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On the Hochschild and Harrison (co)homology of C_∞-algebras and applications to string topology

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Abstract. We study Hochschild (co)homology of commutative and associative up to homotopy algebras with coefficient in a homotopy analogue of symmetric bimodules. We prove that Hochschild (co)homology is equipped with λ-operations and Hodge decomposition generalizing the results in [**GS1**] and [**Lo1**] for strict algebras. The main application is concerned with string topology: we obtain a Hodge decomposition compatible with a non-trivial BV-structure on the homology $H_*(LX)$ of the free loop space of a triangulated Poincaré-duality space. Harrison (co)homology of commutative and associative up to homotopy algebras can be defined similarly and is related to the weight 1 piece of the Hodge decomposition. We study Jacobi-Zariski exact sequence for this theory in characteristic zero. In particular, we define (co)homology of relative A_∞-algebras, *i.e.*, A_∞-algebras with a C_∞-algebra playing the role of the ground ring. We also give a relation between the Hodge decomposition and homotopy Poisson-algebras cohomology.

The Hochschild cohomology and homology groups of a commutative and associative k-algebra A, k being a unital ring, have a rich structure. In fact, when M is a symmetric bimodule, Gerstenhaber-Schack [**GS1**] and Loday [**Lo1**] have shown that there are λ-operations $(\lambda^k)_{k\geq 1}$ inducing so-called γ-rings structures on Hochschild cohomology groups $HH^*(A,M)$ and homology groups $HH_*(A,M)$. In characteristic zero, these operations yield a weight-decomposition called the Hodge decomposition whose pieces are closely related to (higher) André-Quillen (co)homology and Harrison (co)homology. These operations have been widely studied for their use in algebra, geometry and their intrinsic combinatorial meaning.

The Hochschild (co)homology of the singular cochain complex of a topological space is a useful tool in algebraic topology and in particular in string topology. In fact, Chas-Sullivan [**CS**] have shown that the (shifted) homology $H_{*+d}(LM)$, where $LM = \mathrm{Map}(S^1, M)$ is the free loop space of a manifold M of dimension d, is a Batalin-Vilkovisky-algebra. In particular, there is an associative graded commutative operation called the loop product. When M is simply connected, there is an isomorphism $H_{*+d}(LM) \cong HH^*(C^*(M), C^*(M))$ which, according to

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Cohen-Jones [**CJ**], identifies the cup-product with the loop product. Alternative proofs of this isomorphism have also been given by Merkulov [**Mer**] and Félix-Thomas-Vigué [**FTV2**]. This isomorphism is based on the isomorphism $H_*(LX) \cong HH^*(C^*(X), C_*(X))$, where X is a simply connected space, and the fact that the Poincaré duality should bring a "homotopy isomorphism" of bimodules $C_*(X) \to C^*(X)$. Since $HH^*(C^*(X), C^*(X))$ is a Gerstenhaber algebra, it is natural to define string topology operations for Poincaré duality topological space X using the Hochschild cohomology of their cochain algebra $C^*(X)$. To achieve this, one needs to work with homotopy algebras, homotopy bimodules and homotopy maps between these structures, even in the most simple cases. This was initiated by Sullivan and his students, see Tradler and Zeinalian papers [**Tr2, TZ, TZ2**]. For example, they show that for nice enough spaces, $HH^*(C^*(X), C^*(X))$ is a BV-algebra.

In fact the cochain complex $C^*(X)$ is "homotopy" commutative since the Steenrod $\cup_1$-product gives a homotopy for the commutator $f \cup g - g \cup f$. This fact motivates us to study Hochschild (co)homology of commutative up to homotopy associative algebras (C_∞-algebras for short) in order to add λ-operations to the string topology picture for nice enough Poincaré duality spaces. These λ-operations have to be somehow compatible with the other string topology operations. We achieve this program in Section 5. In particular we prove that if X is a triangulated Poincaré duality space, the Hochschild cohomology of its cochain algebra is a BV-algebra equipped with λ-operations commuting with the BV-differential and filtered with respect to the product, see Theorem 5.7.

Besides string topology there are other reasons to study C_∞-algebras and cohomology theories associated to their deformations, i.e. Hochschild and Harrison. Actually, associative structures up to homotopy (A_∞-structures for short), introduced by Stasheff [**St**] in the sixties, have become more and more useful and popular in mathematical physics as well as algebraic topology. A typical situation is given by the study of a chain complex with an associative product inducing a graded commutative algebra structure on homology. The quasi-isomorphism class of the algebraic structure usually retains more information than the homology. In many cases, it is possible to enforce the commutativity of the product at the chain level at the price of relaxing associativity. For instance, in characteristic zero, according to Tamarkin [**Ta**], the Hochschild cochain complex $C^*(A, A)$ of any associative algebra A has a C_∞-structure. The same is true for the cochain algebra of a space [**Sm**], also see Lemma 5.7 below. It is well-known that these structures retain more information on the homotopy type of the space than the associative one, for instance see [**Ka**]. Moreover the commutative, associative up to homotopy algebras are quite common among the A_∞-ones and deeply related to the theory of moduli spaces of curves [**KST**]. In fact an important class of examples is given by the formal Frobenius manifolds in the sense of Manin [**Ma**].

In this paper, we study Hochschild (co)homology of C_∞-algebras with value in general bimodules. The need for this is already transparent in string topology, notably to get functorial properties. Note that we work in characteristic free context (however usually different from 2) in order to have as broad as possible homotopy applications. In particular we do not restrain ourself to the rational homotopy framework. In characteristic zero, a similar approach (but different application) to string topology has been studied by Hamilton-Lazarev [**HL**], see Section 6 for details.

To construct λ-operations, we need to define homotopy generalizations of symmetric bimodules. Notably enough, the appropriate symmetry conditions are not the same for homology and cohomology. These "homotopy symmetric"structures are called C_∞-bimodules and C_∞^{op}-bimodules structures respectively. We define and study λ-operations and Hodge decomposition for Hochschild (co)homology of C_∞-algebras. In particular, the λ-operations induce an augmentation ideal spectral sequence yielding important compatibility results between the Gerstenhaber algebra and γ-ring structures in Hochschild cohomology. In characteristic zero, the Hodge decomposition of Poisson algebras is related to Poisson algebras homology [**Fr1**]. We generalize this result in the homotopy framework.

We study Harrison (co)homology of C_∞-algebras and prove that, if the ground ring k contains the field $\mathbb{Q}$ of rational numbers, the weight 1 piece of the Hodge decomposition coincides with Harrison (co)homology. For strictly commutative algebras, the result is standard [**GS1, Lo1**].

It is well-known that, in characteristic zero, for unital flat algebras, Harrison (co)homology coincides with André-Quillen (co)homology (after a shift of degree). In that case, a sequence $K \to S \to R$ gives rise to of the change-of-ground-ring exact sequence, often called the Jacobi-Zariski exact sequence. We obtain a homotopy analogue of this exact sequence. Our approach is to define relative A_∞ and C_∞-algebras,*i.e.*, A_∞ and C_∞-algebras for which the "ground ring" is also a C_∞-algebra. We define Hochschild and Harrison (co)homology groups for these relative homotopy algebras. These definitions are of independent interest. Indeed, recently, several categories of strictly associative and commutative ring spectra have arisen providing exciting new constructions in homotopy theory, for instance see [**EKMM, MMSS**]. Our constructions of Hochschild and Harrison (co)homology of relative homotopy algebras are algebraic, chain complex level, analogues of topological Hochschild/André-Quillen (co)homology of an R-ring spectrum, where R is a commutative ring spectrum.

Here is the plan of the paper. In section 1 we recall and explain the basic property of A_∞-algebras and their Hochschild cohomology. We give some details, not so easy to find in the literature, for the reader's convenience. In Section 2 we recall the definition of C_∞-algebras, introduce our notion of a C_∞-bimodule, generalizing the classical notion of symmetric bimodule, and then of Harrison (co)homology. We also study some basic properties of these constructions. In Section 3 we establish the existence of λ-operations, Hodge decompositions in characteristic zero and study some of their properties. In Section 4, we study the homotopy version of Jacobi-Zariski exact sequence for Harrison (co)homology and establish a framework for the study of A_∞-algebras with a C_∞-algebra as "ground ring". In the last section we apply the previous machinery to string topology and prove that there exists λ-operations compatible with a BV-structure on $HH^*(C^*(X), C^*(X))$ for X a triangulated Poincaré duality space. The last section is devoted to some additional remarks (without proof) and questions.

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Notations :

- in what follows k will be a commutative unital ring and $R = \bigoplus R^j$ a $\mathbb{Z}$-graded k-module. All tensors products will be over k unless otherwise stated and/or sub scripted.
- We use a cohomological grading for our k-modules with the classical convention that a homological grading is the opposite of a cohomological one. In other words $H_i := H^{-i}$ as graded modules. A (homogeneous) map of degree k between graded modules V^*, W^* is a map $V^* \to W^{*+k}$.
- When $x_1, \dots, x_n$ are elements of a graded module and σ a permutation, the *Koszul sign* is the sign $\pm$ appearing in the equality $x_1 \dots x_n = \pm x_{\sigma(1)} \dots x_{\sigma(n)}$ which holds in the symmetric algebra $S(x_1, \dots, x_n)$.
- We use Sweedler's notation $\delta(x) = \sum x^{(1)} \otimes x^{(2)}$ for a coproduct δ.
- A *strict* up to homotopy structure will be one given by a classical differential graded one.
- The algebraic structures "up to homotopy" appearing in this paper are always uniquely defined by sequences of maps $(D_i)_{i\geq 0}$. Such maps will be referred to as *defining maps*, for instance see Remark 1.5.

Contents

1. Hochschild (co)homology of an A_∞-algebra with values in a bimodule

In this section we recall the definitions and fix notation for A_∞-algebras and bimodules as well as their Hochschild (co)homology. For convenience of the reader, we also recall some "folklore" results which might not be found so easily in the literature and are needed later on.

1.1. A_∞-algebras and bimodules. The *tensor coalgebra* of R is $T(R) = \bigoplus_{n\geq 0} R^{\otimes n}$ with the deconcatenation coproduct

$$\delta(x_1, \dots, x_n) = \sum_{i=0}^{n} (x_1, \dots, x_i) \otimes (x_{i+1}, \dots, x_n).$$

The suspension sR of R is the graded k-module $(sR)^i = R^{i+1}$ so that a degree $+1$ map $R \to R$ is equivalent to a degree 0 map $R \to sR$.

Let V be a graded k-module. The *tensor bicomodule* of V over the tensor coalgebra $T(R)$ is the k-module $T^R(V) = k \oplus T(R) \otimes V \otimes T(R)$ with structure map

$$\begin{aligned}\delta^V(x_1,\dots,x_n,v,y_1,\dots,y_m) &= \sum_{i=1}^{n}(x_1,\dots,x_i)\otimes(x_{i+1},\dots,x_n,v,y_1,\dots,y_m)\\ &\oplus\sum_{k=1}^{m}(x_1\dots,x_n,v,y_1,\dots,y_i)\otimes(y_{i+1},\dots,y_m).\end{aligned}$$

If D is a coderivation of $T(R)$, then a coderivation of $T(V)$ to $T^R(V)$ over D is a map $\Delta : T^R(V) \to T^R(V)$ such that

$$(D\otimes \mathrm{id} + \mathrm{id}\otimes\Delta)\oplus(\Delta\otimes\mathrm{id}+\mathrm{id}\otimes D)\circ\delta^V = \delta^V\circ\Delta. \tag{1.1}$$

We denote $A^\perp(R) = \oplus_{n\geq 1} sR^{\otimes n}$ the coaugmentation

$$0 \to k \to T(sR) \to A^\perp(R) \to 0$$

and abusively write δ for its induced coproduct.

DEFINITION 1.1.
- *An A_∞-algebra structure on R is a coderivation D of degree 1 on $A^\perp(R)$ such that $(D)^2 = 0$.*
- *An A_∞-bimodule over R structure on M is a coderivation D^R_M of degree 1 on $A^\perp_R(M) := T_{sR}(sM)$ over D such that $(D^R_M)^2 = 0$.*
- *A map between two A_∞-algebras R, S is a map of graded differential coalgebras $A^\perp(R) \to A^\perp(S)$.*
- *A map between two A_∞-bimodules M, N over R is a map of graded differential bicomodules $A^\perp_R(M) \to A^\perp_R(N)$.*

Henceforth, R-bimodule will stand for A_∞-bimodule over an A_∞-algebra R.

NOTATION 1.2. We will denote "$a \otimes m \otimes b \in A^\perp_R(M)$" a generic element in $A^\perp_R(M)$. That is, $a, b \in A^\perp(R)$, $m \in M$ and $a\otimes m\otimes b$ stands for the corresponding element in $A^\perp(R) \otimes sM \otimes A^\perp(R) \subset A^\perp(R)$.

REMARK 1.3. These definitions are the same as the definitions given by algebras over the minimal model of the operad of associative algebras and their bimodules and goes back to the pioneering work [**St**].

REMARK 1.4. Coderivations on $A^\perp(R)$ are the same as coderivations on $T(sR)$ that vanishes on $k \subset T(sR)$.

It is well-known that a coderivation D on $T(sR)$ is uniquely determined by a simpler system of maps $(\tilde{D}_i : sR^{\otimes i} \to sR)_{i\geq 0}$. The maps $\tilde{D}_i$ are given by the composition of D with the projection $T(sR) \to sR$. The coderivation D is the sum of the lifts of the maps $\tilde{D}_i$ to $A^\perp(R) \to A^\perp(R)$. More precisely, for $x_1,\dots,x_n \in sR$,

$$D(x_1,\dots,x_n) = \sum_{i\geq 0}\sum_{j=0}^{n-i} \pm\, x_1\otimes\dots\otimes\tilde{D}_i(x_{j+1},\dots,x_{j+i})\otimes\dots\otimes x_n \tag{1.2}$$

where $\pm$ is the sign $(-1)^{|D_i|(|x_1|+\dots+|x_j|)}$. Furthermore, there are isomorphisms of graded modules $\mathrm{Hom}(sR^{\otimes i}, sR) \ni \tilde{D}_i \longmapsto D_i \in s^{1-i}\,\mathrm{Hom}(R^{\otimes i}, R)$ defined by

$$D_i(r_1,\dots,r_i) = (-1)^{i|\tilde{D}_i|+\sum_{k=1}^{i-1}(k-1)|r_k|}\tilde{D}_i(sr_1,\dots,sr_i). \tag{1.3}$$

Note that the signs are given by the Koszul rule for signs. It follows that a coderivation D on $T(sR)$ is uniquely determined by a system of maps $(D_i : R^{\otimes i} \to R)_{i\geq 0}$. Such a coderivation D is of degree k if and only if each D_i is of degree $k+1-i$. According to Remark 1.4 above, a coderivation D on $A^{\perp}(R)$ is one on $T(sR)$ such that $D_0 = 0$.

REMARK 1.5. We call the maps $(D_i : R^{\otimes i} \to R)_{i\geq 0}$ the *defining maps* of the associated coderivation $D : T(sR) \to T(sR)$. We also use similar terminology for all other kind of coderivations appearing in the rest of the paper.

Similarly, a coderivation D^A_R on $A^{\perp}_R(M)$ (over D with $D_0 = 0$) is given by a system of maps $(D^M_{i,j} : R^{\otimes i} \otimes M \otimes R^{\otimes j} \to M)_{i,j\geq 0}$. All of these properties are formal consequences of the co-freeness of the tensor coalgebra (in the operadic setting). Also a very detailed and down-to-earth account is given in [**Tr1**].

REMARK 1.6. Given a coderivation D of degree 1 on $A^{\perp}(R)$ defined by a system of maps $\big(D_i : R^{\otimes i} \to R\big)_{i\geq 0}$, it is well-known [**St**] that the condition $(D)^2 = 0$ is equivalent to an infinite number of equations quadratic in the D_i's. Namely, for $n \geq 1$, $r_1, \dots r_n \in R$,

$$\sum_{i+j=n+1} \sum_{k=0}^{i-1} \pm D_i(r_1, \dots, r_k, D_j(r_{k+1}, \dots, r_{k+j}), r_{k+j+1}, \dots, r_n) = 0. \tag{1.4}$$

In particular, if $D_1 = 0$, Equation (1.4) implies that D_2 is an associative multiplication on R.

There are similar identities for the defining maps $(D^M_{i,j} : R^{\otimes i} \otimes M \otimes R^{\otimes j} \to M)_{i,j\geq 0}$ of an A_∞-bimodule [**Tr1**]. It is trivial to show that, when $D_1 = 0$ and $D^M_{00} = 0$, D^M_{10} and D^M_{01} respectively endows M with a structure of left and right module over the algebra (R, D_2).

EXAMPLE 1.7. Any A_∞-algebra (R, D) is a bimodule over itself with structure maps given by $D^R_{i,j} = D_{i+1+j}$.

REMARK 1.8. Similarly to coderivations, a map of graded coderivation $F : A^{\perp}(R) \to A^{\perp}(S)$ is uniquely determined by a simpler system of maps $(F_i : R^{\otimes i} \to R)_{i\geq 1}$, where F_i is induced by composition of F with the projection on S. The details are similar to those of Remark 1.4 and left to the reader. The maps F_i are referred to as the defining maps of F.

1.2. Hochschild (co)homology. Let (R, D) be an A_∞-algebra and M an R-bimodule. We call a coderivation from $k \oplus A^{\perp}(R) = T(sR)$ into $A^{\perp}_R(M)$ a *coderivation of R into M*. By definition it is a map $f : T(sR) \longrightarrow A^{\perp}_R(M)$ such that

$$\delta^M \circ f = (\mathrm{id} \otimes f + f \otimes \mathrm{id}) \circ \delta.$$

As in Remark 1.5, such a coderivation is uniquely determined by a collection of maps $(f_i : R^{\otimes i} \to M)_{i\geq 0}$ where the f_i are induced by the projections onto sM of the map f restricted to $sR^{\otimes i}$ (for instance see [**Tr1**]).

DEFINITION 1.9. *The Hochschild cochain complex of an A_∞-algebra (R, D) with values in an R-bimodule M is the space $s^{-1}\mathrm{CoDer}(R, M)$ of coderivations of R into M equipped with differential b given by*

$$b(f) = D_M \circ f - (-1)^{|f|} f \circ D.$$

It is classical that b is well-defined and $b^2 = 0$ see [**GJ2**]. We denote $HH^*(A, M)$ its cohomology which is called the Hochschild cohomology of R with coefficients in M.

Example 1.10. Let (R, m, d) be a differential graded algebra and (M, l, r, d_M) be a (differential graded) A-bimodule. Then R has a structure of an A_∞-algebra and M a structure of A_∞-bimodule over R given by maps:

$$D_1 = d, \qquad D_2 = m \quad \text{and} \quad D_i = 0 \text{ for } i \geq 3; \tag{1.5}$$

$$D^M_{0,0} = d_M, \quad D^M_{1,0} = l, \quad D^M_{0,1} = r \text{ and } D^M_{i,j} = 0 \text{ for } i + j \geq 1. \tag{1.6}$$

The converse is true: if R, M are, respectively, an A_∞ algebra and a R-bimodule with $D_{i\geq 3} = 0$ and $D^M_{i,j} = 0$ $(i + j \geq 1)$, then the identities (1.5), (1.6) define a differential graded algebra structure on R and a bimodule structure M. We call this kind of structure a *strict* homotopy algebra or a *strict* homotopy bimodule.

We have seen that the k-module $\mathrm{CoDer}(R, M)$ is isomorphic to the k-module $\mathrm{Hom}\left(\bigoplus_{n\geq 0} R^{\otimes n}, M\right)$ by projection on sM, i.e. the map $f \mapsto (f_i : R^{\otimes i} \to M)_{i\geq 0}$. Thus the differential b induces a differential on $\mathrm{Hom}\left(\bigoplus_{n\geq 0} R^{\otimes n}, M\right)$, which for a homogeneous map $f : R^{\otimes n} \to M$, is given by the sum $b(f) = \alpha(f) + \beta(f)$ where $\alpha(f) : R^{\otimes n} \to M$ and $\beta(f) : R^{\otimes n+1} \to M$ are defined by

$$\begin{aligned}
\alpha(f)(a_1, \dots, a_n) &= (-1)^{|f|+1} d_M(f(a_1, \dots, a_n)) \\
&\quad + \sum_{i=0}^{n-1} (-1)^{i+|a_1|+\cdots+|a_i|} f(a_1, \dots, d(a_{i+1}), \dots, a_n) \\
\beta(f)(a_0, \dots, a_n) &= (-1)^{|f||a_0|+|f|} l(a_0, f(a_1, \dots, a_n)) \\
&\quad + (-1)^{n+|a_0|+\cdots+|a_{n-1}|+|f|} r(f(a_0, \dots, a_{n-1}), a_n) \\
&\quad - \sum_{i=0}^{n-1} (-1)^{i+|a_0|+\cdots+|a_i|+|f|} f(a_0, \dots, m(a_i, a_{i+1}), \dots, a_n).
\end{aligned}$$

Hence $\alpha+\beta$ is the differential in the standard bicomplex giving the usual Hochschild cohomology of a differential graded algebra [**Lo2**]. Consequently Definition 1.9 coincides with the standard one for strict A_∞-algebras, that is the one given by the standard complex.

It is standard that the identity $(D)^2 = 0$ restricted to A yields that $(D_1)^2 = 0$ and $|D_1| = |D| = 1$. Therefore, (R, D_1) is a chain complex whose cohomology will be denoted $H^*(R)$. Moreover the linear map $D_2 : R^{\otimes 2} \to R$ passes to the cohomology $H^*(R)$ to define an associative algebra structure. Similarly D^M_{00} is a differential on M and $H^*(M)$ has a bimodule structure over $H^*(R)$ induced by D^M_{10}, and D^M_{01}. The link between the cohomology of $H^*(A)$ and the one of A is given by the following spectral sequence.

Proposition 1.11. *Let (R, D) be an A_∞-algebra and (M, D^M) an R-bimodule with R, M, $H^*(R)$, $H^*(M)$ flat as k-modules. There is a converging spectral sequence*

$$E_2^{p,q} = HH^{p+q}(H^*(R), H^*(M))^q \Longrightarrow HH^*(R, M).$$

The subscript q in $HH^*(H^*(R), H^*(M))^q$ stands for the piece of internal degree q in the group $HH^*(H^*(R), H^*(M))$ (the internal degree is the degree coming from the grading of $H^*(R)$).

