\bullet}(A,M)$ are canonically symmetric $CH_{X_\bullet}(A)$-bimodules. Similarly, we have

Corollary 4.28 *Let A be a CDGA, M a A-module. There are natural (in A, M, $X_\bullet$, $Y_\bullet$) quasi-isomorphisms*

$$EZ : CH_{Y_\bullet}(CH_{X_\bullet}(A), CH_{X_\bullet}(A,M)) \overset{\simeq}{\longrightarrow} CH_{Y_\bullet \times X_\bullet}(A,M) \quad (4.27)$$
$$EZ^* : CH^{Y_\bullet \times X_\bullet}(A,M) \overset{\simeq}{\longrightarrow} CH^{Y_\bullet}(CH_{X_\bullet}(A), CH^{X_\bullet}(A,M)) \quad (4.28)$$
$$EZ_\wedge : CH_{Y_\bullet}(CH_{X_\bullet}(A),A) \otimes_{CH_{Y_\bullet}(A)} M \overset{\simeq}{\longrightarrow} CH_{Y_\bullet \wedge X_\bullet}(A,M) \quad (4.29)$$
$$EZ^\wedge : CH^{Y_\bullet \wedge X_\bullet}(A,M) \overset{\simeq}{\longrightarrow} Hom_{CH_{Y_\bullet}(A)}\big(CH_{Y_\bullet}(CH_{X_\bullet}(A),A),M\big) \quad (4.30)$$

of $CH_{Y_\bullet}(CH_{X_\bullet}(A))$-modules. Further, if M is an A-CDGA, then the maps (4.27) *and* (4.29) *are quasi-isomorphisms of CDGAs.*

Proof. One can apply a similar technique as for proposition 4.27. Or simply note that by definition 3.9 and proposition 4.27 to get the natural quasi-isomorphisms

$$\begin{aligned}
&CH_{Y_\bullet}(CH_{X_\bullet}(A), CH_{X_\bullet}(A,M))\\
&\qquad\cong CH_{Y_\bullet}(CH_{X_\bullet}(A))\otimes_{CH_{X_\bullet}(A)} CH_{X_\bullet}(A,M)\\
&\qquad\cong CH_{Y_\bullet}(CH_{X_\bullet}(A))\otimes_{CH_{X_\bullet}(A)} \big(CH_{X_\bullet}(A)\otimes_A M\big)\\
&\qquad\cong CH_{Y_\bullet}(CH_{X_\bullet}(A))\otimes_A M \overset{EZ\otimes_A id}{\longrightarrow} CH_{Y_\bullet\times X_\bullet}(A)\otimes_A M\\
&\qquad\cong CH_{Y_\bullet\times X_\bullet}(A,M)
\end{aligned}\tag{4.31}$$

and

$$\begin{aligned}
CH^{Y_\bullet\times X_\bullet}(A,M) &\cong Hom_A\big(CH_{S^1_\bullet\times X_\bullet}(A),M\big)\\
&\overset{EZ^*}{\longrightarrow} Hom_A\big(CH_{Y_\bullet}(CH_{X_\bullet}(A)),M\big)\\
&\cong Hom_A\big(CH_{X_\bullet}(A)\otimes_{CH_{X_\bullet}(A)} CH_{Y_\bullet}(CH_{X_\bullet}(A)),M\big)\\
&\cong Hom_{CH_{X_\bullet}(A)}\big(CH_{Y_\bullet}(CH_{X_\bullet}(A)),Hom_A\big(CH_{X_\bullet}(A),M\big)\big)\\
&\cong CH^{Y_\bullet}\big(CH_{X_\bullet}(A),CH^{X_\bullet}(A,M)\big).
\end{aligned}\tag{4.32}$$

By statement 3 in Theorem 3.24 and Lemma 3.10, we also have

$$\begin{aligned}
&CH_{Y_\bullet}(CH_{X_\bullet}(A),A)\otimes_{CH_{Y_\bullet}(A)} M\\
&\qquad\cong CH_{Y_\bullet}(CH_{X_\bullet}(A),CH_{X_\bullet}(A)) \underset{CH_{Y_\bullet}(A)\otimes_A CH_{X_\bullet}(A)}{\otimes} M\\
&\qquad\xrightarrow{EZ \underset{CH_{Y_\bullet}(A)\otimes_A CH_{X_\bullet}(A)}{\otimes} id} CH_{Y_\bullet\wedge X_\bullet}(A) \underset{CH_{Y_\bullet\vee X_\bullet}(A)}{\otimes} M\\
&\qquad\cong CH_{Y_\bullet\wedge X_\bullet}(A,M).
\end{aligned}\tag{4.33}$$

The last quasi-isomorphism is obtained similarly by combining the construction of the previous two ones. Further, the last claim in the corollary is a consequence of Lemma 3.10. □

Definition 4.29 (Eilenberg-Zilber model for Hochschild (co)chains) Let $X_\bullet$ and $Y_\bullet$ be simplicial sets; A a CDGA and M an A-module.

1. The Eilenberg Zilber model for Hochschild (co)chains of the product $X_\bullet\times Y_\bullet$ are respectively

$$C^{EZ}_{Y_\bullet\times X_\bullet}(A,M) := CH_{Y_\bullet}(CH_{X_\bullet}(A),CH_{X_\bullet}(A,M))$$

and $C_{EZ}^{Y_\bullet\times X_\bullet}(A,M) := CH^{Y_\bullet}(CH_{X_\bullet}(A),CH^{X_\bullet}(A,M))$.

2. The Eilenberg Zilber model for Hochschild (co)chains of the smash product $X_\bullet \wedge Y_\bullet$ are respectively

$$C^{\wedge}_{Y_\bullet, X_\bullet}(A, M) := CH_{Y_\bullet}(CH_{X_\bullet}(A), A) \otimes_{CH_{Y_\bullet}(A)} M$$

and $C_{\wedge}^{Y_\bullet, X_\bullet}(A, M) := Hom_{CH_{Y_\bullet}(A)}\big(CH_{Y_\bullet}(CH_{X_\bullet}(A), A), M\big)$.

Taking $Y_\bullet = S^1_\bullet$, we obtain a model for Hochschild (co)chains of suspensions.

These models are functorial in all arguments and are naturally $CH_{Y_\bullet}(CH_{X_\bullet}(A))$-modules (for the actions induced by the $CH_{Z_\bullet}(A)$-module structure on $CH_{Z_\bullet}(A, M)$ for any simplicial set $Z_\bullet$ and A-module M). Corollary 4.28 then shows that

Proposition 4.30 *The Eilenberg-Zilber models are models for the derived functors* $\mathbf{CH}_{|Y_\bullet| \times |X_\bullet|}(A, M)$, $\mathbf{CH}^{|Y_\bullet| \times |X_\bullet|}(A, M)$, $\mathbf{CH}_{|Y_\bullet| \wedge |X_\bullet|}(A, M)$ *and* $\mathbf{CH}^{|Y_\bullet| \wedge |X_\bullet|}(A, M)$ *equipped with all the functoriality provided by Proposition 3.18 and Proposition 3.23.*

Remark 4.31 The functoriality implies that one can iterate the Eilenberg-Zilber model. Namely,

$$CH_{Z_\bullet}(CH_{Y_\bullet}(CH_{X_\bullet}(A), A) \otimes_{CH_{Y_\bullet}(A)} A, A) \otimes_{CH_{Z_\bullet}(A)} M$$

is a functorial model for $\mathbf{CH}_{|X_\bullet| \times |Y_\bullet| \times |Z_\bullet|}(A, M)$. Likewise,

$$Hom_{CH_{Z_\bullet}(A)}\big(CH_{Z_\bullet}(CH_{Y_\bullet}(CH_{X_\bullet}(A), A) \otimes_{CH_{Y_\bullet}(A)} A, A), M\big)$$

is a functorial model for $\mathbf{CH}^{|X_\bullet| \times |Y_\bullet| \times |Z_\bullet|}(A, M)$. We denote respectively $CH_{\wedge}^{X_\bullet, Y_\bullet, Z_\bullet}(A, M)$ and $CH^{\wedge}_{X_\bullet, Y_\bullet, Z_\bullet}(A, M)$ these models.

The advantage of the Eilenberg-Zilber model is that it allows us to apply our constructions of Section 4.4 to any product or suspension. Indeed, replacing A by $CH_{X_\bullet}(A)$ and M by $CH_{X_\bullet}(A, M)$ (which have the correct algebraic structures by Section 3.2), the maps $\varphi^{k,\bullet}$ (4.16) gives us maps on the Eilenberg-Zilber models for products:

$$\varphi^{k,\bullet}_{EZ} : CH^{S^d_\bullet}(CH_{X_\bullet}(A), CH^{X_\bullet}(A, M)) \to CH^{S^d_\bullet}(CH_{X_\bullet}(A), CH^{X_\bullet}(A, M)) \quad (4.34)$$
$$\varphi^{k}_{\bullet EZ} : CH_{S^d_\bullet}(CH_{X_\bullet}(A), CH_{X_\bullet}(A, M)) \to CH_{S^d_\bullet}(CH_{X_\bullet}(A), CH_{X_\bullet}(A, M)). \quad (4.35)$$

satisfying, in view of Lemma 3.12 and Lemma 4.12,

$$\varphi^{k,\bullet}_{EZ} \circ \varphi^{l,\bullet}_{EZ} = \varphi^{kl,\bullet}_{EZ}, \qquad \varphi^{p}_{\bullet EZ} \circ \varphi^{q}_{\bullet EZ} = \varphi^{pq}_{\bullet EZ}.$$

To define similar operations on the Eilenberg-Zilber model for suspensions, recall the canonical quasi-isomorphisms $CH_{pt_\bullet}(B, M) \overset{\simeq}{\to} M$ (example 3.8) and

$CH_{X_\bullet}(k,M) \overset{\cong}{\to} M$ (example 3.13) fitting in a commutative diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[rrrd, bend right, "="'] & CH_{X_\bullet}(k,M) \arrow[r, "(k\to A)_*"] \arrow[rrd, bend right] & CH_{X_\bullet}(A,M) \arrow[r, "(X_\bullet\to pt_\bullet)_*"] & CH_{pt_\bullet}(A,M) \arrow[d, "\simeq"] \\ & & & M. \end{tikzcd}$$

In particular for $X_\bullet = S^d_\bullet$, it implies that

Lemma 4.32 *The canonical map $CH_{S^d_\bullet}(A,M)) \to CH_{pt_\bullet}(A,M) \overset{\cong}{\to} M$ is a map of γ-rings (with zero multiplication), where M is equipped with the trivial γ-ring structure given by $\varphi^k_M := M \overset{id}{\to} M$ (for all $k \geq 0$).*

By Theorem 4.17 and Theorem 4.25.2, the maps $\varphi^k_\bullet$ are functorial and it follows that they induced well-defined maps

$$\begin{aligned} \varphi^k_\wedge := \varphi^k_\bullet \otimes \varphi^k_M : CH_{S^d_\bullet}(CH_{X_\bullet}(A),A) \underset{CH_{S^d_\bullet}(A)}{\otimes} M \\ \longrightarrow CH_{S^d_\bullet}(CH_{X_\bullet}(A),A) \underset{CH_{S^d_\bullet}(A)}{\otimes} M. \end{aligned} \tag{4.36}$$

on the Eilenberg-Zilber model $C^\wedge_{S^d_\bullet,X_\bullet}(A,M)$ for suspensions. Similarly, we have maps

$$\begin{aligned} \varphi^{k\wedge} := (\varphi^k_\bullet)^* : Hom_{CH_{S^d_\bullet}(A)}\big(CH_{S^d_\bullet}(CH_{X_\bullet}(A),A),M\big) \\ \to Hom_{CH_{S^d_\bullet}(A)}\big(CH_{S^d_\bullet}(CH_{X_\bullet}(A),A),M\big) \end{aligned} \tag{4.37}$$

These maps satisfy

$$\varphi^k_\wedge \circ \varphi^l_\wedge = \varphi^{kl}_\wedge, \qquad \varphi^p_\wedge \circ \varphi^q_\wedge = \varphi^{pq}_\wedge$$

since the maps $\varphi^k_\bullet$ do.

As usual we denote with bold letter $\boldsymbol{\varphi}^{k\wedge}$, $\boldsymbol{\varphi}^k_\wedge$, $\boldsymbol{\varphi}^{kEZ}$ and $\boldsymbol{\varphi}^k_{EZ}$ the maps induced by the operations $\varphi^{k\wedge}$, $\varphi^k_\wedge$, $\varphi^{k,EZ}$, $\varphi^k_{\bullet EZ}$ on the various (co)homology functors (which is possible since these maps are induced by maps of spaces and thus preserves quasi-isomorphisms).

Theorem 4.33 *Let $X_\bullet$ be a pointed simplicial set. The operations $\varphi^{k\wedge}$, $\varphi^k_\wedge$, $\varphi^{k,EZ}$, φ^k_{EZ} defined on the Eilenberg-Zilber models (Definition 4.29) (co)chains of the suspensions $S^1_\bullet \times X_\bullet$ or product $S^1_\bullet \times X_\bullet$ are functorial with respect to maps of algebras and modules, making these cochain (resp. chain) complexes into functors*

$$\mathsf{Mod}^{\mathsf{CDGA}} \longrightarrow (\gamma,0) - \mathsf{Mod}^{\mathsf{CDGA}}, \qquad \textit{(resp.)} \quad \mathsf{Mod}_{\mathsf{CDGA}} \longrightarrow (\gamma,0) - \mathsf{Mod}_{\mathsf{CDGA}}$$

from $\mathsf{Mod}^{\mathsf{CDGA}}$ (resp. $\mathsf{Mod}_{\mathsf{CDGA}}$) to γ-rings with zero multiplication in the category $\mathsf{Mod}^{\mathsf{CDGA}}$ resp. $\mathsf{Mod}_{\mathsf{CDGA}}$).

Moreover

1. *After passing to Hochschild cohomology functor, one has*

$$\boldsymbol{\varphi}^{k\wedge} \cong \boldsymbol{\lambda}^{k^d} : \mathbf{CH}^{S^d\wedge|X_\bullet|}(A,M) \to \mathbf{CH}^{S^d\wedge|X_\bullet|}(A,M).$$

   Similarly, one has $\boldsymbol{\varphi}^k_\wedge = \boldsymbol{\lambda}^{k^d}$, $\boldsymbol{\varphi}^k_{EZ} = \boldsymbol{\lambda}^{k^d}$ $\boldsymbol{\varphi}^{kEZ} = \boldsymbol{\lambda}^{k^d}$ *on the corresponding Hochschild (co)homology functors associated to suspension and products.*

2. *If k is of characteristic* 0*, then there is an natural splitting*

$$CH_\wedge^{S^d_\bullet,X_\bullet}(A,M) = \prod_{j\geq 0} CH_\wedge^{S^d_\bullet,X_\bullet,(j)}(A,M)$$

   where the sub-chain complexes $CH_\wedge^{S^d_\bullet,X_\bullet,(j)}(A,M)$ *are* $\ker(\varphi^{k\wedge} - k^j.\mathrm{id})$ *(for all $k\geq 0$).*

3. *If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, there is a natural decomposition*

$$CH_\wedge^{S^d_\bullet,X_\bullet}(A,M) = \bigoplus_{0\leq n\leq p-1} CH^{S^d_\bullet\wedge X_\bullet,(j)}(A,M)$$

   with each $\varphi^{n\bullet}$ acting by multiplication by n^j on $CH^{S^d_\bullet\wedge X_\bullet,(j)}(A,M)$*.*

4. *The above two decompositions yields similar ones on Hochschild cohomology groups and one has* $H_*\big(CH^{S^d_\bullet\wedge X_\bullet,(\ell)}(A,M)\big) = 0$ *if $\ell \neq jd$. Denoting*

$$HH^{S^d\wedge|X_\bullet|,(j)}(A,M) := H^*\big(CH^{S^d_\bullet\wedge X_\bullet,(jd)}(A,M)\big),$$

   one has an natural splitting

$$HH^{S^d\wedge|X_\bullet|}(A,M) = \prod HH^{S^d\wedge|X_\bullet|,(j)}(A,M).$$

   Further one also has an identification

$$\begin{aligned} HH^{S^d\wedge|X_\bullet|,(j)}(A,M) &= H^*\Big(\ker\big(\mathbf{CH}^{S^d\wedge|X_\bullet|}(A,M) \overset{\boldsymbol{\lambda}^k - k^j.id}{\longrightarrow} \mathbf{CH}^{S^d\wedge|X_\bullet|}(A,M)\big)\Big) \\ &= H^*\Big(\ker\big(CH^{S^d_\bullet\wedge X_\bullet}(A,M) \overset{\varphi^k - k^{jd}.id}{\longrightarrow} CH^{S^d_\bullet\wedge X_\bullet}(A,M)\big)\Big). \end{aligned}$$

5. *In both cases, one has cochains* isomorphisms

$$CH^{S^d_\bullet\wedge X_\bullet,(0)}(A,M) \cong (pt_*)^*(CH^{pt_\bullet}(A,M)) \cong M.$$

   In particular the Hodge filtration on the cohomology groups satisfies $HH^{S^d\wedge|X_\bullet|,(0)}(A,M) \cong M$ *and all other pieces* $HH^{S^d\wedge|X_\bullet|,(j)}(A,M)$ *are not in weight* 0 *with respect to the skeletal filtration.*

6. *All the above statements 2 to 5 are also true for the Eilenberg-Zilber (co)chains* $C^\wedge_{Y_\bullet,X_\bullet}(A,M)$, $C^{EZ}_{Y_\bullet\times X_\bullet}(A,M)$ *and* $C_{EZ}^{Y_\bullet\times X_\bullet}(A,M)$*.*

Proof. We have seen that the various operations $\varphi^k_{\bullet EZ}$, $\varphi^{k\bullet}_{EZ}$, $\varphi^{k\wedge}$, $\varphi^k_\wedge$ satisfy the identity defining a λ-ring structure with trivial multiplication. The other statement corresponding to $\varphi^k_{\bullet EZ}$, $\varphi^{k\bullet}_{EZ}$ are derived immediately from Theorem 4.17 applied to the algebra $CH_{X_\bullet}(A)$ and the dg-module A as well as Theorem 3.24.

For the suspension case, applying the functoriality given by proposition 4.27, we have a commutative diagram

$$\begin{tikzcd}
CH_{\Delta_\bullet(S^1\wedge|X_\bullet|)}(A,M) \arrow[r, "\lambda^j_\bullet"] & CH_{\Delta_\bullet(S^1\wedge|X_\bullet|)}(A,M) \\
CH_{\Delta_\bullet(S^1)\wedge\Delta_\bullet(|X_\bullet|)}(A,M) \arrow[u, "\simeq"'] \arrow[r, "(\lambda^j\wedge id)_*"] & CH_{\Delta_\bullet(S^1)\wedge\Delta_\bullet(|X_\bullet|)}(A,M) \arrow[u, "\simeq"] \\
CH_{\Delta_\bullet(S^1)}(CH_{\Delta_\bullet(|X_\bullet|)}(A),A) \underset{CH_{\Delta_\bullet(S^1)}(A)}{\otimes} M \arrow[u, "EZ_\wedge", "\simeq"'] \arrow[r, "\lambda^j_\bullet\otimes id"'] & CH_{\Delta_\bullet(S^d)}(CH_{\Delta_\bullet(|X_\bullet|)}(A),A) \underset{CH_{\Delta_\bullet(S^1)}(A)}{\otimes} M \arrow[u, "\simeq", "EZ_\wedge"']
\end{tikzcd}$$

where the upper vertical arrows are induced by the canonical map of simplicial sets $\Delta_\bullet(S^1)\wedge\Delta_\bullet(|X_\bullet|)\to\Delta_\bullet(S^1\wedge|X_\bullet|)$. This gives a description of the power operations on the Eilenberg-Zilber model of the suspension. Passing to homology, applying Theorem 4.25 and Theorem 3.24, we deduce the diagram

$$\begin{tikzcd}
\mathbf{CH}_{S^1\wedge|X_\bullet|}(A,M) \arrow[r, "\boldsymbol{\lambda}^j"] & \mathbf{CH}_{S^1\wedge|X_\bullet|}(A,M) \\
\mathbf{CH}_{S^1}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^1}(A)} M \arrow[u, "\simeq"] \arrow[r, "\boldsymbol{\lambda}^j\otimes id"'] & \mathbf{CH}_{S^1}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^1}(A)} M \arrow[u, "\simeq"'] \\
\mathbf{CH}_{S^1}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^1}(A)} M \arrow[u, "\simeq"] \arrow[r, "\boldsymbol{\varphi}^j_\wedge=\boldsymbol{\varphi}^j\otimes id"'] & \mathbf{CH}_{S^1}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^1}(A)} M. \arrow[u, "\simeq"']
\end{tikzcd}$$

is commutative (the last line is given by the map (4.36)). Using once again Theorem 4.25 and noticing that $S^d\wedge|X_\bullet|\cong S^1\wedge(S^{d-1}\wedge|X_\bullet|)$ we obtain similarly that a commutative diagram:

$$\begin{tikzcd}
\mathbf{CH}_{S^1\wedge(S^{d-1}\wedge|X_\bullet|)}(A,M) \arrow[r, "\boldsymbol{\lambda}^{j^d}"] & \mathbf{CH}_{S^1\wedge(S^{d-1}\wedge|X_\bullet|)}(A,M) \\
\mathbf{CH}_{S^d}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^d}(A)} M \arrow[u, "\simeq"] \arrow[r, "\boldsymbol{\lambda}^{j^d}\otimes id"'] & \mathbf{CH}_{S^d}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^d}(A)} M \arrow[u, "\simeq"'] \\
\mathbf{CH}_{S^d}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^d}(A)} M \arrow[u, "\simeq"] \arrow[r, "\boldsymbol{\varphi}^j_\wedge=\boldsymbol{\varphi}^j\otimes id"'] & \mathbf{CH}_{S^d}(\mathbf{CH}_{|X_\bullet|}(A),A)\otimes_{\mathbf{CH}_{S^d}(A)} M. \arrow[u, "\simeq"']
\end{tikzcd}$$

We thus have proved claim 1 for the Hochschild homology of suspension, that is $\boldsymbol{\varphi}^k_\wedge \cong \boldsymbol{\lambda}^{k^d}$ as self-maps of $\mathbf{CH}_{S^d\wedge|X_\bullet|}(A,M)$. The cohomology statement $\boldsymbol{\varphi}^{k\wedge} \cong \boldsymbol{\lambda}^{k^d}$ is obtained straightforwardly by dualizing it.

Then the rest of the proof is similar to the one of Theorem 4.17 and Theorem 4.25. $\square$

Since the d-sphere is an iterated suspension $S^d \cong S^1 \wedge S^{d-1}$, Theorem 4.33 implies that the map (4.36)

$$\begin{aligned}\varphi^{k\wedge} = (\varphi^k_\bullet)^* : Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^{d-1}_\bullet}(A),A),M\big)\\ \to Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^{d-1}_\bullet}(A),A),M\big)\end{aligned}$$

yields a γ-ring structure on the Eilenberg-Zilber model for the standard model of S^d. We compare it with the one given by Theorem 4.14.