Proof: There is a decreasing filtration of cochain complex $F^{*\geq 0}C^*(R, M)$ of $\mathrm{CoDer}(R, M)$ where $F^pC^*(R, M)$ is the subspace of coderivation f such that

$$f(R^{\otimes n}) \subset \bigoplus_{p+i+j\leq n} R^{\otimes i} \otimes M \otimes R^{\otimes j}.$$

The filtration starts at F_0 because any coderivation f is determined by maps $R^{\otimes i\geq 1} \to M$. It is thus a bounded above and complete filtration. Hence, it yields a cohomological converging spectral sequence computing $HH^*(R, M)$. The maps D_i and $D^M_{j,k}$ lower the degree of the filtration unless $i = 1$, $j = k = 0$. Consequently the differential on the associated graded is the one coming from the inner differentials D_1 and $D^M_{0,0}$. It follows by Künneth formula, that

$$E_1^{**} \cong \mathrm{CoDer}\big(A^\perp(H^*(R)), A^\perp_{H^*(R)}(H^*(M))\big).$$

The differential on the E_1^{**} term is induced by D_2, D^M_{10}, D^M_{01}. These operations give $H^*(R)$ a structure of associative algebra and $H^*(M)$ a bimodule structure. Hence the differential d^1 on E_1^{**} is the same as the differential defining the Hochschild cohomology of the graded algebra $H^*(R)$ with values in $H^*(M)$. Now, Example 1.10 implies that $E_2^{*,*} = HH^*(H^*(R), H^*(M))$. ■

The Hochschild cohomology $HH^*(R, R)$ of any A_∞-algebras R has the structure of a Gerstenhaber algebra as was shown in [**GJ2**]. The product of two elements $f, g \in C^*(R, R)$ (with defining maps (f_n), (g_m)) is the coderivation $\mu(f, g)$ defined by

$$\mu(f,g)(a_1,\dots,a_n) = \sum_{j\geq 2, r_1, r_2\geq 0} \pm(a_1\otimes\dots\otimes D_j(\dots f_{r_1},\dots,g_{r_2},\dots)\otimes\dots a_n). \tag{1.7}$$

In the formula the sign $\pm$ is the Koszul sign. There is also a degree 1 bracket defined by $[f, g] = f\widetilde{\circ}g - (-1)^{(|f|+1)(|g|+1)}g\widetilde{\circ}f$ where

$$f\widetilde{\circ}g(a_1,\dots,a_n) = \sum_{i,j} \pm(a_1\otimes\dots\otimes f_i(\dots g_j,\dots)\otimes\dots a_n).$$

PROPOSITION 1.12. *Let R be an A_∞-algebra and take $M = R$ as a bimodule. Then $(HH^*(R, R), \mu, [\,,\,])$ is a Gerstenhaber algebra and the spectral sequence $E^{**}_{m\geq 2}$ is a spectral sequence of Gerstenhaber algebras.*

Proof: The fact that the product μ and the bracket $[\,,\,]$ make $HH^*(R, R)$ a Gerstenhaber algebra is well-known [**GJ2, Tr2**]. Also see Remark 1.13 below for a sketch of proof.

The product map $\mu : F^pC^* \otimes F^qC^* \to F^{p+q}C^*$ and bracket $[\,,\,] : F^pC^* \otimes F^qC^* \to F^{p+q-1}C^*$ are filtered maps of cochain complexes. Thus both operations survive in the spectral sequence. At the level E_0 of the spectral sequence, the product μ boils down to

$$\mu(f,g)(a_0,\dots,a_n) = \sum a_0\dots\otimes D_2(f(\dots),g(\dots))\otimes\dots\otimes a_n$$

which, after taking the homology for the differential d_0, identifies with the usual cup product in the Hochschild cochain complex $\mathrm{Hom}(H^*(R)^{\otimes *}, H^*(R))$ through the isomorphism between coderivations and homomorphisms. Similarly the bracket coincides with the one introduced by Gerstenhaber in the Hochschild complex of $H^*(R)$. The Leibniz relation hence holds at level 2 and on the subsequent levels. ■

Remark 1.13. Actually, the product structure is the reflection of a A_∞-structure on $C^*(R,R)$. It is easy to check that the maps $\gamma_i : C^*(R,R)^{\otimes i} \to C^*(R,R)$ defined by

$$\gamma_i(f^1,\dots,f^i)(a_1,\dots,a_n) = \sum_{j\geq i, r_1,\dots,r_i\geq 1} \pm(a_1\otimes\dots\otimes D_j(\dots f^1_{r_1},\dots,f^2_{r_2},\dots \\ \dots,f^i_{r_i},\dots)\otimes\dots\otimes a_n)$$

together with $\gamma_1 = b$, the Hochschild differential, give a A_∞-structure to $C^*(R,R)$. Thus the map $\mu=\gamma_2$ gives an associative algebra structure to $HH^*(R,R)$. Moreover it is straightforward to check that the Jacobi relation for $[\,,\,]$ is satisfied on $C^*(R,R)$. The Leibniz identity and the commutativity of the product are obtained as in Gerstenhaber fundamental paper [**Ge**].

The Hochschild homology of an A_∞-algebra R was first defined in [**GJ1**]. Let M be an R-bimodule and $b: M\otimes T(sR)\to M\otimes T(sR)$ be the map

$$\begin{aligned} b(m,a_0,\dots,a_n) &= \sum_{p+q\leq n} \pm D^M_{p,q}(a_{n-p+1},\dots,a_n,m,a_1\dots,a_q)\otimes a_{q+1}\cdots a_{n-p} \\ &\quad + \sum_{i+j\leq n} \pm m\otimes a_1\otimes\cdots D_{j+1}(a_i,\dots,a_{i+j})\otimes a_{i+j+1}\otimes\cdots a_n. \end{aligned}$$

Definition 1.14. *The Hochschild homology $HH_*(R,M)$ of an A_∞-algebra (R,D) with values in the bimodule (M,D^M) is the homology of $(M\otimes T(sR),b)$.*

The fact that $b^2=0$ follows from a straightforward computation or from Lemma 1.16 below.

Remark 1.15. Recall that we use a cohomological grading for R,M. Thus a cycle $x\in M^i\subset M\otimes T(sR)$ gives an element $[x]\in HH_{-i}(R,M)$ in homological degree $-i$.

Given a bimodule M over R, there is a map $\gamma_M : M\otimes T(sR)\to T^{sR}(sM)$ defined by

$$\gamma_M = \tau\circ(s\,\mathrm{id}\otimes\delta)$$

where τ is the map sending the last factor of $M\otimes T(sR)\otimes T(sR)$ to the first of $T(sR)\otimes M\otimes T(sR)$.

Lemma 1.16. *Given any coderivation ∂ of $T^{sR}(sM)$ over D, there is a unique map $\overline{\partial}: M\otimes T(sR)\to M\otimes T(sR)$ that makes the following diagram commutative:*

$$\begin{array}{ccc} M\otimes T(sR) & \xrightarrow{\gamma_M} & T^{sR}(sM) \\ \overline{\partial}\downarrow & & \downarrow\partial \\ M\otimes T(sR) & \xrightarrow{\gamma_M} & T^{sR}(sM). \end{array}$$

Proof: The map $\overline{\partial}$ is the sum $\sum \overline{\partial}^{[i]}$, where $\overline{\partial}^{[i]}$ takes value in $M \otimes sA^{\otimes i}$. By induction on i, it is straightforward that

$$\begin{aligned}\gamma_M\big(\overline{\partial}^{[i]}\big) &= \textstyle\sum_{i=0}^{n} \Bigg(\displaystyle\sum_{p+q=n-m} a_{j+1} \otimes \ldots \otimes \partial_{p,q}(a_{n-p+1}, \ldots, a_n, m, a_1, \ldots \\ &\qquad \ldots, a_q) \otimes a_{q+1} \otimes \ldots \otimes a_i + \sum_{j=i+1}^{m-1} \pm a_i \otimes \ldots \otimes D_{n-m+2}(a_j, \\ &\qquad a_{j+1}, \ldots, a_{j+n-m+1}) \otimes \ldots \otimes a_n \otimes m \otimes a_1 \ldots \otimes a_i \\ &\qquad + \sum_{j=0}^{i-n+m-1} \pm a_i \otimes \ldots \otimes a_n \otimes m \otimes a_1 \ldots \otimes D_{n-m+2}(a_j, \ldots \\ &\qquad \ldots, a_{j+n-m+1}) \otimes \ldots \otimes a_i \Bigg).\end{aligned}$$

It follows that the map $\overline{\partial}$ exists and satisfies

$$\begin{aligned}\overline{\partial}(m, a_0 \otimes \ldots \otimes a_n) &= \sum_{p+q \leq n} \pm \partial_{p,q}(a_{n-p+1}, \ldots, a_n, m, a_1 \ldots, a_q) \otimes a_{q+1} \otimes \cdots \\ &\qquad \cdots \otimes a_{n-p} + \sum_{i+j \leq n} \pm m \otimes a_1 \otimes \cdots \otimes D_{j+1}(a_i, \ldots \\ &\qquad \cdots, a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_n.\end{aligned}$$

■

EXAMPLE 1.17. For $\partial = D_M^R$, $(D_M^R)^2$ is the trivial coderivation, hence $(\overline{D_M^R})^2 = \overline{(D_M^R)^2} = 0$ and $\overline{D_M^R}$ is a codifferential. Moreover $D_M^R \circ \gamma_M = \gamma_M \circ b$, thus $\overline{D_M^R} = b$ and $b^2 = 0$.

EXAMPLE 1.18. Let (R, d, m) be a differential graded algebra and (M, d_M, l, r) a strict R-bimodule. The only non-trivial defining maps are $D_1 = d$, $D_2 = m$, $D_{00}^M = d_M$, $D_{01}^M = r$, $D_{10}^M = l$. Hence one has

$$\begin{aligned}b(m \otimes a_1 \otimes \ldots \otimes a_n) &= d_M(m) \otimes a_1 \otimes \cdots \otimes a_n + \sum \pm m \otimes \cdots da_i \ldots \otimes a_n \\ &\quad + r(m, a_1) \otimes a_2 \cdots \otimes a_n + \pm l(a_n, m) \otimes a_1 \cdots \otimes a_n \\ &\quad + \sum \pm m \otimes a_1 \cdots m(a_i, a_{i+1}) \cdots \otimes a_n\end{aligned}$$

which is the usual Hochschild boundary for a differential graded algebra. Thus Definition 1.14 is equivalent to the standard one for strict algebras and bimodules.

THEOREM 1.19. *Let R be an A_∞-algebra and M an R-bimodule, flat as k-modules. There is a converging spectral sequence*

$$E^2_{pq} = HH_{p+q}(H^*(R), H^*(M))_q \Longrightarrow HH_{p+q}(R, M).$$

The subscript q in $HH_n(A, B)_q$ stands for the piece of $HH_n(A, B)$ of internal homological degree q (thus of internal cohomological degree $-q$).
Proof: Consider the filtration $F_{p\geq 0}C_*(R, M) = \bigoplus_{i\leq p} M \otimes sR^{\otimes i}$ dual to the filtration of Proposition 1.11. It is an exhaustive bounded below filtration of chain complex thus it gives a converging homology spectral sequence. Now the result follows as in the proof of Proposition 1.11. ■

2. C_∞-algebras, C_∞-bimodules, Harrison (co)homology

In this section we introduce the key definition of C_∞-bimodules and also recall the Harrison (co)homology of C_∞-algebras for which there is not so much published account.

2.1. Homotopy symmetric bimodules. Commutative algebras are associative algebras with additional symmetry. Similarly a C_∞-algebra could be seen as a special kind of A_∞-algebra. Indeed, this is the point of view we adopt here. The *shuffle product* makes the tensor coalgebra $(T(V), \delta)$ a bialgebra. It is defined by the formula

$$\mathrm{sh}(x_1 \otimes \dots, \otimes x_p, x_{p+1} \otimes \dots \otimes x_{p+q}) = \sum \pm x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(p+q)}$$

where the summation is over all the (p, q)-shuffles, that is to say the permutation of $\{1, \dots, p{+}q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p{+}1) < \cdots < \sigma(p{+}q)$. The sign $\pm$ is the sign given by the Koszul sign convention. A $(p_1, \dots, p_r)$-shuffle is a permutation of $\{1, \dots, p_1 + \cdots + p_r\}$ such that $\sigma(p_1 + \cdots + p_i + 1) < \cdots < \sigma(p_1 + \cdots + p_i + p_{i+1})$ for all $0 \leq i \leq r - 1$.

A B_∞*-structure* on a k-module R is given by a product M^B and a derivation D^B on the (shifted tensor) coalgebra $A^\perp(R)$ such that $(A^\perp(R), \delta, M^B, D^B)$ is a differential graded bialgebra [**Ba**]. A B_∞-algebra is in particular an A_∞-algebra whose codifferential is D^B.

Definition 2.1.
- *A C_∞-algebra is an A_∞-algebra (R, D) such that the coalgebra $A^\perp(R)$, equipped with the shuffle product and the differential D, is a B_∞-algebra.*
- *A C_∞-map between two C_∞-algebras R, S is an A_∞-algebra map $R \to S$ which is also a map of algebras with respect to the shuffle product.*

In particular there is a faithful functor from the category of C_∞-algebras to the category of A_∞-algebras. Moreover a A_∞ algebra defined by maps $D_i : R^{\otimes i} \to R$ is a C_∞-algebra if and only if, for all $n \geq 2$ and $k + l = n$, one has

$$D_n(\mathrm{sh}(x_1 \otimes \dots \otimes x_k, y_1 \otimes \dots \otimes y_l)) = 0. \tag{2.8}$$

Example 2.2. According to Example 1.10 and identity (2.8), any differential graded commutative algebra (R, m, d) has a natural C_∞-structure given by $D_1 = d$, $D_2 = m$ and $D_i = 0$ for $i \geq 3$.

Remark 2.3. Definition 2.1 is taken from [**GJ2**]. In *characteristic zero*, a more classical and equivalent one is to say that a C_∞-algebra structure on R is given by a degree 1 differential on the cofree Lie coalgebra $C^\perp(R) := \mathrm{coLie}(sR)$. The equivalence between the two definitions follows from the fact that $\mathrm{coLie}(sR) = A^\perp(R)/\mathrm{sh}$ is the quotient of $A^\perp(R)$ by the image of the shuffle multiplication $\mathrm{sh} : A^{\perp \geq 1}(R) \otimes A^{\perp \geq 1}(R) \to A^\perp(R)$. For arbitrary characteristic, Definition 2.1 is slightly weaker (see Example 2.4 below) than the one given by the operad theory, namely by a (degree 1) codifferential on $C^\perp(R)$.

Example 2.4. Since the universal enveloping coalgebra of a cofree Lie coalgebra $\mathrm{coLie}(V)$ is the tensor coalgebra $(T(V), \delta, \mathrm{sh})$ equipped with the shuffle product, a degree 1 differential on the cofree Lie coalgebra $C^\perp(R) := \mathrm{coLie}(sR)$ canonically yields a C_∞-algebra structure on R. We call such a C_∞-structure a *strong* C_∞-algebra structure. Note that strict C_∞-algebras are strong C_∞-algebras. Over a ring k containing $\mathbb{Q}$, all C_∞-algebras are strong (Remark 2.3).

A bimodule over a C_∞-algebra is a bimodule over this C_∞-algebra viewed as an A_∞-one. However this notion does not capture all the symmetry conditions of a C_∞-algebra. In the following sections we will need up to homotopy generalization of symmetric bimodules.

Definition 2.5. *A C_∞-bimodule structure on M is a bimodule over (R, D) such that the structure maps D^M_{ij} satisfy, for all $n \geq 1$, $a_1, \dots, a_n \in R$, $x \otimes m \otimes y \in A^\perp_R(M)$, the following relation*

$$(2.9)\quad \sum_{i+j=n} \pm D^M_{(i+|x|)(j+|y|)}\big(\mathrm{sh}(x, a_1 \otimes \dots \otimes a_i), m, \mathrm{sh}(y, a_{i+1} \otimes \dots \otimes a_n)\big) = 0.$$

The sign $\pm$ is the Koszul sign of the two shuffle products multiplied by the sign $(-1)^{(|a_1|+\dots+|a_i|+i)(|m|+1)}$. With Sweedler's notation associated to the coproduct structure of $T(sR)$, identity (2.9) reads as

$$(2.10)\qquad \sum \pm D^M\big(\mathrm{sh}(x, a^{(1)}), m, \mathrm{sh}(y, a^{(2)})\big) = 0.$$

Example 2.6. Let (R, m, d_R) be a graded commutative differential algebra and (M, l, r, d_M) a graded differential R-bimodule. Then M has a bimodule structure as explained in the previous section. Moreover this bimodule structure is a C_∞-bimodule structure if and only $l(m, a) = (-1)^{|a||m|} r(a, m)$ for all $m \in M, a \in R$, that is, M is symmetric in the usual sense.

Example 2.7. If (R, D) is a C_∞ algebra such that $D_1 = 0$, then $D_2 : R \otimes R \to R$ is associative (Remark 1.6) and graded commutative by Equation (2.8). Thus (R, D_2) is a graded commutative algebra. Furthermore, if (M, D^M) is a C_∞-bimodule such that $D^M_{00} = 0$, Equation (2.10) implies that D^M_{01} and D^M_{10} give a structure of symmetric bimodule over (R, D_2) to M.

Example 2.8. Any C_∞-algebra is a C_∞-bimodule over itself. This follows from identity (2.8) and the observation that, for any $a \otimes r \otimes b \in A^\perp_R(R)$ and $x \in A^{\perp\, * \geq 1}(R)$, one has

$$\sum \pm\big(\mathrm{sh}(a, x^{(1)}) \otimes r \otimes \mathrm{sh}(b, x^{(2)})\big) = \mathrm{sh}(a \otimes r \otimes b, x).$$

A C_∞-bimodule is a left (and right by commutativity) module over the shuffle bialgebra. The module structure is the map $\rho : A^\perp_R(M) \otimes A^\perp(R) \to A^\perp_R(M)$ given by the composition

$$A^\perp_R(M) \otimes A^\perp(R) \xrightarrow{\mathrm{id}\otimes\delta} A^\perp_R(M) \otimes A^\perp(R) \otimes A^\perp(R) \xrightarrow{(\mathrm{sh}\otimes\mathrm{id}\otimes\mathrm{sh})\circ\tau} A^\perp_R(M)$$

where the map τ

$$A^\perp(R) \otimes M \otimes A^\perp(R) \otimes A^\perp(R) \otimes A^\perp(R) \xrightarrow{\tau} A^\perp(R) \otimes A^\perp(R) \otimes M \otimes A^\perp(R) \otimes A^\perp(R)$$

is the permutation of the two $A^\perp(R)$ factors sitting in the middle.

Proposition 2.9. *A R-bimodule M is a C_∞-bimodule if and only if $A^\perp_R(M)$ is a differential module over the shuffle bialgebra $A^\perp(R)$. That is to say if the* *following diagram commutes*

$$\begin{array}{ccc} A_R^\perp(M)\otimes A^\perp(R) & \xrightarrow{\rho} & A_R^\perp(M) \\ D^M\otimes \mathrm{id}+\mathrm{id}\otimes D\downarrow & & \downarrow D^M \\ A_R^\perp(M)\otimes A^\perp(R) & \xrightarrow{\rho} & A_R^\perp(M). \end{array}$$

Proof: The compatibility with the coalgebra structure is part of the definition of a R-bimodule. It remains to prove the compatibility with the product. Let's denote by $x\bullet y$ the shuffle product $\mathrm{sh}(x,y)$. First we have to check that ρ defines an action of $(A^\perp(R),\mathrm{sh})$. Using that sh is a coalgebra map it is equivalent, for all $u,x,y\in A^\perp(R)$, $m\in M$, to :

$$(u\bullet x^{(1)})\bullet y^{(1)}\otimes m\otimes (v\bullet x^{(2)})\bullet y^{(2)} = u\bullet (x^{(1)}\bullet y^{(1)})\otimes m\otimes v\bullet (x^{(2)}\bullet y^{(2)})$$

which holds by associativity of sh.

Since D is both a coderivation and derivation, one has, for all $a,b,x\in A^\perp(R)$, $m\in M$,

$$D^M\big(\rho(a,m,b,x)\big)=\sum a^{(1)}\bullet x^{(1)}\otimes D^M_{**}\left(a^{(2)}\bullet x^{(2)},m,x^{(3)}\bullet b^{(1)}\right)\otimes b^{(2)}\bullet x^{(4)}$$
$$+\rho\left(D^M\otimes \mathrm{id}+\mathrm{id}\otimes D\right)(a,m,b,x).$$

The sum is over all decompositions $\delta^3(x)=\sum x^{(1)}\otimes x^{(2)}\otimes x^{(3)}\otimes x^{(4)}$ such that $x^{(2)}$ or $x^{(3)}$ is not in k. But then the difference $D^M\circ\rho-\rho\circ\left(D^M\otimes \mathrm{id}+\mathrm{id}\otimes D\right)$ is 0 if and only if D^M satisfies the defining conditions of a C_∞-bimodule. ■

The strict notion of a symmetric bimodule is self-dual. However this is not true for its homotopy analog. Thus we will also need the dual version of a C_∞-bimodule, that we call a C_∞^{op}-bimodule.

Definition 2.10. *Let (R,D) be a C_∞-algebra. A C_∞^{op}-bimodule structure on M is an R-bimodule structure, such that the structure maps D^M_{ij} satisfy for all $n\geq 1$, $a_1,\dots,a_n\in R$, $x\otimes m\otimes y\in A_R^\perp(M)$:*

$$\sum_{i+j=n}\pm D^M_{(j+|x|)(i+|y|)}\big(\mathrm{sh}(x,a_{i+1}\otimes\ldots\otimes a_n),m,\mathrm{sh}(y,a_1\otimes\ldots\otimes a_i)\big)=0.$$

As in Definition 2.5, the sign is given by the Koszul rule for signs.

Example 2.11. If M is a strict symmetric bimodule (over a strict algebra) then it is a C_∞^{op}-bimodule.

Example 2.12. A C_∞-algebra has no reason to be a C_∞^{op}-bimodule in general. However its dual is always an C_∞^{op}-bimodule. More precisely, let (R,D) be a C_∞-algebra and write $R^\star=\mathrm{Hom}(R,k)$ for the dual module of R. Then the maps $D^{R^\star}_{kl}: R^{\otimes k}\otimes R^\star\otimes R^{\otimes l}\to R^\star$ given by

$$D^{R^\star}_{kl}(r_1,\dots,r_k,f,r_{k+1},\dots,r_{k+l})(m)=\pm f\big(D_{k+l+1}(r_{k+1},r_{k+l},m,r_1,\dots,r_k)\big)$$

yields an A_∞-bimodule structure on $R^\star$, see [**Tr1**] Lemma 3.9. The equation

$$\sum_{i+j=n}\pm D^{R^\star}_{(j+|x|)(i+|y|)}\big(\mathrm{sh}(x,a_{i+1}\otimes\ldots\otimes a_n),f,\mathrm{sh}(y,a_1\otimes\ldots\otimes a_i)\big)=0.$$

is equivalent to

$$\sum_{i+j=n}\pm D^{R}_{(j+|x|)(i+|y|)}\big(\mathrm{sh}(y,a_1\otimes\ldots\otimes a_i),m,\mathrm{sh}(x,a_{i+1}\otimes\ldots\otimes a_n)\big)=0.$$

which is satisfied because R is a C_∞-bimodule (Definition 2.5).