Lemma 4.34 *The Eilenberg-Zilber maps*

$$EZ_\wedge : CH_{S^1_\bullet}(CH_{S^{d-1}_\bullet}(A),A)\otimes_{CH_{S^1_\bullet}(A)} M \xrightarrow{\simeq} CH_{S^1_\bullet\wedge S^{d-1}_\bullet}(A,M) \quad \text{and}$$
$$EZ^\wedge : CH^{S^1_\bullet\wedge S^{d-1}_\bullet}(A,M) \xrightarrow{\simeq} Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^{d-1}_\bullet}(A),A),M\big)$$

are γ-rings maps. In particular, we have a commutative diagram

$$\begin{array}{ccc}\mathbf{CH}^{S^1\wedge S^{d-1}}(A,M) & \xrightarrow{\boldsymbol{\varphi}^{k\wedge}} & \mathbf{CH}^{S^1\wedge S^{d-1}}(A,M)\\ \cong\downarrow & & \downarrow\cong\\ \mathbf{CH}^{S^d}(A,M) & \xrightarrow[\boldsymbol{\lambda}^k]{} & \mathbf{CH}^{S^d}(A,M)\end{array}$$

in cohomology and a similar one in homology.

Proof.The second claim is just the first one after passing to the ∞-categories functors. From the first part of the proof of Theorem 4.33, we have that

$$EZ_\wedge \circ \lambda^k_\bullet \otimes id = (\lambda^k\wedge id)_* \circ EZ_\wedge.$$

Since the map $\lambda^k : S^d \cong S^1\wedge S^{d-1} \to S^1\wedge S^{d-1}$ is precisely $\lambda^k\wedge id$ (see definition (4.8)) the result follows using identity (4.22) in the case $d=1$. The proof is the same in cohomology. $\square$

Now we also realize the γ-ring maps ψ^k on the standard model $S^d_\bullet \cong S^1_\bullet \wedge\cdots S^1_\bullet$ for S^d (§ 4.4) with the (iterated) Eilenberg-Zilber model for $S^d_\bullet$.

When $d=2$, we have the model $Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^1_\bullet}(A),A),M\big)$ and the map $\varphi^{k\wedge} := (\varphi^k_\bullet)^*$. But on the source of this Hom space, we also have the γ-ring map $\varphi_*^k : CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A) \to CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A)$. It follows that we have an induced map:

$$\begin{aligned}\varphi^{k\wedge}_{(2)} := (\varphi^k_\bullet)^* \circ (\varphi^k_*)^* : Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A), M\big) \\ \to Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A), M\big). \end{aligned} \tag{4.38}$$

Similarly, on the chains, we have

$$\begin{aligned}\varphi^k_{\wedge(2)} := \varphi^k_\wedge \circ ((\varphi^k_\bullet)_* \otimes id) : CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A) \underset{CH_{S^1_\bullet}(A)}{\otimes} M \\ \longrightarrow CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A) \underset{CH_{S^1_\bullet}(A)}{\otimes} M.\end{aligned} \tag{4.39}$$

From remark 4.31, we can iterate this construction to get maps

$$\begin{aligned}\varphi^{k\wedge}_{(d)} := (\varphi^k_\bullet)^* \circ (\varphi^k_*)^* : Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A), M\big) \\ \to Hom_{CH_{S^1_\bullet}(A)}\big(CH_{S^1_\bullet}(CH_{S^1_\bullet}(A), A), M\big). \end{aligned} \tag{4.40}$$

and

$$\varphi^k_{\wedge(d)} := CH^{\wedge}_{S^1_\bullet,\cdots,S^1_\bullet}(A, M) \to CH^{\wedge}_{S^1_\bullet,\cdots,S^1_\bullet}(A, M). \tag{4.41}$$

In Section 4.4, we have defined operations ψ^k on $CH^{S^d_\bullet}(A, M)$ and $CH_{S^d_\bullet}(A, M)$.

Proposition 4.35 *The following diagrams are commutative:*

$$\begin{tikzcd} CH^{\wedge}_{S^1_\bullet,\cdots,S^1_\bullet}(A,M) \arrow[r, "\varphi^k_{\wedge(d)}"] \arrow[d, "EZ_\wedge"', "\cong"] & CH^{\wedge}_{S^1_\bullet,\cdots,S^1_\bullet}(A,M) \arrow[d, "\cong"', "EZ_\wedge"] \\ CH_{S^d_\bullet}(A,M) \arrow[r, "\varphi^k"'] & CH_{S^d_\bullet}(A,M) \end{tikzcd},$$

$$\begin{tikzcd} CH_\wedge^{S^1_\bullet,\cdots,S^1_\bullet}(A,M) \arrow[r, "\varphi^{k\wedge}_{(d)}"] & CH_\wedge^{S^1_\bullet,\cdots,S^1_\bullet}(A,M) \\ CH^{S^d_\bullet}(A,M) \arrow[r, "\varphi^k"'] \arrow[u, "\cong", "EZ^\wedge"'] & CH^{S^d_\bullet}(A,M). \arrow[u, "EZ^\wedge", "\cong"'] \end{tikzcd}$$

In other words, under the Eilenberg-Zilber quasi-isomorphism, the γ-ring maps φ^k becomes the $\varphi^{k\wedge}_{(d)}$ maps.

Proof. Note that by definition, we have

$$\varphi^j_{\wedge(d)} = \varphi^j_*(\varphi^j_*(\cdots(\varphi^j_* \otimes \varphi^k_A)\cdots) \otimes \varphi^k_M.$$

The first part of the proof of Theorem 4.33 and identity (4.22) shows that we have a commutative diagram

$$\begin{tikzcd}
CH_{\Delta_\bullet(S^1)\wedge\cdots\wedge\Delta_\bullet(S^1)}(A,M) \arrow[r, "(\lambda^j\wedge\cdots\wedge\lambda^j)_*"] & CH_{\Delta_\bullet(S^1)\wedge\cdots\wedge\Delta_\bullet(S^1)}(A,M) \\
CH_{S^1_\bullet\wedge\cdots\wedge S^1_\bullet}(A,M) \arrow[u, "\simeq"'] & CH_{S^1_\bullet\wedge\cdots\wedge S^1_\bullet}(A,M) \arrow[u, "\simeq"] \\
CH^{\wedge}_{S^1_\bullet,\cdots,S^1_\bullet}(A,M) \arrow[u, "EZ_\wedge", "\simeq"'] \arrow[r, "\varphi^j_{\wedge(d)}"'] & CH^{\wedge}_{S^1_\bullet,\cdots,S^1_\bullet}(A,M) \arrow[u, "\simeq", "EZ_\wedge"']
\end{tikzcd}$$

where the vertical maps are the composition of the iterated Eilenberg-Zilber map provided by Proposition 4.30 with iteration of the canonical maps $C_{S^1_\bullet}(A,M) \to C_{\Delta_\bullet(S^1)}(A,M)$ induced by the simplicial structure of $S^1_\bullet$ as in Remark 3.20. Together with Theorem 4.17, this proves the result after passing to ∞-categories and thus that the diagrams in the proposition are commutative up to homotopies. In order to prove the that the diagrams of the Proposition are strictly commutative, we use the combinatorial description of the map φ^k. For $d=1$, the result is immediate by definition. For $d>1$, we apply Lemma 4.16 (recall that we are working with normalized (co)chain complexes) to identify $\varphi^j_{\wedge(d)} = \varphi^j_*(\varphi^j_*(\cdots(\varphi^j_*)\cdots)$ with $\varphi^j : CH^{S^d_\bullet}(A,M) \to CH^{S^d_\bullet}(A,M)$. $\square$

5 Additional ring structures for Higher Hochschild cohomology

5.1 The wedge and cup product

Let $A \xrightarrow{f} B$ be a map of CDGAs. Note that it makes B into an A-algebra as well as an $A\otimes A$-algebra (since the multiplication $A\otimes A \to A$ is an algebra morphism). The excision axiom 3.24.2 implies

Lemma 5.1 ([Gi3]) *Let M be an A-module and X, Y be pointed topological spaces. There is a natural equivalence*

$$\boldsymbol{\mu} : \mathbb{R}Hom_{A\otimes A}\left(\mathbf{CH}_X(A)\otimes \mathbf{CH}_Y(A), M\right) \xrightarrow{\simeq} \mathbf{CH}^{X\vee Y}(A,M)$$

which is represented for any simplicial models $X_\bullet$, $Y_\bullet$ of X,Y by a natural quasi-isomorphism

$$\mu : Hom_{A\otimes A}\left(CH_{X_\bullet}(A)\otimes CH_{Y_\bullet}(A), M\right) \xrightarrow{\cong} CH^{X_\bullet\vee Y_\bullet}(A,M)$$

of $CH_{X_\bullet}(A)\otimes CH_{Y_\bullet}(A)$-dg-modules.

Proof. Recall that $CH_{X_\bullet}(A) = \mathcal{L}(A,A)^{dg}(X_\bullet)$ is the CDGA obtained as the realization out of the simplicial cdga $\mathcal{L}(A,A)(X_\bullet)$ (Section 3.1). The natural isomorphism μ is the composition of (the map induced by) the Alexander-Whitney quasi-isomorphism[25] (of $A\times A$-modules)

$$\big(\mathcal{L}(A,A)(X_\bullet)\otimes\mathcal{L}(A,A)(Y_\bullet)\big)^{dg} \xrightarrow{\simeq} CH_{X_\bullet}(A)\otimes CH_{Y_\bullet}(A)$$

with the cochain complex *isomorphism*

$$\begin{aligned} Hom_{A\otimes A}\Big(\big(\mathcal{L}(A,A)(X_\bullet)\otimes\mathcal{L}(A,A)(Y_\bullet)\big)^{dg}, M\Big) &\xrightarrow{\cong} Hom_{A\otimes A}\big(CH_{X_\bullet\coprod Y_\bullet}(A), M\big)\\ \xrightarrow{\cong} Hom_A\big(A\otimes_{A\otimes A} CH_{X_\bullet\coprod Y_\bullet}(A), M\big) &\xrightarrow{\cong} Hom_A\big(CH_{X_\bullet\vee Y_\bullet}(A), M\big)\\ &\xrightarrow{\cong} CH^{X_\bullet\vee Y_\bullet}(A,M) \qquad (5.1)\end{aligned}$$

where the second line is given by the fact that the $A\otimes A$-action on M is the one induced by the one of A along the multiplication map $A\otimes A\to A$ and the adjunction formula $Hom_A(f^*(N),M)\cong Hom_B(N,f_*(M))$ for any map $F:B\to A$ of cdgas. $\square$

When M is in fact an A-algebra, we can use Lemma 5.1 to define the *wedge product* of Hochschild cochains (which we first introduced in [Gi3, Section 3]) as the $(A\otimes A$-)linear map

$$\begin{aligned}\mu_\vee : CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet}(A,B) &\longrightarrow Hom_{A\otimes A}\Big(CH_{X_\bullet}(A)\otimes CH_{Y_\bullet}(A), B\otimes B\Big)\\ &\xrightarrow{(m_B)_*} Hom_{A\otimes A}\Big(CH_{X_\bullet}(A)\otimes CH_{Y_\bullet}(A), B\Big)\cong CH^{X_\bullet\vee Y_\bullet}(A,B) \qquad (5.2)\end{aligned}$$

where the first map is given by the tensor products $(f,g)\mapsto f\otimes g$ of functions and the second is induced by the multiplication of B.

Proposition 5.2 *The wedge product is a cochain map and is associative, meaning that the following diagram is commutative*

$$\begin{tikzcd} CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet}(A,B)\otimes CH^{Z_\bullet}(A,B) \arrow[r, "\mu_\vee\otimes id"] \arrow[d, "id\otimes\mu_\vee"'] & CH^{X_\bullet\vee Y_\bullet}(A,B)\otimes CH^{Z_\bullet}(A,B) \arrow[d, "\mu_\vee"] \\ CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet\vee Z_\bullet}(A,B) \arrow[r, "\mu_\vee"] & CH^{X_\bullet\vee Y_\bullet\vee Z_\bullet}(A,B) \end{tikzcd},$$

and commutative meaning that $\mu_\vee\circ\tau = {\tau_{X_\bullet,Y_\bullet}}_*\circ\mu_\vee$ where $\tau_{X_\bullet,Y_\bullet}: X_\bullet\vee Y_\bullet\cong Y_\bullet\vee X_\bullet$ is the canonical isomorphism and $\tau: M\otimes N\cong N\otimes M$ is the permutation of the two factors.

25. in other words the quasi-isomorphism from $CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet}(A,B)$ to the chain complex associated to the diagonal cosimplicial space $\big(CH^{X_n}(A,B)\otimes CH^{Y_n}(A,B)\big)_{n\in\mathbb{N}}$.

Proof. This is a consequence of the naturality of μ in Lemma 5.1 with respect to spaces, the fact that the Alexander-Whitney map is colax and the explicit formula (5.2). $\square$

The category of pointed simplicial sets (resp. pointed topological spaces) has a symmetric monoidal structure given by the wedge product

$$\bigvee : (X_{i,\bullet})_{i\in I} \quad\mapsto\quad \bigvee_{i\in I} X_{i,\bullet} \tag{5.3}$$

which induces a symmetric monoidal structure on its associated ∞-category as well as on the opposite of the above categories. By functoriality of Hochschild cochains (Theorem 4.14), we see that Proposition 5.2 means that $\mu_\vee$ makes the Hochschild functor $X_\bullet \mapsto CH^{X_\bullet}(A,B)$ into a lax monoidal functor from $(\mathsf{sSet}_*{}^{op}, \otimes^\vee)$ to $(\mathsf{k}\text{-}\mathsf{Mod}^{dg}, \otimes)$.

Corollary 5.3 *The map $\mu_\vee$ passes to the derived category to exhibit the Hochschild homology functor $\mathbf{CH}^{(-)}(A,B)$ into a lax symmetric monoidal ∞-functor*

$$(\mathbf{Top}_*{}^{op}, \vee) \longrightarrow (\mathsf{k}\text{-}\mathbf{Mod}, \otimes) : \bigotimes_{i\in I} \mathbf{CH}^{X_i}(A,B) \xrightarrow{\boldsymbol{\mu}_\vee} \mathbf{CH}^{\bigvee X_i}(A,B).$$

Further any map of cdga's $A \to A'$ or $B \to B'$ yields a lax monoidal natural transformation.

Proof. Proposition 5.2 shows that the rule

$$(X_0,\dots,X_n) \quad\mapsto\quad \Big(\mu_\vee^{\circ n} \quad:\quad \bigotimes CH^{\Delta_\bullet(X_i)}(A,B) \quad\longrightarrow\quad CH^{\bigvee \Delta_\bullet(X_i)}(A,B)\Big)$$

is a natural transformation between the Γ objects satisfying the Segal condition $(X_i) \mapsto \bigvee X_i$ and $(N_i) \mapsto \bigotimes N_i$. By homotopy invariance of Hochschild cochains (Corollary 3.4) and of the tensor product over k, this functor passes to the associated ∞-category to give an natural transformation between the claimed symmetric monoidal transformation since the canonical map $\bigvee \Delta_\bullet(X_i) \to \Delta_\bullet(\bigvee X_i)$ is a functorial (with respect to pointed maps of spaces) weak-equivalence of simplicial sets. $\square$

Example 5.4 Let $X_\bullet, Y_\bullet$ be finite simplicial sets models of X, Y. We assume that we identify X_i with the set $\{1,\dots,\#X_i\}$ with base point 1. We identify similarly the sets Y_j. The wedge product $\mu_\vee$ is then combinatorially described as the composition of the Alexander-Whitney map with the linear map given, for any $f \in CH^{X_n}(A,B) = Hom_A(A^{\otimes \# X_n}, B)$, $g \in CH^{Y_n}(A,B) = Hom_A(A^{\otimes \# Y_n}, B)$ by

$$\widetilde{\mu}(f,g)(a_1,a_2,\dots a_{\#X_n}, b_2,\dots,b_{\#Y_n}) = a_1.f(1,a_2,\dots a_{\#X_n}).g(1,b_2,\dots,b_{\#Y_n})$$

(here a_1 this corresponds to the element indexed by the base point of $X_n \vee Y_n$).

Thus, for any $f \in CH^{X_p}(A,B) = Hom_A(A^{\otimes \# X_p}, B)$, $g \in CH^{Y_q}(A,B) = Hom_A(A^{\otimes \# Y_q}, B)$, we have

$$\begin{aligned}\mu_\vee(f,g)&(a_1, a_2, \dots a_{\# X_{p+q}}, b_2, \dots, b_{\# Y_{p+q}}) \\ &= \widetilde{\mu}\big(f(d_{p+1} \circ \cdots d_{p+q}(a_1, \dots, a_{\# X_{p+q}})).g((d_0)^{\circ p}(1, b_2, \dots, b_{\# Y_{p+q}}))\big) \quad (5.4)\end{aligned}$$

where the d_i are the face maps of the (respective) simplicial structure.

If a space X is further endowed with an (homotopy) co-associative diagonal $\delta\colon X \to X \vee X$, then we can compose the wedge product with the map induced by the diagonal to give the Hochschild cohomology $CH^X(A,B)$ over X an (homotopy) associative algebra structure:

Definition 5.5 Let $\delta\colon X \to X \vee X$ be a continuous map. The *cup-product* over X is the composition

$$\cup_X \colon \mathbf{CH}^X(A,B) \otimes \mathbf{CH}^X(A,B) \xrightarrow{\boldsymbol{\mu}_\vee} \mathbf{CH}^{X\vee X}(A,B) \xrightarrow{\delta^*} \mathbf{CH}^X(A,B).$$

Proposition 5.6 *Assume (X,δ) is an E_1-coalgebra in pointed spaces. Then the cup product extend to give an E_1-algebra structure to the Hochschild cohomology $\mathbf{CH}^X(A,B)$ which is functorial with respect to maps of pointed spaces cdga's A and maps of A-cdgas B.*

In other words, the Hochschild cohomology functor gives rise to a functor:

$$E_1\text{-Alg}(\mathsf{Top}_*) \times \mathbf{CDGA}(\mathsf{Mod}_{\mathsf{CDGA}}) \longrightarrow E_1\text{-Alg}(\mathsf{k}\text{-}\mathbf{Mod})$$

Proof. By assumption we have an E_1-coalgebra $X \mapsto \bigvee_{i\in I} X$ in $\mathbf{Top}_*$ hence a symmetric monoidal functor $\delta : Ass \to End_{\mathbf{Top_*}^{op}}(X)$. Since $\mathbf{CH}$ is contravariant with respect to maps of spaces we get an ∞-functor $\delta^* : Ass \to End_{\mathsf{k}\text{-}\mathsf{Mod}^{dg}}(\mathbf{CH}^X(A,B))$. By Corollary 5.3, this functor is also symmetric monoidal. The naturality is a consequence of the naturality of Hochschild cochains and the explicit description of $\mu_\vee$. $\square$

In general, one can use simplicial approximation of the diagonal to compute the cup-product on an explicit combinatorial model. They will in general be associative only up to homotopy.

Example 5.7 A standard example of space with a diagonal is given by the spheres $X = S^n$. Actually, one can check (see Lemma 5.8 below) that for $d = 1$, the cup-product $\cup_{S^1}$ is (homotopy) equivalent to the usual cup-product for Hochschild cochains as in [G] and for $n = 2$, $\cup_{S^2}$ is (homotopy equivalent to) the Riemann sphere product as defined in [GTZ]. Note that the diagonal $S^n \to S^n \vee S^n$ becomes more commutative as n-increases. This can be use to lift the cup-product to E_n-algebra structure as we show in the next section.

Specifying the construction of the wedge product to example 3.17, we get the chain maps

i) $m_{st} : CH^{S^d_\bullet}(A,B) \otimes CH^{S^d_\bullet}(A,B) \to CH^{S^d_\bullet \vee S^d_\bullet}(A,B)$;

ii) $m_{sm} : CH^{(S^d_{sm})_\bullet}(A,B) \otimes CH^{(S^d_{sm})_\bullet}(A,B) \to CH^{(S^d_{sm})_\bullet \vee (S^d_{sm})_\bullet}(A,B)$.

Recall from section 4.4 the simplicial map $p_{(k)} : \mathrm{sd}_k(S^d_\bullet) \to \bigvee_{\{1,\dots k\}^d} S^d_\bullet$. Element indexing the wedge are tuples $(j_1,\dots,j_d)$ of elements in k, see Figure 3 and the proof of Lemma 4.20. We consider the projection $i_{(k)} : \bigvee_{\{1,\dots k\}^d} S^d_\bullet \to \bigvee_{\{1,\dots k\}} S^d_\bullet$ which maps every non-diagonal[26] sphere $S^d_\bullet$ to the point $pt_\bullet$. Also recall the map $\mathcal{D}^\bullet_k$ (4.4) which induces the inverse of the homeomorphism $D_k : |S^d_\bullet| \xrightarrow{\cong} |\mathrm{sd}_k(S^d_\bullet)|$.

Lemma 5.8 *The composition*

$$\begin{aligned}\cup_{S^d_\bullet} : \underline{CH}^{S^d_\bullet}(A,B) \otimes \underline{CH}^{S^d_\bullet}(A,B) &\xrightarrow{m_{st}} \underline{CH}^{S^d_\bullet \vee S^d_\bullet}(A,B)\\ &\xrightarrow{i^*_{(2)}} \underline{CH}^{\bigvee_{\{1,\dots 2\}^d} S^d_\bullet}(A,B) \xrightarrow{p^*_{(2)}} \underline{CH}^{\mathrm{sd}_2(S^d_\bullet)}(A,B) \xrightarrow{\mathcal{D}^\bullet_2} \underline{CH}^{S^d_\bullet}(A,B)\end{aligned}$$

is a model for the cup-product of spheres.

If $d = 1$, this product restricted to normalized cochains is the standard Hochschild cochain cup-product [G].

Proof. By Lemma 4.20, we have that $i_{(k)} \circ p_{(k)}$ is a model for the composition $|i_{(k)}| \circ pinch^{(k)}_{(d)} \circ D_k$. The composition $|i_{(k)}| \circ pinch^{(k)}_{(d)}$ is homotopic to the pinching map $pinch^{(k)}$ defined in Section 4.3. Hence the fact that the composition is a model for the cup-product now follows from Lemma 4.9.