The argument of Example 2.12 can be generalized to prove

Proposition 2.13. *The dual $M^\star = \mathrm{Hom}(M,k)$ of any C_∞-bimodule is a C_∞^{op}-bimodule. The dual of any C_∞^{op}-bimodule is a C_∞-bimodule.*

The (operadic) notion of strong C_∞-algebras (Example 2.4) gives rise to the notion of *strong* C_∞-bimodules which form a nice subclass of the C_∞-bimodules because, under suitable freeness assumption, they are also C_∞^{op}-bimodules, see Proposition 2.16 below. Let (R,D) be a strong C_∞-algebra.

Definition 2.14. *A strong C_∞-bimodule structure on M over a strong C_∞-algebra (R,D) is a structure of strong C_∞-algebra on $R\oplus M$, given by a codifferential D^M on $T(sR\oplus sM)$, satisfying*

1. *$D_n^M(x_1,\dots,x_n) = 0$ if at least two of the x_is are in M, $D_n^M(x_1,\dots,x_n) \in M$ if exactly one of the x_is is in M and $D_n^M(x_1,\dots,x_n)\in R$ if all x_is are in R;*
2. *the restriction of D^M to $T(sR)$ is equal to the differential D defining the (strong) C_∞-structure on R.*

In particular, a strong C_∞-bimodule structure on M is uniquely determined by maps $D^M_{p,q} : R^{\otimes p}\otimes M\otimes R^{\otimes q}\to M$. Furthermore, these defining maps $D^M_{p,q}$ satisfy the relation (2.9), thus M is a C_∞-bimodule. A strong C_∞-algebra is a strong C_∞-bimodule over itself (with defining map as in Example 1.7).

Remark 2.15. If k contains $\mathbb{Q}$, any C_∞-bimodule is strong. This follows from Remark 2.3, Proposition 2.9 and the proof of Proposition 2.16 below.

Moreover when M is free over k, one has

Proposition 2.16. *Let M and R be free over k, and (R,D) be a strong C_∞-algebra (see Example 2.4). If M is a strong C_∞-bimodule over (R,D), it is a C_∞^{op}-bimodule and its dual $M^\star$ is C_∞-bimodule.*

Proof: Denote $D : T(sR)\to T(sR)$ the differential defining the A_∞-structure and D^M the bimodule one. By duality and Proposition 2.13, it is sufficient to prove that a strong C_∞-bimodule M is also a C_∞^{op}-bimodule. Let us show that it is enough to prove the result for the canonical bimodule structure over R. Note that there is a splitting

$$(R\oplus M)^{\otimes i} \cong X\oplus R^{\otimes i}\oplus \bigoplus_{j=0}^{i} R^j\otimes M\otimes R^{i-1-j}.$$

Here $X\subset (R\oplus M)^{\otimes i}$ is the submodule generated by tensors with at least two components in M. Consider the maps $B_i : (R\oplus M)^{\otimes i}\to R\oplus M$ defined to be zero on X, D_i on $R^{\otimes i}$ and $D^m_{j,i-1-j}$ on $R^j\otimes M\otimes R^{i-1-j}$. It is straightforward to check that the maps $(B_i)_{i\geq 1}$ give an A_∞-structure on $R\oplus M$ iff M is an A_∞-bimodule. Moreover, if M is a strong C_∞-bimodule, $R\oplus M$ is a strong C_∞-algebra and it remains to prove the statement for a strong C_∞-algebra.

Let R be a strong C_∞-algebra equipped with its canonical (strong) bimodule structure over itself (Example 2.8). We have to prove that R is a C_∞^{op}-bimodule. Denote $\overline{D} : T(sR)\to sR$ the projection of the differential $D : T(sR)\to T(sR)$. Since D defines a C_∞-structure, $\overline{D}$ factors through the free Lie coalgebra $\mathrm{CoLie}(sR)\to$

sR. By hypothesis its dual $\overline{D}^\star$ is a map $(sR)^\star \to Lie\big((sR)^\star\big)$ the free Lie algebra on $(sR)^\star$. Since $R \hookrightarrow R^{\star\star}$ is injective, it is enough to prove that for any $F : (sR)^\star \to Lie\big((sR)^\star\big)$, $f \in (sR)^\star$ $a_1, \dots, a_n \in R$ and $x \otimes m \otimes y \in A^\perp_R(R)$, one has

$$F(f)\big(\mathrm{sh}(x, a_{i+1} \otimes \dots \otimes a_n), m, \mathrm{sh}(y, a_1 \otimes \dots \otimes a_i)\big) = 0. \tag{2.11}$$

We can work with homogeneous component and use induction, the result for order one component of $F(f)$ being trivial. Thus we can assume $F(f) = [G, H]$ and G, H satisfies identity (2.11). Then, writing z for the term to which we apply $F(f)$, we find

$$\begin{aligned} F(f)(z) \quad = \quad & \sum G(x^{(1)} \bullet a^{(2)}) H(x^{(2)} \bullet a^{(3)}, m, y \bullet a^{(1)}) \\ & + \sum G(x \bullet a^{(3)}, m, y^{(1)} \bullet a^{(1)}) H(y^{(2)} \bullet a^{(2)}) \\ & - \sum H(x^{(1)} \bullet a^{(2)}) G(x^{(2)} \bullet a^{(3)}, m, y \bullet a^{(1)}) \\ & - \sum H(x \bullet a^{(3)}, m, y^{(1)} \bullet a^{(1)}) G(y^{(2)} \bullet a^{(2)}). \end{aligned}$$

By definition G and H vanish on shuffles, thus all the terms of the first line for which $x^{(1)}$ and $a^{(2)}$ are non trivial are zero. Moreover H satisfies identity (2.11). Thus all the terms for which $a^{(2)}$ is trivial also cancel out. The same analysis works for line 4. Thus for lines 1 and 4 we are left to the terms for which $x^{(1)}$ is trivial and $a^{(2)}$ is not. Those terms cancels out each other by commutativity of k. Line 2 and 3 cancels out by a similar argument. ■

REMARK 2.17. In particular if k is a characteristic zero field, C_∞ and C_∞^{op} bimodules coincide.

REMARK 2.18. The author realized that C_∞ and C_∞^{op} should coincide under quite general hypothesis while reading [**HL**]. The proof of Proposition 2.16 is taken from Lemma 7.9 in [**HL**].

Proposition 2.9 can be dualized too. A C_∞^{op}-bimodule is a left (and right by commutativity) module over the shuffle bialgebra through the opposite action $\widetilde{\rho}$. The map $\widetilde{\rho} : A^\perp_R(M) \otimes A^\perp(R) \to A^\perp_R(M)$ is the composition

$$A^\perp_R(M) \otimes A^\perp(R) \xrightarrow{\mathrm{id} \otimes t \circ \delta} A^\perp_R(M) \otimes A^\perp(R) \otimes A^\perp(R) \xrightarrow{(\mathrm{sh} \otimes \mathrm{id} \otimes \mathrm{sh}) \circ \tau} A^\perp_R(M)$$

where the map τ

$$A^\perp(R) \otimes M \otimes A^\perp(R) \otimes A^\perp(R) \otimes A^\perp(R) \xrightarrow{\tau} A^\perp(R) \otimes A^\perp(R) \otimes M \otimes A^\perp(R) \otimes A^\perp(R)$$

is the permutation of the middle $A^\perp(R)$ factors and t the transposition. Now dualizing the argument of Proposition 2.9 yields

PROPOSITION 2.19. *A R-bimodule M is a C_∞^{op}-bimodule if and only if $A^\perp_R(M)$ is a differential module over the shuffle bialgebra $A^\perp(R)$ for the action $\widetilde{\rho}$. That is to say if the following diagram commutes*

$$\begin{array}{rcl} A^\perp_R(M) \otimes A^\perp(R) & \xrightarrow{\widetilde{\rho}} & A^\perp_R(M) \\ D^M \otimes \mathrm{id} + \mathrm{id} \otimes D \downarrow & & \downarrow D^M \\ A^\perp_R(M) \otimes A^\perp(R) & \xrightarrow{\widetilde{\rho}} & A^\perp_R(M). \end{array}$$

The equation satisfied by the defining maps D^M_{10} and D^M_{01} for being a C_∞ or a C^{op}_∞-bimodule are the same, namely

$$\forall x \in R, m \in M \quad D^M_{10}(x,m) = (-1)^{|x||m|} D^M_{01}(m,x).$$

From this observation follows the obvious

PROPOSITION 2.20. *If M is either a C_∞ or a C^{op}_∞-bimodule over R, then $H^*(M)$ is a symmetric $H^*(R)$-module.*

2.2. Harrison (co)homology with values in bimodules. In this section we define Harrison (co)homology for a C_∞-algebras. For simplicity, in this section, we restrict attention to the case where k is a field and C_∞-algebras and bimodules are strong. In particular, applying Proposition 2.16 all modules are C_∞ and C^{op}_∞-bimodules.

We first deal with cohomology. Thus let (R,D) be a (strong) C_∞-algebra and let (M,D^M) be a (strong) C_∞-bimodule over R. Recall that a coderivation $f \in \mathrm{CoDer}\big(T(sR), A^\perp_R(M)\big)$ is determined by its projection $f^i : R^{\otimes i\geq 0} \to M$. Denote $\mathrm{BDer}(R,M)$ the subspace of $\mathrm{CoDer}\big(T(sR), A^\perp_R(M)\big)$ of coderivations f such that the f_i vanishes on the module generated by the shuffles i.e.,

$$f_i\big(\mathrm{sh}(x,y)\big) = 0 \quad \text{ for } \quad i \geq 2, x \in R^{k\geq 1}, y \in R^{i-k\geq 1}.$$

LEMMA 2.21. *The map $b(f) = D^M \circ f - (-1)^{|f|} f \circ D$ sends $\mathrm{BDer}(R,M)$ to itself and satisfies $b^2 = 0$.*

Proof: We already know that b maps coderivations into coderivations. Let $x \in R^{\otimes k\geq 1}, y \in R^{\otimes l\geq 1}$ and $f \in \mathrm{BDer}(R,M)$.

$$\begin{aligned} b(f)_i\big(x \bullet y\big) &= \sum \pm D^M_{i^{(1)} i^{(3)}}\big(x^{(1)} \bullet y^{(1)}, f_{i^{(2)}}(x^{(2)} \bullet y^{(2)}), x^{(3)} \bullet y^{(3)}\big) \\ &= \sum \pm D^M_{i^{(1)} i^{(3)}}\big(x^{(1)} \bullet y^{(1)}, f_{i^{(2)}}(x^{(2)}), x^{(3)} \bullet y^{(2)}\big) \\ &\quad + \pm D^M_{i^{(1)} i^{(3)}}\big(x^{(1)} \bullet y^{(1)}, f_{i^{(2)}}(y^{(2)}), x^{(2)} \bullet y^{(3)}\big) \\ &= 0. \end{aligned}$$

The first line follows from the definition of $\mathrm{BDer}(R,M)$ and the other because M is a C_∞-bimodule.

The last statement follows from

$$\begin{aligned} b^2(f) &= D\big(D^M \circ f - (-1)^{|f|} f \circ D\big) - (-1)^{|f|+1}\big(D^M \circ f - (-1)^{|f|} f \circ D\big) D \\ &= (-1)^{|f|+1} D^M \circ f \circ D + (-1)^{|f|} D^M \circ f \circ D \\ &= 0. \end{aligned}$$

■

DEFINITION 2.22. *Let (R,D) be a strong C_∞-algebra and (M,D^M) be a strong C_∞-bimodule over R, the Harrison cohomology $Har^*(R,M)$ of R with values in M is the cohomology of the complex $CHar^*(R,M) := \mathrm{BDer}(R,M)$ with differential $b(f) = D^M \circ f - (-1)^{|f|} f \circ D$.*

EXAMPLE 2.23. Let R be a non-graded commutative algebra and M a symmetric R-bimodule. Then the space of coderivations $\mathrm{BDer}(R,M)$ is isomorphic to $\mathrm{Hom}(T(R)/\mathrm{sh}, M)$ and is concentrated in positive degrees. Thus, as in Example 1.10, the Harrison cohomology coincides with the usual one for strictly commutative algebras in degree ≥ 1.

Remark 2.24. The notation BDer is chosen to put emphasis on the fact that the Harrison cochain complex $\mathrm{BDer}(R, M)$ is a space of B_∞-derivation. More precisely: if R has a structure of B_∞-algebra and $A^\perp_R(M)$ is a differential graded module over the bialgebra $(T(sR), \delta, M^B)$, a derivation of B_∞-algebra from R to M is a map $f : T(sR) \to A^\perp_R(M)$ which commutes with δ and M i.e.

$$\delta^M \circ f = (\mathrm{id} \otimes f + f \otimes \mathrm{id}) \circ \delta \quad \text{and}$$
$$f \circ M^B = \rho(\mathrm{id} \otimes f + f \otimes \mathrm{id}).$$

When R is a C_∞-algebra and M a C_∞-bimodule over R, $\mathrm{BDer}(R, M)$ is the space of B_∞-derivations from R to M for the B_∞-structure given by the shuffle product and the coderivation given by ρ.

We now define Harrison homology. Let R be a C_∞-algebra and M a C_∞-R-bimodule. We denote $T(sR)/\mathrm{sh}$ the quotient of the shifted tensor coalgebra $T(sR)$ by the image of the shuffle product map $A^\perp(sR) \otimes A^\perp(sR) \to T(sR)$. Reasoning as in the first part of the proof of Lemma 2.21 yields

Lemma 2.25. *Let R be a C_∞-algebra and M a R-bimodule. If M is a C_∞-bimodule, the Hochschild differential $b : M \otimes T(sR) \to M \otimes T(sR)$ passes to the quotient $M \otimes T(sR)/\mathrm{sh}$.*

Definition 2.26. *Let (R, D) be a strong C_∞-algebra and (M, D^M) a strong C_∞-bimodule over R, the Harrison homology $Har_*(R, M)$ of R with values in M is the homology of the complex $(CHar_*(R, M) := M \otimes C^\perp(R), b)$.*

Example 2.27. If R, M are respectively a commutative algebra and a symmetric bimodule, the complex $CHar_*(R, M)$ is the usual Harrison chain complex, up to degree 0 terms.

Proposition 2.28. *Let R be a strong C_∞-algebra and M a strong C_∞-bimodule over R, flat as k-modules. There are converging spectral sequences*

$$E_2^{pq} = Har^{p+q}(H^*(R), H^*(M))_q \Longrightarrow Har^{p+q}(R, M),$$
$$E^2_{pq} = Har_{p+q}(H^*(R), H^*(M))_q \Longrightarrow Har_{p+q}(R, M).$$

Proof: The shuffle product preserves the grading of $T(sR)$ by tensor powers. Thus the filtration F_pC_* of Proposition 1.19 induces a filtration on the Harrison chain complex $CHar_*(R, M)$. Similarly the filtration F^pC^* restricts to the Harrison cochain complex. Now, the proof of Proposition 1.11 applies also in these cases. ■

Remark 2.29. It is of course possible to work with more general ground ring k and general C_∞-algebras and bimodules. In that case, we have to assume that M is a C^{op}_∞-bimodule in statements relative to homology (and a C_∞-bimodule in statement relative to cohomology). Henceforth we shall do so without further comments when there is no risk for confusion, for instance in Theorem 3.1.

3. λ-operations and Hodge decomposition

This section is devoted to the definition and study of the Hodge decomposition for Hochschild cohomology of C_∞-algebras. We first recall quickly some basic facts about λ-rings. The λ-*operations* are standard maps that exists on the Hochschild and cyclic (co)homology of a commutative algebra [**GS1, Lo1**]. They yield a Hodge decomposition in characteristic zero and give a structure of γ-ring with trivial multiplication to the (co)homology groups. A *γ-ring with trivial multiplication* $(A,(\lambda^k))$ is a k-module A equipped with linear maps $\lambda^n : A \to A$ $(n \geq 1)$ such that λ^1 is the identity map and

$$\lambda^p \circ \lambda^q = \lambda^{pq}.$$

A γ-ring with trivial multiplication $(A,(\lambda^n))$ has a canonical decreasing filtration $F^\gamma_\bullet A$. For $n \geq 1$, denote $\gamma^n = \sum_{i=0}^{n-1}\binom{k-1}{i}(-1)^{k-i-1}\lambda^{k-i}$. It is standard that λ^k acts as multiplication by k^n on each associated graded module $\mathrm{Gr}^{(n)}A = F^\gamma_n A/F^\gamma_{n+1}A$. In many cases this filtration splits A into pieces $A^{(i)}$ which are the n^i-eigenspaces of the maps λ^n, see [**Lo1**] for more details.

The tensor coalgebra $(T(sR),\delta,\bullet)$ is a graded bialgebra, indeed a Hopf algebra, which is commutative as an algebra. Thus there exist maps $\psi^p : T(sR) \to T(sR)$ defined by

$$\psi^p \quad = \quad \mathrm{sh}^{p-1} \circ (\delta)^{p-1} \tag{3.12}$$

which yield a γ-ring with trivial multiplication structure on $T(sR)$ [**Lo2, Lo3, Pa**]. These maps induce the γ-ring structure and Hodge structure in Hochschild (co)homology.

When the ground ring k contains the rational numbers $\mathbb{Q}$, there is a family of orthogonal idempotents $e^{(i)} : T(sR) \to T(sR)$ such that the tensor coalgebra $T(sR) = \bigoplus_{n\geq 0} e^{(i)}(T(sR))$ with $e^{(0)}(T(sR)) = k$ and

$$e^{(1)}(T(sR)) = T(sR)/T(sR) \bullet T(sR) \cong T(sR)/\mathrm{sh}$$

is the set of indecomposable of the shuffle bialgebra $T(sR)$. Furthermore, the idempotents $e^{(i)}$ are linear combinations of the maps ψ^n and $e^{(i)}(T(sR))$ is the n^i-eigenspaces of the map ψ^n.

3.1. Hochschild cohomology decomposition.

In this section we study the λ-operations on Hochschild cohomology of a C_∞-algebra R with values in a C_∞^{op}-bimodule M.

A coderivation $f \in \mathrm{CoDer}(R,M)$ is uniquely defined by its components $f_i : R^{\otimes i \geq 0} \to M$. Thus, for $n \geq 1$, we obtain the coderivation

$$\lambda^n(f) := \big(f_i \circ \psi^n/_{R^{\otimes i}}\big)_{i\geq 0}$$

defined by the maps $f_i \circ \psi^n : R^{\otimes i} \to M$.

THEOREM 3.1. *Let (R,D) be a C_∞-algebra and (M,D^M) be a C_∞^{op}-bimodule over R.*

1. *The maps $(\lambda^i)_{i\geq 0}$ give a γ-ring with trivial multiplication structure to the Hochschild cochain complex* $(\mathrm{CoDer}(R,M),b)$ *and the Hochschild cohomology $HH^*(R,M)$.*
2. *If k contains $\mathbb{Q}$, there is a natural Hodge decomposition*

$$HH^*(R,M) = \prod_{n\geq 0} HH^*_{(n)}(R,M)$$

into eigenspaces for the maps λ^n. Moreover $HH^*_{(1)}(R,M) \cong Har^*(R,M)$ and $HH^*_{(0)}(R,M) \cong H^*(M)$.

(3) *If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, there is a natural Hodge decomposition*

$$HH^*(R,M) = \bigoplus_{0\leq n\leq p-1} HH^*_{(n)}(R,M)$$

with each λ^n acts by multiplication by n^i on $HH^*_{(i)}(R,M)$. There is a natural linear map $Har^*(R,M)\to HH^*_{(1)}(R,M)$ inducing an isomorphism $HH^*_{(1)}(R,M)^{q\geq *-p+1}\cong Har^*(R,M)^{q\geq *-p+1}$.

Proof: The identity $\lambda^i\circ\lambda^j=\lambda^{ij}$ is immediate from $\psi^j\circ\psi^i=\psi^{ij}$. Moreover $\lambda^1(f)=f$. To prove the first part of the theorem it remains to check that the maps λ^n are chain complex morphisms. By Definition 2.1, the differential D is both a derivation and a coderivation. Thus the differential D commutes with the maps ψ^p. Hence with the Eulerian idempotents when they are defined. By definition of the differential b, it is sufficient to prove, for $p\geq 1$, that

$$\mathrm{pr}\left(D^M(f(\psi^{p+1}))-D^M(\psi^{p+1}(f))\right)=0$$

where $\mathrm{pr}:T(sR)\to sR$ is the canonical projection. Let x be in $R^{\otimes n}$. Since the shuffle product $\bullet$ is a map of coalgebra, one has

$$\begin{aligned}\mathrm{pr}\left(D^M\circ f(\psi^{p+1}(x))\right) &= \sum D^M_{**}\big(\pm x^{(1)}\bullet x^{(4)}\cdots\bullet x^{(3p-2)}\otimes f_*\Big(x^{(2)}\bullet x^{(5)}\bullet\\ &\qquad \cdots\bullet x^{(3p-1)}\Big)\otimes x^{(3)}\bullet x^{(6)}\bullet\cdots\bullet x^{(3p)}\big)\end{aligned}$$

where the sum is over all possible indexes (recall that we are using Sweedler's notation). By definition 2.10, the terms where $x^{(3)}$ or $x^{(4)}$ are not in k cancel out each others (fixing all other components, it follows immediately from the definition). The same argument works for the terms $x^{(3i)}$ or $x^{(3i+1)}$, $i\leq p-1$. Thus we are left with

$$\begin{aligned}\mathrm{pr}\left(D^M\circ f(\psi^{p+1}(x))\right) &= \sum D^M_{**}\big(\pm x^{(1)}\otimes f_*\Big(x^{(2)}\bullet x^{(3)}\bullet\cdots\bullet x^{(p+1)}\Big)\otimes x^{(p+2)}\big)\\ &= \mathrm{pr}\left(D^M(f\circ\psi^{p+1}(x))\right)\end{aligned}$$

and the first part of the theorem follows.

If $k\supset\mathbb{Q}$, the idempotents $e^{(k)}$ are defined. By the first part of the theorem the Hochschild cochain space splits as the product

$$C^*(R,M)=\prod_{i\geq 0} C^*_{(k)}(R,M)$$

where $C^*_{(i)}(R,M)$ is the subspace of coderivation f whose defining maps f_i factors through $e^{(k)}(T(sR))$. We write $C^*_{(i)}(R,M)=\mathrm{CoDer}\big(e^{(k)}(T(sR)),A^{\perp}_R(M)\big)$ by abuse of notation. This yields the Hochschild cohomology decomposition. It is standard that $\psi^n=\sum_{i\geq 0}n^ie^{(i)}$. Moreover

$$(C^*_{(0)}(R,M),b)=(\mathrm{CoDer}(k,A^{\perp}_R(M)),b)\cong\big(\mathrm{Hom}(k,M),D^M_{00}\big).$$

By section 2.2, Harrison cohomology is well defined. As $e^{(1)}(T(sR))\cong T(sR)/\mathrm{sh}$, one has

$$C^*_{(1)}(R,M)\cong(\mathrm{Hom}(T(sR)/\mathrm{sh},A^{\perp}_R(M)),b)\cong(\mathrm{BDer}(R,M),b)=CHar^*(R,M).$$

If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, the operators $\overline{e_n^{(i)}} = \sum_{m\geq 0} e_n^{(i+(p-1)m)} : R^{\otimes n} \to R^{\otimes n}$ are well defined for $1 \leq i \leq p-1$ and $n \geq 1$, see [**GS2**, Section 5]. Denote $\overline{e^{(i)}}$ the map induced by the operators $\overline{e_n^{(i)}}$ for n varying. Note that $\overline{e_n^{(i)}} = e_n^{(i)}$ for $n \leq p-1$. As above, the Hochschild differential commutes with the operators $\overline{e^{(i)}}$, thus the Hochschild cochain complex splits as

$$C^*(R,M) = \bigoplus_{0\leq i\leq p-1} \overline{C^*_{(i)}}(R,M)$$

where $\overline{C^*_{(i)}}(R,M) = \mathrm{CoDer}(\overline{e^{(i)}}(T(sR)), A_R^{\perp}(M))$. By [**GS2**], $(\overline{e^{(1)}}(T(sR))$ lies in the quotient of the cotensor coalgebra $T(sR)/\mathrm{sh}$. It follows that we have a canonical map $Har^*(R,M) \to HH^*_{(1)}(R,M)$ (see Definition 2.22) which is an isomorphism when restricted to components of external degree $q \geq * - p + 1$ because $\overline{e_n^{(1)}} = e_n^{(1)}$ for $n \leq p-1$. Since $p \in k$ is null, reasoning as above we get that the complexes $\overline{C^*_{(i\geq 1)}}(R,M)$ are n^i-eigenspaces for λ^i. ■

Remark 3.2. The γ-ring structure given by Proposition 3.1.(1) gives rise to the canonical filtration of complexes $F^{\gamma}_{\bullet}(\mathrm{CoDer}(R,M),b)$. Hence there is a spectral sequence

$$E^{\gamma}{}_1^{p,q} = H^{p+q}(F^{\gamma}_q F^{\gamma}_{q+1}) \Longrightarrow HH^{p+q}(R,M).$$

Denote $F^{n,(q)}_{ind}(R,M) := \mathrm{Im}(H^n(F^{\gamma}_q) \to HH^{p+q}(R,M))$, the induced filtration on $HH^*(R,M)$. The argument of [**Lo1**, Théorème 3.5] shows that $E_1^{p,2} \cong Har^p(R,M)$ and $F^{n,(q)}_{ind}(R,M)^{*\geq n-q+2} \cong 0$, $F^{n,(1)}_{ind}(R,M) \cong HH^n(R,M)$.

Example 3.3. By Proposition 2.19, the Hochschild cohomology $HH^*(R,R^{\star})$ always has a γ-ring structure. When R is free and R is strong, $HH^*(R,R)$ is also a γ-ring according to Proposition 2.16.

Remark 3.4. The splitting of the differential graded bialgebra $T(sR)$ (with the differential giving the C_∞-structure and the shuffle product) used in the proof of Theorem 3.1 is also the one given in [**WGS**] for the shuffle bialgebra.

Theorem 3.1 applies to strict algebras. For a strict commutative algebra R, we denote by Ω^*_R the graded exterior algebra of the graded Kähler differential R-module Ω^1_R. The decompositions given by Theorem 3.1 agree with the classical ones for strict algebras according to

Proposition 3.5. *Let (R,d) be a differential graded commutative algebra and M a symmetric bimodule. Then there exist λ-operations on $HH^*(R,M)$. If $k \supset \mathbb{Q}$, the λ-operations yield a Hodge decomposition of the Hochschild cohomology of R:*

$$HH^n(R,M) = \prod_{i\geq 0}^{n} HH^n_{(i)}(R,M) \text{ for } n \geq 1$$

Moreover one has:

i) *If R is unital and $k \supset \mathbb{Q}$, $HH^n_{(j)}(R,M) \supset H^{n-j}(\mathrm{Hom}_R(\Omega^j_R,M),d^*)$ for $n \geq 1$, $j \geq 0$, this inclusion being an isomorphism if R is smooth;*

ii) *If R and M are non-graded, then the decomposition coincides with the one of Gerstenhaber and Schack* [**GS1, GS2**].

Proof: By Example 2.11, we know that M is a C_∞^{op}-bimodule over R. By Example 1.10, the Hochschild cochain complex of R as a C_∞-algebra is isomorphic to its usual Hochschild complex as an associative algebra. Through this isomorphism the operation λ^i becomes

$$f \in \mathrm{Hom}(R^{\otimes n}, M) \mapsto f \circ \psi^i. \tag{3.13}$$

Thus when $k \supset \mathbb{Q}$, the induced splitting coincides with the one of [**GS1**] and *ii*) is proved. Theorem 3.1 implies that the Hochschild cohomology of R (with its canonical C_∞-structure) admits a Hodge type decomposition. We know from Example 1.10 that this cohomology coincides with the usual Hochschild cohomology.

When R is furthermore unital, there is a canonical isomorphism of cochain complexes

$$\mathrm{Hom}(R^{\otimes n}, M) \cong \mathrm{Hom}_R(R \otimes R^{\otimes n}, M)$$

where the differential on the right is dual to the Hochschild differential on the Hochschild complex $C_*(R,R) = R \otimes R^{\otimes *}$. There is the well-known canonical map $R \otimes T(sR) \xrightarrow{\pi} \Omega^*_R$ given by $\pi(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 \partial a_1 \ldots \partial a_n$ which is a map of complexes. The differential on Ω^*_R is the one induced by the inner differential $d : R \to R$. Hence we get a chain map

$$\pi^* : (\mathrm{Hom}_R(\Omega^j_R, M), d^*) \to (\mathrm{CoDer}(R, M), b).$$

It is known that $\Omega^i_R \cong R \otimes e^{(i)}(R^{\otimes i})$ [**Lo2**]. Thus the chain map π^* splits and identifies $\mathrm{Hom}_R(\Omega^j_R, M), d^*)$ as a subcomplex of $C^*_{(i)}(R, M)$. Also, when R is smooth, the map π_* is a quasi-isomorphism and Ω^1_R is projective over R, thus π^* is a quasi-isomorphism too. ■

REMARK 3.6.
- Proposition 3.5 applies to *non unital* algebras.
- If R and M are non-graded and moreover flat over $k \supset \mathbb{Q}$, then assertion *ii*) implies that
$$HH^n_{(i)}(R, M) \cong AQ^{n-i}_i(R/k, M)$$
where $AQ^*_k(R/k, M)$ is the higher André-Quillen cohomology of R with coefficients in M.

Recall that for any C_∞-algebra (R, D^C), the map $D_1 : R \to R$ is a differential and that we denote $H^*(R)$ the homology of (R, D_1). Similarly, for an R-bimodule M, $H^*(M)$ is the homology of (M, D^M_{00}). According to Proposition 1.11, there is a spectral sequence abutting to $HH^*(R, M)$. It is in fact a spectral sequence of γ-rings.

PROPOSITION 3.7. *Let (R, D) be a C_∞-algebra with R, $H^*(R)$ flat as a k-module and M be a C_∞^{op}-bimodule with $M, H^*(M)$ flat.*
- *The spectral sequence $E^{*,*}_2 = HH^*(H^*(R), H^*(M)) \Longrightarrow HH^*(R, M)$ is a spectral sequence of γ-rings (with trivial multiplication).*
- *If $k \supset \mathbb{Q}$, then the above spectral sequence splits into pieces*
$$AQ^{n-i}_{(i)}(H^*(R), H^*(M)) \Longrightarrow HH^*_{(i)}(R, M).$$

Proof: The spectral sequence of Proposition 1.11 is induced by the filtration $F^*C^*(R, M)$. The maps λ^n preserves the filtration, thus the γ-ring structure passes to the spectral sequence. The E^{**}_1 term corresponds to the Hochschild cochain complex of the commutative algebra $H^*(R)$ with values in the symmetric bimodule $H^*(M)$. Example 1.10 ensures that the induced operations $\lambda^n : E_1^{**} \to E_1^{**}$ corresponds to Gerstenhaber-Schack standard ones on the cochain complex, hence in its cohomology E_2^{**}.

When $k \supset \mathbb{Q}$, the Hochschild cochain complex splits as a direct product of complexes $\prod_{i\geq 0} C^*_{(k)}(R,M)$. Furthermore, this splitting is induced by the Eulerian idempotent, which are linear combination of the maps λ^n. Thus the filtration on $C^*(R,M)$ is identified with a product of filtered complexes $F^\bullet C^*_{(i)}(R,M)$. As above, we find that the level $E_1^{**}C^*_{(i)}(R,M)$ is given by the weight i part $C^*_{(i)}(H^*(R),H^*(M))$ of $C^*(H^*(R),H^*(M))$. The flatness and rational assumptions ensures that the cohomology of $C^*_{(i)}(H^*(R),H^*(M))$ is the André-Quillen cohomology $AQ^{n-i}_{(i)}(H^*(R),H^*(M))$ see [**Lo2**], Chapter 3. ∎

Remark 3.8. One easily checks that, when R is flat over $k \supset \mathbb{Q}$, the weight 1 part of the spectral sequence coincides with the Harrison cohomology spectral sequence 2.28. Notice that the spectral sequences also splits with respects to the partial Hodge decomposition if $k \supset \mathbb{Z}/p\mathbb{Z}$.

Let $F : (S,B) \to (R,D)$ be a map of A_∞-algebras and (M,D^M) be an R-bimodule. Let B^M be the coderivation of $A^\perp_S(M)$ given by the defining maps

$$\left(B^M_{pq} : S^{\otimes p}\otimes M\otimes S^{\otimes q} \stackrel{F\otimes \mathrm{id}\otimes F}{\longrightarrow} A^\perp_R(M) \stackrel{D^M}{\longrightarrow} A^\perp_R(M) \stackrel{\mathrm{pr}}{\longrightarrow} M\right)_{p,q\geq 0}.$$

Lemma 3.9. *The coderivation B^M endows M with a structure of S-bimodule. Furthermore, if F is a map of C_∞-algebras and M is a C^{op}_∞-bimodule, then M is also a C^{op}_∞-bimodule over S.*

Proof: We have to check that $(B^M)^2=0$. For $x\in A^\perp_R(M)$, the tensor $B^M\big(B^M(x)\big)$ is a sum of three kinds of elements: the ones which only involve the maps B_n, the ones which only involve the maps B^M_{nm} $(m,n\geq 0)$ and the ones which involve one B_n and one B^M_{pq} $(n,p,q\geq 0)$. Since $B\circ B=0$ the sum of elements of the first kind vanishes. Since the degrees $|B|=|B^M|=1$ are odd, the sum of elements of the third kind is also null. The sum of terms of the second kind vanishes because the projections on M satisfies

$$\mathrm{pr}\left(\sum B^M_{nm}\circ(F\otimes \mathrm{id}\otimes F)\,(B^M(x))\right) = \mathrm{pr}\left(D^M\circ D^M(F\otimes\mathrm{id}\otimes F)\right)=0.$$

When F is a C_∞-map, each of its components F_i vanish on shuffles. It follows that the coalgebra map $F : A^\perp(S)\to A^\perp(R)$ is also a map of algebras (for the shuffle product). Furthermore, according to Proposition 2.19, $A^\perp_R(M)$ is a differential module over the shuffle algebra $A^\perp(R)$. Thus $A^\perp_S(M)$ is a differential module over the shuffle algebra $A^\perp(S)$. Hence M is a C_∞-bimodule over S. ∎

Proposition 3.10. *Let (R,D) be an A_∞-algebra, (M,D^M) an R-bimodule and (S,B) be a A_∞-algebras.*

- *If there is an A_∞-map $F:(S,B)\to(R,D)$, then there is a linear map*

$$F^* : HH^*(R,M)\to HH^*(S,M)$$

which is an isomorphism if $F_1 : (S,d_S)\to(R,d_R)$ is a quasi-isomorphism.

- *If* (N, D^N) *is an* R*-bimodule and* $\phi : (M, D^M) \to (N, D^N)$ *is a* A*-bimodule map, then there is a linear map*
$$\phi_* : HH^*(R, M) \to HH^*(R, N)$$
which is an isomorphism when $\phi_1 : (M, d^M) \to (N, d^N)$ *is a quasi-isomorphism.*
- *When* R*,* S *are* C_∞*-algebras,* M*,* N *are* C_∞*-bimodules and* F*,* ϕ C_∞*-morphisms, then* F^* *and* ϕ_* *are maps of* γ*-rings.*

Proof: Lemma 3.9 implies that M has an S-bimodule structure given by the map B^M. The morphism of differential coalgebras F induces a morphism of coderivations $F^* : \mathrm{CoDer}(R, M) \to \mathrm{CoDer}(S, M)$. For any x in $C^*(R, M) = \mathrm{CoDer}(R, M)$, one has,

$$\begin{aligned} b(F(x)) &= B^M(x(F)) - x(F(B)) \\ &= B^M(x)(F) - x(D(F)) \end{aligned}$$

hence F is a morphism of complex. If F_1 is a quasi-isomorphism, then $\widetilde{F_1}$ is an isomorphism at the page 1 of the spectral sequences associated to $HH^*(R, M)$, $HH^*(S, M)$ by Proposition 1.11. The first assertion is proved. The second one is analogous using the application $\phi_* : F \in C^*(R, M) \mapsto \phi \circ f \in C^*(R, N)$ instead of F^*.

Moreover when F is a C_∞-map, then M is a C_∞-bimodule by Lemma 3.9 and the λ-operations already commutes with F^* and ϕ_* at the cochain level; the compatibility is proved as in Theorem 3.1. ■

A C_∞-algebra (R, D) is said to be *formal* if there is a morphism
$$F : (H^*(R), D_2) \to (R, D)$$
of C_∞-algebras with F_1 a quasi-isomorphism.

Corollary 3.11. *Let* (R, D) *be a formal* C_∞*-algebra, free as a* k*-module. If there is a* C_∞*-map* $F : (H^*(R), D_2) \to (R, D)$ *with* F_1 *a quasi-isomorphism, then there is a natural isomorphism of* γ*-rings:*
$$HH^*\big(R, R\big) \cong HH^*\big(H^*(R), H^*(R)\big)$$
and if $k \supset \mathbb{Q}$*, an isomorphism of Hodge decomposition*
$$HH^*_{(n)}\big(R, R\big) \cong HH^*_{(n)}\big(H^*(R), H^*(R)\big) \text{ for } n \geq 0.$$

Proof: We denote by $\phi : H^*(R) \to R$ the morphism of C_∞-bimodule induced by F. Proposition 3.10 yields a zigzag
$$HH^*\big(R, R\big) \xrightarrow{\widetilde{F^*}} HH^*\big(H^*(R), R\big) \xleftarrow{\widetilde{\phi^*}} HH^*\big(H^*(R), H^*(R)\big)$$
where the arrows are isomorphisms of γ-rings. Hence the result. ■

Remark 3.12. The definition of *formality* that we use here is quite strong. However, it is enough for our purpose. In the literature, one might find the definition that (R, D) is formal if $(H^*(R), D_2)$ and (R, D) are connected by a chain of C_∞-quasi-isomorphisms. When k is a characteristic zero field, these two definitions agree since one can check that C_∞-quasi-isomorphisms are invertible. This is also the case over any field if one only considers strong C_∞-algebras. Details are left to the reader.

Proposition 3.13. *Let (R, D) be a C_∞-algebra and $F_1 : H^*(R) \to R$ a quasi-isomorphism inducing the product structure on $H^*(R)$. If*

$$Har^n(H^*(R), H^*(R))_{\leq 2-n} = 0 \text{ for } n \geq 1$$

then R is formal.

Proof: The techniques of [**GH2**] for homotopy Gerstenhaber algebras apply *mutatis mutandis* to C_∞-algebras as well. Thus, given a quasi-isomorphism F_1 : $H^*(R) \to R$, there is a C_∞-structure $(H^*(R), B)$ and a C_∞-quasi-isomorphism $G : (H^*(R), B) \to (R, D)$ such that $B_1 = 0, B_2 = D_2$ and $G_1 = F_1$. When the Harrison cohomology is concentrated in bidegree $(1, *)$, for the bigrading induced by the tensor power of maps and internal degree of $H^*(R)$, there is a C_∞-morphism $H : (H^*(R), D_2) \to (H^*(R), B)$ with H_1 being the identity map. The composition of this two C_∞-maps gives the formality map. ■

Example 3.14. By a deep result of Tamarkin [**Ta**], it is now well-known that the Hochschild cochain complex of any associative algebra A, over a characteristic zero ring, has a G_∞-structure, hence a C_∞-one, which is (non-canonically) induced by the cup-product and the braces of [**GV**]. For the algebra $A = C^\infty(X)$ of smooth functions on a manifold X, the Hochschild cochain complex $C^*(A, A)$ of multilinear and multidifferential operators on A is a formal C_∞-algebra. Its cohomology is $\Gamma = \Gamma(X, \Lambda TX)$ the polyvector fields on X. We can apply Proposition 3.13 and then Corollary 3.11 (because the Harrison cohomology of Γ vanish) to find

$$\begin{aligned} HH^*_{(j)}(C^*(A, A), C^*(A, A)) &\cong HH^*_{(j)}\big(\Lambda^*\Gamma(X, \Lambda TX), \Lambda^*\Gamma(X, \Lambda TX)\big) \\ &\cong \mathrm{Hom}_\Gamma(\Omega^j_\Gamma, \Gamma) \\ &\cong \Lambda^j s\Omega_\Gamma. \end{aligned}$$

The last step follows from the Jacobi-Zariski exact sequence applied to the smooth algebra Γ that leads to $\Omega_\Gamma \cong \Gamma \otimes_R \Omega_R \oplus s(\Omega_R)^\star$. Moreover, the Hochschild chain complex $C_*(A, A)$ is a C_∞^{op}-bimodule by Proposition 2.19. From the previous argument one easily gets

$$HH^*_{(j)}(C^*(A, A), C_*(A, A)) \cong \mathrm{Hom}_\Gamma(\Omega^j_\Gamma, \Omega^*_A).$$

Example 3.15. When formality does not hold, Proposition 3.7 can be used to study $HH^*\big(C^*(R, R), C^*(R, R)\big)$. For instance, Let R be a semi-simple separable algebra, then $HH^*(R) = HH^0(R) = Z(R)$ the center of R. It follows that the spectral sequence E_1^{**} is concentrated in bidegree $(*, 0)$ hence collapses. Thus one has

$$HH^*\big(C^*(R, R), C^*(R, R)\big) \cong HH^*\big(Z(R), Z(R)\big)$$

which is an isomorphism of Gerstenhaber algebras and γ-rings on the associated graded to the canonical filtration of $A^\perp(R)$.

Proposition 3.16. *Let k be a characteristic zero field and (R, D) be a C_∞-algebra with $D_1 = 0$. Assume that there is an element $1 \in R^0$ which is a unit for D_2. Let N be a C_∞^{op}-module.*

- *If (R, D_2) is smooth, for any $n \geq 0$, one has*

$$HH^*_{(n)}(R, N) \cong \mathrm{Hom}_R(\Omega^n_R, N).$$

- *If (R, D_2) is not necessarily smooth but satisfies $D_3(1, x, y) = D_3(y, 1, x) = 0$, then*
$$HH^*_{(n)}(R, N) \supset \mathrm{Hom}_R(\Omega^n_R, N).$$

Proof: Let $\pi_n : M \otimes R^{\otimes n} \to M \otimes_{(R,D_2)} \Omega^n_{(R,D_2)}$ be the canonical surjection. It factors through $M \otimes e^{(n)}(R^{\otimes n})$. The C_∞-differential D commutes with $e^{(n)}$. Moreover the map $D_{i\geq 2}\big(R^{\otimes n}\big) \subset R^{\otimes \leq n-1}$. Thus, as π_* factors through $M \otimes e^{(n)}(R^{\otimes n})$, the map $\pi_* : (C_*(R, M), b) \to (M \otimes_{(R,D_2)} \Omega^*_{(R,D_2)}, 0)$ is a chain map. This is a generalization of a well-known fact for strict algebras [**Lo2**, chapter 4]. Therefore we obtain a morphism of cochain complexes
$$(\mathrm{Hom}_R(\Omega^n_R, N), 0) \hookrightarrow (C^*(R, N), b).$$
The filtration by the exterior power of $\mathrm{Hom}_R(\Omega^*_{(R,D_2)}, R)$ yields a spectral sequence computing $\mathrm{Hom}_R(\Omega^*_{(R,D_2)}, R)$. The complex map π_* yields a map between this spectral sequence and the one of Proposition 1.11 for $HH^*(R, M)$. When (R, D_2) is smooth, the map π_1 is an isomorphism at page 1 by the Hochschild Kostant Rosenberg theorem, thus on the abutment. If D_3 vanishes when one of the variable is the unit, then the anti-symmetrization map $\varepsilon_n : \Omega^n_{(R,D_2)} \to M \otimes R^{\otimes n}$ is well defined modulo a boundary of $(M \otimes T(sR), b)$, as for strict commutative algebras. Thus we have a well defined map
$$\varepsilon_* : HH^*(R, M) \to \mathrm{Hom}_R(\Omega^n_R, N)$$
which is a section of π_* up to multiplication by non zero scalars [**Lo2**]. ■

Example 3.17. A C_∞-algebra satisfying $D_1 = 0$ is called a minimal C_∞-algebra. Formal Frobenius algebras are a huge class of examples [**Ma**].

3.2. Decomposition in Hochschild homology. In this section we define and study the Hodge decomposition for Hochschild homology of C_∞-algebras. We denote $\overline{\lambda}^p : M \otimes T(sR) \to M \otimes T(sR)$ the map $\mathrm{id} \otimes \psi^p$ for $p \geq 1$, where ψ^p is defined by formula (3.12).

Theorem 3.18. *Let R be a C_∞-algebra and M an C_∞-bimodule over R.*

(1) *The maps $(\overline{\lambda}^i)_{i\geq 1}$ define a γ-ring (with trivial multiplication) structure on the Hochschild complex $(M \otimes T(sR), b)$ and on Hochschild homology $HH_*(R, M)$.*
(2) *If k contains $\mathbb{Q}$, there is a Hodge decomposition*
$$HH_*(R, M) = \bigoplus_{i\geq 0} HH^{(i)}_*(R, M)$$
into eigenspaces for the maps $\overline{\lambda}^n$. Moreover $HH^{(1)}_*(R, M) \cong Har_*(R, M)$ and $HH^{(0)}_*(R, M) \cong H^*(M, D^M_{00})$.
(3) *If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, there is a Hodge decomposition*
$$HH_*(R, M) = \bigoplus_{0\leq n\leq p-1} HH^{(n)}_*(R, M)$$
with $\overline{\lambda}^n$ acting by multiplication by n^i on $HH^{(i)}_*(R, M)$. Furthermore, there is a natural linear map $HH^{(1)}_*(R, M) \to Har_*(R, M)$ inducing an isomorphism $HH^{(1)}_*(R, M)^{*\leq p-1-n} \cong Har_*(R, M)^{*\leq p-1-n}$.