Assume now $d = 1$. Then $i_{(2)} = id$ and, for $f \in \underline{CH}^{S^1_p}(A,B)$, $g \in \underline{CH}^{S^1_q}(A,B)$, from example 5.4, one finds

$$\begin{aligned} &m_{st}(f,g)(a_0, a_1 \dots, a_{p+q}, b_1, \dots b_{p+q})\\ &\qquad = \pm a_0 \cdot a_{p+1} \cdots a_{p+q} f(1, a_1 \dots, a_p) \cdot b_1 \cdots b_p \cdot g(1, b_{p+1}, \dots, b_{p+q}) \end{aligned} \tag{5.5}$$

where the sign is the Koszul-Quillen sign. Hence

$$\begin{aligned} &p^*_{(2)} \circ i^*_{(2)} \circ m_{st}(f,g)(a_0, a_1 \dots, a_{p+q}, a_{2p+2q+1})\\ &= \pm a_0 \cdot a_{p+1} \cdots a_{p+q} f(1, a_1 \dots, a_p) \cdot a_{p+q+1} \cdots a_{2p+q+1} \cdot g(1, a_{2p+q+2}, \dots, a_{2p+2q+1}) \end{aligned} \tag{5.6}$$

where the sign is again the Koszul-Quillen sign. Since the cochains f, g vanishes if any of their entries (but the first one) is a scalar, from formula (4.4) and the previous one, we obtain that

$$f \cup_{S^1_\bullet} g(a_0, \dots, a_{p+q}) = a_0 \cdot f(1, a_1, \dots, a_p) \cdot g(1, a_{p+1}, \dots a_{p+q})$$

which concludes the proof. $\square$

26. by a diagonal sphere, we mean a component indexed by a tuple for which all the j_i are the same. Which are precisely those on the diagonal cubes in Figure 3.

One can also work with the singular model for the sphere. Indeed, let

$$j : k[\Delta_\bullet(S^d \vee S^d)] \to k[\Delta_\bullet(S^d) \vee \Delta_\bullet(S^d)]$$

be a quasi-inverse of the canonical (inclusion) map $\Delta_\bullet(S^d) \vee \Delta_\bullet(S^d) \hookrightarrow \Delta_\bullet(S^d \vee S^d)$. Explicitly, for $\sigma : \Delta^{n\geq 1} \to S^d \vee S^d$, one can take $j(\sigma) = \sigma_1 \vee \mathrm{cst} + \mathrm{cst} \vee \sigma_2$ where σ_i are the respective projections on each factor and cst is the constant map to the base point of S^d.

Lemma 5.9 *The map*

$$\begin{aligned} m_{sg} : C^{\Delta_\bullet(S^d)}(A,B) \otimes C^{\Delta_\bullet(S^d)}(A,B) &\xrightarrow{\mu_\vee} C^{\Delta_\bullet(S^d)\vee\Delta_\bullet(S^d)}(A,B) \\ &\xrightarrow{j^*} C^{\Delta_\bullet(S^d \vee S^d)}(A,B) \xrightarrow{pinch_2^*} C^{\Delta_\bullet(S^d)}(A,B) \end{aligned}$$

is a model for the cup-product.

Proof. It follows from Proposition 5.6 and the commutativity of the diagram

$$\begin{tikzcd} \Delta_\bullet|\Delta_\bullet(S^d) \vee \Delta_\bullet(S^d)| \arrow[d, "\cong"'] & \Delta_\bullet(S^d) \vee \Delta_\bullet(S^d) \arrow[l, "\eta"'] \arrow[d] \\ \Delta_\bullet(|\Delta_\bullet(S^d)| \vee |\Delta_\bullet(S^d)|) \arrow[r, "\Delta_\bullet(\beta\vee\beta)"] & \Delta_\bullet(S^d \vee S^d) \end{tikzcd}$$

whose horizontal arrows are the canonical ones induced by the adjunction between realization and singular chains (that is $\beta|\Delta_\bullet(Y)| \to Y$ is the counit of the adjunction). □

5.2 The universal E_n-algebra structure lifting the cup-product

In [Gi3], we extended the above cup-product for spheres S^n (Definition 5.5) into an E_n-algebra structure (at the level of cochains). This result is actually a version of *higher Deligne conjecture for morphisms* of CDGAs, *i.e.*, an explicit construction of Lurie's notion of (derived) *centralizers* of a map of CDGAs in the category of E_n-algebras, see [Lu3, GTZ3]. We below recall the construction for cdgas and then explain how to interpret it in terms of convolutions and generalize it to (iterated) suspensions and products.

5.2.1 The E_n-structure of Hochschild (co)homology over S^n

Let $\mathcal{C}_n = (\mathcal{C}_n(r))_{r\geq 0}$ be the usual n-dimensional little cubes operad, as an operad of topological spaces, and respectively $\mathcal{C}_n^{nu} = (\mathcal{C}_n^{nu}(r))_{r\geq 0}$ its non-unital version. Recall that $\mathcal{C}_n(k)$ is the configuration space of (resp. non-empty) k n-dimensional open cubes in I^n. We let $\mathbb{E}_d$ stands for its enveloping symmetric monoidal category, viewed as a symmetric monoidal ∞-category. More precisely, it is the ∞-category associated to the topological category whose objects are disjoint unions of finitely many copies of the unit cube I^n and morphisms are the spaces of rectilinear embeddings [27] (resp. $\mathbb{E}_d^{nu}$ is given by rectilinear embeddings which are surjective on connected components). The monoidal structure is given by disjoint union. In other words, $\mathbb{E}_d$ is the ∞-operad governing E_n-algebras in the sense of [Lu3]; an explicit model for those algebras in the category of chain complexes being given by algebras over $C_*(\mathcal{C}_n)$, the singular chains on the little cube operad.

A key (and defining) property of this operad is that any element $c \in \mathcal{C}_d(k)$ defines a map $pin_c : S^d \to \bigvee_{i=1\dots k} S^d$ by collapsing the complement of the interiors of the cubes to the base point. The maps pin_c assemble together to give a continuous map

$$pin : \mathcal{C}_d(k) \times S^d \longrightarrow \bigvee_{i=1\dots k} S^d. \tag{5.7}$$

These maps in turn gives maps, for every objects k

$$pin : \mathbb{E}_d(k,1) \times S^d \longrightarrow \bigvee_{i=1\dots k} S^d \tag{5.8}$$

where we identify an natural number i with the set $\{1,\dots,i\}$. , ℓ $\bigvee_{i=1\dots\ell} S^d$

Note also that the map pin preserve the base point of S^d hence pass to the pointed category Top_* in all cases.

We start by giving the ∞-categorical construction of the E_n-algebra lift of the product before detailing its explicit combinatorial incarnation. The natural equivalences (3.11) and the pinch map (5.8) yield a space morphism

$$\begin{aligned}
Pin_A^* : \mathrm{Map}_{\mathbf{CDGA}}(\mathbf{CH}_{\bigvee_{i=1}^k S^d}(A), \mathbf{CH}_{\bigvee_{i=1}^k S^d}(A))
&\cong \mathrm{Map}_{\mathbf{Top}}\big(\bigvee_{i=1\dots k} S^d, \mathrm{Map}_{\mathbf{CDGA}}(A, \mathbf{CH}_{\bigvee_{i=1}^k S^d}(A))\big)\\
&\xrightarrow{pin^*} \mathrm{Map}_{\mathbf{Top}}\big(\mathbb{E}_d(k,1)\times S^d, \mathrm{Map}_{\mathbf{CDGA}}(A, \mathbf{CH}_{\bigvee_{i=1}^k S^d}(A))\big)\\
&\cong \mathrm{Map}_{\mathbf{Top}}\big(\mathbb{E}_d(k,1), \mathrm{Map}_{\mathbf{CDGA}}(\mathbf{CH}_{S^d}(A), \mathbf{CH}_{\bigvee_{i=1}^k S^d}(A))\big).
\end{aligned}$$

We denote

$$pin^*_{S^d,A} : \mathbb{E}_d(k,1) \to \mathrm{Map}_{\mathbf{CDGA}}(\mathbf{CH}_{S^d}(A), \mathbf{CH}_{\bigvee_{i=1}^k S^d}(A)) \tag{5.9}$$

the image $Pin_A^*(id)$ of the identity morphism of $\mathbf{CH}_{\bigvee_{i=1}^k S^d}(A)$. Note that this map is functorial in A since Pin_A^* is. Since pin preserves the base point, $pin^*_{S^d,A}$ takes value in the space $\mathrm{Map}_{A-\mathbf{CDGA}}\big(\mathbf{CH}_{\bigvee_{i=1}^{\ell} S^d}(A), \mathbf{CH}_{\bigvee_{i=1}^k S^d}(A)\big)$ of A-linear cdgas. Dualizing over A, from Definitions 3.9 and 3.19 we get

27. that is continuous maps which are embeddings, which, restricted to each cube is obtained by a translation and and an homothety in each of the n^{th} direction of the cube I^n

Definition 5.10 We define

$$pin_*^{S^d,A} : \mathbb{E}_d(k,1) \to \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}(\mathbf{CH}^{\bigvee_{i=1}^k S^d}(A,M), \mathbf{CH}^{S^d}(A,M))$$

the dual of $pin^*_{S^d,A}$ with values in a A-module M.

When $M = B$ is an unital A-cdga, we can precompose these maps with the the maps $\boldsymbol{\mu}_\vee$ from the previous section to get

$$\begin{aligned}\mathbb{E}_d(k,1) &\stackrel{pin_*^{S^d,A}}{\longrightarrow} \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}(\mathbf{CH}^{\bigvee_{i=1}^k S^d}(A,B), \mathbf{CH}^{S^d}(A,B)) \\ &\stackrel{(\boldsymbol{\mu}_\vee)^*}{\longrightarrow} \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}\big((\mathbf{CH}^{S^d}(A,B))^{\otimes k}, \mathbf{CH}^{S^d}(A,B)\big) \end{aligned} \quad (5.10)$$

Theorem 5.11 *Let B be a unital A-CDGA. The map* (5.10) *makes $\mathbf{CH}^{S^d}(A,B)$ into an E_d-algebra functorially*[28] *in A and B.*

If B is non-unital, then $\mathbf{CH}^{S^d}(A,B)$ inherits an $\mathbb{E}_d^{un}$-algebra structure. The underlying E_1-structure is the cup-product of Definition 5.5.

In particular, for $d > 1$, the induced cup-product on the cohomology groups $HH_\bullet^{S^n}(A,B)^{\otimes 2} \to HH_\bullet^{S^n}(A,B)$ is commutative.

Proof. Since we have an equivalence of ∞-categories E_n-**Alg** $\cong$ $Fun^{\otimes}\big(\mathbb{E}_d, \mathrm{End}_{\mathsf{k}\text{-}\mathsf{Mod}^{dg}}(\mathbf{CH}^{S^d}(A,B))\big)$ [Lu3, F], it is enough to see that the above map induces such a symmetric monoidal functor. On objects, we define it as $k \mapsto (\mathbf{CH}^{S^d}(A,B))^{\otimes k}$ (which is essentially forced since we want it monoidal). We now extend (5.10) to define the functor on morphisms. By definition of the envelopping category of an operad, we have weak equivalences

$$\coprod_{\varphi:k\to\ell}\prod_{i=1}^{\ell} \mathbb{E}_d(\varphi^{-1}(i),1) \cong \mathbb{E}_d(k,\ell)$$

so that the map (5.10) yields

$$\begin{aligned}\mathbb{E}_d(k,\ell) &\stackrel{\coprod_\varphi \prod_{i=1}^\ell pin_*^{S^d,A}}{\longrightarrow} \coprod_{\varphi}\prod_{i=1}^{\ell} \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}(\mathbf{CH}^{\bigvee_{i=1}^{\varphi^{-1}(i)} S^d}(A,B), \mathbf{CH}^{S^d}(A,B)) \\ &\stackrel{\coprod\prod(\boldsymbol{\mu}_\vee)^*}{\longrightarrow} \coprod_{\varphi}\prod_{i=1}^{\ell} \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}\big((\mathbf{CH}^{S^d}(A,B))^{\otimes\varphi^{-1}(i)}, \mathbf{CH}^{S^d}(A,B)\big) \\ &\longrightarrow \coprod_{\varphi} \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}\big(\bigotimes_{i=1\dots\ell} (\mathbf{CH}^{S^d}(A,B))^{\otimes\varphi^{-1}(i)}), \bigotimes_{i=1\dots\ell} \mathbf{CH}^{S^d}(A,B)\big) \\ &\cong \coprod_{\varphi} \mathrm{Map}_{\mathsf{k}\text{-}\mathbf{Mod}}\big((\mathbf{CH}^{S^d}(A,B))^{\otimes k}, (\mathbf{CH}^{S^d}(A,B))^{\otimes \ell}\big) \\ &\longrightarrow \mathrm{End}_{\mathsf{k}\text{-}\mathbf{Mod}}(\mathbf{CH}^{S^d}(A,B)).\end{aligned}$$

28. in other words, $\mathbf{CH}^{S^d}(-,-)$ is a functor $\mathbf{CDGA}(\mathbf{Mod}_{\mathsf{CDGA}}) \to E_n$-$\mathbf{Alg}(\mathsf{k}\text{-}\mathbf{Mod})$

That the collection of maps $pin_*^{S^d,A}$ defines a symmetric monoidal ∞-functor $\mathbb{E}_d \to \mathrm{End}_{\text{k-}\mathbf{Mod}}(\mathbf{CH}^{S^d}(A,B)$, follows from the fact that the equivalence (3.11) is natural with respect to maps of spaces and composition of cdgas maps. We thus get the claimed E_d-algebra structure on $\mathbf{CH}^{S^d}(A,B)$. Further the functor is natural in A and B since Pin_A^* is functorial in A and $Hom_A(-,B)$ is functorial in B.

One recovers the cup product by considering a standard diagonal configuration $c_{st,2} : I^d \coprod I^d \to I^d$.

The definition for non-unital algebras is similar except that we have to restrict to maps with are surjective on the connected component of each morphism space of the ∞-operad. The naturality follows from the fact that $\mathbf{CH}^{S^d}(-,-)$ is a ∞-functor from $\mathbf{Mod}_{\mathsf{CDGA}}$ to $\mathbf{Mod}_{\mathsf{CDGA}}$ by Proposition 3.18 and that all pinching maps are pointed. $\square$

Example 5.12 If $A = k$, there is a canonical equivalence of E_n-algebras $\mathbf{CH}^{S^n}(k,B) \cong B$ (which actually is the restriction of an equivalence of CDGAs) since we have an natural equivalence $\mathbf{CH}_X(k) \cong k$ of cdgas for any space X.

A contrario if $B = k$ (and its A-cdga structure is induced by an augmentation of A), the induced structure is more complicated. Indeed, one can show that then the E_n-algebra structure of $\mathbf{CH}^{S^n}(A,k)$ is the dual of the E_n-coalgebra structure given by the n-times iterated Bar construction $Bar^{(n)}(A)$ (see [Fr2, F, Lu2, GTZ3]).

Let us be more precise about the naturality. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be maps of CDGAs so that $A \xrightarrow{g\circ f} C$ is also a CDGA map. In particular, we have the pushforward functor $f_* : B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$. By Theorem 3.24.3 (using the decomposition of a sphere in two hemisphere), we have an natural equivalence

$$\mathbf{CH}_{D^d}(A) \bigotimes_{\mathbf{CH}_{S^{d-1}}(A)}^{\mathbb{L}} \mathbf{CH}_{D^d}(A) \xrightarrow{\simeq} \mathbf{CH}_{S^d}(A). \tag{5.11}$$

Applying the duality functor (and canonical equivalence $\mathbf{CH}_{D^d}(-) \cong id$ of endofunctors of $\mathbf{CDGA}$):

$$\mathbb{R}Hom_A(-,B) \cong \mathbb{R}Hom_A(-,\mathbf{CH}_{D^d}(B)) \cong \mathbb{R}Hom_{\mathbf{CH}_{D^d}(A)}(-,\mathbf{CH}_{D^d}(B))$$

we deduce a canonical equivalence

$$\begin{aligned}\mathbf{CH}^{S^d}(A) \xrightarrow{\simeq} \mathbb{R}Hom_{\mathbf{CH}_{D^d}(A)}\Big(\mathbf{CH}_{D^d}(A) \bigotimes_{\mathbf{CH}_{S^{d-1}}(A)}^{\mathbb{L}} \mathbf{CH}_{D^d}(A), \mathbf{CH}_{D^n}(B)\Big)&\\ \cong \mathbb{R}Hom_{\mathbf{CH}_{S^{d-1}}(A)}\big(\mathbf{CH}_{D^d}(A), \mathbf{CH}_{D^d}(B)\big)&\end{aligned} \tag{5.12}$$

Using equivalence (5.12) and the pushforward $\mathbf{CH}_{S^{d-1}}(f)_* : \mathbf{CH}_{S^{d-1}(B)}\text{-}\mathbf{Mod} \to \mathbf{CH}_{S^{d-1}(A)}\text{-}\mathbf{Mod}$ we can define the (derived) *composition of Hochschild cohomology* *over spheres*:

$$
\begin{aligned}
&\mathbf{CH}^{S^n}(A,B)\otimes\mathbf{CH}^{S^n}(B,C)\\
&\cong \mathbb{R}Hom_{\mathbf{CH}_{S^{d-1}}(A)}\big(\mathbf{CH}_{D^d}(A),\mathbf{CH}_{D^d}(B)\big)\otimes\mathbb{R}Hom_{\mathbf{CH}_{S^{d-1}}(B)}\big(\mathbf{CH}_{D^d}(B),\mathbf{CH}_{D^d}(C)\big)\\
&\stackrel{id\otimes\mathbf{CH}_{S^{d-1}}(f)_*}{\longrightarrow} \mathbb{R}Hom_{\mathbf{CH}_{S^{d-1}}(A)}\big(\mathbf{CH}_{D^d}(A),\mathbf{CH}_{D^d}(B)\big)\otimes\mathbb{R}Hom_{\mathbf{CH}_{S^{d-1}}(A)}\big(\mathbf{CH}_{D^d}(B),\mathbf{CH}_{D^d}(C)\big)\\
&\qquad\stackrel{\circ}{\longrightarrow}\mathbb{R}Hom_{\mathbf{CH}_{S^{d-1}}(A)}\big(\mathbf{CH}_{D^d}(A),\mathbf{CH}_{D^d}(C)\big)\cong CH^{S^d}(A,C). \qquad (5.13)
\end{aligned}
$$

where the arrow $\stackrel{\circ}{\longrightarrow}$ is (derived) composition of $\mathbf{CH}_{S^{d-1}}(A)$-modules morphisms.

Remark 5.13 Under the equivalence of Proposition 3.25, the derived composition identifes with the composition

$$\mathbb{R}Hom_A^{E_n}\big(A,B\big)\otimes\mathbb{R}Hom_B^{E_n}\big(B,C\big)\stackrel{\circ}{\longrightarrow}\mathbb{R}Hom_A^{E_n}\big(A,C\big)$$

in the categories of E_n-modules over A.

The following is proved in [GTZ3].

Lemma 5.14 *The derived composition* $\mathbf{CH}^{S^d}(A,B)\otimes\mathbf{CH}^{S^d}(B,C)\to\mathbf{CH}^{S^d}(A,C)$ *is a map of E_d-algebras.*

Sketch of Proof. The E_d-algebra structure on the tensor product $\mathbf{CH}^{S^d}(A,B)\otimes\mathbf{CH}^{S^d}(B,C)$ is induced by the diagonal maps $\mathcal{C}_d(r)\to\mathcal{C}_d(r)\times\mathcal{C}_d(r)$ in **Top**. The equivalence (5.12) is represented at the cochain complexes level by the quasi-isomorphism

$$CH^{\partial S^d_\bullet}(A,B)\cong Hom^{left}_{CH_{\partial I^d_\bullet}(A)}\left(CH_{I^d_\bullet}(A),CH_{I^d_\bullet}(B)\right). \qquad (5.14)$$

(hence we take, as a model for S^{d-1}, the boundary of the standard cube $\partial I^d_\bullet$). The advantage of that model is that $CH_{I^d_\bullet}(A)$ is a cofibrant module over $CH_{\partial I^d_\bullet}(A)$. Then we can apply the same construction as the map (5.13) to get a chain map

$$CH^{S^d_\bullet}(A,CH_{I^d_\bullet}(B))\otimes CH^{S^d_\bullet}(B,CH_{I^d_\bullet}(C))\longrightarrow CH^{S^d_\bullet}(A,CH_{I^d_\bullet}(C)) \qquad (5.15)$$

representing the derived composition of Hochschild cohomology over the d-sphere S^d. The E_d-algebra structure on higher Hochschild cochains is induced by the pinching map, which itself is induced by inclusions of (configurations of) cubes in the right hand side of the equivalence (5.14), *i.e.* the definition of the little d-cubes operadic structure as expalined in § 5.2.2. It becomes straightforward to check that the derived composition (5.13) preserves the $\mathcal{C}_n$-action (also see [GTZ3]). □

The E_d-algebra structure we exhibited actually satisfies an universal property. Assume that our A-cdga structure on B is induced by a map $f:A\to B$ of cdgas (that is the A-module structure on B is the pullback of the canonical B-module structure on itself along a cdga map f). If B is unital, then the module structure is necessarily induced by such a map which is defined as $a \mapsto a \cdot 1_B$.

Following Lurie [Lu3], the (derived) **centralizer** of an E_n-algebra map $f : A \to B$ is the *universal* E_n-algebra $\mathfrak{z}(f)$ endowed with a morphism of E_n-algebras $\kappa : A \otimes \mathfrak{z}(f) \to B$ making the following diagram

$$\begin{tikzcd} & A \otimes \mathfrak{z}(f) \arrow[dr, "\kappa"] & \\ A \arrow[ur, "id\otimes 1_{\mathfrak{z}(f)}"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{5.16}$$

commutative in E_n**-Alg**. Its existence is proved in [Lu3].

Remark 5.15 There is also a notion of non-unital centralizer which can be described in terms of moduli problem associated to $f : A \to B$, that is the functor $E_n\textbf{-Alg}^{art} \to \textbf{Top}$ whose value on an artinian E_n-algebra R is $\mathrm{Map}_{E_n\textbf{-Alg}}(A \otimes R, B)_f$ the space of maps whose reduction to $\mathrm{Map}_{E_n\textbf{-Alg}}(A \otimes k, B)$ is precisely f.

The E_d-algebra structure provided by Theorem 5.11 coincides for cdga maps $f : A \to B$ with the one in [GTZ3] since they are derived in the same way from the wedge product.

Proposition 5.16 ([GTZ3]) *Let $A \stackrel{f}{\to} B$ be a map of CDGAs. Then $CH^{S^d}(A, B)$ (equipped with the structure given by Theorem 5.11 is the centralizer of f in the category of E_d-algebras.*

Sketch of Proof. This is proved in [GTZ3] for E_d-algebras maps. Let us sketch the proof for cdgas. By naturality of the E_d-algebra structure (Lemma 5.14) and Example 5.12 below, there is an natural evaluation map $\mathrm{eval} : A \otimes \mathbf{CH}^{S^d}(A, B) \to B$ which is a map of E_d-algebras making the following diagram

$$\begin{tikzcd} & A \otimes CH^{S^n}(A, B) \arrow[dr, "\mathrm{eval}"] & \\ A \arrow[ur, "id\otimes 1"] \arrow[rr, "f"] & & B \end{tikzcd}$$

commutative in E_n**-Alg**.