Proof: The proof is similar to the one of Theorem 3.1. The only slight difference is the compatibility of the maps ψ^{p+1} ($p \geq 1$) with the differential D^M. For $m \in M, x \in T(sR)$ one has

$$\begin{aligned} D^M\Big(\overline{\lambda}^{p+1}(x)\Big) &= \sum D^M_{**}\Big(\pm\, x^{(3)} \bullet x^{(6)} \cdots \bullet x^{(3p)} \otimes m \otimes x^{(1)} \bullet x^{(4)} \bullet \\ &\quad \cdots \bullet x^{(3p-2)}\Big) \otimes x^{(2)} \bullet x^{(5)} \bullet \cdots \bullet x^{(3p-1)} \end{aligned}$$

All the terms for which $x^{(3i+1)}$ or $x^{(3i)}$ is non trivial ($1 \leq i \leq p-1$) vanish by definition of a C_∞-module. Thus

$$\begin{aligned} D^M\big(\overline{\lambda}^{p+1}(x)\big) &= \pm D^M_{**}\Big(x^{(p+2)} \otimes m \otimes x^{(1)}\Big) \otimes x^{(2)} \bullet x^{(3)} \bullet \cdots \bullet x^{(p+1)} \\ &= \overline{\lambda}^{p+1}(D^M(x)). \end{aligned}$$

■

EXAMPLE 3.19. When R is a differential graded commutative algebra and M a symmetric bimodule, the γ-ring structures and Hodge decompositions given by Theorem 3.18 coincides with the classical ones [**Lo1, Vi**].

REMARK 3.20. Similarly to Remark 3.2, the γ-ring structure given by Theorem 3.18.(1) gives rise to a canonical filtration of complexes $F^\gamma_\bullet(M \otimes T(sR), b)$ and thus a spectral sequence $E^{\gamma}{}^1_{p,q} = H_{p+q}(F^\gamma_q F^\gamma_{q+1}) \Longrightarrow HH_{p+q}(R, M)$. The induced filtration $F^{ind}_{n,(q)}(R, M) := \mathrm{Im}(H^n(F^\gamma_q) \to HH^{p+q}(R, M))$ satisfies $E^1_{p,2} \cong Har_p(R, M)$ and $F^{ind}_{n,(q)}(R, M)^{*\geq q-2-n} \cong 0$, $F^{ind}_{n,(1)}(R, M) \cong HH_n(R, M)$.

PROPOSITION 3.21. *Let (R, D) be a C_∞-algebra with R, $H^*(R)$ flat as a k-module and M be a C^{op}_∞-bimodule such that M and $H^*(M)$ are flat.*

- *The spectral sequence $E^2_{*,*} = HH_*(H^*(R), H^*(M)) \Longrightarrow HH_*(R, M)$ (see Theorem 1.19) is a spectral sequence of γ-rings (with trivial multiplication).*
- *If $k \supset \mathbb{Q}$, the spectral sequence splits into pieces*

$$AQ^{(i)}_{n-i}(H^*(R), H^*(M)) \Longrightarrow HH^{(i)}_*(R, M).$$

Proof: The proof is dual to the one of Theorem 1.19 and Proposition 3.7 using the dual filtration $F_iC_*(R, M) = M \otimes R^{\otimes * \leq i}$. ■

REMARK 3.22. One easily checks that when R is flat over $k \supset \mathbb{Q}$, the weight 1 part of the spectral sequence coincides with the Harrison homology spectral sequence of Proposition 2.28.

When the bimodule M is the C_∞-algebra R, the Hochschild complex is a C_∞-algebra. For $i \geq 2$, let $B_i : \big(R \otimes T(sR)\big)^{\otimes i} \to R \otimes T(sR)$ be the map defined, for $r_k \otimes x_k \in R \otimes T(sR)$, $k = 1 \dots i$, by

$$B_i(r_1 \otimes x_1, \dots, r_i \otimes x_i) = \sum_{j \geq i} \pm D_j\big(x_1^{(3)} \otimes r_1 \otimes x_1^{(1)} \triangleleft \cdots \triangleleft x_i^{(3)} \otimes r_i \otimes x_i^{(1)}\big) \otimes x_1^{(2)} \bullet \cdots \bullet x_i^{(2)}$$

where $x \otimes r_1 \otimes y \triangleleft x' \otimes r_2 \otimes y'$ is obtained from the shuffle product $x \otimes r_1 \otimes y \bullet x' \otimes r_2 \otimes y'$ by taking only shuffles such that r_1 appears before r_2. Take the Hochschild differential b for B_1 and write $B : C_*(R, R) \otimes T(sC_*(R, R)) \to C_*(R, R) \otimes T(sC_*(R, R))$ for the codifferential induced by the maps B_i.

PROPOSITION 3.23.
- *$(C_*(R,R),B)$ is a C_∞-algebra. In particular B_2 induces a structure of commutative algebra on $HH_*(R,R)$.*
- *$B_i(\overline{\lambda^k}^{\otimes i}) = \overline{\lambda^k}(B_i)$ and in particular the operations $\overline{\lambda^k}$ are multiplicative in Hochschild homology.*
- *The spectral sequence $E^2_{*,*} = HH_*(H^*(R),H^*(R)) \Longrightarrow HH_*(R,R)$ is a spectral sequence of algebras equipped with multiplicative operations $\overline{\lambda^k}$.*

Proof: By commutativity of the shuffle product, the vanishing of the maps $B_{i\geq 2}$ on shuffles amounts to the vanishing of

$$D_{j\geq p+q}(x_1^{(3)}r_1x_1^{(1)} \lhd \ldots \lhd x_p^{(3)}r_px_p^{(1)} \bullet y_1^{(3)}s_1y_1^{(1)} \lhd \ldots \lhd y_q^{(3)}s_qy_q^{(1)})$$

which follows since R is a C_∞-algebra. Since

$$D(x_1 \bullet \cdots \bullet x_i) = \sum \pm x_1 \bullet \cdots \bullet D(x_j) \bullet \cdots \bullet x_i,$$

the vanishing of B^2 is equivalent to the equation

$$\sum \pm D_*\Big(x_1^{(3)}r_1x_1^{(1)} \lhd \ldots \lhd x_i^{(3)}r_ix_i^{(1)} \lhd x_{i+1}^{(4)} \bullet \cdots \bullet x_j^{(4)} D_*\big(x_{i+1}^{(5)}r_{i+1}x_{i+1}^{(1)} \lhd \ldots$$
$$\ldots x_j^{(5)}r_jx_j^{(1)}\big)x_{i+1}^{(2)} \bullet \cdots \bullet x_j^{(2)} \lhd x_{j+1}^{(3)}r_{j+1}x_{j+1}^{(1)} \lhd \ldots \lhd x_n^{(3)}r_nx_n^{(1)}\Big) = 0.$$

This is the A_∞-equation (1.4) applied to

$$x_1^{(3)}r_1x_1^{(1)} \lhd \ldots \lhd x_n^{(3)}r_nx_n^{(1)}$$

up to terms like

$$D_*(\ldots D_*(x_i^{(1)} \bullet x_{i+1}^{(4)}r_{i+1}x_{i+1}^{(1)} \lhd \ldots x_j^{(4)}r_jx_j^{(1)}) \ldots)$$

which are trivial by Definition 2.1. Thus the Hochschild complex $(C_*(R,R),B)$ is a C_∞-algebra. In particular its homology for the differential $B_1 = b$ is a commutative algebra.

The maps $\overline{\lambda}^k$ are algebras morphisms (with respect to the shuffle product). As in the proof of Theorem 3.18 (which is the B_1-case), we obtain that $B_i(\overline{\lambda^k}^{\otimes i}) = \overline{\lambda^k}(B_i)$. Furthermore, the map B_i $(i \geq 1)$ preserves the filtration $F^*(C_*(R,R))$. It follows that the spectral sequence E^1_{**} is a spectral sequence of commutative algebras. On the page E_1, the product is given by the usual shuffle product on the Hochschild complex of $H^*(R)$. Since the $\overline{\lambda}^k$-operations on page 2 commutes with the shuffle product, the result follows. ■

REMARK 3.24. When R is a strict commutative algebra, one recovers the usual shuffle product of [**GJ1**].

The functorial properties of Hochschild cohomology holds for homology as well.

PROPOSITION 3.25. *Let (R,D) be an A_∞-algebra, (M,D^M) an R-bimodule and (S,B) be an A_∞-algebra.*
- *An A_∞-morphism $F:(S,B)\to(R,D)$ induces a natural linear map*

  $$F_* : HH_*(S,M) \to HH_*(R,M)$$

  which is an isomorphism if $F_1:(S,B_1)\to(R,D_1)$ is a quasi-isomorphism.

- *Let* (N, D^N) *be an* R*-bimodule and let* $\phi : (M, D^M) \to (N, D^N)$ *be an* R*-bimodule map. There is a natural linear map* $\phi_* : HH_*(R, M) \to HH_*(R, N)$ *which is an isomorphism if* $\phi_1 : (M, D^M_{00}) \to (N, D^N_{00})$ *is a quasi-isomorphism.*
- *Moreover when* R*,* S *are* C_∞*-algebras,* M*,* N C_∞*-bimodules and* F*,* ϕ C_∞*-morphisms, then* F_* *and* ϕ_* *are maps of* γ*-rings.*

Example 3.26. Recall that, for an associative algebra over a field of characteristic zero, the Hochschild cochain complex $C^*(A, A)$ is a C_∞-algebra (Example 3.14). It is moreover formal when $A = C^\infty(X)$. Thus Proposition 3.25 gives an isomorphism

$$HH_*(C^*(R, R), C_*(R, R)) \cong \Gamma(X, \Lambda^*(TX)^\star) \otimes_\Gamma \Omega^*_\Gamma.$$

Recall that if (R, D) is a C_∞-algebra such that $D_1 = 0$, then (R, D_2) is a graded commutative algebra (Example 2.7)

Proposition 3.27. *Let* k *be a characteristic zero field,* (R, D) *be a* C_∞*-algebra such that* $D_1 = 0$ *and* D_2 *unital, and* M *be a* C_∞*-module.*

- *If* (R, D_2) *is smooth, one has*
$$HH^{(n)}_*(R, M) \cong M \otimes_{(R,D_2)} \Omega^n_{(R,D_2)}.$$
- *If* R *is not necessarily smooth but* $D_3(1, x, y) = D_3(y, 1, x) = 0$*, then*
$$HH^{(n)}_*(R, M) \supset M \otimes_{(R,D_2)} \Omega^n_{(R,D_2)}.$$

3.3. The augmentation ideal spectral sequence. In this section, we generalize results of [**WGS**] in the context of C_∞-algebras. In particular, we study the compatibility between the Hodge decomposition and the Gerstenhaber structure, see Theorem 3.31 below.

Convention 3.28. In this section, the ground ring k is either torsion free or a $\mathbb{Z}/p\mathbb{Z}$-algebra. Moreover all k-modules are assumed to be flat.

The (signed) shuffle bialgebra $T(sR)$ has a canonical augmentation $T(sR) \to k \oplus sR$. We wrote $I(sR)$ for its augmentation ideal. There is a decreasing filtration

$$\dots \subset I(sR)^n \subset I(sR)^{n-1} \subset \dots \subset I(sR)^1 \subset I(sR)^0 = T(sR).$$

This filtration induces a filtration of Hochschild (co)chain spaces

$$\dots \subset M \otimes I(sR)^n \subset M \otimes I(sR)^{n-1} \subset \dots \subset M \otimes I(sR)^1 \subset M \otimes I(sR)^0 = C_*(R, M),$$

$$\begin{aligned} C^*(R, M) = &\mathrm{CoDer}(I(sR)^0, A^{\perp}_R(M)) \to \mathrm{CoDer}(I(sR)^1, A^{\perp}_R(M)) \to \dots \\ &\dots \to \mathrm{CoDer}(I(sR)^{n-1}, A^{\perp}_R(M)) \to \mathrm{CoDer}(I(sR)^n, A^{\perp}_R(M)) \to \dots. \end{aligned}$$

We called these filtration *the augmentation ideal filtration.*

Lemma 3.29. *Let* R *be a* C_∞*-algebra,* M *and* N *respectively a* C_∞*-bimodule and a* C^{op}_∞*-bimodule over* R*. The augmentation ideal filtration of* $C_*(R, M)$ *and* $C^*(R, N)$ *are filtration of (co)chain complexes.*

Proof: Since the augmentation ideal filtration is induced by the shuffle product, the result follows as in the proofs of Theorems 3.1, 3.18. ■

By Lemma 3.29, there are *augmentation ideal* spectral sequences

$$I^1_{pq}(R,M) = H_{p+q}(M\otimes I(sR)^p/I(sR)^{p+1}), \tag{3.14}$$
$$I_1^{pq}(R,N) = H^{p+q}(\mathrm{CoDer}(I(sR)^p/I(sR)^{p+1}, A_R^{\perp}(N)). \tag{3.15}$$

Proposition 3.30. *Let R be a C_∞-algebra, M and N two R-bimodules which are respectively a C_∞-bimodule and a C_∞^{op}-bimodule.*

1. *The spectral sequence $I^1_{pq}(R,M)$ converges to $HH_{p+q}(R,M)$ and the spectral sequence $I_1^{pq}(R,N)$ converges to $HH^{p+q}(R,N)$ if R ,M are concentrated in non-negative degrees and N in non-positive degrees.*
2. *Let k be a field. Then $I_1^{1*}(R,N)\cong Har^{*+1}(R,N)$ and $I^1_{1*}(R,M)\cong Har_{*+1}(R,M)$.*
3. *When R is free, $I_1^{1*}(R,R)$ is a spectral sequence of Gerstenhaber algebras.*

Proof: It follows from the combinatorial observations of [**WGS**], Section 3 and 4. The only difficulty is to check that all the constructions are compatible with the C_∞-differential. This is straightforward since the differentials D, D^M, D^N (defining the algebra and bimodules structures) are coderivations for the coproduct and moreover compatible with the filtrations (Lemma 3.29). ■

Theorem 3.31. *Let k be a field and R be a strong C_∞-algebra (see Example 2.4).*

- *The Harrison cohomology $Har^*(R,R) = HH^*_{(1)}(R,R)$ is stable by the Gerstenhaber bracket.*
- *If $k\supset\mathbb{Q}$, the cup-product and Gerstenhaber bracket are filtered for the Hodge filtration $\mathcal{F}_pHH^*(R,R)=\bigoplus_{n\le q}HH^*_{(n)}(R,R)$, in the sense that*

$$\mathcal{F}_pHH^*(R,R)\cup\mathcal{F}_qHH^*(R,R)\subset\mathcal{F}_{p+q}HH^*(R,R) \text{ and}$$

$$[\mathcal{F}_pHH^*(R,R),\mathcal{F}_qHH^*(R,R)]\subset\mathcal{F}_{p+q-1}HH^*(R,R)$$

Proof: Since k is a field, the convention 3.28 is satisfied and furthermore, Theorem 3.1 gives a Hodge decomposition if k is of characteristic zero or a partial Hodge decomposition if k is of positive characteristic. Moreover, the identification of the Harrison cohomology also follows from Theorem 3.1. By Proposition 2.16, R is a C_∞^{op}-bimodule over itself. Let g be in $C^*_{(1)}(R,M)$. Since each component $g_i: R^{\otimes i}\to R$ vanishes on shuffles, we obtain $g(x\bullet y)=g(x)\bullet y+(-1)^{|x||g|}x\bullet g(y)$. Thus, for $f,g\in C^*_{(1)}(R,M)$,

$$\begin{aligned}\mathrm{pr}([f,g](x\bullet y)) &= \mathrm{pr}\big(f(g(x)\bullet y)+\pm f(x\bullet g(y))-\pm g(f(x)\bullet y)+\pm g(x\bullet f(y))\big)\\ &= 0.\end{aligned}$$

Hence $[f,g]\in C^*_{(1)}(R,M)$.

When $k\supset\mathbb{Q}$ it is well-known that there is an isomorphism of algebras $T(sR)\cong S(e^{(1)}(sR))$ where the product on $T(sR)$ is the shuffle product. Furthermore $e^{(i)}(sR)=S^i(e^{(1)}(sR))$. Hence, the filtration $\mathcal{F}_qHH^*(R,R)$ is the filtration induced by the augmentation ideal filtration in cohomology. Let $f\in\mathcal{F}_pHH^*(R,R)$ and $g\in\mathcal{F}_qHH^*(R,R)$; we have to prove that the defining maps $(f\cup g)_m(x_1\bullet\cdots\bullet x_n)=0$ for $n\ge p+q$, $m\ge 1$ and $x_i\in e^{(1)}(sR)$. The argument is similar to the first part of the proof. Indeed,

$$\begin{aligned}(f\cup g)_m(x_1\bullet\cdots\bullet x_n) &= \sum_{i+j+k=m+2} \pm D_k\big(x_1^{(1)}\bullet\cdots\bullet x_n^{(1)}\otimes f_i(x_1^{(2)}\bullet\cdots\bullet x_n^{(2)}) \\ &\qquad \otimes x_1^{(3)}\bullet\cdots\bullet x_n^{(3)}\otimes g_j(x_1^{(4)}\bullet\cdots\bullet x_n^{(4)})\otimes x_1^{(5)}\bullet\cdots\bullet x_n^{(5)}\big).\end{aligned}$$

Since $m\geq p+q$, we can assume that there is an index l such that $x_l^{(2)}=1=x_l^{(4)}$ (if not either $f_i(x_1^{(2)}\bullet\cdots\bullet x_n^{(2)})$ or $g_j(x_1^{(4)}\bullet\cdots\bullet x_n^{(4)})$ is zero). It follows that $D_k\big(x_1^{(1)}\bullet\cdots\otimes f_i(x_1^{(2)}\bullet\cdots\bullet x_n^{(2)})\otimes\cdots\otimes g_j(x_1^{(4)}\bullet\cdots\bullet x_n^{(4)})\cdots\bullet x_n^{(5)}\big)$ is equal to

$$D_k\big(\big(x_1^{(1)}\bullet\cdots\otimes f_i(x_1^{(2)}\bullet\cdots\bullet x_n^{(2)})\otimes\cdots\otimes g_j(x_1^{(4)}\bullet\cdots\bullet x_n^{(4)})\cdots\bullet x_n^{(5)}\big)\bullet x_l\big)$$

which is zero since D_k vanishes on shuffles. Hence, $f\cup g\in\mathcal{F}_{p+q}HH^*(R,R)$. A similar argument shows that $[f,g]\in\mathcal{F}_{p+q-1}HH^*(R,R)$. ■

Remark 3.32. Theorem 3.31 applies in particular to differential graded commutative algebras. For non-graded algebras it was first proved in [**BW**]. A careful analysis of the proof of Theorem 3.31 shows that it holds whenever R is free over a ground ring k which contains either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ (p a prime).

Remark 3.33. Note that when the spectral sequence $I_1^{pq}(R,R)$ converges, the second assertion in Theorem 3.31 follows immediately from Proposition 3.30.

3.4. Hodge decomposition and cohomology of homotopy Poisson algebras. In this section, for simplicity, we work over a ground ring containing $\mathbb{Q}$. If P is a Poisson algebra, the Hodge decomposition of the Hochschild (co)homology of the underlying commutative algebra identifies with the first page of a spectral sequence computing its Poisson cohomology [**Fr2**]. We want to prove that this result makes sense for homotopy Poisson algebras (P_∞-algebras for short) as well. We briefly recall the definition of P_∞-algebras and refer to [**Gi**] for more details.

Definition 3.34. *Let R be a k-module and $P^\perp(R):=S\big(\mathrm{coLie}(sR)\big)$ be the symmetric coalgebra on the cofree Lie coalgebra over sR. A P_∞-algebra structure on R is given by a coderivation ∇ of degree 1 on $P^\perp(R)$ such that $\nabla^2=0$. A map of P_∞-algebra $(R,\nabla)\to(S,\nabla')$ is a graded differential coalgebra map $P^\perp(R)\to P^\perp(S)$.*

The "coalgebra"-structure of $P^\perp(R)$ is obtained by the sum of the symmetric coproduct (i.e. the free cocommutative one) and the lift as a coderivation of the Lie coalgebra cobracket (see [**Gi**] for an explicit formula). As for A_∞-algebras, a P_∞-structure on R is uniquely defined by maps $\nabla_{p_1,\dots,p_n}:R^{\otimes p_1}\otimes\ldots\otimes R^{\otimes p_n}\to R$ such that $\nabla(x_1,\dots,x_n)$ ($x_i\in R^{\otimes p_i}$) is antisymmetric with respect to the coordinates x_i and vanish if one of the x_i is a shuffle. There is a forgetful functor from the category of P_∞-algebras to the one of C_∞-algebras. It is determined by considering only the map $D_n:R^{\otimes n}\to R$ which restricts to $\mathrm{coLie}(sR)$. When (R,∇) is a P_∞-algebra, we denote $\mathrm{CoDer}(R,R)$ the k-module of coderivations of $P^\perp(R)$.

Definition 3.35. *The cohomology of the P_∞-algebra (R,∇) is the cohomology $HP^*(R,R)$ of the complex $\mathrm{CoDer}(R,R)$ equipped with the differential $[-,\nabla]$. More precisely, one has*

$$[f,\nabla]=f\circ\nabla-(-1)^{|f|}\nabla\circ f \textit{ for } f\in\mathrm{CoDer}(R,R).$$

PROPOSITION 3.36. *Let (P,∇) be a P_∞-algebra, such that P is a C_∞^{op}-bimodule over itself. There is a converging spectral sequence*

$$E_1^{pq} = HH_{(p)}^{q+p}(P,P) \Longrightarrow HP^{p+q}(P,P). \tag{3.16}$$

Proof: Since $k \supset \mathbb{Q}$, the cofree Lie coalgebra $\mathrm{coLie}(sP)$ is isomorphic to the indecomposable space $e^{(1)}(T(sP))$ and there is an isomorphism $S^n(e^{(1)}) \cong e^{(n)}$ induced by the shuffle product. Hence

$$P^\perp(P) = \bigoplus_{n\geq 1} S^n \mathrm{coLie}(sP) \cong \bigoplus S^n(e^{(1)}(A^\perp(P))) \cong \bigoplus e^{(n)}(T(sP))).$$

The space $P^\perp(P)$ is filtered by the symmetric power of $\bigoplus S^n\mathrm{coLie}(sP)$. On the associated graded module E_0, the differential reduces to the coderivation defined by the maps $(\nabla_n : R^{\otimes n} \to R)_{n\geq 1}$. Clearly, this is the coderivation defining the C_∞-structure of P. Since P is assumed to be a C_∞-bimodule over its underlying C_∞-algebra structure, the differential $\left[-,\sum_{n\geq 1}\nabla_n\right]$ preserves the decomposition $P^\perp(P) = \bigoplus e^{(n)}(A^\perp(P)))$. It follows that E_1^{p*} is the cohomology of the complex $\mathrm{CoDer}\left(e^{(p)}\left(A^\perp(P)\right), A^\perp(P)\right)$, where the coderivations are taken with respect to the coalgebra structure of $A^\perp(P)$, equipped with the Hochschild differential given by the underlying C_∞-algebra structure of P. Thus $E_1^{p*} \cong HH^*_{(q)}(P,P)$, see the proof of Theorem 3.1. ■

EXAMPLE 3.37. Let $(P, m, [\ ;\ ])$ be a Poisson algebra. The maps $D_2 = m$ and $D_{1,1} = [\ ;\ ]$ endow P with its canonical P_∞-structure and the cohomology of $\mathrm{CoDer}(P^\perp(P), P^\perp(P))$ is its Poisson cohomology $HP^*(P,P)$. Proposition 3.36 implies that there is a spectral sequence converging to $HP^*(P,P)$ whose E_1 term is the Hochschild cohomology of P. This spectral sequence is the dual of the one found by Fresse [**Fr1, Fr2**].