Now let $\mathfrak{z}$ be an E_n-algebra, endowed with a E_n-algebra map $\phi : A \otimes \mathfrak{z} \to B$ fitting in a commutative diagram

$$\begin{tikzcd} & A \otimes \mathfrak{z} \arrow[dr, "\phi"] & \\ A \arrow[ur, "id\otimes 1_{\mathfrak{z}}"] \arrow[rr, "f"] & & B\,. \end{tikzcd} \tag{5.17}$$

By adjunction (in k-**Mod**), the map ϕ has a (derived) adjoint $\theta_\phi : \mathfrak{z} \to \mathbb{R}Hom(A,B)$. Since ϕ is a map of E_n-algebras and diagram (5.17) is commutative, one check that θ_ϕ factors through a map

$$\begin{aligned}
&\widetilde{\theta_\phi}\colon \mathfrak{z} \cong k\otimes \mathfrak{z} \\
&\xrightarrow{1_{\mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,A)}\otimes id} \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,A)\otimes \mathfrak{z} \cong \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,A)\otimes \mathbb{R}Hom_{\mathbf{Mod}^k_{E_n}}(k,\mathfrak{z}) \\
&\longrightarrow \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A, A\otimes \mathfrak{z}) \xrightarrow{\phi_*} \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,B) \cong \mathbf{CH}^{S^n}(A,B). \qquad (5.18)
\end{aligned}$$

The last equivalence is provided by Proposition 5.16. It follows from naturality of the E_n-algebra structure of Hochschild cohomology over S^n that this composition $\widetilde{\theta_\phi}\colon \mathfrak{z} \to \mathbf{CH}^{S^n}(A,B)$ is actually a map of E_n-algebras. Further, by definition of θ_ϕ, the identity

$$\mathrm{eval}\circ \big(id_A\otimes \theta_\phi\big) \;=\; \phi$$

holds. Now, the uniqueness of the map $\widetilde{\theta_\phi}$ follows quite easily from the fact that the composition

$$\begin{aligned}
&\mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,B) \cong \mathbb{R}Hom_k^{E_n}\Big(k, \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,B)\Big) \\
&\quad\xrightarrow{1_{\mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,A)}\otimes id} \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,A)\otimes \Big(k, \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,B)\Big) \\
&\qquad\longrightarrow \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}\Big(A, A\otimes \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,B)\Big) \\
&\qquad\qquad\xrightarrow{ev_*} \mathbb{R}Hom_{\mathbf{Mod}^A_{E_n}}(A,B) \qquad (5.19)
\end{aligned}$$

is the identity map. Hence $\mathbf{CH}^{S^n}(A,B)$ satisfies the universal property of the derived center $\mathfrak{z}(f)$. $\Box$

Remark 5.17 One can check in a similar (but slightly more complicated) way that, when B is non-unital (and one uses the well-defined normalized cochain complex $\ker\big(\underline{CH}_{I^n_\bullet}(k\oplus B) \to \underline{CH}_{pt_\bullet}(k\oplus B) \cong k\oplus B \to k\big)$ as a replacement for $\mathbf{CH}_{I^n_\bullet}(B)$), $\mathbf{CH}^{S^d}(A,B)$ is the non-unital centralizer of f.

Following Lurie, we now recall a very nice consequence of the centralizer functoriality. By Lemma 5.14 above, the derived composition

$$\mathbf{CH}^{S^n}(A,A)\otimes \mathbf{CH}^{S^n}(A,A) \;\longrightarrow\; \mathbf{CH}^{S^n}(A,A) \qquad (5.20)$$

is a homomorphism of E_n-algebras (with unit given by the identity map 1_A) which is further (homotopy) associative and unital (with unit 1_A). In other words it makes $\mathbf{CH}^{S^n}(A,A)$ an object of E_1-Alg(E_n-Alg).

By the ∞-category version of Dunn Theorem [Du, Lu3], there is an equivalence of ∞-categories E_1-Alg(E_n-Alg) $\cong E_{n+1}$-Alg. Thus the multiplication (5.20) lifts the E_n-algebra structure of $\mathbf{CH}^{S^n}(A,A)$ to an E_{n+1}-algebra structure:

Corollary 5.18 (Higher Deligne Conjecture) *Let A be a CDGA. There is a natural E_{n+1}-algebra structure on $\mathbf{CH}^{S^n}(A,A)$ whose underlying E_n-algebra structure is the one given by Theorem 5.11. In particular, the underlying E_1-algebra structure is given by the standard cup-product.*

5.2.2 The combinatorial description of the centralizer of cdga maps

We now give an explicit combinatorial model for the E_d-structure given by Theorem 5.11. To do this, we first start with a model of (3.11) at the simplicial cdga level. Let $f : X \times Y \to Z$ be a map of topological spaces. Applying the singular set functor, we get the simplicial set morphism

$$\Delta_\bullet(X) \times \Delta_\bullet(Y) \cong \Delta_\bullet(X \times Y) \xrightarrow{f_*} \Delta_\bullet(Z). \tag{5.21}$$

If $L \in \Gamma - Mod$, we then get a k-module morphism

$$k[\Delta_\bullet(X)] \otimes \mathcal{L}(A,A)(\Delta_\bullet(Y)) \xrightarrow{\mathcal{L}(f_*,A)} \mathcal{L}(A,A)(\Delta_\bullet(Z)) \tag{5.22}$$

defined, in simplicial degree n, as the colimit, over all finite $(\tau_k : \Delta^n \to Y)_{k\in K}$ subsets of $\Delta_n(Y)$, of

$$\Big(\sum \lambda_j(\Delta^n \stackrel{\sigma_j}{\to} X), \bigotimes_K a^{\tau_k}\Big) \mapsto \sum_j \lambda_j\, \iota_j\Big(\bigotimes_{\gamma\in f_*(\sigma_j\times K)} \Big(\prod_{\sigma_j\times\tau_k\in f_*^{-1}(\gamma)} a^{(\sigma_j\times\tau_k)}\Big)\Big) \tag{5.23}$$

where $\iota_j : \mathcal{L}(A,A)(f_*(\sigma_j \times K)) \hookrightarrow \mathcal{L}(A,A)(\Delta_\bullet(Z))$ is the canonical map from a finite subset to the colimit defining $\mathcal{L}(A,A)(\Delta_\bullet(Z))$

Lemma 5.19 *The map* (5.22) *is a simplicial k-module morphism*

Proof. Since $k[\Delta_\bullet(X)] \otimes \mathcal{L}(A,A)(\Delta_\bullet(Y)) \cong k[\Delta_\bullet(X)] \otimes \operatorname*{colim}_{q:L\to\Delta_\bullet(Y)} L(A,A)(L)$ and f_* is a simplicial set morphism, the result boils down to check that the tensor product$\bigotimes_{\gamma\in f_*(\sigma_j\times K)}\Big(\prod_{\sigma_j\times\tau_k\in f_*^{-1}(\gamma)} a^{(\sigma_j\times\tau_k)}\Big)$ is compatible with face and degeneracies operations on the source and target. This follows from the fact that $\mathcal{L}(A,A)(L) = \bigotimes_{l\in L} a^l$ is a functor from finite sets to k-dg-modules. $\square$

Composing with The Eilenberg-Zilber map we get a chain complex morphism

$$f_* : C_*(X) \otimes CH_{\Delta_\bullet(Y)}(A) \longrightarrow CH_{\Delta_\bullet(Z)}(A). \tag{5.24}$$

where we use f_* as an abusive notation which is reasonable in view of the following

Lemma 5.20 *Given a commutative diagram*

$$\begin{tikzcd} X \times Y \arrow[r, "f"] \arrow[d, "p\times q"'] & Z \arrow[d, "r"] \\ X' \times Y' \arrow[r, "f'"] & Z' \end{tikzcd}$$

the following induced square of chain complexes

$$\begin{tikzcd} C_*(X) \otimes CH_{\Delta_\bullet(Y)}(A) \arrow[r, "f_*"] \arrow[d, "p_* \otimes q_*"'] & CH_{\Delta_\bullet(Z)}(A) \arrow[d, "r_*"] \\ C_*(X') \otimes CH_{\Delta_\bullet(Y')}(A) \arrow[r, "f'_*"] & CH_{\Delta_\bullet(Z')}(A) \end{tikzcd}$$

is commutative.

If, Y, Z are pointed and for any $x \in X$, the induced map $f : \{x\} \times Y \to Z$ is pointed, then the map (5.24) *is a dg-A-module morphism.*

Proof. The first claim is a consequence of the fact that $\mathcal{L}(A, A)$ is a functor from sets to dg-k-modules and the naturality of the map (5.21) and Eilenberg-Zilber chain map.

The A-module structure is given by the A-module structure on Hochschild chains on a pointed simplicial set both at source and target. In other words by multiplication on the tensor factor indexed by the constant function from the simplex with value the base point of respectively Y and Z. The assumption implies that for any $\sigma : \Delta^n \to X$, $f_*(\sigma, \tau)$ is the base point of $Hom(\Delta^n, Z)$ (that is the constant map to the base point of Z) when τ is the base point of $Hom(\Delta^n, Y)$. Then, formula (5.23) shows that pointed, then the map (5.22) maps the tensor factor corresponding to the base point into a product of element which are in the factor indexed by the base point of $\Delta_\bullet(Z)$. Hence the A-module is preserved, and we already know it is a dg-map. $\square$

Assuming (Y, y_0), (Z, z_0) are pointed and $f : X \times Y \to Z$ maps $f(X \times \{y_0\}) = \{z_0\}$, Lemma 5.20 implies that we can apply $Hom_A(-, M)$ to the map (5.24) to get a chain map

$$f^* : C_*(X) \otimes CH^{\Delta_\bullet(Z)}(A, M) \longrightarrow CH^{\Delta_\bullet(Y)}(A, M) \tag{5.25}$$

for any A-module M.

Now we will apply this to the map (5.7) $f = pin$ which preserves the base-point as in the above assumption. This gives us the following chain map.

$$\begin{aligned} pinch^*_{S^n,r} : C_*\big(\mathcal{C}_n(r)\big) \otimes \Big(CH^{\Delta_\bullet(S^n)}(A, B)\Big)^{\otimes r} & \\ \xrightarrow{(\mu_\vee)^{(r-1)}} C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\bigvee_{i=1}^r \Delta_\bullet(S^n)}\big(A, B\big) & \\ \xrightarrow{j^*} C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\Delta_\bullet\left(\bigvee_{i=1}^r S^n\right)}\big(A, B\big) & \\ \xrightarrow{pinch^*} CH^{S^n}(A, B) & \end{aligned} \tag{5.26}$$

in $\mathsf{k\text{-}Mod}^{dg}$; here $(\mu_\vee)^{(r-1)}$ is the iteration of the wedge product (5.2) and

$$j : k[\Delta_\bullet(\bigvee_{i=1}^r S^n)] \longrightarrow k[\bigvee_{i=1}^r \Delta_\bullet(S^n)] \tag{5.27}$$

is the linear map $\sum_{k=1}^r pr_k$ obtained by summing all the projections pr_k : $\Delta_\bullet\big(\bigvee_{i=1}^r S^n\big) \stackrel{proj_k\circ -}{\longrightarrow} \bigvee_{i=1}^r \Delta_\bullet(S^n)$ of a singular simplex of the wedge on each of its component (as in Lemma 5.9).

Theorem 5.21 *Let B be a A-cdga. The collection of maps* (5.26) *gives a structure of $C_*\big(\mathcal{C}_n\big)$-algebra*[29] *to $CH^{\Delta_\bullet(S^n)}(A,B)$ which is functorial with respect to A and B.*

This structure is a model for the $\mathbb{E}_n$-algebra structure of $\mathbf{CH}^{S^n}(A,B)$ given by Theorem 5.11.

In the above Theorem and definition of the maps (5.26), we can replace the little cubes operad by the little disk operad $\mathcal{D}_n$, that is the operad consisting of configurations of euclidean open disks[30] inside the open unit disk od $\mathbb{R}^n$. The proofs and constructions goes on *mutatis mutandis.*

Proof. Let us prove the first claim first. Note that the second claim will *a priori* only imply that the maps (5.26) gives rise to an homotopy E_n-algebra structure on the cochains $CH^{\Delta_\bullet(S^n)}(A,B)$.

We have already seen that all the involved map whose composition is the map (5.26) are chain maps. So we are left to prove that it is compatible with the operadic composition. By definition of the operad structures, we have commutative squares of chain complexes

$$\begin{tikzcd}
\mathcal{C}_n(r)\times\mathcal{C}_n(s)\times S^n \arrow[r, "id\times pinch"] \arrow[d, "\circ_i\times id"'] & \mathcal{C}_n(r)\times\bigvee_{i=1}^s S^n \arrow[d, "pinch\times id_{\bigvee_{j\neq i} S^n}"] \\
\mathcal{C}_n(r+s-1)\times S^n \arrow[r, "pinch"] & \bigvee_{i=1}^{r+s-1} S^n
\end{tikzcd}$$

for every $i = 1\dots r$. Dualizing Lemma 5.20 (that is applying $Hom_A(-,B)$) we obtain the commutative squares

$$\begin{tikzcd}
C_*(\mathcal{C}_n(r))\otimes C_*(\mathcal{C}_n(s))\otimes CH^{\Delta_\bullet(\bigvee_{i=1}^{r+s-1} S^n)}(A,B) \arrow[r, "id\otimes(pinch\times id_{\bigvee_{j\neq i} S^n})^*"] \arrow[d, "\circ_{i*}\otimes id"'] & C_*(\mathcal{C}_n(r))\otimes CH^{\Delta_\bullet(\bigvee_{i=1}^{r} S^n)}(A,B) \arrow[d, "pinch^*"] \\
\mathcal{C}_n(r+s-1)\otimes CH^{\Delta_\bullet(\bigvee_{i=1}^{r+s-1} S^n)}(A,B) \arrow[r, "pinch^*"] & CH^{\Delta_\bullet(S^n)}(A,B).
\end{tikzcd} \tag{5.28}$$

Then the compatibility follows from Proposition 5.2 and the associativity of j^* as well.

It remains to prove that this map describes the E_n-algebra structure of Theorem 5.11. By adjunction the simplicial set map (5.21) gives a simplicial set morphism

$$ev_f : \Delta_\bullet(X) \longrightarrow \mathrm{Map}_{\mathsf{sSet}}(\Delta_\bullet(Y),\Delta_\bullet(Z)) \tag{5.29}$$

29. in particular of E_n-algebra in cochain complexes
30. instead of rectangles parallel to the axes

defined by sending a simplex $\sigma : \Delta^n \to X$ to the simplicial set morphism

$$ev_f(\sigma) : \Delta^n \times \Delta_\bullet(Y) \xrightarrow{1\times\widetilde{\sigma}} \Delta_\bullet(X) \times \Delta_\bullet(Y) \xrightarrow{f} \Delta_\bullet(Z) \tag{5.30}$$

where $\widetilde{\sigma} : \Delta^n \to \Delta_\bullet(X)$ is the canonical map sending the non-degenerate n-simplex of Δ^n to $\sigma \in \Delta_n(X)$ (see [GJ]). Applying the Hochschild chain functor 3.6 we obtain a simplicial map

$$ev_{f_*} : \Delta_\bullet(X) \longrightarrow \mathrm{Map}_{\mathsf{CDGA}}(CH_{\Delta_\bullet(Y)}(A), CH_{\Delta_\bullet(Z)}(A)). \tag{5.31}$$

Forgetting the algebra structure, we obtain a linear map $\Delta_\bullet(X) \longrightarrow \mathrm{Map}_{\mathsf{k\text{-}Mod}^{dg}}(CH_{\Delta_\bullet(Y)}(A), CH_{\Delta_\bullet(Z)}(A))$. By adjunction (and applying the singular chain functor), we obtain a chain map

$$C_*(X) \otimes CH_{\Delta_\bullet(Y)}(A) \longrightarrow CH_{\Delta_\bullet(Z)}(A). \tag{5.32}$$

Formula (5.30) and Definition 3.6 shows that for $f = pin : \mathcal{C}_n(r) \times S^n \to \bigvee_{i=1}^r S^n$, the map obtained from (5.32) by applying the duality $Hom_A(-, B)$ functor is the chain map $pinch^*_{S^n,r}$ (this dual is well defined by Lemma 5.20 again).

On the other hand we have the weak equivalence $C_*(\mathrm{Map}_{\textsf{k}\textbf{-Mod}}(A, B) \cong Hom_{dg}(A, B))$ with the cochain complex of non-negatively graded chain complexes maps from A to B equipped with the standard differential $d(f) := d_B \circ f - (-1)^{|f|} f \circ d_A$. Under this weak equivalence, $pinch^*_{S^n,r}$ is the image by the exponential law

$$\begin{aligned}
&\mathrm{Map}_{\textsf{k}\textbf{-Mod}}(C_*((\mathcal{C}_n(r)) \otimes CH_{\Delta_\bullet(S^d)}(A), CH_{\Delta_\bullet(\bigvee_{i=1}^r S^d)}(A)) \\
&\qquad \cong \mathrm{Map}_{\textsf{k}\textbf{-Mod}}\big(C_*((\mathcal{C}_n(r)), \mathrm{Map}_{\textsf{k}\textbf{-Mod}}(CH_{\Delta_\bullet(S^d)}(A), CH_{\Delta_\bullet(\bigvee_{i=1}^r S^d)}(A))\big)
\end{aligned}$$

in $\textsf{k}$**-Mod** of $Hom_A(ev_{f_*}, B)$; that is the image of the (A-linear) dual with value in B of the map (5.31). By lemma 5.22 below applied to the pinching map (5.7), this later map is a model for the map of Theorem 5.11 as claimed since the little cube operad is an E_n-operad. $\square$

Lemma 5.22 *The map* (5.29) *represents in* **Top** *the image of the identity along the following composition:*

$$\begin{aligned}
&Map_{\mathbf{CDGA}}(\mathbf{CH}_Z(A), \mathbf{CH}_Z(A)) \\
&\qquad\qquad \cong\ Map_{\mathbf{Top}}\big(Z,\ Map_{\mathbf{CDGA}}(A, \mathbf{CH}_Z(A))\big) \\
&\qquad\qquad \xrightarrow{-\circ f}\ Map_{\mathbf{Top}}\big(X \times Y,\ Map_{\mathbf{CDGA}}(A, \mathbf{CH}_Z(A))\big) \\
&\qquad\qquad\qquad \cong\ Map_{\mathbf{Top}}\big(X,\ Map_{\mathbf{CDGA}}(\mathbf{CH}_Y(A), \mathbf{CH}_Z(A))\big).
\end{aligned}$$

Proof. The rule $f \mapsto ev_f$ is the bijection corresponding to the exponential law $\mathrm{Hom}_{\mathsf{sSet}}(\Delta_\bullet(X) \times \Delta_\bullet(Y), \Delta_\bullet(Z)) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta_\bullet(X), \mathrm{Map}_{\mathsf{sSet}}(\Delta_\bullet(Y), \Delta_\bullet(Z))$ ([GJ]) and $\mathbf{CH}_{(-)}(A)$ is a symmetric monoidal ∞-functor exhibiting the tensor structure of **CDGA** over **sSet** by Theorem 3.24. $\square$

Remark 5.23 For $n > 1$, Theorem 5.21 gives an explicit homotopy, that is a $\cup_1$-product for the commutativity of the cup product; and more generally iterated $\cup_i$ product up to $i = n - 1$.

5.3 The $O(d)$-equivariance of the universal E_d algebra structure on Hochschild cochomology over spheres

In this section we lift the E_d-structure of Theorem 5.11 to a structure of unoriented[31] E_d-algebra and it is a smooth (and weakly homotopy equivalent) version of what is simply called a $Disk_d$ algebra in the terminology of Ayala-Francis [AF, F]. For $d = 2$, this is the same (by formality and transfer of structure) as an homotopy BV-algebra structure together with an involution of the algebra.

Following [SW], if G is a (topological) group acting on an operad $\mathcal{O}$ in topological spaces, we can define the G-framed analogue of $\mathcal{O}$ define as the operad

$$\Big((\mathcal{O} \rtimes G)(r) := \mathcal{O}(r) \times G^r\Big)_r .$$

The symmetric group action is diagonal (and acting by permutation on G^r). The operad structure map $(\mathcal{O} \rtimes G)(r) \times (\mathcal{O} \rtimes G)(i_1) \times \cdots \times (\mathcal{O} \rtimes G)(i_r) \to (\mathcal{O} \rtimes G)(i_1 + \cdots + i_r)$ is extended from the one on $\mathcal{O}$ by the formula

$$\big((x, \underline{g}), (y_1, \underline{h_1}), \dots, (y_r, \underline{h_r})\big) \mapsto \big(\mu_{\mathcal{O}}(x, g_1 \cdot y_1, \dots, g_r \cdot y_r), g_1\underline{h_1}, \dots, g_r\underline{h_r}\big).$$

Here $\underline{g} = (g_1, \dots, g_r)$ is a tuple (and similarly for $\underline{h_i}$), $\mu_{\mathcal{O}}$ is the operadic composition in $\mathcal{O}$ and $g_j\underline{h_j}$ is the diagonal action og g_j on the components of the tuple $\underline{h_j} \in G^{i_j}$. By [SW], an $(\mathcal{O} \rtimes G)$-operad is the same as an $\mathcal{O}$-algebra in the category of G-spaces.

The framed operad we are mainly interested is the framed little disk operad $\mathcal{D}_n \rtimes O(n)$ of little disks together with an orthogonal transformation. The action of the orthogonal group on disks is the rotation or reflexion action on the disk fixing the center. Note that (the ∞-category of) algebras over this operad are the same as (the ∞-category of) unoriented E_n-algebras in the sense of [Gi4]. Let us also denote the operad of (topological) d-disks[32] algebras by $(\mathrm{Disk}_d(r))_{r \geq 0}$ where

$$\mathrm{Disk}_d(r) = \mathrm{Emb}\Big(\coprod_{i=1\dots r} \mathbb{R}^d, \mathbb{R}^d\Big)$$

is the space of topological embeddings of r-many disjoint pairwise copies of the euclidean space $\mathbb{R}^d$ in itself. The operadic structure is of course given by composition of embeddings (similarly to the operad of cubes), see [AF].

Fixing a base point on S^d, we have an natural action of $O(d)$ and $\mathrm{Homeo}_{pt}(S^d) \cong \mathrm{Homeo}(\mathbb{R}^n)$ on S^d preserving the base point. The $O(d)$-action is simply obtained by rotation along the axis through the center and the base point. By functoriality of the Hochschild chain functor, we thus get induced actions on $CH^{\Delta_\bullet(S^d)}(A, B)$.

Lemma 5.24 *The above action of $O(d)$ on S^d is the same as the action induced by the standard euclidean action of $O(d)$ on $\mathbb{R}^d$ on the quotient $S^d \cong D^d/\partial D^d$*

31. we follow the convention of [Gi4]; it is called framed in [SW]
32. a disk is open here

Proof. The natural action of $O(d)$ on the $\mathbb{R}^d$ preserves the unit disk and the unit sphere. Hence it passes to the quotient $S^d = D^d/\partial D^d$ fixing the base point of S^d and its antipodal point corresponding to the (image by the quotient map of the) center of D^d. $\square$

Corollary 5.25 *Let $f : A \to B$ be a cdga map.*

1. *The E_d-algebra structure on $CH^{\Delta_\bullet(S^d)}(A,B)$ given by Theorem 5.21 is equivariant with respect to $O(d)$.*

   *In particular, it lifts to a structure of algebra over the framed little disk operad $\mathcal{D}_d \rtimes O(d)$ and further $\mathbf{CH}^{S^d}(A,B)$ is in E_d^{unor}-**Alg** so that the E_d-centralizer of a cdga map $f : A \to B$ is canonically an unoriented E_d-algebra.*
2. *The above structure on $CH^{\Delta_\bullet(S^d)}(A,B)$ lifts to a structure of algebras over* Disk_d *which is* $\mathrm{Homeo}(\mathbb{R}^d)$*-invariant. In particular $\mathbf{CH}^{S^d}(A,B)$ is an d-disk algebra in the sense of [AF].*
3. *The γ-ring structures maps $\lambda^{k,\bullet}$ (and therefore $\boldsymbol{\lambda}^k$) of Definition 4.13 are $O(d)$-equivariant.*

Proof. By [SW], the fact that the structure maps $pinch^*_{S^d,r}$ admits a lift to a structure of framed little disk algebra follows from their equivariance. Namely, it is sufficient to check that, for $g \in C_*(O(d))$, $c \in C_*(\mathcal{D}_n(r))$, $f_1, \dots, f_r \in CH^{\Delta_\bullet(S^d)}(A,B)$, one has

$$g \cdot pinch^*_{S^d,r}(c \otimes f_1 \otimes \cdots \otimes f_r) = \sum pinch^*_{S^d,r}(g_{(0)} \cdot c \otimes g_{(1)} \cdot f_1 \otimes \cdots \otimes g_{(r)} \cdot f_r)$$

where $\sum g_{(0)} \otimes \cdots \otimes g_{(r)}$ is the iterated diagonal. The latter follows from the commutativity of the following diagram of spaces

$$\begin{tikzcd} S^d \arrow[r, "Pin_c"] & \bigvee_{i=1\dots r} S^d \arrow[d, "\bigvee g\cdot"] \\ S^d \arrow[u, "g\cdot"] \arrow[r, "Pin_{g\cdot c}"] & \bigvee S^d \end{tikzcd} \tag{5.33}$$

where $c \in \mathcal{D}_d(r)$ and $g \in O(d)$. It follows that $CH^{\Delta_\bullet(S^d)}(A,B)$ is a $\mathcal{D}_d \rtimes O(d)$-algebra hence $\mathbf{CH}^{S^d}(A,B)$ is in E_d^{unor}-**Alg** by Theorem 5.21.

For the second claim of the corollary, we first note that the operads $\mathcal{D}_d$ and $\mathcal{C}_d$ are suboperads of the Disk operad $\mathrm{Disk}_d = (\mathrm{Disk}_d(r))_{r\geq 0}$ described above. Indeed, the first operad is obtained from Disk_d by restricting to those embeddings which are obtained by dilatation and translation of each copies of $\mathbb{R}^d$ (together with a standard homeomorphism between the open unital disk and $\mathbb{R}^d$). The second operad is obtained similarly but by restricting to those embeddings which are rectilinear (meaning that they are composition of translation and dilatation in each direction given by the canonical basis of $\mathbb{R}^d$; in particular they send a rectangle with axis parallel to the axes of the unit cube to a rectangle with axis still parallel to those of the unit cube).

Identifying S^d with the Alexandroff compactification of $\mathbb{R}^d$ (the base point being the point at ∞) yields a pinching map

$$pinch^{top}_{S^d,r} : \mathrm{Disk}_d(r) \times S^d \longrightarrow \bigvee_{i=1\dots r} S^d \tag{5.34}$$

which maps the complements of the images of the embedding (lying in $\mathbb{R}^d = S^d \setminus \{\infty\}$) to the point at infinity of S^d (that is, its base point). It is a continuous map. Hence we can define analogues for that operad $Disk_d$ of Disks of the maps (5.26) and the maps (5.10) are well defined and the proofs of Theorem 5.11 and 5.21 apply mutatis mutandis. This gives an action of the the singular chains of the topological operad $Disk_d$ on $CH^{\Delta_\bullet(S^d}(A,B)$ which is a model for the structure of $\mathbf{CH}^{S^d}(A,B)$ as an object of the ∞-category $Disk_d{-}\mathbf{Alg}$. The equivariance follows from the commutativity of diagram

$$\begin{tikzcd} S^d \arrow[r, "Pin^{top}_c"] & \bigvee_{i=1\dots r} S^d \arrow[d, "\vee g\cdot"] \\ S^d \arrow[u, "g\cdot"] \arrow[r, "Pin^{top}_{g\cdot c}"] & \bigvee S^d \end{tikzcd} \tag{5.35}$$

for any $c \in \mathrm{Emb}(\coprod_{i=1\dots r} \mathbb{R}^d, \mathbb{R}^d)$ and $g \in \mathrm{Homeo}(\mathbb{R}^d) \cong \mathrm{Homeo}_*(S^d)$. This proves the second claim.

For the last claim, we note that that λ^k splits as $S^d \overset{pinch^{(k)}}{\longrightarrow} S^d \vee \cdots \vee S^d \overset{\vee id}{\longrightarrow} S^d$ (see Section 4.3) hence the statement reduces to the first one and the commutativity of diagram (5.33). $\square$

Example 5.26 As a consequence, $\mathbf{CH}^{S^2}(A,B)$ has a canonical homotopy BV-algebra structure. It will be interesting to explicitly describe the induced BV-operator on its cohomology. When $f : A \to k$ is an augmentation, then we obtain an homotopy BV-structure on the iterated Bar construction $Bar^{(2)}(A)$ of A. If $A = \Omega^*(X)$ where X is 2-connected, then we have an equivalence

$$\mathbf{CH}^{S^2}(A,k) \cong C_*(\Omega^2 X)$$

by Theorem 7.2 and [GTZ3]. By functoriality of the iterated integral map, the induced action is given by the standard $O(2)$-action on $\Omega^2 X$ and thus, we recover the standard BV-structure of the singular chains $C_*(\Omega^2 X)$ on a 2-fold based loop space.

Remark 5.27 The category of algebras over the operad Disk_d of disks in Corollary 5.25 is also weakly equivalent to the one of algebras over the semi-direct product $\mathcal{E}_d \rtimes \mathrm{Homeo}(\mathbb{R}^d)$ of the operad of *framed embeddings* $\mathcal{E}_d(r) = \mathrm{Emb}^{framed}(\coprod_{i=1\dots r} \mathbb{R}^d, \mathbb{R}^d)$ with the group of homeomorphisms of $\mathbb{R}^d$ (see [AF, Gi4] for details).

Example 5.28 (The universal unoriented E_d-algebra structure on semi-reduced suspensions) Let X be a non-empty topological space. We defined its d-fold semi-reduced suspension to be the topological space

$$\widetilde{\Sigma}^d X := I^d \times X/(\partial I^d \times X).$$

Note that $\widetilde{\Sigma} X$ is the quotient of the (unreduced) suspension of

$$\Sigma(X) = I \times X/\begin{cases}(1,x) \simeq (1,y) \\ (0,x) \simeq (0,y)\end{cases} \quad \text{for all } x, y \in X$$

obtained by identifying the points $[(0,x)]$ with the point $[(1,y)]$ in ΣX. On the other hand, if X is pointed (with x_0 the base point), then the reduced (d-fold iterated) suspension $S^d \wedge X$ is a quotient of the semi-reduced d-fold suspension:

$$S^d \wedge X \cong \widetilde{\Sigma}^d X/(I^d \times \{x_0\} \simeq \infty)$$

where we write ∞ for the point given by the class of $\partial I^d \times X$ in $\widetilde{\Sigma}^d X$.

If $f : A \to B$ is a CDGA map, the canonical (and unique) map $X \to pt$ induces the CDGA morphism

$$f \circ pt_* : \mathbf{CH}_X(A) \to \mathbf{CH}_{pt}(A) \cong A \xrightarrow{f} B$$

which is represented by the CDGA morphism $\underline{CH}_{X_\bullet}(A) \to \underline{CH}_{pt_*}(A) \cong A \xrightarrow{f} B$ for any simplicial set model $X_\bullet$ of X (see example 3.8).

Corollary 5.29 *Let $f : A \to B$ be a CDGA map and X a non-empty space.*

1. *The Hochschild cohomology $\mathbf{CH}^{\widetilde{\Sigma}^d X}(A, B)$ is naturally equivalent to the centralizer $\mathfrak{z}(\mathbf{CH}_X(A) \overset{f\circ pt_*}{\to} B)$ and is naturally an object of E_d^{unor}-**Alg** and a multiplicative γ-ring with trivial multiplication.*
2. *The γ-ring structures maps $\boldsymbol{\lambda}^k$ given by claim 1 are $O(d)$-equivariant.*

Proof. By the excision property of Theorem 3.24, we have an equivalence in**CDGA**

$$\mathbf{CH}_{\widetilde{\Sigma}^d X}(A) \cong A \overset{\mathbb{L}}{\underset{\mathbf{CH}_{\partial I^d \times X}(A)}{\otimes}} \mathbf{CH}_{I^d \times X}(A) \cong A \overset{\mathbb{L}}{\underset{\mathbf{CH}_X(A)}{\otimes}} \mathbf{CH}_{S^d \times X}(A)$$

since the semi-reduced suspension is also the quotient $\widetilde{\Sigma}^d X \cong S^d \times X/\{*\} \times X$ where we denote by $* = [\partial I^d]$ the base point of S^d (given by the class of ∂I^d in the quotient $S^d \cong I^d/\partial I^d$). From it we deduce an equivalence

$$\begin{aligned}\boldsymbol{\alpha}_{X,d} : \mathbf{CH}^{\widetilde{\Sigma}^d X}(A, B) &\cong \mathbb{R}Hom_A\big(\mathbf{CH}_{\widetilde{\Sigma}^d X}(A), B\big) \\ &\cong \mathbb{R}Hom_A\big(A \overset{\mathbb{L}}{\underset{\mathbf{CH}_X(A)}{\otimes}} \mathbf{CH}_{S^d \times X}(A), B\big) \\ &\cong \mathbb{R}Hom_{\mathbf{CH}_X(A)}\big(\mathbf{CH}_{S^d \times X}(A), B\big) \qquad (5.36)\end{aligned}$$

in $\mathbf{CH}_X(A)$-**Mod**.

By the Eilenberg-Zilber equivalence (4.27) (also see Theorem 3.24 and Corollary 4.28), we also have an equivalence

$$\boldsymbol{\beta}_{X,d} : \mathbb{R}Hom_{\mathbf{CH}_X(A)}\big(\mathbf{CH}_{S^d\times X}(A),\, B\big) \cong \mathbb{R}Hom_{\mathbf{CH}_X(A)}\big(\mathbf{CH}_{S^d}(\mathbf{CH}_X(A)),\, B\big) \\ \cong \mathbf{CH}^{S^d}(\mathbf{CH}_X(A),\, B). \quad (5.37)$$

Composing the maps (5.36) and (5.37), we obtain the equivalence

$$\boldsymbol{\beta}_{X,d}\circ\boldsymbol{\alpha}_{X,d} : \mathbf{CH}^{\widetilde{\Sigma}^d X}(A,B) \xrightarrow{\cong} \mathbf{CH}^{S^d}(\mathbf{CH}_X(A),\, B)$$

where the $\mathbf{CH}_X(A)$-module structure on B is induced by the map $f\circ pt_*$. Hence the right hand side is the centralizer $\mathfrak{z}(f\circ pt_*)$. The two claims now follow from Corollary 5.25 applied to $\mathbf{CH}^{S^d}(\mathbf{CH}_X(A),\, B)$. $\square$

From the proof of Corollary 5.29, we see that, if $X_\bullet$ is a simplicial set model of a non-empty space X, then a cochain model for the unoriented E_d-algebra structure of $\mathbf{CH}^{\widetilde{\Sigma}^d X}(A,B)$ is given by the $CH_{X_\bullet}(A)$-module

$$Hom_{CH_{X_\bullet}(A)}\big(CH_{\Delta_\bullet(S^d)}(\mathbf{CH}_{X_\bullet}(A)),\, B\big) \cong CH^{\Delta_\bullet(S^d)}(CH_{X_\bullet}(A), B). \quad (5.38)$$

Then Corollary 5.25 implies that the latter inherits a structure of algebras over Disk_d which is $\mathrm{Homeo}(\mathbb{R}^d)$-invariant.

6 Applications of Higher Hochschild-Kostant-Rosenberg Theorem

The classical Hochschild-Kostant-Rosenberg Theorem is a powerful result to compute Hochschild (co)homology of a smooth algebra. It also gives a nice description of the pieces of the Hodge decomposition. In this section, we explain how to generalize it to higher analogues.

6.1 Statement of HKR Theorem

If X is a space, we denote $\widetilde{H}_*(X)$ its reduced homology coalgebra. Recall that a space is formal if its cochain algebra is quasi-isomorphic to its cohomology as a CDGA. This includes all spheres, suspensions, Lie group or Kähler varieties.

Theorem 6.1 *Assume X is a formal space of finite type in each degree. And let $(Sym(V),d) \xrightarrow{\cong} A$ be a cofibrant resolution*[33] *of A. There are natural (in A, M) equivalences*

$$\mathbf{CH}_X(A) \xrightarrow{\cong} (Sym(V\otimes H_*(X)), d_X); \quad (6.1)$$

$$\mathbf{CH}_X(A,M) \xrightarrow{\cong} (M\otimes_{Sym(V)} Sym(V\otimes H_*(X), d_X)) \quad (6.2)$$

33. Any A is quasi-isomorphic to a semi-free one of the form $(Sym(V), d)$ and by quasi-invariance of Hochschild chains it is enough to compute the left hand side of (6.1) for the later cdgas.

respectively in **CDGA** *and in* $\mathbf{CH}_X(A)$-***Mod***. *Further, if* $f : X \to Y$ *is a* formal map[34] *we have a commutative diagrams (respectively in* **CDGA** *and* $\mathbf{CH}_X(A)$-***Mod****):*

$$\begin{tikzcd} \mathbf{CH}_X(A) \arrow[r, "\cong"] \arrow[d, "f_*"'] & Sym(V \otimes H_*(X)) \arrow[d, "Sym(id \otimes H_*(f))"] \\ \mathbf{CH}_Y(A) \arrow[r, "\cong"] & Sym(V \otimes H_*(Y)), \end{tikzcd} \tag{6.3}$$

$$\begin{tikzcd} \mathbf{CH}_X(A, M) \arrow[r, "\cong"] \arrow[d, "f_*"'] & M \otimes_{Sym(V)} Sym(V \otimes H_*(X)) \arrow[d, "Sym(id \otimes H_*(f))"] \\ \mathbf{CH}_Y(A) \arrow[r, "\cong"] & M \otimes_{Sym(V)} Sym(V \otimes H_*(Y)), \end{tikzcd} \tag{6.4}$$

The differential d_X in the right hand sides of (6.1) and (6.2) are induced by the inner differential of A (as well as M as the usual differential of a tensor product of complexes) as follows. For $v \in V$, we denote

$$d(v) := \sum v_{(1)} \cdots v_{(n)}$$

its differential; that is $v_{(1)} \cdots v_{(n)} \in Sym^n(V)$ is the weight n summand of $d(v)$ and is thus by definition a finite sum of monomials of total degree n in V. Following Sweedler's notations, we write

$$\Delta^{(n-1)}(\alpha) = \sum \alpha_{(1)} \cdots \otimes \alpha_{(n)}$$

where $\Delta^{(n-1)}$ is the iterated coproduct in the commutative coalgebra $H_*(X)$.

Then the differential d_X on $Sym(V \otimes H_*(X))$ is the unique derivation extending the map given for any $v \otimes \alpha \in V \otimes H_*(X)$ by

$$d_X(v \otimes \alpha) := \sum (v_{(1)} \otimes \alpha_{(1)}) \cdots (v_{(n)} \otimes \alpha_{(n)}). \tag{6.5}$$

Proof.We refer to [GiRo] for details. We will mainly use this results for spheres, in which case the result is a corollary of [P, Section 4.7]. In general, the result essentially follows from Proposition 3.3 applied to a dg cocommutative coalgebra model for chains obtained by dualizing the ones on cochains. This yields a (zigzag of) weak equivalences of right Γ-modules $\widetilde{C}_*(X) \simeq {}^{co}\mathcal{L}(H_*(X))$.

Then one computes the tensor product

$${}^{co}\mathcal{L}(H_*(X)) \otimes_\Gamma \mathcal{L}(Sym(V)) \cong Sym(V \otimes H_*(X))$$

which is an explicit computation done[35] in [P]. The differential on d on $Sym(V)$ makes $\mathcal{L}(Sym(V))$ a dg-Γ-module which induces the claimed differential on the above tensor product of right and left Γ-module.

34. we recall that it means that f is quasi-isomorphic to $H_*(f) : H^*(Y) \to H^*(X)$ in **CDGA**; in particular X and Y are formal

35. it can also be deduced from the fact that Sym is a left adjoint hence commutes with colimits

Further, if $f : X \to Y$ is formal, then there is a commutative diagram

$$\begin{tikzcd} \widetilde{C}_*(X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & {}^{co}\mathcal{L}(H_*(X)) \arrow[d, "H_*(f)"] \\ \widetilde{C}_*(Y) \arrow[r, "\cong"] & {}^{co}\mathcal{L}(H_*(Y)) \end{tikzcd}$$

in the ∞-category of right Γ-modules. The commutative diagrams (6.3) and (6.4) are deduced from this diagram and the first part of the proof. $\square$

Remark 6.2 For surfaces, the HKR theorem has first been proved in [GTZ]. A variant for Chevalley-Eilenberg complexes has been given in [TW].

Corollary 6.3 *Assume X is a formal space of finite type in each degree. And let $(Sym(V), d) \xrightarrow{\cong} A$ be a cofibrant resolution of A. There are natural (in A, M) equivalences in $\mathbf{CH}_X(A)$-**Mod**.*

$$\mathbf{CH}^X(A, M) \xleftarrow{\cong} M \otimes_{\mathrm{Sym}(V)} Sym_{\mathrm{Sym}(V)}(V^\vee \otimes \widetilde{H}^*(X)). \tag{6.6}$$

The differential is computed as for the Hochschild chains.

Proof. One applies theorem 6.1 and Lemma 3.11. $\square$

Remark 6.4 Given our assumption on A, we have that $Sym(V) \otimes V[1]$ is a model for $\mathbb{L}\Omega^1(A)$, the derived functor of Kähler forms. In other words, it computes the André-Quillen chain complex. In particular Theorem 6.1 can be restated as

$$\mathbf{CH}_X(A) \xrightarrow{\cong} Sym_A(\mathbb{L}\Omega^1(A) \otimes \widetilde{H}_*(X)).$$

Recall that $\Omega^1(A)$ is the module generated by symbols $ad(b)$, $a, b \in A$ satisfying the relations that d is k-linear and $d(ab) = ad(b) + bd(a)$; it is also called the Kähler forms. The differential on A extends to a differential on $\Omega^1(A)$ in the obvious way. It is canonically an A-module (with action $x \otimes ad(b) \mapsto xad(b)$ by multiplication on the left). The (left) derived Kähler forms functor $\mathbb{L}\Omega^1(A)$ is canonically quasi-isomorphic to the dg-module $\Omega^1(\mathrm{Sym}(V))$ for any cofibrant resolution $(\mathrm{Sym}(V), d) \to A$; indeed a quasi-isomorphism of cdgas between *semi-free* cdgas induces a quasi-isomorphism between their Kähler forms. In other words, the HKR theorem identifies the Hochschild homology over a formal space X with a twisted by X version of André-Quillen homology.

Similarly, one has

$$\mathbf{CH}^X(A, M) \xleftarrow{\cong} Sym_A(\mathbb{R}Der(A, M) \otimes \widetilde{H}^*(X)).$$

Here $Der(A, M)$ is the dg-A-module of derivations of A into M. Again if A, B are is *semi-free*, a quasi-isomorphism $f : A \to B$ of cdgas between them induces a quasi-isomorphism $f^* : Der(B, M) \to Der(A, M)$ of dg-A-modules. Hence $\mathbb{R}Der(A, M)$ is canonically equivalent to $Der(Sym(V), M)$ for any cofibrant resolution $(\mathrm{Sym}(V), d) \to A$ of a cdga A.

Example 6.5 (Smooth algebras) We now give an explicit map describing the HKR equivalence for $X = S^d_\bullet$ the standard model for the sphere (see example 3.17) for smooth algebras. This will be useful in § 6.2. We left to the reader the task to generalize the formulas for more general spaces such as $X \times S^d_\bullet$.

We start with a general notation. Recall that $CH_{S^d_n}(A) = A \otimes A^{\otimes n^d}$; an element of which is a sum of tensors $a_0 \otimes \bigotimes_{(i_1,\dots,i_d)\in\{1,\dots n\}^d} a_{i_1,\dots i_d}$.

Definition 6.6 For $b \in A$ and $j_1, \dots, j_d \in \{1, \dots n\}$, we denote $E_{j_1,\dots,j_d}(b)$ the tensor

$$E_{j_1,\dots,j_d}(b) := 1 \otimes \bigotimes_{(i_1,\dots,i_d)\in\{1,\dots n\}^d} b_{i_1,\dots i_d}$$

for which $b_{j_1,\dots j_d} = b$ and the other ones are equal to 1. In other words it is the tensor which is one everywhere except in the position $(j_1, \dots, j_d)$ for which it is equal to b.

Since we are considering normalized chain complexes, $E_{j_1,\dots,j_d}(b) = 0$ unless $n = 0, d$. We will, unless otherwise stated, only use this notation for $n = d$.

For the remainder of this example, we will consider the case where the differential of A is null, that is of a graded algebra A. Let V be a graded vector space. We denote $s_d : V \to V[d]$ the desuspension functor. In other word an element of $V[d]$ of (cohomological) degree i will be written as $s_d(v)$ for an unique $v \in V$ of degree $i + d$.