EXAMPLE 3.38. Let $\mathfrak{g}$ be a Lie algebra. The free Poisson algebra on the Lie algebra $\mathfrak{g}$ is $S^*\mathfrak{g}$. Indeed the Lie bracket of $\mathfrak{g}$ extends uniquely on $S^*\mathfrak{g}$ in such a way that the Leibniz rule is satisfied. Since $S^*\mathfrak{g}$ is free as an algebra (thus smooth), according to Proposition 3.5.i), the term E_1 of spectral sequence (3.16) is equal to

$$\begin{aligned} HH^*_{(p)}(S^*\mathfrak{g}, S^*\mathfrak{g}) \cong HH^p_{(p)}(S^*\mathfrak{g}, S^*\mathfrak{g}) &\cong \mathrm{Hom}_{S^*\mathfrak{g}}(\Omega^p_{S^*\mathfrak{g}}, S^*\mathfrak{g}) \\ &\cong S^p\mathrm{Hom}(\mathfrak{g}, S^*\mathfrak{g}). \end{aligned}$$

It follows that the spectral sequence collapses at level 2 with $E_2^n = H^n_{\mathrm{Lie}}(\mathfrak{g}, S^*\mathfrak{g})$ where H^*_{Lie} stands for Lie algebra cohomology.

EXAMPLE 3.39. A C_∞-algebra is a P_∞-algebra by choosing all other defining maps to be trivial. Let R be a P_∞-algebra with $\nabla_{p_1,\ldots,p_n} = 0$ for $n \geq 2$. Then spectral sequence (3.16) collapses at level 1 since the maps $\nabla_{p_1,\ldots,p_n}$ are null for $n \geq 2$. Hence

$$HP^*(R,R) \cong \bigoplus_{p\geq 0} HH^*_{(p)}(R,R).$$

4. An exact sequence à la Jacobi-Zariski

It is well-known that if $K \to S \to R$ is a sequence of strict commutative rings with unit, there is an exact sequence relating the André Quillen (co)homology groups of R viewed as a K-algebra with the ones of S viewed as a K-algebra and R viewed as a S-algebra. This sequence is called the Jacobi-Zariski exact sequence (or the transitivity exact sequence). Under flatness hypothesis and if the rings contained $\mathbb{Q}$, the André Quillen (co)homology corresponds to Harrison (co)homology with degree shifted by one. In particular the exact sequence holds for the Harrison groups of Definitions 2.22, 2.26. We prove here a similar result for C_∞-algebras with units. We first study the category of A_∞ or C_∞-algebras over a C_∞-one.

4.1. Relative A_∞-algebras. In order to make sense of a Jacobi-Zariski exact sequence, we shall first define the notion of a C_∞-algebra over another one. The theory makes sense for A_∞-algebras over an A_∞-algebra as well so that we start working in this more general context. Let $T^\perp(R/S)$ be the coalgebra

$$T^\perp(R/S) \quad := \quad T(R \oplus S) = \bigoplus_{n,p_0,\dots,p_n \geq 0} S^{\otimes p_0} \otimes R \otimes S^{\otimes p_1} \otimes \dots \otimes R \otimes S^{p_n}. \tag{4.17}$$

The coalgebra map is the one on $T(R \oplus S)$, that is to say

$$\delta_{R/S}(s_1^{p_0}, \dots, s_{k_{p_0}}^{p_0}, a_1, \dots, a_n, s_1^{p_n}, \dots, s_{k_{p_n}}^{p_n}) = \\ \sum (s_1^{p_0}, \dots, a_j, s_1^{p_j}, \dots, s_{\ell_{p_j}}) \otimes (s_{\ell_{p_j}+1}^{p_j}, \dots, a_n, s_1^{p_n}, \dots, s_{\ell_{p_n}}^{p_n}).$$

The coalgebra $T^\perp(sR/sS)$ is co-augmented

$$k \hookrightarrow T^\perp(sR/sS) \twoheadrightarrow A^\perp(sR/sS).$$

It is straightforward that $\delta_{R/S}$ restricts to $T(S)$ and $T(R)$, hence the following lemma.

Lemma 4.1. *Both $A^\perp(S)$ and $A^\perp(R)$ are subcoalgebras of $A^\perp(R/S)$.*

Denote $A_+^\perp(R/S)$ the subspace of $A^\perp(R/S)$ which contains at least one factor R so that

$$T^\perp(sR/sS) = T(sS) \oplus A_+^\perp(R/S), \quad A^\perp(R/S) = A^\perp(S) \oplus A_+^\perp(R/S).$$

Definition 4.2. *Let (S, B) be an A_∞-algebra and R a k-module.*

- *An S-algebra structure on R is an A_∞-algebra structure on $R \oplus S$ such that the natural inclusion $S \to S \oplus R$ and the natural projection $S \oplus R \to S$ are maps of A_∞-algebras.*
- *A C_∞-algebra over S structure on R is a C_∞-algebra structure on $R \oplus S$ such that the natural inclusion $S \to S \oplus R$ and the natural projection $S \oplus R \to S$ are maps of C_∞-algebras.*

The natural inclusion $S \to S \oplus R$ is the coalgebra map $F : T(sS) \to T(s(S \oplus R))$ with defining maps $F_1 = S \hookrightarrow S \oplus R$ and $F_{i \geq 2} = 0$ (see Remark 1.8). The natural projection $S \oplus R \to S$ is the map $G : T(s(S \oplus R)) \to T(sS)$ with defining maps $G_1 = S \oplus R \twoheadrightarrow S$ and $G_{i \geq 2} = 0$.

In terms of coderivations Definition 4.2 means

PROPOSITION 4.3. *Let (S,B) be an A_∞-algebra and R a k-module. A structure of S-algebra on R is uniquely determined by a coderivation $D_{R/S}$ on $A^\perp(R/S)$ such that*

i): $D_{R/S}(A^\perp(S)) \subset A^\perp(S)$ *and* $(D_{R/S})_{/A^\perp(S)} = B$;
ii): $D_{R/S}(A_+^\perp(R/S)) \subset A_+^\perp(R/S)$;
iii): $(D_{R/S})^2 = 0$.

If, in addition, (S,B) is a C_∞-algebra, R is a C_∞-algebra over S if $(R, D_{R/S})$ is an S-algebra such that the codifferential $D_{R/S}$ on $A^\perp(R/S)$ is a derivation for the shuffle product on $T^\perp(sR/sS)$.

In plain English condition **i)** means that the codifferential $D_{R/S}$ restricts to $A^\perp(S)$ and that this restriction $D_{R/S}/A^\perp(S)$ is equal to B.
Proof: We already know that an A_∞-structure on $R\oplus S$ is given by a coderivation of square zero. The claim **i)** and **ii)** follows from the fact that the natural inclusion and natural projection are maps of A_∞-algebras. The statement for C_∞-structure is an immediate consequence of Definition 2.1. ∎

REMARK 4.4. The definition 4.2 put emphasis on homotopy algebras over a fixed A_∞ or C_∞-structure (S,B). However, it makes perfect sense to study coderivation $D_{R/S} : A^\perp(R/S) \to A^\perp(R/S)$ with $(D_{R/S})^2 = 0$, restricting to $A^\perp(S)$ and with $D_{R/S}(A_+^\perp(R/S)) \in A_+^\perp(R/S)$. Such a coderivation $D_{R/S}$ restricts into a codifferential on $A^\perp(S)$, hence yielding an A_∞-structure on S. Moreover $(R, D_{R/S})$ is a $(S, D_{R/S}/_{A^\perp(S)})$-algebra in the sense of Definition 4.2.

LEMMA 4.5. *A structure of S-algebra on R is uniquely determined by maps*

$$D_{p_0,\dots,p_n} : S^{\otimes p_0}\otimes R\otimes S^{\otimes p_1}\otimes\ldots\otimes R\otimes S^{\otimes p_n} \to R \quad (n\geq 1)$$

satisfying, for all $s_k = a_1^k\otimes\cdots a_{p_k}^k \in S^{\otimes p_k}$ $(k=0\ldots n)$ and $r_1,\ldots r_n\in R$,

$$\sum_{i+j=n-1}\sum_{q=0}^{n-1-i} \pm \quad D_{k_0,\dots,k_i}\big(s_0, r_1,\ldots,r_i, s_q^{(1)}, D_{\ell_0,\dots,\ell_j}\big(s_q^{(2)}, r_{q+1},\ldots$$

$$\ldots, r_{q+j}, s_{q+j}^{(1)}\big), s_{q+j}^{(2)}, r_{q+j+1},\ldots,r_n,s_n\big) = 0.$$

Note that in the formula above, we use Sweedler's notation $s^{(1)}\otimes s^{(2)}$ for the deconcatenation coproduct of $s\in S^{\otimes m}$. The indexes $k_0,\ldots,k_i$ and $\ell_0,\ldots,\ell_j$ are uniquely unambiguously defined by the sequences of elements to which they apply.
Proof: According to Remark 1.4, a structure of S-algebra on R is uniquely determined by maps

$$D_{p_0,\dots,p_n} : S^{\otimes p_0}\otimes R\otimes S^{\otimes p_1}\otimes\ldots\otimes R\otimes S^{\otimes p_n} \to R\oplus S \quad (n\geq 0).$$

The requirement $D_{R/S}(A_+^\perp(R/S)) \subset A_+^\perp(R/S)$ forces the composition of the maps $D_{p_0,\dots,p_n}$ with the projection on S to be trivial for $n\geq 1$. For $n=0$, the maps $D_{p_0} = B_{p_0} : S^{\otimes p_0}\to S$ are determined by the A_∞-structure of S (Proposition 4.3). The formula follows from Remark 1.6. ∎

REMARK 4.6. Let (R,D) be an S-algebra. Lemmas 4.5 and 4.1 imply that the maps $D_n = D_{0,\dots,0} : R^{\otimes n}\to R$ define a coderivation $\widetilde{D}$ of $A^\perp(R)$. Since $D^2=0$, it follows that $\widetilde{D}^2 = 0$. Hence R is an A_∞-algebra. Moreover it is a C_∞-algebra if R is a C_∞-algebra over S.

The notion of an A_∞*-bimodule over an* S*-algebra* R (R/S*-bimodule* for short) is the same as in Definition 1.1 with $A^\perp(R)$ replaced by $A^\perp(R/S)$. In other words, a structure of R/S-bimodule on M is given by a codifferential on $A^\perp_{R\oplus S}(M)$.

Remark 4.7. According to Section 1.1, an R/S-bimodule structure on M is determined by maps

$D^M_{p_0,\dots,p_n|q_0,\dots,q_m} : S^{\otimes p_0} \otimes R \otimes S^{\otimes p_1} \otimes \dots \otimes S^{p_n} \otimes M \otimes S^{\otimes q_0} \otimes R \otimes \dots \otimes S^{q_m} \to M$

where $\{p_0, \dots, p_n\}$, $\{q_0, \dots, q_m\}$ are allowed to be the empty set $\emptyset$.

Similarly an A_∞*-morphism of* A_∞*-algebras over* S (S-A_∞*-morphism* for short) is a map of A_∞-algebra $F : R \oplus S \to R' \oplus S$ such that the composition

$$S \to S \oplus R \xrightarrow{F} R' \oplus S$$

is the natural inclusion and the composition

$$R \oplus S \xrightarrow{F} R' \oplus S \to S$$

is the natural projection. Equivalently, it is a map of A_∞-algebra $F : A^\perp(R/S) \to A^\perp(R'/S)$ such that F restricts to $A^\perp(S)$ as the identity and $F(A^\perp_+(R/S)) \subset A^\perp_+(R'/S)$. A C_∞*-morphism over* S is an S-A_∞-morphism such that its defining maps $F_{p_0,\dots,p_n}$ satisfies $F_{p_0,\dots,p_n}(x \bullet y) = 0$, i.e. vanish on shuffles.

Lemma 4.8. *An* A_∞*-morphism* $A^\perp(R/S) \to A^\perp(R'/S)$ *is uniquely determined by maps*

$$F_{p_0,\dots,p_n} : S^{\otimes p_0} \otimes R \otimes \dots \otimes R \otimes S^{\otimes p_n} \to R' \qquad (n \geq 1)$$

such that the unique coalgebra map F *defined by the system* ($S^{\otimes p_0} \otimes R \otimes \dots \otimes R \otimes S^{\otimes p_n} \overset{F_{p_0,\dots,p_n}}{\to} R' \hookrightarrow R' \oplus S$) *is a map* $F : (A^\perp(R/S), D_{R/S}) \to (A^\perp(R'/S), D_{R'/S})$ *of differential coalgebras.*

Proof: According to Section 1.1, a map of coalgebras $F : A^\perp(R/S) \to A^\perp(R'/S)$ is uniquely determined by maps

$$F_{p_0,\dots,p_n} : S^{\otimes p_0} \otimes R \otimes S^{\otimes p_1} \otimes S^{\otimes p_1} \dots R \otimes S^{\otimes p_n} \to R' \oplus S.$$

The requirement $F/_{C^\perp(S)} = \mathrm{id}$ implies that $F_1 : S \to R \oplus S$ is the canonical inclusion $S \hookrightarrow R' \oplus S$ and that $F_n : S^{\otimes n} \to R \oplus S$ is trivial. Moreover $F(A^\perp_+(R/S)) \subset A^\perp_+(R'/S)$ implies that the other defining maps take values in $R \subset R \oplus S$. ∎

Proposition 4.9. *If* R *is an* S*-algebra, then* R *is canonically an* S*-bimodule. Moreover, if* R *is a* C_∞*-algebra, then* R *is a* C_∞*-bimodule over* S*.*

Proof: We denote $D_{p_0,\dots,p_n}$ the map defining the S-algebra structure. Note that there is an inclusion $T^S(R) \overset{i}{\hookrightarrow} A^\perp(R/S)$ and that $(i \otimes i) \circ \delta^R = \delta(i)$. Thus the restriction $D^R_{p,q} := D_{p,q}$ defines a coderivation from $T(sR)$ to $A^\perp_S(R)$ of square zero; hence a canonical S-bimodule structure. When S is a C_∞-algebra, and R a C_∞-algebra over R, the vanishing of $D_{R/S} : A^\perp(R \oplus S) \supset T^S(R) \to A^\perp(R \oplus S)$ on shuffles is equivalent to Definition 2.5. ∎

Remark 4.10. Later on, we also will have to deal with different "ground" homotopy structures on S at the same time. Thus, for two C_∞-algebras (S, B), (S', B'), we define **an** A_∞**-morphism** $C^\perp(R/S) \to C^\perp(R'/S')$ to be a map of differential coalgebras such that F restricts to $A^\perp(S)$ yielding an A_∞-morphism

$(A^\perp(S), B) \to (A^\perp(S'), B')$. We further require that $F(A^\perp_+(R/S)) \subset A^\perp_+(R'/S')$. Such a map is uniquely determined by the maps of Lemma 4.8 together with maps $F_n : S^{\otimes n} \to S'$ (the proof is the same).

In terms of Definition 4.2, such a map is an A_∞-algebra morphism $(R\oplus S, D) \to (R' \oplus S', D')$ such that the composition

$$S \to R \oplus S \to R' \oplus S' \to S'$$

is a prescribed A_∞-map $F : S \to S'$ and moreover F commutes with natural inclusions and projections i.e. the following diagrams commutes

$$\begin{tikzcd} (S,B) \arrow[r] \arrow[d] & (R\oplus S, D) \arrow[d] \\ (S',B') \arrow[r] & (R'\oplus S', D'), \end{tikzcd} \qquad \begin{tikzcd} (R\oplus S, D) \arrow[r] \arrow[d] & (S,B) \arrow[d] \\ (R'\oplus S', D') \arrow[r] & (S',B'). \end{tikzcd}$$

Low degrees identities satisfied by an A_∞-algebra over a C_∞-algebra : Let (S, B) be a C_∞-algebra and (R, D) an A_∞-algebra over S.

- The condition $(D)^2 = 0$ implies that the degree one map $D_{0,0} : R \to R$ is a differential that we denote d_R. We also denote $d_S = B_1 : S \to S$.
- The maps $D_{0,1} : R \otimes S \to R$, $D_{1,0} : S \otimes R \to R$ and $D_{0,0,0} : R \otimes R \to R$ are degree 0 maps. Moreover $D_{0,0,0}$ is graded commutative if and only if R is a C_∞-algebra and $D_{1,0}(s, a) = (-1)^{|a|\cdot|s|}D_{0,1}(a, s)$.
- Restricted to $S\otimes R$, the condition $(D)^2 = 0$ implies that $D_{1,0} : S\otimes R \to R$ is a map of differential graded modules.
- Denote by the single letter d the differential induced by d_R and d_S on $A^\perp(R \oplus S)$. The identities satisfied by $D_{p_0,\ldots,p_n}$ on $A^\perp(R \oplus S)^{\leq 3}$ are

$$\begin{aligned} d_R(D_{0,0,0,0}(a, b, c)) + D_{0,0,0,0}(d_R(a, b, c)) &= D_{0,0}(D_{0,0}(a, b), c) \\ &\quad -D_{0,0}(a, D_{0,0}(b, c)) \\ d_R(D_{0,1,0})(a, s, b) + D_{0,1,0}(d(s, a, b)) &= D_{0,0,0}(D_{0,1}(a, s), b) \\ &\quad +D_{0,0,0}(a, D_{1,0}(s, b)) \\ d_R(D_{1,0,0})(s, a, b) + D_{1,0,0}(d(s, a, b)) &= D_{1,0}(s, D_{0,0}(a, b)) \\ &\quad +D_{0,0}(D_{1,0}(s, a), b) \\ d_R(D_{2,0}(s, t, a)) + D_{2,0}(d(s, t, a)) &= D_{1,0}(D_{S2}(s, t), a) \\ &\quad +D_{1,0}(s, D_{1,0}(t, a)) \end{aligned}$$

  plus the equations similar to the last two ones involving $D_{0,1,0}$, $D_{0,0,1}$ and $D_{0,2}$ instead of $D_{1,0,0}$ and $D_{2,0}$.

These identities imply the following Proposition.

Proposition 4.11. *Let R be an algebra over the C_∞-algebra S and M an R/S-bimodule. Then $H^*(R)$ is an associative $H^*(S)$-algebra, which is graded commutative if R is a C_∞-algebra. Moreover $H^*(M) := H^*(M, D^M_{\emptyset|\emptyset})$ is a bimodule over the $H^*(S)$-algebra $H^*(R)$.*

4.2. Relative A_∞-algebras over strict C_∞-algebras. When S is a commutative algebra, one can take $k = S$ as ground ring. In particular, Definition 1.1 gives the notion of *S-linear A_∞-algebra* (we also say A_∞-algebra in the category of S-modules); such a structure is a codifferential on $A^{S\perp}(R) := \bigoplus_{n\geq 0} R^{\otimes_S n}$, see

Section 1.1, Definition 1.1. We have to make sure that this definition is equivalent to Definition 4.2, where S is equipped with its canonical A_∞-algebra structure. This is the aim of the next Proposition and of Proposition 4.20 below.

PROPOSITION 4.12. *Let (S,d,m) be a strict C_∞-algebra and (R,D) be an S-linear A_∞-algebra.*

i): *R has a natural structure of A_∞-algebra over S (in the sense of Definition 4.2) given by*

$$D_{0,0}=d_R,\quad D_{1,0}(s,a)=s.a=\pm D_{0,1}(a,s),$$
$$D_{0,\dots,0}(a_1,\dots,a_n)=D_n(a_1,\dots,a_n).$$

ii): *If (M,D^R_M) is an S-linear R-bimodule, the maps*

$$D^M_{\emptyset|\emptyset}=D^M_{0,0},\qquad D^M_{\emptyset|1}(m,s)=m.s,\qquad D^M_{1|\emptyset}(s,m)=s.m,$$
$$D^M_{0,\dots,0|0,\dots,0}(r_1,\dots,r_p,m,r'_1,\dots,r'_q)=D^M_{p,q}(r_1,\dots,r_p,m,r'_1,\dots,r'_q)$$

give M the structure of an R/S-bimodule.

iii): *Let (S,d,m) be a strict C_∞-algebra and R be an S-module. Assume that (R,D) is a C_∞-algebra over S such that*

$$D_{1,0}(s,a)=s.a=(-1)^{|a|.|s|}D_{0,1}(a,s).$$

Then the defining maps $D_{0,\dots,0}$ are S-multilinear; hence defined a structure of S-linear algebra on R.

Proof: For *i*), we need to prove that $(D_{R/S})^2=0$, which reduces to the identities,

$$\begin{aligned}D^R_n(a_1,\dots,a_i.s,a_{i+1},\dots,a_n) &= D^R_n(a_1,\dots,a_i,s.a_{i+1},\dots,a_n)\quad 1\leq i\leq n-1\\ D^R_n(a_1,\dots,a_n.s) &= D^R_n(a_1,\dots,a_n).s\\ D^R_n(s.a_1,\dots,a_n) &= s.D^R_n(a_1,\dots,a_n)\end{aligned}$$

These identities follows by S-linearity.

For *ii*), the fact that $(D^M)^2=0$ reduces to the S-linearity of the maps $D^M_{p,q}$ and the vanishing of $(D^M)^2$ as in *i*).

The low degrees identities of A_∞-algebras over a C_∞-algebra of Section 4.1 imply that the maps $D_{0,\dots,0}$ are S-linear. Then *iii*) follows easily. ■

EXAMPLE 4.13. It follows from Proposition 4.12 that if (S,d,m) is a strict C_∞-algebra and (R,d_R,m_R) is a strict commutative S-algebra, then R is a C_∞-algebra over S with structure maps $D_{p_0,\dots,p_n}=0$ except for

$$D_0=d_R,\quad D_{0,0}=m_R,\ \text{and}\ D_{0,1}(r,s)=r.s,\ D_{1,0}(s,r)=s.r$$

for all $(r,s)\in R\otimes S$. Reciprocally, if R is an S-linear C_∞-algebra whose only nontrivial structure maps are $D_0,D_{0,1},D_{1,0},D_{0,0}$ then R is a strict S-algebra. This follows easily from the low degrees relations satisfied by a C_∞-algebra over S, see Section 4.1.