Let $\varepsilon^d_{HKR} : Sym(V \oplus V[d]) \to CH_{S^d_\bullet}(Sym(V))$ be the *unique* graded commutative $Sym(V)$-algebra map defined, for any $s_d(v) \in V[d]$ by

$$\varepsilon^d_{HKR}(s_d(v)) := \frac{1}{d!} \sum_{\sigma\in\Sigma_d} (-1)^\sigma E_{\sigma^{-1}(1),\dots,\sigma^{-1}(d)}(v) \in CH_{S^d_d}(Sym(V)). \qquad (6.7)$$

Lemma 6.7 *Let V be a graded space. The map $\varepsilon^d_{HKR} : Sym(V \oplus V[d]) \to CH_{S^d_\bullet}(Sym(V))$ is a CDGA quasi-isomorphism and is equivalent to the HKR equivalence of Theorem 6.1*[36].

If A is a graded algebra, the map

$$\varepsilon^d_{HKR} : Sym_A(\Omega^1(A)[d]) \longrightarrow CH_{S^d_\bullet}(A)$$

defined as the unique graded commutative A-algebra map satisfying

$$\varepsilon^d_{HKR}(d(b)) := \frac{1}{d!} \sum_{\sigma\in\Sigma_d} (-1)^\sigma E_{\sigma^{-1}(1),\dots,\sigma^{-1}(d)}(b) \in HH_{S^d_d}(A)$$

is well defined in homology. It is further an algebra isomorphism $Sym_A(\Omega^1(A)[d]) \cong HH_{S^d}(A)$ if A is smooth.

36. Note that $Sym(V)$ is endowed with the zero differential

Proof. First, note that $Sym(V\oplus V[d])\cong Sym(V\otimes H_*(S^d))$. There is a canonical cycle $\tau:\Delta^d\to\Delta^d/\partial\Delta^d$ representing the generator of $H_d(S^d)$ and we have a projection $C_*(S^d)\to k*\oplus k\tau=H_*(S^d)$ which is a quasi-isomorphism ($*$ is the base point of S^d). It follows that the unique commutative algebra map $Sym(V\oplus V[d])\to CH_{\Delta_\bullet(S^d)}(Sym(V))$ defined by $s_d(v)\mapsto v\otimes\bigotimes_{\tau\neq\sigma\in\Delta_d(S^d)}1$ is a chain map and represents the HKR equivalence of Theorem 6.1. Since a cube I^d can be partitioned into $d!$ many standard d-simplices τ_σ, we obtain a splitting of the canonical cycle τ as the chain $\widetilde{\tau}:\sum_{\sigma\in\Sigma_d}\tau_\sigma$. Now, recall that $S^d_\bullet=I^d_\bullet/\partial I^d_\bullet$. The canonical unit of the adjunction map gives us, for any $(i_1,\dots,i_d)\in\{1,\dots,n\}^d$ the map $\eta_{(i_1,\dots,i_d)}:\Delta^n\to I^d/\partial I^d$ given, for $t=(t_1\leq\cdots\leq t_n)\in\Delta^n$ by

$$\eta_{(i_1,\dots,i_d)}(t):=(t_{i_1},\dots,t_{i_d}).$$

Therefore the HKR equivalence is equivalent to the unique $Sym(V)$-algebra map $Sym(V\oplus V[d])\to CH_{\Delta_\bullet(S^d)}(Sym(V))$ defined by

$$s_d(v)\mapsto\frac{1}{d!}\sum_{\sigma\in\Sigma_d}v\cdot\otimes\bigotimes_{\eta_{(i_1,\dots,i_d)}\neq\sigma\in\Delta_d(S^d)}1$$

By functoriality, the map $\eta_*:CH_{S^d_\bullet}(Sym(V))\to CH_{\Delta_\bullet(S^d)}(Sym(V))$ is a CDGA homomorphism and the above map is

$$\eta_*(\sum_{\sigma\in\Sigma_d}(-1)^\sigma E_{\sigma^{-1}(1),\dots,\sigma^{-1}(d)}(v))=\varepsilon^d_{HKR}(s_d(v)).$$

Therefore the map ε^d_{HKR} is a CDGA map representing the HKR equivalence.

For the second part, one first checks that $E_{1,\dots,d}(ab)-bE_{1,\dots,d}(a)-aE_{1,\dots,d}(b)$ is equal to the differential of the tensor $\bigotimes\alpha_{j_1,\dots,j_d}\in CH_{S^d_{d+1}}(A)$ given by $\alpha_{1,2,\dots,d}=a$, $\alpha_{1,2,\dots,d-1,d+1}=b$ and the other components equal to 1. This proves (after symmetrizing) the the map is well defined in homology since each term $E_{1,\dots,d}(x)$ is a cocycle, for any x in A as can be checked explicitly. Taking a cofibrant resolution, one sees as in Lemma 6.11 that this induced map corresponds to the composition

$$Sym_A(\Omega^1(A)[d])\longrightarrow Sym_A(\mathbb{L}\Omega^1(A)[d])\xrightarrow{HKR_*}HH_{S^d}(A)$$

When A is smooth, the first map is an equivalence hence the result. $\square$

6.2 HKR isomorphism and Hodge decomposition

We now relate the HKR isomorphisms from Section 6.1 with the Hodge filtrations on the various (co)chains functors.

Recall from Example 4.6 that $Sym(V\oplus V[d])$ and $Sym(V\oplus V^\vee[d])$ are endowed with canonical dg-multiplicative-γ-ring with zero multiplication structure for which V is of pure weight 0 and $V[d]$ or $V^\vee[d]$ are of pure weight 1. More generally if U and W are graded modules and d is a differential on $Sym(U\oplus W)$ such that $d(W)\subset W\otimes Sym(U)$, then $(Sym(U\oplus W),d)$ has a dg-multiplicative-γ-ring with zero multiplication structure for which U is on weight 0 and W in weight 1.

Corollary 6.8 *Assume X is a formal space of finite type in each degree. And let $(Sym(V), d) \xrightarrow{\cong} A$ be a cofibrant resolution of A.*

The HKR quasi-isomorphisms yields natural (in A and M) equivalences

1.
$$HKR : \mathbf{CH}_{S^d\times X}(A) \xrightarrow{\cong} Sym\big((V\otimes H_*(X)) \oplus (V\otimes H_*(X))[d]\big), \qquad (6.8)$$
$$HKR : \mathbf{CH}_{S^d\wedge X}(A) \xrightarrow{\cong} Sym\big(V \oplus (V\otimes \widetilde{H}_*(X))[d]\big) \qquad (6.9)$$

*of dg-*multiplicative *γ-ring with trivial multiplication,*

2.
$$\mathbf{CH}_{S^d\times X}(A,M) \xrightarrow{\cong} M \underset{Sym(V)}{\otimes} Sym\big((V\otimes H_*(X)) \oplus (V\otimes H_*(X))[d]\big), \qquad (6.10)$$

$$\mathbf{CH}_{S^d\wedge X}(A,M) \xrightarrow{\cong} M \underset{Sym(V)}{\otimes} Sym\big(V \oplus (V\otimes \widetilde{H}_*(X))[d]\big) \qquad (6.11)$$

*of dg-γ-ring with trivial multiplication in $\mathbf{CH}_{S^d\times X}(A)$-**Mod** and $\mathbf{CH}_{S^d\wedge X}(A)$-**Mod** respectively*

3. *as well as*
$$\mathbf{CH}^{S^d\times X}(A,M) \xleftarrow{\cong} M \underset{Sym(V)}{\otimes} Sym\big((V^\vee\otimes \widetilde{H}^*(X)\oplus V)\oplus(V^\vee\otimes H^*(X))[-d]\big), (6.12)$$
$$\mathbf{CH}^{S^d\wedge X}(A,M) \xleftarrow{\cong} M \underset{Sym(V)}{\otimes} Sym\big(V \oplus (V^\vee\otimes \widetilde{H}^*(X))[-d]\big) \qquad (6.13)$$

*of dg-γ-ring with trivial multiplication in $\mathbf{CH}_{S^d\times X}(A)$-**Mod** and $\mathbf{CH}_{S^d\wedge X}(A)$-**Mod** respectively.*

The pure weight 1 part in the right hand sides are given by the shifted by d or $-d$ component. The other components are of weight 0.

The above theorem applies in particular for $X = pt$ to give models of Higher order Hochschild (co)chains for spheres and their compatibility with Hodge decomposition; in particular we recover the results of [P] and [Gi3].

Proof. We use the normalized (co)chain complexes as models for the derived functors. Let us consider the case 1.

By Proposition 4.30 and Definition 4.29, we have an equivalence of cdgas

$$\mathbf{CH}_{S^d\times X}(A) \cong \mathbf{CH}_{S^d}(\mathbf{CH}_X(A)) \cong \mathbf{CH}_{S^d}(Sym(V\otimes H_*(X)))$$

where the last equivalence is given by HKR equivalence (Theorem 6.1) and naturality of the Hochschild functor (Proposition 3.18). Since S^d is formal with cohomology algebra $H^*(S^d) \cong k\oplus k[-d]$, we deduce the equivalence (6.8) of cdgas. Of course one can also simply apply the HKR quasi-isomorphism to the formal space $S^d \times X$. The two quasi-isomorphisms obtained this way are equivalent by the naturality of Corollary 4.28.

The map $\lambda^k \wedge id : S^d = S^1 \wedge S^{d-1} \to S^1 \wedge S^{d-1} = S^d$ is formal for $d > 0$ since it is a suspension. It is also formal for $d = 1$ since the homotopy type of a map from S^1 to itself is determined by its degree. Hence, we can apply the commutativity of diagram (6.3) to the above reasonning to obtain

$$\begin{tikzcd} \mathbf{CH}_{S^d\times X}(A) \arrow[r, "HKR"] \arrow[d, "\boldsymbol{\lambda}^k"'] & Sym\big((V\otimes H_*(X)) \oplus (V\otimes H_*(X))[d]\big) \arrow[d, "Sym(id\oplus id\otimes H_*(\lambda^k))"] \\ \mathbf{CH}_{S^d\times X}(A) \arrow[r, "HKR"] & Sym\big((V\otimes H_*(X)) \oplus (V\otimes H_*(X))[d]\big). \end{tikzcd}$$

Since $\lambda^k : S^d \to S^d$ is of degree k, we obtain that $HKR \circ \boldsymbol{\lambda}^k \circ HKR^{-1}$ restricted to $(V \otimes H_*(X))[d]$ is multiplication by k and the identity on the restriction to $Sym(V \otimes H_*(X))$. This proves 1 since Theorem 4.25 already proves that $\mathbf{CH}_{S^d\times X}(A)$ is a multiplicative γ-ring with zero multiplication.

The proof of 2. is similar. Namely, we use Proposition 4.30, Definition 4.29 and again Theorem 6.1 for X and S^d to get equivalences of cdgas:

$$\begin{aligned} \mathbf{CH}_{S^d\wedge X}(A) &\cong \mathbf{CH}_{S^d}(\mathbf{CH}_X(A), A) \underset{\mathbf{CH}_{S^d}(A)}{\otimes} A \\ &\cong \mathbf{CH}_{S^d}(Sym(V\otimes H_*(X)), Sym(V)) \underset{\mathbf{CH}_{S^d}(Sym(V))}{\otimes} Sym(V) \\ &\cong Sym(V) \underset{Sym(V\otimes H_*(X))}{\otimes} Sym(V\otimes H_*(X) \oplus (V\otimes H_*(X))[d]) \underset{Sym(V\oplus V[d])}{\otimes} Sym(V) \\ &\cong Sym\big(V \oplus (V\otimes \widetilde{H}_*(X))[d]\big). \end{aligned}$$

As before, by formality of λ^k, this equivalence becomes the cdga map induced by $id \oplus id \otimes H_*(\lambda^k)$ on the right hand side of (6.9), hence the multiplication by k on the component $(V \otimes \widetilde{H}_*(X))[d]$ and the identity on V. The proofs of the remaining cases are exactly similar. The module structures are obtained from the similar statement in the HKR Theorem 6.1 and Theorem 4.14, Theorem 4.25. $\square$

Remark 6.9 (Puzzle decomposition) A corollary of the HKR theorem is the so-called puzzle and puzzling meaning of the groups arising in the decomposition [P]. Indeed, the right hands of the (various) HKR maps in Corollary 6.8 are rather similar except for the shift by d, the dimension of the sphere S^d. In particular for d odd, the groups appearing in the Hodge decomposition (for any A and over any suspension $S^d \wedge X$ or product $S^d \times X$) are those in the Hodge decomposition for $d = 1$ but they are dispatched in different degrees. The same is true for d even with the groups appearing in the decomposition for $d = 2$.

For d-odd and $X = pt$, theses groups are (shifted) higher André-Quillen (co)homology groups. For d-even they correspond to a symmetric version of higher André-Quillen (co)homology groups in view of remark 6.4. For $X = pt$, this was noticed in homology by Pirashvili in [P]. Note that for $d = 1$, the Hodge decomposition coincides with the classical one [GS], [L1].

We now give an explicit expression for the HKR map in the case of a semi-free CDGA $(Sym(V), d)$ for the standard sphere model $S^d_\bullet$ (example 3.17). Recall that $CH_{S^d_n}(A)$ is spanned by tensors of the form $a_0 \otimes \bigotimes_{(i_1,\dots,i_d)\in\{1,\dots n\}^d} a_{i_1,\dots i_d}$. Also recall (from example 6.5) that we write $s_d : V \to V[d]$ the canonical map. We extend it as the derivation of $Sym(V \oplus V[d])$ by setting $s_d(V[d]) = 0$.

We define a map $\pi_{HKR} : CH_{S^d_\bullet}(Sym(V)) \to (Sym(V \oplus V[d]), d)$ as follows:

- π_{HKR} is set to be zero on $CH_{S^d_n}(Sym(V))$ if $n \neq dj$, that is if n is a multiple of d;
- it is the canonical inclusion on $CH_{S^d_0}(Sym(V)) = Sym(V)$;
- for $n = d(j+1)$ $(j \geq 0)$, we set

$$\begin{aligned}\pi_{HKR}\left(a_0 \otimes \bigotimes_{(i_1,\dots,i_d)\in\{1,\dots n\}^d} a_{i_1,\dots i_d}\right) \\ = \sum_{\sigma_0,\dots,\sigma_j\in\Sigma_d} \left(\prod_{\tau\in K_{\sigma_0,\dots,\sigma_j}} a_{\tau_1,\dots,\tau_d}\right)\cdot\Big((-1)^{\sigma_0} s_d(a_{\sigma_0^{-1}(1),\dots,\sigma_0^{-1}(d)})\Big)\cdots \\ \cdots\Big((-1)^{\sigma_j} s_d(a_{jd+\sigma_j^{-1}(1),\dots,jd+\sigma_j^{-1}(d)})\Big) \qquad (6.14)\end{aligned}$$

  where $K_{\sigma_0,\dots,\sigma_j}$ is the set of all other possible indices [37].

In plain english, the map π_{HKR} is the product for all diagonal d-cubes of size d of the canonical derivation s_d applied to each "permutation entry" together with the product of all other elements (which belongs to $Sym(V)$).

Remark 6.10 The explicit HKR map (6.14) has another interpretation using the small model $S^d_{sm\bullet}$ (as in Example 3.17) as follows. We have a map from $CH_{S^d_{smd}}(A) \cong A \otimes A \to \Omega^1(A)[d]$ which maps $a \otimes b$ to $ad(b)$. On the other hand we have a simplicial set projection $S^d_\bullet \to S^d_{sm\bullet}$. Concretely, it is obtained by the natural decomposition of I^d into $d!$-many simplices (obtained, for every permutation $\sigma \in \Sigma_d$, by choosing the simplex with coordinates $0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \leq 1$ where the x_i's are the coordinates in the standard cube) and then identifying to each other all these simplices, and their boundary to a point. The composition $CH_{S^d_d}(A) \to CH^{S^d_{smd}}(A) \to \Omega^1(A)[d]$ is precisely the map π_{KHR} together with the isomorphism $\Omega^1(Sym(V)) \cong Sym(V) \otimes V$.

Lemma 6.11 *The map $\pi_{HKR} : CH_{S^d_\bullet}(Sym(V)) \to (Sym(V \oplus V[d]), d)$ is a CDGA quasi-isomorphism representing the HKR equivalence.*

Its dual $\pi^*_{HKR} := Hom_{Sym(V)}(\pi_{HKR}, Sym(V)) : (Sym(V \oplus V^\vee[-d]), d) \to CH^{S^d_\bullet}(Sym(V))$ is a quasi-isomorphism of dg-$(Sym(V), d)$-modules representing the cohomology HKR equivalence.

Proof. Once again, we use the normalized (co)chain complexes as models for the derived functors. Since $s_d(1) = 0$, all terms which do not permute the diagonal

37. that is those *not* of the form $\ell d + \sigma_\ell^{-1}(1), \dots, \ell d + \sigma_\ell^{-1}(d)$

factors are maps to zero. From there follows the fact that π_{HKR} is a CDGA map. By construction, the map commutes with the inner differential of $Sym(V)$. To check it is a chain map thus reduces to prove that the composition

$$\underline{CH}_{S^d_{jd+1}}(\mathrm{Sym}(V)) \overset{\sum \pm (d_r)_*}{\longrightarrow} \underline{CH}_{S^d_{jd}}(\mathrm{Sym}(V)) \overset{\pi_{HKR}}{\longrightarrow} (Sym(V \oplus V[d]), d)$$

is zero which is straightforward applying $s_d(a \cdot b) = as_d(b) + bs_d(a)$ to exactly one permutation for each $r = 0 \ldots jd+1$ except for $d = 0$ and $d = jd+1$.

By construction, the map π_{HKR} is thus a bigraded chain map. Considering the filtration with respect to the inner degree of $Sym(V)$, we obtain a spectral sequence. On the page E_1, the map π_{HKR} is a chain map

$$\underline{CH}_{S^d_\bullet}(Sym(V)) \overset{\pi_{HKR}}{\longrightarrow} Sym(V \oplus V[d])$$

where both symmetric algebras are endowed with the zero differential. At the level of the page E_1 we have the quasi-isomorphism $\varepsilon^d_{HKR} : Sym(V \oplus V[d]) \to \underline{CH}_{S^d_\bullet}(Sym(V))$ from Lemma 6.7, which represents the HKR equivalence. Now one checks that

$$\pi_{HKR} \circ \varepsilon^d_{HKR} = id_{Sym(V \oplus V[d])}. \tag{6.15}$$

Since both maps are algebra maps, it is enough to check it on V and $V[d]$ for which it is straightforward. It follows that π_{HKR} as well is a quasi-isomorphism on the page E_1, hence a quasi-isomorphism as claimed.

We obtain the cohomological statement by dualizing π_{HKR} which is a $Sym(V)$-linear map since it is a cdga map. $\square$

One obtains versions of π_{HKR} for arbitrary modules as coefficient by tensoring it with M over $Sym(V)$. Further, one can construct similar maps for $S^d \times X$ and $S^d \wedge X$. Details are left the reader for this last two cases.

6.3 Compatibility of Hodge decomposition with the algebra structure in cohomology and induced $Pois_{n+1}$-algebra structure

When $M = A$ with its standard module structure, the higher Hochschild cochains $\mathbf{CH}^{S^d}(A, A)$ for spheres are more than modules; they also have a multiplicative structure induced y the cup-product, see Section 5.1 and Definition 5.5 below.

The fact that for $d > 1$, the (homotopy) commutativity of the cup-product $\cup_{S^d}$ (of example 5.7) can be induced by a base-point preserving homotopy implies

Lemma 6.12 *Assume $d > 1$ and let B be an unital A-CDGA. Then, the diagram*

$$\begin{tikzcd} \mathbf{CH}^{S^n}(A,B)^{\otimes 2} \arrow[r, "\boldsymbol{\lambda}^k \otimes \boldsymbol{\lambda}^k"] \arrow[d, "\cup_{S^n}"'] & \mathbf{CH}^{S^n}(A,B)^{\otimes 2} \arrow[d, "\cup_{S^n}"] \\ \mathbf{CH}^{S^n}(A,B) \arrow[r, "\boldsymbol{\lambda}^k"] & \mathbf{CH}^{S^n}(A,B) \end{tikzcd}$$

is commutative in $\mathbb{E}_1-\mathbf{Alg}$ *and* $\boldsymbol{\lambda}^k$ *is a map in the* ∞*-category* E_1*-**Alg**. In particular,*

$$\lambda^k(f) \cup_{S^n} \lambda^k(g) = \lambda^k(f \cup_{S^n} g)$$

for all $f, g \in HH^{S^n}(A,B)$*.*

Recall [BW] that this result is false for $d=1$.

Proof. Let us recall that the power map φ^k is the composition $(pinch^{(k)}_{(n)})^* \circ (\bigvee_{i=1}^k id)^*$, see § 4.4 and identity (4.22). The, by Theorem 4.17, it is enough to prove the statement for φ^k instead of λ^k. Thus the statement is implied by the fact that the following diagrams (6.16), (6.17) and (6.18) are homotopy commutative when $*=0$.

The diagram

$$\begin{tikzcd}
C_*\big(\mathcal{C}_n(r)\big) \otimes \big(CH^{\Delta_\bullet(S^n)}(A,B)\big)^{\otimes r} \arrow[r, "(\mu_\vee)^{(r-1)}"] \arrow[d, "id \bigotimes ((\bigvee_{i=1}^k id)^*)^{\otimes r}"'] & C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\bigvee_{i=1}^r \Delta_\bullet(S^n)}\big(A,B\big) \arrow[d, "id \otimes \bigvee_{i=1}^r (\bigvee_{i=1}^k id)^*"] \\
C_*\big(\mathcal{C}_n(r)\big) \otimes \Big(CH^{\Delta_\bullet(\bigvee_{i=1}^k S^n)}(A,B)\Big)^{\otimes r} \arrow[r, "(\mu_\vee)^{(r-1)}"] \arrow[d, "id \otimes (pinch^{(k)\,*}_{(n)})^{\otimes r}"'] & C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\bigvee_{i=1}^r \Delta_\bullet(\bigvee_{i=1}^k S^n)}\big(A,B\big) \arrow[d, "id \otimes \bigvee_{i=1}^r \Delta_\bullet (pinch^{(k)}_{(n)})^*"] \\
C_*\big(\mathcal{C}_n(r)\big) \otimes \big(CH^{\Delta_\bullet(S^n)}(A,B)\big)^{\otimes r} \arrow[r, "(\mu_\vee)^{(r-1)}"] & CH^{\bigvee_{i=1}^r \Delta_\bullet(S^n)}\big(A,B\big)
\end{tikzcd} \tag{6.16}$$

is homotopy commutative by naturality of the wedge product (Corollary 5.3) with respect to the two maps of topological spaces $\bigvee_{i=1}^k id : \bigvee_{i=1}^k S^n \to S^n$ and $pinch^{(k)}_{(n)} : S^n \to \bigvee_{i=1}^k S^n$. It is in fact strictly commutative at the dg-level by naturality of Hochschild cochains.