REMARK 4.14. Let S be a strict A_∞-algebra and R a strict S-bimodule together with a pairing of differential graded module $\nu: R\otimes R\to R$ left linear in the first variable, right linear in the second and satisfying $\nu(r.s,r')=\nu(r,s.r')$. Then there is an S-linear A_∞-structure on R given by

$$D_{0,0}=d_R,\quad D_{1,0}(s,a)=s.a,\quad D_{0,1}(a,s)=a.s,\quad D_{0,0,0}=\nu.$$

Proposition 4.12.i) also holds in the case where S is a strict A_∞-algebra by requiring that R is an A_∞-algebra in the category of S-bimodules.

4.3. Weakly unital homotopy algebras. The standard Jacobi-Zariski exact sequence holds for unital algebras. Its C_∞-analogue in Section 4.5 also requires unitality assumption. Details on unital A_∞ and C_∞-algebras can be found in [**Tr2, HL2, HL3**]. In fact, we only need weaker unitality assumptions. A **weakly unital** A_∞-algebra (R, D) is an A_∞-algebra equipped with a distinguished element $1 \in R^0$ that satisfies $D_2(1, a) = D_2(a, 1) = a$ for any $a \in R$. Thus unital A_∞-algebras (in the sense of [**Tr2, HL2, HL3**]) are in particular weakly unital. A weakly unital C_∞-algebra is a C_∞-algebra which is weakly unital as an A_∞-algebra.

CONVENTION 4.15. Henceforth, when we write R has a unit, we mean R is weakly unital.

REMARK 4.16. The fact that D_1 is a derivation for D_2 implies that $D_1(1) = 0$. in other words, a weak unit is necessarily a cocycle (for D_1).

REMARK 4.17. If the A_∞-algebra R has unit, then $H^*(R)$ is a unital algebra.

EXAMPLE 4.18. A strict A_∞-algebra is weakly unital if and only if it is a differential graded associative algebra with unit (in the usual sense). For instance if A is a unital associative algebra, then its Hochschild cochain complex $C^*(A, A)$ is weakly unital with unit given by the unit of A viewed as an element of $C^0(A, A)$. Also the cochain complex $C^*(X)$ of a topological or simplicial set X is weakly unital.

We need to extend the definition of weak unitality to the relative setting. Let S be a weakly unital C_∞-algebra, with (weak) unit 1_S. Let R be an S-algebra. The **S-algebra R is said to be weakly unital** if the element $0 \oplus 1_S \in R \oplus S$ is weak unit for the A_∞-algebra $R \oplus S$.

Assume S is a strict unital C_∞-algebra and (R, D^R) is an S-linear A_∞-algebra. The action of the unit $1_S \in S$ is trivial on R, thus $0 \oplus 1_S$ is a weak unit for $R \oplus S$. Therefore we obtain

PROPOSITION 4.19. *Let (S, d, m) be a strict unital C_∞-algebra and (R, D^R) be an S-linear A_∞-algebra. Then R, equipped with the A_∞-algebra structure over S given by Proposition 4.12, is weakly unital if and only if (R, D^R) is weakly unital as an S-linear A_∞-algebra.*

4.4. (Co)homology groups for relative C_∞-algebras. Let M be an R/S-bimodule and let $T^\perp(sR/sS)$ be the coalgebra defined by Equation (4.17). The *Hochschild (co)homology groups* of the S-algebra R with values in M are the (co)homology groups of the (co)chain complexes

$$\big(C^*(R/S, M), b\big) := \mathrm{CoDer}\big(T^\perp(sR/sS), A^\perp_{R\oplus S}(M)), b\big), \tag{4.18}$$

$$\big(C_*(R/S, M), b\big) := M \otimes T^\perp(sR/sS), b\big). \tag{4.19}$$

The differential on the complex $C^*(R/S, M)$ is the Hochschild differential on $C^*(R \oplus S, M) \cong C^*(R/S, M)$ corresponding to the A_∞-algebra structure of $R \oplus S$ (see Definition 4.2). The differential on $C_*(R/S, M)$ is defined similarly. When R is a C_∞-algebra and M a $C_\infty^{(op)}$-bimodule, $R \oplus S$ is automatically a C_∞-algebra and we can define the Harrison (co)chain complexes

$$\begin{aligned} &(4.20) \qquad \big(CHar^*(R/S,M),b\big) &:=& \quad \mathrm{BDer}\big(R\oplus S,M),b\big) \\ &(4.21) \qquad \big(CHar_*(R/S,M),b\big) &:=& \quad M\otimes C^{\perp}(sR\oplus sS),b\big). \end{aligned}$$

The (co)homology groups of the complexes (4.18), (4.19),(4.20) and (4.21) are denoted $HH^*(R/S,M)$, $HH_*(R/S,M)$, $Har^*(R/S,M)$ and $Har_*(R/S,M)$, respectively.

When S is a strict C_∞-algebra, and R is an S-linear A_∞-algebra, we denote $HH^*_S(R,M)$ and $HH^S_*(R,M)$ the Hochschild (co)homology groups of R over the ground ring S,*i.e.*, those given by Definitions 1.9 and 1.14. Similarly we will denote $Har^*_S(R,M)$, $Har^S_*(R,M)$ the Harrison (co)homology groups.

Proposition 4.20. *Let S be a strict (unital) commutative algebra, R be a A_∞-algebra and M, N R-bimodules which are S-linear and flat over S. There are natural isomorphisms*

$$HH^*(R/S,M)\overset{\sim}{\leftarrow} HH^*_S(R,M):h^*,\quad h_*:HH_*(R/S,M)\overset{\sim}{\to} HH^S_*(R,M).$$

If R is a C_∞-algebra, M a C_∞^{op}-bimodule, N a C_∞-bimodule, then h_* and h^* are isomorphisms of γ-rings. Furthermore, there are natural isomorphisms

$$Har^*(R/S,M)\cong Har^*_S(R,M),\quad Har_*(R/S,N)\cong Har^S_*(R,N).$$

Proof: Let D be the codifferential on $A^{S\perp}(R)$ defining the S-linear A_∞-structure on R. Let $D_{R/S}$ be the codifferential on $A^{\perp}(R/S)$ defining the A_∞-algebra over S structure on R. There is an obvious projection

$$h:A^{\perp}(R/S)\longrightarrow\bigoplus_{n\geq 0}R^{\otimes n}\longrightarrow\bigoplus_{n\geq 0}R^{\otimes_S n}=A^{S\perp}(R).$$

This map induces a complex morphism $\mathrm{id}\otimes h: M\otimes A^{\perp}(R/S)\to M\otimes A^{S\perp}(R)$ by S-linearity of the structure morphisms $D_{0,\dots,0}$.

Filtrating $M\otimes A^{\perp}(R/S)$ by the powers of R, we get a spectral sequence converging to $HH_*(R/S,M)$ whose E^1 term is the homology of $A^{\perp}(R/S)$ for the differential given by $D_0=d_R$, $D_{1,0}=l$, $D_{0,1}=r$ and the multiplication $S\otimes S\to S$. In particular the differential restricted to $\bigoplus_{n\geq 0}R\otimes S^{\otimes n}\otimes R$ coincides with the one in the double Bar construction $B(R,S,R)$. Since R is S-flat, the Bar construction $B(R,S,R)$ is quasi-isomorphic to $R\otimes_S R$. Hence

$$E^1_{**}\cong H^*(M)\otimes_S H^*(R)\otimes_S\ldots\otimes_S H^*(R).$$

The filtration by the powers of R of $M\otimes A^{S\perp}(R)$ yields also a spectral sequence (see Proposition 3.21) with isomorphic E_1-term. Moreover, the map h_1 is an isomorphism at page 1 hence is an isomorphism. The cohomology statement is analogous.

Clearly h is a map of coalgebra. Moreover, when R is a C_∞-algebra, it is a map of algebras (with respect to the shuffle product). Thus h commutes with the maps ψ^k inducing the γ-ring structures in (co)homology. It also implies that h factors through the quotient by the shuffles hence the result for Harrison (co)homology. ■

A homomorphism $F:S\to R$ of commutative algebras induces a canonical structure of commutative S-algebra on R. The following Proposition is the up to homotopy analogue. First we fix some notation:

Notation: If $F : (S, D^S) \to (R, D)$ is a morphism of C_∞-algebras, we denote $F^{[i]}$ the composition $A^\perp(S) \xrightarrow{F} A^\perp(R) \xrightarrow{\mathrm{pr}} R^{\otimes i}$, that is to say the component of F which lies in the i-th power of R.

PROPOSITION 4.21. *Let $F : (S, D^S) \to (R, D^R)$ be a C_∞-map. Then R has a structure of a C_∞-algebra over S given by the maps*

$$D_{p_0,\dots,p_n}(x_0, r_1, \dots, x_n) = \sum_{i=i_0+\cdots+i_n+n} D_i^R(F^{[i_0]}(x_0), r_1, \dots, r_n, F^{[i_n]}(x_n)).$$

Proof: According to Lemma 4.5, we have to prove that the coderivation $D_{R/S}$, induced by $D_{p_0,\dots,p_n}$, is of square 0. Since D^S is of square 0 and the degree of $D_{R/S}$ is 1 we find that $(D_{R/S})^2(x_0, r_1, \dots, x_n)$ is equal to

$$\begin{aligned}\sum \pm D_i^R &\Big(F^{[i_0]}(x_0), r_1, \dots, D_j^R\big(F^{[j_k]}(x_k), \dots, F^{[j_{k+l}]}(x_{k+l})\big), \dots, r_n, F^{[i_n]}(x_n)\Big)\\ &+ \sum \pm D_i^R \Big(F^{[i_0]}(x_0), r_1, \dots, F^{[i_p]}(D^S(x_p)), r_p, \dots, r_n, F^{[i_n]}(x_n)\Big)\\ &= (D^R)^2\big(\sum F^{[i_0]}(x_0) \otimes r_1 \otimes \cdots \otimes r_n \otimes F^{[i_n]}(x_n)\big) = 0\end{aligned}$$

The last step follows from $D^R \circ F = F \circ D^S$. ■

EXAMPLE 4.22. Let S be strict and R be a strict unital associative S-algebra. If R is unital, then there is a ring map $F : S \to R$ and we have $\nu(s, r) = F(s).r$ where ν denotes the S-action. We also denote $F : (S, D^S) \to (R, D)$ the associated A_∞-morphism. The structure of A_∞-algebra over S given by Proposition 4.21 is the same than the one given by Proposition 4.12.i) applied to R viewed as an A_∞-algebra in the category of S-modules.

EXAMPLE 4.23. Let S be any (weakly unital) C_∞-algebra. There is a C_∞-map $F : (S, D^S) \to (S, D^S)$ given by $F_1 = \mathrm{id}$, $F_{n>1} = 0$. In particular S has a canonical C_∞-structure over S (which is the canonical one if S is strict by the previous example). Corollary 4.25 below states that these structure is (co)homologically trivial as expected.

PROPOSITION 4.24. *Let M be an R/S-bimodule. Assume R, S, M and their cohomology groups are k-flat. There are converging spectral sequences*

$$E_2^{**} = HH^*(H^*(R)/H^*(S), H^*(M)) \Rightarrow HH^*(R/S, M)$$

and

$$E^2_{**} = HH_*(H^*(R)/H^*(S), H^*(M)) \Rightarrow HH_*(R/S, M).$$

Proof: The spectral sequences are given by the filtration by the power of S. ■

COROLLARY 4.25. *Let S be a weakly unital C_∞-algebra and let M be an S/S-bimodule. There are isomorphisms*

$$HH_*(S/S, M) = H^*(M), \qquad HH^*(S/S, M) = H^*(M).$$

Proof: Applying Proposition 4.24, it is sufficient to consider the case of $H^*(S)$, that is of a strict algebra. According to Proposition 4.20, the later case is the well-known computation of Hochschild (co)homology of the ground algebra [**Lo2**]. ■

4.5. The Jacobi-Zariski exact sequence. In this section all C_∞-algebras are supposed to be *weakly unital*.

THEOREM 4.26. *Let $K \to S \to R$ be a sequence of weakly unital C_∞-maps, with K, S, R and their cohomology k-flat. Then there is a long exact sequence*

$$\begin{aligned} \cdots \to Har_*(S/K, M) \to Har_*(R/K, M) \to Har_*(R/S, M) \\ \to Har_{*-1}(S/K, M) \to Har_{*-1}(R/K, M) \to Har_{*-1}(R/S, M) \to \dots \end{aligned}$$

and also a long exact sequence in cohomology

$$\begin{aligned} \cdots \to Har^*(R/S, M) \to Har^*(R/K, M) \to Har^*(S/K, M) \\ \to Har^{*+1}(R/S, M) \to Har^{*+1}(R/K, M) \to Har^{*+1}(S/K, M) \to \dots \end{aligned}$$

where M is a C_∞-bimodule over R in homology, respectively a C_∞^{op}-bimodule over R in cohomology.

To prove Theorem 4.26, we use the following lemma.

LEMMA 4.27. *Let $K \xrightarrow{F} S \xrightarrow{G} R$ be a sequence of C_∞-maps.*

- **i):** *There is a C_∞-morphism $C^\perp(S/K) \xrightarrow{\overline{G}} C^\perp(R/K)$ given by the defining maps $\overline{G}_1 = \mathrm{id}$, $\overline{G}_{p_0 \geq 2} = 0$ and for $n \geq 2$*

$$\overline{G}_{p_0,\dots,p_n}(x_0, s_1, \dots, x_n) = \sum_{i=i_0+\dots+i_n+n} G_i(F^{[i_0]}(x_0), s_1, \dots, s_n, F^{[i_n]}(x_n)).$$

- **ii):** *There is a C_∞-morphism $C^\perp(R/K) \xrightarrow{\overline{F}} C^\perp(R/S)$ given by*

$$\overline{F}_{p_0} = F_{p_0}, \quad \overline{F}_{00} = \mathrm{id} \text{ and the other maps } \overline{F}_{p_0,\dots,p_n} = 0.$$

Proof: One has

$$\begin{aligned} D_{R/K}(\overline{G}(x_0, s_1, \dots, x_n)) &= \sum D\Big(G \circ F(x_0^{(1)}), G\big(F(x_0^{(2)}), s_1, \dots, x_k^{(1)}\big), \\ &\qquad F(x_k^{(2)}), \dots, G\big(F(x_l^{(2)}), s_l, \dots, x_n^{(1)}\big), F(x_n^{(2)})\Big) \\ &= \sum G\big(D^S(F(x_0^{(1)}), F(x_0^{(2)}), s_1, \dots, x_k^{(1)}, \dots, F(x_n^{(2)})\big) \\ &= \overline{G}(D_{S/K}(x_0, s_1, \dots, x_n)). \end{aligned}$$

It proves i). The proof of ii) is similar. ■

Proof of Theorem 4.26: Let $F : C^\perp(K) \to C^\perp(S)$, $G : C^\perp(S) \to C^\perp(R)$ be two C_∞-maps. By Lemma 4.27, they induce chain maps

$$M \otimes C^\perp(S/K) \xrightarrow{G} M \otimes C^\perp(R/K) \text{ and } M \otimes C^\perp(R/K) \xrightarrow{F} M \otimes A^\perp(R/S).$$

Let $c(G)$ be the cone of the chain map G. That is to say

$$c(G) := M \otimes C^\perp(R/K) \oplus M \otimes C^\perp(S/K)[1].$$

In particular we have an exact sequence

$$\begin{aligned} \cdots \to Har_*(S/K, M) \to Har_*(R/K, M) \to H_*(c(G)) \to \\ Har_{*-1}(S/K, M) \to Har_{*-1}(R/K, M) \to \dots \end{aligned}$$

The homology spectral sequence will follow once we prove that there is a natural isomorphism $H_*(c(G)) \cong Har_*(R/S, M)$. The morphism F induces a chain map

$$M \otimes C^\perp(R/K) \oplus M \otimes C^\perp(S/K)[1] \xrightarrow{i} M \otimes C^\perp(R/S) \oplus M \otimes C^\perp(S/S)[1].$$

The target of i is isomorphic to the cone $c(F)$ of $M \otimes C^\perp(S/S) \to M \otimes C^\perp(R/S)$. There is also the inclusion of chain complexes

$$M \otimes C^\perp(R/S) \xrightarrow{j} M \otimes C^\perp(R/S) \oplus M \otimes C^\perp(S/S)[1].$$

Corollary 4.25 implies that $Har_*(S/S)$ is trivial, thus $H_*(c(F)) \cong Har_*(R/S, M)$ The spectral sequences of Proposition 4.24 also yield converging spectral sequences for $c(G)$ and $c(F)$. Applying the Jacobi Zariski exact sequence for strict commutative unital rings, we get that, at page 1 of the spectral sequences, the map i^1 is a quasi-isomorphism. Similarly the map j^1 is an isomorphism at page 1. It follows that i and j are quasi-isomorphisms, hence $H_*(c(G)) \cong Har_*(R/S, M)$ as claimed.

The existence of the cohomology exact sequence is proved in the same way. ■

5. Applications to string topology

In this section we apply the machinery of previous sections to string topology. We assume that our ground ring k is a field of characteristic different from 2.

Let X be a topological space, the singular cochain $C^*(X)$ is an associative differential graded algebra (thus an A_∞-algebra) and the singular chains $C_*(X)$ forms a differential graded coalgebra. String topology is concerned about algebraic structures on Hochschild (co)homology of singular cochains because of

THEOREM 5.1 (Jones [**Jo**]). *If X is simply connected, then there are isomorphisms*

$$HH^*(C^*(X), C_*(X)) \cong H_*(LX),$$

$$HH_*(C^*(X), C^*(X)) \cong H^*(LX).$$

Degree issues: one has to be careful that the isomorphisms in Theorem 5.1 above are isomorphisms preserving the *cohomological degree*. As $x \in H_i(LX)$ has cohomological degree $-i$, the isomorphism reads as $HH^{-i}(C^*(X), C_*(X)) \cong H_i(LX)$ and similarly in Hochschild homology. Note that our convention for the degree of Hochschild cohomology is the opposite of the one in [**FTV**].

5.1. C_∞-structures on cochain algebras.

The chain coalgebra $C_*(X)$ and cochain algebra $C^*(X)$ are not (co)commutative. Nevertheless the existence of Steenrod's $\cup_1$-product leads to the existence of natural C_∞-(co)algebras structures. The definition of C_∞-coalgebras is dual to C_∞-algebras. More precisely

- A A_∞-*coalgebra structure* on a k-module R is given by a square zero derivation ∂ of degree -1 on $A_\perp(R) := \prod_{i\geq 1}(sR)^{\otimes n}$, the completed tensor algebra equipped with the (continuous) concatenation

  $$\mu(sx_1 \dots sx_p, sy_1, \dots sy_q) = sx_1 \otimes sx_2 \otimes \dots sy_{q-1} \otimes sy_q.$$

  Coderivations on $A_\perp(R)$ are in one-to-one correspondence with family of maps $\partial^i : R \to R^{\otimes i}$ by dualizing the argument of Remark 1.5.

- The shuffle coproduct is defined by

$$\Delta^{sh}(sx_1\dots sx_n) = \sum \pm\big(sx_{\sigma^{-1}(1)}\otimes\cdots\otimes sx_{\sigma^{-1}(p)}\big)\otimes\big(sx_{\sigma^{-1}(p+1)}\otimes\cdots \cdots\otimes sx_{\sigma^{-1}(n)}\big)$$

where the sum is over shuffles $\sigma \in S_n$, making $A_\perp(R)$ a commutative bialgebra. A *C_∞-coalgebra* is an A_∞-coalgebra (R,∂) such that $(R,\Delta^{sh},\mu,\partial)$ is a differential graded bialgebra (in other words a B_∞-coalgebra).

It is easy to define A_∞-coalgebras maps, A_∞-comodules and their C_∞-analogs in the same way [**TZ**].

Proposition 5.2. *Let k be a field of characteristic zero. There exists a natural C_∞-coalgebra structure on $C_*(X)$ and C_∞-algebra structure on $C^*(X)$, with $C_*(X)$ being a C_∞^{op}-module over $C^*(X)$, such that ∂^1 and D_1 are the singular differentials and, furthermore, the induced (co)algebras structures on $H_*(X)$, $H^*(X)$ are the usual ones.*

Proof: The singular cochains $C^*(X)$ are equipped with a brace algebra structure [**GV**] and thus a B_∞-structure. By a fundamental result of Tamarkin [**Ta, GH1**], a B_∞-structure yields a C_∞-structure, (which is the restriction of a G_∞-structure), with defining maps $D_i : C_*(X)^{\otimes i} \to C_*(X)$. Furthermore, D_1 is the usual differential on singular cochains and D_2 induces the cup-product in cohomology. The dual of the defining maps $D_i : C_*(X)^{\otimes i} \to C_*(X)$ yield a C_∞-coalgebra structure on $C_*(X)$. Moreover $C_*(X)$ inherits a C_∞^{op}-comodule structure by Proposition 2.19. Alternatively, one can use an acyclic models argument as in [**Sm**]. ■

Remark 5.3. The Proposition above holds for non-simply connected spaces. For simply connected X, Rational homotopy theory gives strict C_∞-structures equivalent to the differential graded algebra $C^*(X)$.

For string topology applications, one needs a Poincaré duality between chains and cochain. We use Tradler's terminology [**Tr1, Tr2**]. Given any A_∞-algebra (R, D_R), a **A_∞-inner product** on R is a bimodule map $G : R \to R^\star$ (where $R^\star = \mathrm{Hom}(R,k)$ is the dual of R). We denote D_R, $D_{R^\star}$, the codifferentials defining the canonical R-module structure of R and $R^\star$. An A_∞-algebra R is said to have a **Poincaré duality structure** if R has an A_∞-inner product together with a bimodule map $F : R^\star \to R$ such that $G : \big(A_R^\perp(R), D_R\big) \leftrightarrows \big(A_R^\perp(R^\star), D_{R^\star}\big) : F$ are quasi-isomorphisms which are quasi-inverse of each others (morphisms are not assumed to be of degree 0).

For *finely triangulated* oriented spaces, one can find C_∞-structures on chains and cochains together with a Poincaré duality. By finely triangulated we mean that the closure of every simplex has the homology of a point. The following Lemma is taken from an appendix of Sullivan [**Su**] together with an application of Tradler and Zeinalian [**TZ**]. We write $C^*(X), C_*(X)$ for the simplicial complexes associated to the triangulation of a space. Hopefully, the context should always makes clear if we are working with singular chains or the ones from a triangulation. We denote by $d : C_*(X) \to C_{*-1}(X)$ the differential and by $\Delta : C_*(X) \to C_*(X)\otimes C_*(X)$ the diagonal. We also write respectively d, $\cup$ for the differential and the cup-product on $C^*(X)$ (induced by Δ).