The diagram

$$\begin{tikzcd}
C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\bigvee_{i=1}^r \Delta_\bullet(S^n)}\big(A,B\big) \arrow[r, "j^*"'] \arrow[d, "id \otimes \bigvee_{i=1}^r (\bigvee_{i=1}^k id)^*"'] & CH^{\Delta_\bullet\left(\bigvee_{i=1}^r S^n\right)}\big(A,B\big) \arrow[d, "id \otimes \bigvee_{i=1}^r (\bigvee_{i=1}^k id)^*"] \\
C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\bigvee_{i=1}^r \Delta_\bullet(\bigvee_{i=1}^k S^n)}\big(A,B\big) \arrow[r, "j^*"'] \arrow[d, "id \otimes \bigvee_{i=1}^r \Delta_\bullet (pinch^{(k)}_{(n)})^*"'] & C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\Delta_\bullet\left(\bigvee_{i=1}^r \left(\bigvee_{i=1}^k S^n\right)\right)}\big(A,B\big) \arrow[d, "id \otimes \bigvee_{i=1}^r (\bigvee_{i=1}^k id)^*"] \\
CH^{\bigvee_{i=1}^r \Delta_\bullet(S^n)}\big(A,B\big) \arrow[r, "j^*"'] & C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\Delta_\bullet\left(\bigvee_{i=1}^r S^n\right)}\big(A,B\big)
\end{tikzcd} \tag{6.17}$$

is also strictly commutative by direct inspection and definition of the map j^*, see § 5.2.2, map (5.27).

The upper square of the following diagram

$$\begin{tikzcd}
CH^{\Delta_\bullet\left(\bigvee_{i=1}^r S^n\right)}(A,B) \arrow[r, "pinch^*"] \arrow[d, "id\otimes\bigvee_{i=1}^r(\bigvee_{i=1}^k id)^*"'] & CH^{\Delta_\bullet(S^n)}(A,B) \arrow[d, "(\bigvee_{i=1}^k id)^*"] \\
C_*\big(\mathcal{C}_n(r)\big)\otimes CH^{\Delta_\bullet\left(\bigvee_{i=1}^r\left(\bigvee_{i=1}^k S^n\right)\right)}(A,B) \arrow[r, "\left(\bigvee_{i=1}^k pinch\right)^*"] \arrow[d, "id\otimes\bigvee_{i=1}^r(\bigvee_{i=1}^k id)^*"'] & CH^{\Delta_\bullet\left(\bigvee_{i=1}^k S^n\right)}(A,B) \arrow[d, "pinch_{(n)}^{(k)\,*}"] \\
C_*\big(\mathcal{C}_n(r)\big)\otimes CH^{\Delta_\bullet\left(\bigvee_{i=1}^r S^n\right)}(A,B) \arrow[r, "pinch^*"] & CH^{\Delta_\bullet(S^n)}(A,B).
\end{tikzcd} \tag{6.18}$$

is also strictly commutative because it is induced by a commutative diagram of topological spaces: on each sphere of the bouquet $\bigvee_{i=1}^k S^n$, $\bigvee_{i=1}^k \circ \bigvee_{i=1}^k pinch$ is just the pinching map $pinch$. the lower square is however not strictly commutative. But since $n>1$, for elements in $C_0(\mathcal{C}_n(r)$, it is homotopy commutative (for a pointed homotopy) because the diagonal $S^n \to \bigvee S^n$ is homotopy commuative (S^n is a E_n-coalgebra in pointed spaces). This is the only place where we need the $n>1$ assumption (and is false without it).

Since the ∞-category E_1**-Alg** and the one of A_∞-algebras (with A_∞-morphisms) are equivalent, the above analysis also shows that to prove that $\boldsymbol{\lambda}^k$ is a map in the ∞-category E_1**-Alg**, it is sufficient to to prove that the pinching map $pinch : S^n \to S^n \bigvee S^n$ extends an A_∞-coalgebra map in $(\mathsf{Top}*, \vee)$. The pinching map extends an an E_2-coalgebra structure on $(\mathsf{Top}*, \vee)$. By the ∞-categorical Dunn Theorem [Lu3], it thus extends as an object of E_1**-coAlg**(E_1**-Coalg**) hence into an homotopy E_1-coalgebra map. Thus λ^k is an homotopy E_1-algebra map for $d>1$. $\square$

The cup-product is in fact part of an homotopy $Pois_{d+1}$-algebra structure on Hochschild cochains (as we have seen in § 5.2.1 and 5.3 since, for $n\geq 1$, the operad of little cubes is formal with cohomology the operad $Pois_n$ of $Pois_n$-algebras). We can actually use the HKR theorem to describe this structure very explicitly. Recall that if A is a dg-algebra, then its derivation $Der(A,A)$ is a dg-Lie algebra. Then for any integer d,

$$Sym_A(Der(A,A)[-d])$$

endowed with the symmetric algebra multiplication and differential induced by the one of A is a dg-$Pois_{d+1}$-algebra. Its Lie bracket is just the Lie bracket of derivations extended to the whole symmetric algebra by the graded Leibniz rule.

Lemma 6.13 *Let $(Sym(V),d)$ be a quasi-free cdga. Assume V is finitely generated in each degree. There is an isomorphism of dg-$Pois_{d+1}$-algebras*

$$Sym(V\oplus V^\vee[-d]) \cong Sym_{Sym(V)}(Der(Sym(V),Sym(V))[-d])$$

where the shifted Lie bracket on the left hand-side is induced by the pairing $V\otimes V^\vee[-d]\to k[-d]$ and the Leibniz rule.

Proof. By freeness of the algebra structure, one has $Der(Sym(V), M) \cong M \otimes_{Sym(V)} Sym(V) \otimes V^\vee$ as a differential graded $Sym(V)$-module. $\square$

For any cdga A and resolution $(Sym(V), d) \xrightarrow{\cong} A$, we have canonical quasi-isomorphisms

$$\begin{aligned} Sym_A(\mathbb{R}Der(A,A)[-d]) &\xleftarrow{\cong} Sym_{Sym(V)}(\mathbb{R}Der(Sym(V), Sym(V))[-d]) \\ &\cong Sym_{Sym(V)}(Der(Sym(V), Sym(V))[-d]) \\ &\cong Sym(V \oplus V^\vee[-d]). \end{aligned} \tag{6.19}$$

From § 6.1 we know that the above equivalence computes the Hodge decomposition of higher order Hochschild; it also computes its algebraic structure:

Theorem 6.14 *Let A be a cdga with finite type cohomology in each degree over a ring of characteristic zero and assume $d > 1$.*

1. *There is an equivalence*

$$\mathbf{CH}^{S^d}(A,A) \cong Sym_A(\mathbb{R}Der(A,A)[-d])$$

of graded dg-$Pois_{d+1}$-algebras where the grading is given by the Hodge grading in Hochschild cohomology such that, if $(Sym(V), d) \xrightarrow{\cong} A$ is a resolution one has an equivalence

$$\mathbf{CH}^{S^d}(A,A) \xleftarrow{\cong} Sym\big(V \oplus (V^\vee)[-d]\big)$$

of graded P_{d+1}-algebras. The underlying multiplicative on the left hand side is the cup-product and the $Pois_{d+1}$-structure on the right hand side is the Schouten one of Lemma 6.13

2. *There is a quasi-isomorphism of P_{d+1}-algebras*

$$\mathbf{CH}^{S^d}(A,A) \cong C^\bullet_{Pois_d}(A,A)$$

where $C^\bullet_{Pois_d}(A,A)$ is the $Pois_{n+1}$-deformation/cohomology complex of A viewed as a $Pois_d$-algebra with zero bracket, endowed with Tamarkin's $Pois_{d+1}$-structure [Ta].

Remark 6.15 The first part of Theorem 6.14 indeed defines an homotopy invariant $Pois_{d+1}$-structure on higher Hochschild cochains lifting the cup-product. It was first stated in [Gi3]. The second part identify the structure with Tamarkin's celebrated deformation structure of the cdga seen as a $Pois_d$-algebra with zero bracket.

Proof. We first consider the first statement for the underlying algebra structure. Below, we consider the normalized cochain complex but use the usual notation for

Hochschild cochains. Let $q : (Sym(V), d) \xrightarrow{\cong} A$ be a resolution. Then A is a $Sym(V)$-CDGA so that we have a quasi-isomorphism of E_1-algebras

$$CH^{S^d_\bullet}(Sym(V), Sym(V)) \xrightarrow[q_*]{\cong} CH^{S^d_\bullet}(Sym(V), A) \xleftarrow[q^*]{\cong} CH^{S^d_\bullet}(A, A)$$

which proves that the cup-product is invariant under equivalences of CDGAs. Hence it is enough to assume A is cofibrant. Now, the Hochschild-Kostant-Rosenberg equivalence gives us the chain map

$$\pi^*_{HKR} : Sym(V \oplus V^\vee[-d]) \xrightarrow{\cong} CH^{S^d_\bullet}(Sym(V), Sym(V)).$$

Now, let $\varepsilon^d_{HKR} : (Sym(V \oplus V[d]), 0) \longrightarrow CH_{S^d_\bullet}(Sym(V))$ be the map defined as in Lemma 6.7. That is, the map induced by the $Sym(V)$ linear commutative algebra map defined on a generator $s_d(v) \in V[d]$ by

$$\varepsilon^d_{HKR}(s_d(b)) := \frac{1}{d!} \sum_{\sigma \in \Sigma_d} (-1)^\sigma E_{\sigma^{-1}(1), \ldots, \sigma^{-1}(d)}(b).$$

This map is well-defined since the right hand side is always a cocycle with respect to the simplicial differential $\sum \pm (d_r)_*$ where d_r are the face maps $S^d_d \to S^d_{d-1}$ of the simplicial structure. Hence it is a chain map if we forget the differential of $Sym(V)$. Applying the duality functor $Hom_{S(V)}(-, S(V))$ and Lemma 6.7, we obtain that $(\varepsilon^d_{HKR})^* : CH_{S^d_\bullet}(Sym(V)) \longrightarrow (Sym(V \oplus V^\vee[-d]), 0)$ is a quasi-inverse of π_{HKR} in the case where the differential is null. The formula for the cup-product (Lemma 5.9) shows that $(\varepsilon^d_{HKR})^*$ is a dg-associative algebra map since we are working with reduced cochains (so that in the shuffle formula defining $\varepsilon^d_{HKR}(a_1, \ldots, a_n)$, only terms putting the $E_{\sigma^{-1}(1), \ldots, \sigma^{-1}(d)}(a_j)$ as a diagonal block matrix, up to a permutation of the diagonal blocks, are non-zero after applying $f_1 \cup_{S^d_\bullet} \cdots \cup_{S^d_\bullet} f_j$). Hence, in the zero case we obtain that both ε^d_{HKR} and therefore its quasi-inverse π^*_{HKR} are A_∞-algebras quasi-isomorphisms.

Returning to the general case, we have that $\pi^*_{HKR} : Sym(V \oplus V^\vee[-d]) \xrightarrow{\cong} CH^{S^d_\bullet}(Sym(V), Sym(V))$ is in fact a graded complex with respect to the grading induced by its Hodge decomposition; in other words with $V^\vee[-d]$ in weight 1. Since the map π^*_{HKR} is $Sym(V)$-linear and preserves the Hodge decomposition (by Corollary 6.8 and Corollary 6.3), this grading is given by the Hodge decomposition on Hochschild cochains. At the page E_1 of this spectral sequence, the quasi-isomorphism π_{HKR} is an homotopy dg-associative algebra map by the previous case; indeed, the map $(\varepsilon^d_{HKR})^*$ is still well defined (since the differential on $V^\vee$ is trivial) and the rest of the argument is strictly the same. It is thus also the case for the associated graded with respect to the Hodge filtration. By Lemma 4.24, we have that the operations φ^k of the γ-ring structure are A_∞-maps, hence so are the projections $e^{(n)}$ on the weight n-pieces of the decomposition. It follows that the quotient map from the complex to its associated graded is also a A_∞-quasi-isomorphism. Hence $CH^{S^d_\bullet}(Sym(V), Sym(V)) \xrightarrow{\pi^*_{HKR}} Sym(V \oplus V^\vee[-d])$ lifts to an A_∞-quasi-isomorphism.

To finish the first claim, using the Leibniz rule, it is now sufficient to prove that the underlying shifted Lie structure on $Sym(V \oplus V^{\vee}[-d])$ induced by the pairing $V \otimes V^{\vee} \to k$ is (homotopy) invariant under the choice of resolutions. This is a consequence of a general statement for André-Quillen cochain complex: for two cofibrant and quasi-isomorphic cdgas, $Der(A, A)$ and $Der(B, B)$ are quasi-isomorphic dg Lie algebras, see [BL, Theorem 2.8] which by Lemma 6.13 is equivalent to our claim.

The second claim is a consequence of the first claim and the formality statement of Calaque-Willwacher [CW]. In fact, the proof of Theorem 2 in [CW] shows that there is an equivalence

$$C^{\bullet}_{Pois_d}(A, A) \cong Sym_A(\mathbb{R}Der(A, A))$$

of P_{d+1}-algebras. The equivalence as chain complexes was an immediate consequence of Proposition 3.25 and the identification between Hochschild and Poisson cochain complexes in [GY]. $\square$

The above proof of part of the theorem relies on a formality theorem of [CW]. It seems also possible to use the combinatorics of Adams operation to get a proof of this equivalence. This requires identifying the image of π_{HKR} as living in $Sym(e^{(d)})$ and to identify (by an explicit quasi-isomorphism) the latter with the convolution of symmetric sequence $Com\{d\} \circ Lie(-)$.

Remark 6.16 Theorem 6.14 is known not to be true for $d = 1$. Namely, the HKR quasi-isomorphism

$$\mathbf{CH}^{S^1}(A, A) \cong Sym_A(\mathbb{R}Der(A, A)[-1])$$

is not an equivalence of algebras in general (though it is for smoth algebras concentrated in degree 0). In that case, the Gerstenhaber structure is only filtered with respect to the Hodge decomposition. See [BW].

6.4 Applications to $Pois_n$-algebras (co)homology

We use the Hodge decomposition to identify a spectral sequence computing the Homology of $Pois_n$-algebras.

We write $Pois_n$ and $uPois_n$ the operads encoding respectively the non-unital and unital differential graded Poisson$_n$-algebras, that is dg- commutative algebras endowed with a cohomological degree $1 - n$ Lie bracket satisfying the Leibniz rule, that is, the Lie bracket is a graded derivation with respect to each variable (for instance see [Fr1, Fr4, Ta, CW]). Finally, let us denote $H^{s+t}_{Pois_n}(R, M)$ and $H^{Pois_n}_{s+t}(R, M)$ for the (co)homology groups of $Pois_n$-algebras, see [CW, Ta, Fr4, Fr1] for the precise definitions.

For ordinary Poisson algebras, that is $Pois_n1$-algebras in our convention, there is another ad hoc definition of Poisson (co)homology (different from the operadic ones in general). We now introduce a higher version of this Poisson (co)homology groups that we call higher Lichnerowicz Poisson (co)homology groups. The shifted Lie algebra structure on R induces a Lie algebra structure on $Der(R, R)[1 - n]$

The *higher Lichnerowicz cochains of a $Pois_n$-algebra* is the cochain complex

$$C^{\bullet}_{\mathcal{LP}_n}(R,M) := Sym_R\big(\operatorname{Der}(R,M)[-n]\big)$$

with differential given by the sum of the internal differential of R and M and the differential

$$\partial_{\mathcal{LP}_n}(F)(r_0\dots,r_n) = \sum_{i=1}^{n}(-1)^{in}\pm[r_i,F(r_0,\dots\hat{r_i},\dots,r_n)] \\ +\sum_{i<j}(-1)^{n(i+j)}\pm F(r_0,\dots[r_i,r_j],\dots r_n) \quad (6.20)$$

where $[-,-]$ is the (degree $1-n$) Lie bracket or action and the $\pm$ signs are given by the Koszul-Quillen rule with respect to the permutations of the graded elements r_i (in the internal gradings of R, M). This differential is thus analogue of Chevalley-Eilenberg cochain complex [38] and the proof it squares to zero is the same.

Similarly we define the *higher Lichnerowicz chains of a $Pois_n$-algebra* as the chain complex

$$C_{\bullet}^{\mathcal{LP}_n}(R,M) := Sym_R\big(M\otimes_R \Omega^1(R)[n]\big)$$

with differential given by the sum of the internal differential of R and M and the differential

$$\partial^{\mathcal{LP}_n}(m\otimes d(r_1)\cdots\otimes d(r_n)) = \sum_{i=1}^{n}(-1)^{ni}\pm[m,r_i]\otimes d(r_1)\cdots\widehat{d(r_i)}\cdots\otimes d(r_n) \\ +\sum_{i<j}(-1)^{n(i+j)}\pm m\otimes d(r_1)\cdots d([r_i,r_j])\cdots\otimes d(r_n). \quad (6.21)$$

The following result is a generalization of (the dual of) a result of Fresse [Fr1]. Ideas related to this spectral sequence also appeared in [RZ].

Corollary 6.17 *Let R be a $Pois_n$-algebra, with $n\geq 1$ and M be a R-module.*

1. *There are weakly convergent spectral sequences*

$$E_1^{s,t} = HH_{(s)}^{S^n,s+t}(R,M) \Longrightarrow \widehat{H}^{s+t}_{Pois_n}(R,M)$$

$$E_1^{s,t} = HH^{(s)}_{S^n,s+t}(R,M) \Longrightarrow \widehat{H}_{s+t}^{Pois_n}(R,M)$$

where on the left R is simply considered as a commutative algebra.

2. *The cohomology (resp. homology) spectral sequence converges strongly to $H^{s+t}_{Pois_n}(R,M)$ if R,M are concentrated in non-negative degrees (resp. to $H_{s+t}^{Pois_n}(R,M)$ if R, M are concentrated in non-positive degrees) Poisson (co)homology groups.*

38. for the Lie algebroid $\Omega^1(R)[1-n]$ deduced from the Lie algebra structure on $R[1-n]$

3. *If $M = R$ and $n > 1$, then the first spectral sequence is a spectral sequence of (graded) $Pois_{n+1}$-algebras.*
4. *There is a chain map from higher Lichnerowicz cochain complex to the $Pois_n$-cochain complexes which is a quasi-isomorphism if R is smooth seen as a commutative algebra.*

   Similarly, there is a chain map from the $Pois_n$-chain complexes to the Lichnerowicz chain complex which is a quasi-isomorphism if R is smooth seen as a commutative algebra.

PROOF. We do the case of cohomology. The homology one is similar. When $M = R$, the complex computing the $Pois_n$-cohomology of R is the underlying chain complex of

$$C^*_{Pois_n}(R,R) := Hom_\Sigma(uPois_n{}^*\{n\}, End_R)[-n] \tag{6.22}$$

where Hom_Σ means the morphisms of symmetric sequences. Here $(-)^*$ is the linear dual, $\{n\}$ is the operadic n-iterated suspension and End_R is the endomorphism operad of R ([CW, Ta]). This complex shifted by n is the convolution between an operad and a cooperad and inherits a dg-Lie algebra structure. The $Pois_n$-algebra structure defines a Maurer-Cartan element in it and the (operadic) $Pois_n$-cohomology is precisely the n-desuspension of this dg-Lie algebra with differential twisted by this Maurer-Cartan element. By construction, this chain complex is computed by the Koszul resolution of the $Pois_n$-operad. And we recall that

$$Pois_n = Com \circ Lie\{n-1\}$$

is the operad given by a distributive law between the operads of commutative algebras and the one of (shifted) Lie algebras. In particular, the later n-Poisson cochain complex $C^*_{Pois_n}(R,R)$ is naturally a filtered complex (by the arity)see [Gi1, DP]; we denote $C^{\bullet,\bullet}_{Pois_n}(R,R)$ the underlying graded object. The filtration of this complex yields a spectral sequence concentrated in the upper half-plane, see [Gi1, Théorème 4.3], hence the convergence condition [We], where

$$\widehat{H}^n_{Pois_n}(R,M) \cong H^n\big(\prod_{s+t=n} C^{p,q}_{Pois_n}(R,M)\big).$$

Since $n > 0$, if $R^{<0} = 0$, then the spectral sequence lies in the first quadrant hence the stronger convergence condition.

By *loc. cit.* the page E_1 computes the Poisson cohomology of R endowed with the zero bracket. In other words, the $Pois_n$-cohomology of the underlying commutative algebra structure of R. We can then use the main result of [CW] (or the computations in [GY, Gi1]) to compute the page E_1. It is isomorphic to $Sym_R(\mathbb{R}Der(R,R)[-n])$ as a graded (with respect to the spectral sequence filtration on one side and symmetric powers filtration in the other side) $Pois_{n+1}$-algebra for $n > 1$. Then using Theorem 6.14. we can identify it with $HH^{S^n,\bullet}(R,R)$ and the symmetric power with the weight of the Hodge decomposition in higher Hochschild cohomology. For $n = 1$, we have the same result but as a module only applying

Theorem 6.3 and Calaque-Willwacher formality theorem [CW]. Taking an arbitrary module amounts to replace $\mathbb{R}Der(R,R)$ by $\mathbb{R}Der(A,M)$ above and we only get a quasi-isomorphism of modules again. This proves the first three claims.

For the last claim, the natural map $Der(R,M) \hookrightarrow \mathbb{R}Der(R,M)$ induces a graded vector space map

$$C^\bullet_{\mathcal{LP}_n}(R,M) = Sym_R\big(\operatorname{Der}(R,M)[-n]\big) \hookrightarrow Sym_R(\mathbb{R}Der(R,M)[-n]) \cong C^*_{Pois_n}(R,M)$$

where the right hand side is identified with symmetric powers of the Harrison cochain complex. More precisely this map is realized as the canonical map

$$\begin{aligned} Sym_R\big(\operatorname{Der}(R,R)[-n]\big) &\hookrightarrow Hom_\Sigma(uCom^*\{n\}, End_R) \\ &\hookrightarrow Hom_\Sigma(uPois_n{}^*\{n\}, End_R) \end{aligned} \tag{6.23}$$

where the first map is given by those elements of the convolution Lie algebra $Hom_\Sigma(uCom^*\{n\}, End_R)$ which take values in derivation in each slot. From there it follows by direct inspection that this map is a chain map. When R is smooth, then the map $Der(R,M) \hookrightarrow \mathbb{R}Der(R,M)$ is a quasi-isomorphism, which proves, by the claim 1 and 2 that the induced map

$$C^\bullet_{\mathcal{LP}_n}(R,M) = Sym_R\big(\operatorname{Der}(R,M)[-n]\big) \hookrightarrow Sym_R(\mathbb{R}Der(R,M)[-n])$$

is a quasi-isomorphism as well at the page 2 of the spectral sequence. Hence claim 4 follows. $\square$

Example 6.18 If $\mathfrak{g}$ is a (dg-)Lie algebra, then $U_n(\mathfrak{g}) = S(\mathfrak{g}[1-n])$ is a $Pois_n$-algebra, which is precisely the E_n-enveloping algebra of $\mathfrak{g}$ see [Kn, FG, CG]. Then Corollary 6.17.4 and Corollary 6.8 yield

$$H^*_{Pois_n}(U_n(\mathfrak{g}), U_n(\mathfrak{g})) \cong H^*_{CE}\big(\mathfrak{g}, U_n(\mathfrak{g})\big).$$

7 Applications to Brane topology

In this section we give an important application of Hochschild theory for spaces in manifold topology and more precisely to show that higher brane topology is compatible with the power maps.

7.1 Higher Hochschild (co)homology as a model for mapping spaces

First, the Hochschild cochains over spaces yields an algebraic model for (sufficiently connected) mapping spaces. In characteristic zero, the relationship is materialized by Chen iterated integrals and is thus highly explicit. We follow the approach described in [GTZ]. Let M be a compact, oriented manifold, and denote by $\Omega_{dR} = \Omega^\bullet_{dR}(M)$ the space of differential forms on M and let $Y_\bullet$ be a simplicial set with geometric realization $Y := |Y_\bullet|$. Denote by $M^Y := \mathrm{Map}_{sm}(Y, M)$ the space of continuous maps from Y to M, which are smooth on the interior of each simplex $Image(\eta(i)) \subset Y$. Chen [Ch, Definition 1.2.1 and 1.2.2] gave a differentiable structure on M^Y specified by sets of plots $\phi : U \to M^Y$, where $U \subset \mathbb{R}^n$ for some n. Plots are those maps whose adjoint $\phi_\sharp : U \times Y \to M$ is continuous on $U \times Y$, and moreover, are smooth on the restriction to the interior of each simplex of Y. Then, one defines a p-form $\omega \in \Omega^p_{dR}(M^Y)$ on M^Y as a collection of p-forms $\omega_\phi \in \Omega^p_{dR}(U)$ (one form for each plot $\phi : U \to M^Y$), which is required to be invariant with respect to smooth transformations of the domain.

Recall that the adjunction between simplicial sets and topological spaces gives, for any simplicial structure of $Y_\bullet$, the simplicial map $\eta\colon Y_\bullet \to S_\bullet|Y_\bullet|$. It is given for $i \in Y_k$ by maps $\eta(i) : \Delta^k \to Y$ in the following way,

$$\eta(i)(t_1 \le \cdots \le t_k) := [(t_1 \le \cdots \le t_k) \times \{i\}] \in \left(\coprod \Delta^\bullet \times Y_\bullet / \sim\right) = Y.$$

From the map η, we can define, for any plot $\phi : U \to M^Y$, a map $\rho_\phi := ev \circ (\phi \times id)$,

$$\rho_\phi : U \times \Delta^k \xrightarrow{\phi \times id} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}, \tag{7.1}$$

where ev is defined as the evaluation map,

$$ev(\gamma : Y \to M, t_1 \le \cdots \le t_k) = \big(\ldots, \big(\gamma \circ \eta(i)\big)(t_1 \le \cdots \le t_k), \ldots, \big)_{i \in Y_k}. \tag{7.2}$$

Now, if we are given forms $a_0, \ldots, a_{y_k} \in \Omega = \Omega^\bullet_{dR}(M)$ on M (one for each element in the set Y_k), , or more precisely a form $a_0 \ldots a_{y_k} \in \big(\Omega_{dR}(M)\big)^{\otimes Y_k}$, the pullback $(\rho_\phi)^*(a_0 \otimes \cdots \otimes a_{y_k}) \in \Omega^\bullet(U \times \Delta^k)$, may be integrated along the fiber Δ^k, and is denoted by

$$\left(\int_{\mathcal{C}} a_0 \ldots a_{y_k}\right)_\phi := \int_{\Delta^k} (\rho_\phi)^*(a_0 \otimes \cdots \otimes a_{y_k}) \quad \in \Omega^\bullet_{dR}(U).$$

The resulting $p = (\sum_i deg(a_i) - k)$-form $\int_{\mathcal{C}} a_0 \ldots a_{y_k} \in \Omega^p_{dR}(M^Y)$ is called the (generalized) *iterated integral* of $a_0, \ldots, a_{y_k}$. The subspace of the space of De Rham forms $\Omega^\bullet(M^Y)$ generated by all iterated integrals is called the space of Chen (generalized) *iterated integrals* $\mathcal{C}hen(M^Y)$ of the mapping space M^Y. In short, we may picture an iterated integral as the pullback composed with the integration along the fiber Δ^k of a form in M^{Y_k},

$$M^Y \xleftarrow{\int_{\Delta^k}} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}$$

Definition 7.1 We define $\mathcal{I}t^{Y_\bullet}_M : CH_{Y_\bullet}(\Omega) \cong \Omega^{\otimes Y_\bullet} \to \mathcal{C}hen(M^Y)$ by

$$\mathcal{I}t^{Y_\bullet}_M(a_0 \otimes \cdots \otimes a_{y_k}) := \int_{\mathcal{C}} a_0 \ldots a_{y_k}. \tag{7.3}$$

The de Rham algebra functor can be extended over $\mathbb{Q}$ and for any topological space by the Sullivan [S1] polynomial de Rham form functor, denoted $\Omega^\bullet : \mathsf{sSet} \longrightarrow \mathsf{CDGA}$. Being invariant under weak-equivalences, this functor canonically gives rise to an ∞-functor $\mathbf{\Omega}^\bullet : \mathbf{Top} \longrightarrow \mathbf{CDGA}$.

Theorem 7.2 ([GTZ]) *The iterated integral map* $\mathcal{I}t_M^{Y_\bullet} : CH_{Y_\bullet}(\Omega^\bullet_{dR}(M)) \to \Omega^\bullet_{dR}(M^Y)$ *is a natural map of CDGAs and lift to an ∞-natural transformation*

$$\mathcal{I}t_{(-)}^{(-)} : \mathbf{CH}_{(-)}(\mathbf{\Omega}^\bullet(-)) \longrightarrow \mathbf{\Omega}^\bullet((-)^{(-)})$$

between ∞-functors $\mathbf{Top}^{op} \times \mathbf{Top} \longrightarrow \mathbf{CDGA}$.

Further, assume that $Y = |Y_\bullet|$ is n-dimensional, i.e. the highest degree of any non-degenerate simplex is n, and assume that the space M is n-connected. Then, $\mathcal{I}t_M^{Y_\bullet}$ is a quasi-isomorphism.

Proof. The first part follows from [GTZ, Lemma 2.2.2], [GTZ, Proposition 2.4.6] and [GTZ, Proposition 2.5.3].
□

Dualizing the construction of iterated integrals, we obtained [GTZ, Corollary 2.5.5],

Corollary 7.3 *Under the assumptions of Theorem 7.2, we have a quasi-isomorphism* $(\mathcal{I}t^{Y_\bullet})^* : C_\bullet(\mathrm{Map}(Y, M)) \to CH^{Y_\bullet}(\Omega, \Omega^*)$.

Explicit examples of iterated integrals are described carefully in [GTZ].

Remark 7.4 Theorem 7.2 and Corollary 7.3 have analogs within the E_∞-algebra context as we prove in [GTZ3]. Also see [U] for a recent related proof.

Let us consider our example of main interest : $Y = S^n$. First we note that the power maps (4.8) $\lambda^k : S^n \to S^n$ from Section 4.3 yields by precomposition and functoriality maps

$$\lambda^{k\,*}_{\ *} : C_\bullet(X^{S^n}) \to C_\bullet(X^{S^n}). \tag{7.4}$$

From identity (4.11) and functoriality of chains and forms, we immediately get

Lemma 7.5 *The maps* $((\lambda^k)^*_*)$ *makes* $C_\bullet(X^{S^n})$ *a γ-ring with trivial multiplication and the maps* $(\lambda^{k**})_{k\geq 1}$ *makes* $\Omega^\bullet(X^{S^n})$ *a multiplicative γ-ring with trivial multiplication.*

Corollary 7.6 *The iterated integral maps*

$$\mathbf{CH}_{S^n}(\mathbf{\Omega}^\bullet(X)) \longrightarrow \mathbf{\Omega}^\bullet\big(X^{S^n}\big)$$
$$C_\bullet(\mathrm{Map}(S^n, X)) \longrightarrow \mathbf{CH}^{S^n}(\Omega(X), \Omega(X)^\vee)$$

are maps of (multiplicative for the first one) γ-rings with trivial multiplication.

Proof. Since the map $\mathcal{I}t_{(-)}^{(-)} : \mathbf{CH}_{(-)}(\mathbf{\Omega}^\bullet(-)) \longrightarrow \mathbf{\Omega}^\bullet((-)^{(-)})$ is a natural transformation, we have that

$$(\lambda^k)^{**}) \circ \mathcal{I}t_X^{S^n} = \mathcal{I}t_X^{S^n} \circ \boldsymbol{\lambda}^k$$

where $\boldsymbol{\lambda}^k : \mathbf{CH}_Y(A, M) \xrightarrow{(\lambda^k)_*} \mathbf{CH}_Y(A, M)$ is the map (4.15). This is actually already true before passing to the derived category:

$$(\lambda^k)^{**}) \circ \mathcal{I}t_X^{\Delta_\bullet(S^n)} = \mathcal{I}t_X^{\Delta_\bullet(S^n)} \circ \lambda_\bullet^k$$

where $\boldsymbol{\lambda}_\bullet^k$ is defined in the beginning of Section 4.3. This yields the first claim and the second one is obtained by dualizing it and replacing $\Omega(X)$ by its dual for the module coefficient. $\square$

7.2 Models for Brane Topology in characteristic zero

We now apply the previous results on higher Hochschild cochains to give an algebraic models for Brane topology [CV], the analogue of string topology for free spheres spaces. Further, we get *chain level* construction. The maps $((\lambda^k)_*^*)$ makes $C_\bullet(X^{S^n})[\dim(X)]$ a γ-ring with trivial multiplication.

We have seen in the previous Section that the singular chains $C_\bullet(X^{S^n})$ inherits a γ-ring structure gien by the power maps (Lemma 7.5). And so does its shift by $\dim(X)$.

Combining Theorem 7.2, Theorem 6.14 and Theorem 5.18 we arrive to our main result, which extends a result first stated in [Gi3].

Theorem 7.7 *Let X be an n-connected Poincaré duality space, with $n \geq 2$. Then the shifted chain complex $C_\bullet(X^{S^n})[\dim(X)]$ has a canonical E_{n+1}-algebra structure satisfying:*

1. *it induces the sphere product [CV, Section 5]*

$$H_p(X^{S^n}) \otimes H_q(X^{S^n}) \to H_{p+q-\dim(X)}(X^{S^n})$$

in homology when X is an oriented closed manifold;

2. *it is compatible with the γ-ring structure, that is, it makes $C_\bullet(X^{S^n})[\dim(X)]$ into an object of $(\gamma, 0)-E_n-\mathbf{Alg}$ (see Remark 4.7),*
3. *one has a canonical equivalence of graded E_{n+1}-algebras*

$$C_\bullet(X^{S^n})[\dim(X)] \cong Sym_A(\mathbb{R}Der(A, A)[-n])$$

for any cdga model $A \cong \Omega(X)$ of X. Here the additional grading is the Hodge grading associated to the γ-ring structure.

Proof. Let $B \xrightarrow{\simeq} \Omega(X)$ be a cochain model for X. Then we have an induced quasi-isomorphism of B-modules $\Omega(X)^\vee \to B^\vee$; and, by Theorem 7.2, we have an equivalence

$$C_\bullet(\mathrm{Map}(S^n, X)) \xrightarrow{\simeq} CH^{S^n_\bullet}(\Omega(X), \Omega(X)^\vee) \\ \xleftarrow{\simeq} CH^{S^n_\bullet}(B, \Omega(X)^\vee) \xrightarrow{\simeq} CH^{S^n_\bullet}(B, B^\vee). \quad (7.5)$$

We have a similar (though through a longer chain) equivalence for a zigzag between B and $\Omega(X)$. Since X is a Poincaré duality space, there is a quasi-isomorphism of E_∞-modules $\chi_X : C^*(X) \to C_*(X)[\dim(X)] \;\cong\; \big(C^*(X)\big)^\vee[\dim(X)]$. We can take B to a Poincaré duality model for $\Omega(X)$ which can, for instance, be taken from [LS]. That is B is a cdga weakly equivalent to $\Omega(X)$ equipped with a quasi-isomorphism

$$\chi_B : B \xrightarrow{\simeq} B^\vee[\dim(X)]$$

of (symmetric) B-modules inducing the Poincaré duality quasi-isomorphism χ_X. Thus, we also get a weak-equivalence

$$CH^{S^n_\bullet}(B, B) \cong Hom_B\big(CH_{S^n_\bullet}(B), B\big) \\ \xrightarrow{(\chi_B)\circ -} Hom_B\Big(CH_{S^n_\bullet}(B), (B)^\vee\Big)[\dim(X)] \\ \cong CH^{S^n_\bullet}\Big(B), (B)^\vee\Big)[\dim(X)]. \quad (7.6)$$

Composing the string of weak equivalences (7.5) and (7.6) we obtain an equivalence

$$C_\bullet(\mathrm{Map}(S^n, X))[\dim(X)] \;\cong\; \mathbf{CH}^{S^n}(B, B) \quad (7.7)$$

in the ∞-category k-$\mathbf{Mod}$.

On the other hand, we know from Theorem 5.18 that $\mathbf{CH}^{S^n}(B, B)$ has an natural (with respect to weak equivalences of cdgas) structure of homotopy E_{n+1}-algebras, which by Theorem 6.14 is quasi-isomorphic to the E_{n+1}-algebra structure induced by formality by the $Pois_{n+1}$-structure on $Sym_B(\mathbb{R}Der(B, B)[-n])$. By transfer of structure, we thus obtain a canonical structure of homotopy P_{n+1}-algebra structure on $C_\bullet(X^{S^n})[\dim(X)]$. From Theorem 6.14, we know that this equivalence is graded with respect to the Hodge decomposition. The existence of the homotopy P_{n+1}-algebra structure and its compatible γ-ring with trivial multiplication structure now follows. In particular, it makes $Sym_B(\mathbb{R}Der(B, B)[-n])$ and thus $C_\bullet(X^{S^n})[\dim(X)]$ an object of the ∞-category of $(\gamma, 0)-Pois_n-\mathbf{Alg}$ (and actually gives a *strict* model of it). In order to finish to prove 2. and 3., we need to check that this γ-ring structure coming from the Hochschild cohomology side is compatible with the one defined on the singular chains by Lemma 7.5. By Corollary 7.6 we have that the γ-ring structure on chains is equivalent to the one given by the maps $\boldsymbol{\lambda}^k$ (from § 4.3) which by Theorem 6.14 are equivalent to the natural one on $Sym_B(\mathbb{R}Der(B, B)[-n])$. This finishes the proof of 2. and 3. The proof of 1. is completely analogous (and slightly easier) to the one in [GTZ3, Section 7.2], replacing singular cochains $C^\bullet(X)$ by the Poincaré duality model B. $\square$

Remark 7.8 In particular the above Theorem 7.7 is a chain level construction of Brane topology operations. However, we only deal with the non-framed version of the E_{n+1}-structure. We conjecture that the structure given above can be lifted to an action of the framed E_{n+1}-operad as in the sense of Section 5.3. Indeed Corollary 5.25 gives such a result for the underlying centralizer structure (in E_n-algebras) but unfortunately does not imply in any way that this structure lifts to an $O(n+1)$-equivariant one in E_1**-Alg**$(E_n$**-Alg**$)$.

Remark 7.9 Theorem 7.7 does not hold in general for $n = 1$, in the sense that the Hodge decomposition does not make $C_*(LX)[\dim(X)]$ a graded E_2-algebra; not even in homology. It is however filtered. It is proved in [Gi2] (also see [FT]), that

— There is a BV-structure on $HH^*(C^*(X), C^*(X))$ and a compatible γ-ring structure.

— If X is simply connected, there is a **BV**-algebra structure on $\mathbb{H}_*(LX) := H_{*+d}(LX)$ and a compatible γ-ring structure. When X is a manifold the underlying Gerstenhaber structure of the **BV**-structure is the Chas-Sullivan one [CS].

By a BV-structure on a graded space H^* and compatible γ-ring structure we mean the following:

1. H^* is both a BV-algebra and a γ-ring.
2. The BV-operator Δ and the γ-ring maps λ^k satisfy $\lambda^k(\Delta) = k\Delta(\lambda^k)$.
3. There is an "ideal augmentation" spectral sequence $J_1^{pq} \Rightarrow H^{p+q}$ of BV algebras.
4. On the induced filtration J_∞^{p*} of the abutment H^*, one has, for any $x \in J_\infty^{p*}$ and $k \geq 1$,
$$\lambda^k(x) = k^p x \bmod J_\infty^{p+1*}.$$
5. If $k \supset \mathbb{Q}$, there is a Hodge decomposition $H^* = \prod_{i\geq 0} H^*_{(i)}$ (given by the associated graded of the filtration J_∞^{**}) such that the filtered space $\mathcal{F}_p H^* := \bigoplus H^*_{(n\leq p)}$ is a filtered BV-algebra.

We conclude with an example of computation

Example 7.10 (Complex projective space $\mathbb{C}P^m$) The complex projective space $\mathbb{C}P^m$ has a Sullivan model $A = (S(V), d)$ generated by x in degree $|x| = 2$ and y in degree $|y| = 2m+1$ with differential $d(x) = 0$ and $d(y) = x^{m+1}$. Note that A is semi-free and that $\mathrm{Der}(A, A)[n]$ is generated by the derivations $\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell$ given by formulas

$$\alpha_\ell(x) = x^\ell, \qquad \alpha_\ell(y) = 0, \tag{7.8}$$
$$\delta_\ell(x) = 0, \qquad \delta_\ell(y) = x^\ell y. \tag{7.9}$$

The degrees in the (shifted) space of derivations $\mathrm{Der}(A,A)[n]$ are thus

$$\begin{aligned}
|\alpha_\ell| &= 2(\ell-1)+n,\\
|\beta_\ell| &= 2(\ell-1)+(2m+1)+n = 2(\ell+m)+n-1,\\
|\gamma_\ell| &= 2\ell-(2m+1)+n = 2(\ell-m)+n-1,\\
|\delta_\ell| &= 2\ell+n.
\end{aligned}$$

The module relations are :

$$x.\alpha_\ell = \alpha_{\ell+1}, \qquad y.\alpha_\ell = \beta_\ell, \qquad x.\gamma_\ell = \gamma_{\ell+1}, \qquad y.\gamma_\ell = \delta_\ell, \tag{7.10}$$

and the basic bracket relations are

$$[\alpha_0,\alpha_0] = [\alpha_0,\gamma_0] = [\gamma_0,\gamma_0] = 0. \tag{7.11}$$

The differential induced by d on $\mathrm{Der}(A,A)$ becomes $D(\rho) = [\gamma_{m+1},\rho]$, which gives the relations $D(\alpha_\ell) = -(m+1)\gamma_{\ell+m}, D(\beta_\ell) = (m+1)\delta_{\ell+m} + \alpha_{\ell+m+1}, D(\gamma_\ell) = 0, D(\delta_\ell) = \gamma_{\ell+m+1}$. Thus D is defined on $\mathrm{Sym}_A(\mathrm{Der}(A,A)[n])$ by taking

$$D(x)=0, \quad D(y)=x^{m+1}, \quad D(\alpha_0) = -(m+1)\gamma_m = -(m+1)x^m\gamma_0, \quad D(\gamma_0)=0,$$

and extending this to $\mathrm{Sym}_A(\mathrm{Der}(A,A)[n])$ as a graded derivation. In particular we have the following relations

$$\begin{aligned}
&[a.\alpha_0^{\odot p}\odot\gamma_0^{\odot q}, b.\alpha_0^{\odot r}\odot\gamma_0^{\odot s}]\\
&\quad = a[\alpha_0^{\odot p}\odot\gamma_0^{\odot q}, b.\alpha_0^{\odot r}\odot\gamma_0^{\odot s}] + (-1)^{\epsilon_1}[a, b.\alpha_0^{\odot r}\odot\gamma_0^{\odot s}]\odot\alpha_0^{\odot p}\odot\gamma_0^{\odot q}\\
&\quad = a[\alpha_0^{\odot p}\odot\gamma_0^{\odot q}, b]\odot\alpha_0^{\odot r}\odot\gamma_0^{\odot s} + (-1)^{\epsilon_1+\epsilon_2}b[a,\alpha_0^{\odot r}\odot\gamma_0^{\odot s}]\odot\alpha_0^{\odot p}\odot\gamma_0^{\odot q}\\
&\quad = a[\alpha_0^{\odot p}\odot\gamma_0^{\odot q}, b]\odot\alpha_0^{\odot r}\odot\gamma_0^{\odot s} - (-1)^{\epsilon_1+\epsilon_2+\epsilon_3}b[\alpha_0^{\odot r}\odot\gamma_0^{\odot s}, a]\odot\alpha_0^{\odot p}\odot\gamma_0^{\odot q},
\end{aligned} \tag{7.12}$$

$$\left\{\begin{array}{cl}
[x^r\alpha_0^{\odot p}, x^s\alpha_0^{\odot q}] & = (ps-qr)x^{r+s-1}\alpha_0^{\odot(p+q-1)},\\
[x^r y\alpha_0^{\odot p}, x^s y\alpha_0^{\odot q}] & = 0,\\
[x^r\alpha_0^{\odot p}\odot\gamma_0, x^s\alpha_0^{\odot q}\odot\gamma_0] & = 0,\\
[x^r y\alpha_0^{\odot p}\odot\gamma_0, x^s y\alpha_0^{\odot q}\odot\gamma_0] & = 0,
\end{array}\right. \tag{7.13}$$

and

$$\left\{\begin{array}{cl}
[x^r\alpha_0^{\odot p}, x^s y\alpha_0^{\odot q}] & = (ps-qr)x^{r+s-1}y\alpha_0^{\odot(p+q-1)},\\
[x^r\alpha_0^{\odot p}, x^s\alpha_0^{\odot q}\odot\gamma_0] & = (ps-qr)x^{r+s-1}\alpha_0^{\odot(p+q-1)}\odot\gamma_0,\\
[x^r\alpha_0^{\odot p}, x^s y\alpha_0^{\odot q}\odot\gamma_0] & = (ps-qr)x^{r+s-1}y\alpha_0^{\odot(p+q-1)}\odot\gamma_0,\\
[x^r y\alpha_0^{\odot p}, x^s\alpha_0^{\odot q}\odot\gamma_0] & = (ps-qr)x^{r+s-1}y\alpha_0^{\odot(p+q-1)}\odot\gamma_0 + x^{r+s}\alpha_0^{\odot(p+q)},\\
[x^r y\alpha_0^{\odot p}, x^s y\alpha_0^{\odot q}\odot\gamma_0] & = -x^{r+s}y\alpha_0^{\odot(p+q)},\\
[x^r\alpha_0^{\odot p}\odot\gamma_0, x^s y\alpha_0^{\odot q}\odot\gamma_0] & = x^{r+s}\alpha_0^{\odot(p+q)}\odot\gamma_0.
\end{array}\right. . \tag{7.14}$$

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