LEMMA 5.4. *Let k be a field of characteristic different from 2 and 3 and X be a triangulated oriented closed space with Poincaré duality such that the closure of every simplex has the homology (with coefficient in k) of a point. There exists a counital C_∞-coalgebra structure on $C_*(X)$ with structure maps $\delta^i : C_*(X) \to C_*(X)^{\otimes i}$ such that*

- **i):** *δ^1 is the simplicial differential and $\delta^2 = \dfrac{1}{2}(\Delta + \Delta^{\mathrm{op}})$;*
- **ii):** *there exists a quasi-isomorphism of A_∞-coalgebras $F : (C_*(X), \delta) \to (C_*(X), d + \Delta)$;*
- **iii):** *the cochains $C^*(X)$ inherits a unital C_∞-structure by duality and there is an A_∞-algebra quasi-isomorphism $F : (C^*(X), D) \to (C^*(X), d + \cup)$;*
- **iv):** *there is a Poincaré duality $C_*(X) \xrightarrow{\Xi} C^*(X)$ of A_∞-modules inducing the Poincaré duality isomorphism in (co)homology.*

Proof: The triangulation of X yields a simplicial complex K^X and a homeomorphism $|K^X| \cong X$. The complex $C_*(X)$ is the simplicial space $C_*(K^X)$. By assumption, the closure of a q-cell (aka q-simplex) of K^X has the homology of a point. Statement $iv)$ is in [**TZ**] as well as A_∞-analogs of $i), iii)$. As in [**TZ**], a map between simplicial complexes is said to be *local* if all simplexes $c \in C_*(X)$ are mapped to $\prod_{i\geq 1} C_*(\overline{c})^{\otimes i}$, where $C_*(\overline{c})$ is the subcomplex generated by the closure $\overline{c}$ of c. By assumption $C_*(\overline{c})$ is *contractible*, *i.e.*, is quasi-isomorphic to k concentrated in degree 0. Let $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$ be a cell approximation to the diagonal. For instance one can take the Alexander-Whitney diagonal. Assertion $iii)$ is obvious consequence of $i)$ and $ii)$. The proof of $i)$ and $ii)$ is essentially contained in [**Su**]. Here we only assume that our field is of characteristic different from 2 and 3. Let us outlined the argument:

Similarly to Example 2.4, a strong C_∞-coalgebra structure on $C_*(X)$ is given by a structure of differential graded Lie algebra on the free Lie algebra $L(X) := \mathrm{Lie}(C_*(X)[1])$ generated by the vector space $C_*(X)[1]$. We denote $\delta : L(X) \to L(X)$ the differential. A strong C_∞-coalgebra is a C_∞-coalgebra. Clearly δ is uniquely determined by its restrictions $\delta^i : C_*(X) \to C_*(X)^{\otimes i}$. Note that, since k is of characteristic different from 2 and 3, the identity $\delta^2 = 0$ is equivalent to $[\delta, \delta] = 0$ and the Jacobi identity for δ is equivalent to $[\delta, [\delta, \delta]] = 0$. We proceed by induction to construct both δ and the quasi-isomorphism $F : (C_*(X), \delta) \to (C_*(X), d + \Delta)$. We define $F_1 = \mathrm{id}$ and $\delta^1 = d$, which are local maps. Thus

$$(F \otimes F) \circ \delta = (d + \Delta) \circ F + O(2)$$

where $O(i)$ means that we restrict to components of $L(X)$ lying in the subspace $\bigoplus_{j\leq i-1} C_*(X)^{\otimes j}$. By $i)$ we have to take $\delta_2 = \dfrac{1}{2}(\Delta + \Delta^{\mathrm{op}})$ which is local and cocommutative, hence with values in $L(X)$. The identity $\delta^2 = 0 + O(3)$ boils down to the fact that Δ is a map of chain complexes. We have to find F_2. We only have to do so locally. The compatibility between F and δ in $O(3)$ is equivalent to

$$[F_2, d] = \frac{1}{2}(\Delta - \Delta^{\mathrm{op}})\,.$$

The right part is a cocycle in the complex of endomorphisms $\big(\mathrm{End}\big(C_*(\overline{\sigma})\big), [-, d]\big)$ for every simplex σ. Since $C_*(\overline{\sigma})$ is contractible, the complex $\big(\mathrm{End}\big(C_*(\overline{\sigma})\big)\big), [-, d]\big)$ has trivial homology and the existence of F_2 follows. Assume by induction that

$\delta_1, \dots, \delta_n,\ F_1, \dots, F_n$ have already been chosen and satisfy $i), ii)$ and $iii)$ up to $O(n+1)$.

We first define $\delta_{n+1} : C_*(X) \to L^n(X)$. By hypothesis we have $[\delta, \delta] = E_{n+1} + O(n+1)$ with $E_{n+1} \subset L^{n+1}(X)$. Since $\delta^2 = \frac{1}{2}[\delta, \delta]$, the Jacobi identity gives $[\delta, [\delta, \delta]] = 0$ and thus

$$[d, [\delta, \delta]] + O(n+2) = [d, E_{n+1}] + O(n+1) = 0 + O(n+2).$$

Thus $[d, E_{n+1}] \subset L^{n+1}(X)$ is equal to zero. Again, the contractibility of each $C(\overline{\sigma}))$ implies that we can find a local map δ_{n+1} such that $E_{n+1} = [d, \delta_{n+1}]$. By definition of E_{n+1}, we have

$$[\delta + \delta_{n+1}, \delta + \delta_{n+1}] = O(n+2)$$

that is $i)$ up to $O(n+2)$.

The induction hypothesis ensures that

$$F(\delta) + (d + \Delta)(F) = G_{n+1} + O(n+2)$$

with $G_{n+1} \subset F^n(X)$ equal to

$$\sum_{2 \le k \le n} F_k(\delta_{n+2-k}) - \Delta(F_n).$$

A straightforward computation of $\rho^2(F) = 0$, where $\rho(F) = F \circ \delta + (d + \Delta) \circ F$, using that $d + \Delta$ gives an A_∞-coalgebra structure on $C_*(X)$, shows that

$$[d, G_{n+1}] + E_{n+1} = 0.$$

Now a map $F_{n+1} : C_*(X) \to C_*(X)^{\otimes n+1}$ makes $F + F_{n+1}$ satisfies $ii)$ up to $O(n+2)$ if and only if

$$[d, F_{n+1}] + \delta_{n+1} + G_{n+1} = 0. \tag{5.22}$$

The map $\delta_{n+1} + G_{n+1}$ is a local cycle by above, hence a local map F_{n+1} could be chosen to satisfy (5.22). This concludes the induction. ■

Remark 5.5. Lemma 5.4 actually holds when $C_*(X)$ is replaced by any simplicial complex in which the closure of any q-cell ($q \ge 0$) is contractible. It seems reasonable that it also holds if X is an oriented regular CW-complex. Note that cellular approximation to the diagonal can be constructed using the same ideas, see [**Su**, Remark A.3]. Also note that the C_∞-structures given by Lemma 5.4 are *not* canonical. Furthermore, the C_∞-structure given by Lemma 5.4 is strong.

5.2. Hodge decomposition for string topology. Hochschild cohomology of singular chains of any space X has a Hodge decomposition according to Proposition 5.2.

Proposition 5.6. *Let k be a characteristic zero field. There exists Hodge decompositions*

$$HH^*(C^*(X), C^*(X)) = \prod_{i \ge 0} HH^*_{(i)}(C^*(X), C^*(X)),$$

$$HH^*(C^*(X), C_*(X)) = \prod_{i \ge 0} HH^*_{(i)}(C^*(X), C_*(X)),$$

$$HH_*(C^*(X), C^*(X)) = \bigoplus_{i \ge 0} HH_*^{(i)}(C^*(X), C^*(X)),$$

$$HH_*(C^*(X),C_*(X)) = \bigoplus_{i\geq 0} HH_*^{(i)}(C^*(X),C_*(X)).$$

The Hodge filtration $\mathcal{F}_i HH^*(C^*(X),C^*(X)) = \bigoplus_{n\leq i} HH^*_{(n)}(C^*(X),C^*(X))$ *is a filtration of Gerstenhaber algebras. Moreover*

$$HH^*_{(0)}(C^*(X),C^*(X)) = H^*(X) = HH_*^{(0)}(C^*(X),C^*(X)),$$

$$HH^*_{(0)}(C^*(X),C_*(X)) = H_*(X) = HH_*^{(0)}(C^*(X),C_*(X)).$$

For $i\geq 1$ *there are spectral sequences*

$$\begin{aligned}
HH^{p+q}_{(i)}(H^*(X),H^*(X))_p &\implies HH^{p+q}_{(i)}(C^*(X),C^*(X))\\
HH^{p+q}_{(i)}(H^*(X),H_*(X))_p &\implies HH^{p+q}_{(i)}(C^*(X),C_*(X))\\
HH_{p+q}^{(i)}(H^*(X),H^*(X))_p &\implies HH_{p+q}^{(i)}(C^*(X),C^*(X))\\
HH_{p+q}^{(i)}(H^*(X),H_*(X))_p &\implies HH_{p+q}^{(i)}(C^*(X),C_*(X)).
\end{aligned}$$

Proof: According to Proposition 5.2, $C^*(X)$ is a C_∞-algebra and $C_*(X)$ is a C_∞^{op}-bimodule. Propositions 2.16, 2.13 ensure that $C_*(X)$ and $C^*(X)$ are both C_∞ and C_∞^{op}-bimodules. Now, the Hodge decompositions follow from Theorems 3.1, 3.18. Since $D_1 : C^*(X) \to C^*(X)$ and $\partial^1 : C_*(X) \to C_*(X)$ are the singular differential, the identification of the weight 0-part is immediate. There is a filtration of Gerstenhaber algebras according to Theorem 3.31. The spectral sequences are given by Propositions 3.7, 3.21. ■

In presence of Poincaré duality for chains, the Hochschild cohomology of the cochain algebra lies in the realm of "string topology". Indeed, there is an isomorphism

$$H_*(LX) \cong HH^*(C^*(X),C_*(X)) \cong HH^*(C^*(X),C^*(X))[d]$$

if X is an oriented manifold of dimension d [**CJ, Mer, FTV2**]. The isomorphism $H_*(LX) \cong HH^*(C^*(X),C^*(X))$ is an isomorphism of algebras with respect to Chas-Sullivan product [**CS**] on the left and the cup product on the right, see [**CJ, Co, Mer**]. When X is a triangulated oriented Poincaré duality space, applying Sullivan's techniques as in Lemma 5.4, Tradler and Zeinalian proved that the Hochschild cohomology

$$HH^*(C^*(X),C^*(X)) \cong HH^*(C^*(X),C_*(X))[d]$$

is a BV-algebra (whose underlying Gerstenhaber algebra is the usual one) [**TZ**]. The intrinsic reason for the existence of this BV structure is that a Poincaré duality is a up to homotopy version of a Frobenius structure and that for Frobenius algebras, the Gerstenhaber structure in Hochschild cohomology is always BV [**Me**]. This result and our preliminary work leads to

Theorem 5.7. *Let k be a field of characteristic different from 2 and 3 and X be a triangulated oriented closed space with Poincaré duality (of dimension d), such that the closure of every simplex has the homology of a point.*

- *There is a BV-structure on $HH^*(C^*(X),C^*(X))$ and a compatible γ-ring structure.*
- *If X is simply connected, there is a BV-algebra structure on $\mathbb{H}_*(LX) := H_{*+d}(LX)$ and a compatible γ-ring structure. When X is a manifold the underlying product of the BV-structure is the Chas-Sullivan loop product.*

By a BV-structure on a graded space H^* and compatible γ-ring structure we mean the following:

(1) H^* is both a BV-algebra and a γ-ring.
(2) The BV-operator Δ and the γ-ring maps λ^k satisfy
$$\lambda^k(\Delta) = k\Delta(\lambda^k).$$
(3) There is an "ideal augmentation" spectral sequence $J_1^{pq} \Rightarrow H^{p+q}$ of BV algebras.
(4) On the induced filtration J_∞^{p*} of the abutment H^*, one has, for any $x \in J_\infty^{p*}$ and $k \geq 1$,
$$\lambda^k(x) = k^p x \bmod J_\infty^{p+1*}.$$
(5) If $k \supset \mathbb{Q}$, there is a Hodge decomposition $H^* = \prod_{i\geq 0} H^*_{(i)}$ (given by the associated graded of the filtration J_∞^{**}) such that the filtered space $\mathcal{F}_p H^* := \bigoplus H^*_{(n\leq p)}$ is a filtered BV-algebra.

As a consequence of Theorem 5.7, $Har^*(C^*(X), C^*(X))$ has an induced Lie algebra structure. Moreover $J_\infty^{0*}/J_\infty^{1*} \cong H_*(X)$ always splits.

Proof: We apply Lemma 5.4 to get a C_∞-algebra structure (given by a differential D) on $C^*(X)$. Assertion $iii)$ of this lemma ensures that there is a quasi-isomorphism of A_∞-algebras $F : (C^*(X), D) \to (C^*(X), d + \cup)$. Proposition 3.10 implies that

$$(5.23)\ HH^*((C^*(X), D), (C^*(X), D)) \cong HH^*((C^*(X), d+\cup), (C^*(X), d+\cup)).$$

Thus we only need to prove the theorem for $C^*(X)$ endowed with its C^∞-structure. The proof of 3.10 shows that the isomorphism (5.23) is the composition of the following isomorphisms:

$$F_* : HH^*((C^*(X), D), (C^*(X), D)) \to HH^*((C^*(X), D), (C^*(X), d+\cup)) \text{ and}$$

$$HH^*((C^*(X), D), (C^*(X), d+\cup)) \leftarrow HH^*((C^*(X), d+\cup), (C^*(X), d+\cup)) : F^*.$$

Since $(C^*(X), d+\cup)$ is an A_∞-algebra, formula (1.7) yields a ring structure on $HH^*((C^*(X), D), (C^*(X), d+\cup))$ and F_* and F^* are rings morphisms. Thus the cohomology $HH^*((C^*(X), D), (C^*(X), D))$ and $HH^*((C^*(X), d+\cup), (C^*(X), d+\cup))$ are isomorphic as rings.

By Theorem 3.1 there is a γ-ring structure on $HH^*(C^*(X), C_*(X))$. The Poincaré duality structure quasi-isomorphism $\Xi : C_*(X) \to C^*(X)$ and Proposition 3.10 implies that there is an isomorphism of γ-rings

$$HH^*(C^*(X), C_*(X)) \cong HH^*(C^*(X), C^*(X)).$$

The compatibility between the γ-ring structure and the Gerstenhaber structure follows from Proposition 3.30. The existence of the BV-structure is asserted by Tradler-Zeinalian [**TZ**] as stated above. Note that the BV-structure identifies with Connes's operator $B^* : HH^*(C^*(X), C^*(X)) \to HH^*(C^*(X), C^*(X))$ through the isomorphism $HH^*(C^*(X), C_*(X)) \cong HH^*(C^*(X), C^*(X))$ [**Tr2**]. It is proved in [**Lo1**] that $kB(\lambda^k) = \lambda^k(B)$ on $T(sR)$. Thus by duality we get the BV-compatibility.

If X is simply connected, Theorem 5.1 ensures that

$$\begin{aligned} H_*(LX) \cong HH^*((C^*(X), d+\cup), (C_*(X), d+\cup)) &\cong HH^*(C^*(X), C_*(X)) \\ &\cong HH^{*-d}(C^*(X), C^*(X)) \end{aligned}$$

where the last isomorphism is induced by naturality and the Poincaré duality quasi-isomorphism Ξ. Thus the BV-structure is transferred to $H_*(LX)$. ■

EXAMPLE 5.8. Let $X = S^3$ with its usual simplicial structure and the associated triangulation and k be a field of characteristic different from 2. Its cochain complex is a C_∞-algebra. The term E_2^{pq} of the spectral sequence 3.7 is

$$HH^{p+q}(H^*(S^3), H^*(S^3)) = HH^{p+q}(k[y], k[y]) \text{ where } |y| = 3.$$

It is a spectral sequence of γ-rings. An easy computation yields that this page of the spectral sequence has a Hodge decomposition where the only non trivial terms are

$$HH^{-2p}_{(p)}(H^*(S^3), H^*(S^3)) = k, \qquad HH^{-2p+3}_{(p)}(H^*(S^3), H^*(S^3)) = k \quad (p \geq 0).$$

The Hodge decomposition above holds even if $\mathrm{char}(k) > 0$. In that case, the computation yields a partial Hodge decomposition with the same terms but with the subscript (p) being taken modulo $\mathrm{char}(k) - 1$ for $p > 0$, i.e.

$$HH^*_{(p)}(H^*(S^3), H^*(S^3)) = HH^*_{(p+n(\mathrm{char}(k)-1))}(H^*(S^3), H^*(S^3))$$

for $1 \leq p \leq \mathrm{char}(k) - 1$. The total degree of an element in $HH^*(H^*(S^3), H^*(S^3))$ enables to split off the various terms of the partial decomposition, thus giving the claimed Hodge decomposition above. The higher differentials necessarily vanish and one finds that $HH^*(C^*(S^3), C^*(S^3))$ has a decomposition

$$HH^{-2p+3}(C^*(S^3), C^*(S^3)) = HH^{-2p+3}_{(p)}(C^*(S^3), C^*(S^3)) = k,$$
$$HH^{-2p}(C^*(S^3), C^*(S^3)) = HH^{-2p}_{(p)}(C^*(S^3), C^*(S^3)) = k$$

where $p \geq 0$. By Theorem 5.7, the BV-operator commutes with the λ-operations and the ring structure is the same as the one of the Hochschild cohomology of its singular cochains (viewed as an associative differential graded algebra). Thus we have an isomorphism of rings

$$\mathbb{H}_*(LS^3) \cong HH^*(C^*(S^3), C^*(S^3)) \cong k[u, v] \text{ with } |u| = 3,\ |v| = -2,$$

see [**FTV**] for example (the degrees are cohomological ones). The weight p-piece of the cohomology is the component $k[u]v^p$. In particular the λ-operations also commute with the loop product and the Hodge decomposition is graded for the BV-structure. An analogous computation using spectral sequence 3.21 gives

$$HH_{-2p}(C^*(S^3), C^*(S^3)) = HH^{(p)}_{-2p}(H^*(S^3), H^*(S^3)) = k \text{ and}$$

$$HH_{-2p-3}(C^*(S^3), C^*(S^3)) = HH^{(p)}_{-2p-3}(H^*(S^3), H^*(S^3)) = k$$

for $p \geq 0$ and other terms are null.

The computation for S^3 are straightforwardly generalized to all spheres. For odd dimensional simply connected spheres one has isomorphism of rings ($n \geq 1$)

$$\mathbb{H}_*(LS^{2n+1}) = HH^*(C^*(S^{2n+1}), C^*(S^{2n+1})) = k[u, v]$$

with $|u| = 2n + 1$, $|v| = -2n$ and the weight p-component of the Hodge decomposition is

$$\mathbb{H}^{(p)}_*(LS^{2n+1}) = HH^*_{(p)}(C^*(S^{2n+1}), C^*(S^{2n+1})) = kv^p \oplus kuv^p.$$

For even dimensional (simply connected) spheres, one has an isomorphism of rings ($n \geq 1$)

$$\mathbb{H}_*(LS^{2n}) = HH^*(C^*(S^{2n}), C^*(S^{2n})) = k[v, w] \oplus k[u]/(u^2)$$

with $|u| = 2n$, $|v| = 2 - 4n$ and $|w| = 1$. The weight p-component of the Hodge decomposition is

$$\mathbb{H}_*^{(p\geq 1)}(LS^{2n}) = kv^p \oplus kwv^{p-1}, \quad \mathbb{H}_*^{(0)}(LS^{2n}) = k[u]/(u^2).$$

In particular the BV-structure is graded with respect to the Hodge decomposition. Furthermore, denoting $s^{-i}k =: k[i]$ the module k concentrated in cohomological degree i (hence homological degree $-i$), the homology spectral sequence yields that the groups

$$H^k(LS^{2n+1}) \cong HH_{-k}(C^*(S^{2n+1}), C^*(S^{2n+1})) \text{ and}$$
$$H^k(LS^{2n}) \cong HH_{-k}(C^*(S^{2n}), C^*(S^{2n}))$$

have Hodge decomposition where the weight p-pieces are

$$HH_*^{(p\geq 0)}(C^*(S^{2n+1}), C^*(S^{2n+1})) = k[2p + 2n + 1] \oplus k[2p] \text{ and}$$

$$HH_*^{(p\geq 1)}(C^*(S^{2n}), C^*(S^{2n})) = k[p(4n - 2) + 2n] \oplus k[p(4n - 2) - 2n + 1].$$

Of course $HH_*^{(0)}(C^*(S^{2n}), C^*(S^{2n})) = H^*(S^{2n}) = k[2n] \oplus k$.

EXAMPLE 5.9. If $\mathrm{char}(k) = 0$, the Harrison (co)homology groups of $C^*(S^n)$ immediately follow from Theorem 3.1 and Example 5.8:

$$Har^*(C^*(S^{2n+1}), C^*(S^{2n+1})) = k[-2n] \oplus k[1],$$
$$Har^*(C^*(S^{2n}), C^*(S^{2n})) = k[2 - 4n] \oplus k[1],$$
$$Har_*(C^*(S^{2n+1}), C^*(S^{2n+1})) = k[2n + 3] \oplus k[2],$$
$$Har_*(C^*(S^{2n}), C^*(S^{2n})) = k[6n - 2] \oplus k[2n - 1]$$

where $k[i]$ still means k concentrated in cohomological degree i. If $\mathrm{char}(k) = p > 0$ then

$$Har^*(C^*(S^{2n+1}), C^*(S^{2n+1})) = \prod_{i\geq 0} k[-2n(pi - i + 1)] \oplus k[1 - 2n(pi - i)],$$

$$Har^*(C^*(S^{2n}), C^*(S^{2n})) = \prod_{i\geq 0} k[(2 - 4n)(pi - i + 1)] \oplus k[1 + i(2 - 4n)(p - 1)].$$

EXAMPLE 5.10. For $X = \mathbb{CP}_n$, the loop homology ring is $H^*(X) = k[x]/(x^{n+1})$ (where $|x| = 2$), see [**CJY**]. When k is of characteristic different from $n + 1$, the spectral sequence 3.7 also collapses at page 2 and a straightforward computation yields an isomorphism of rings

$$H_*(L\mathbb{CP}_n) \cong HH^*(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) \cong k[u, v, w]/(u^{n+1}, u^n v, u^n w)$$

where $|u| = 2$, $|v| = -2n$ and $|w| = 1$. Furthermore the Hodge decomposition is given by

$$H_*^{(p\geq 1)}(L\mathbb{CP}_n) = \big(v^p k[u] \oplus wv^{p-1} k[u]\big)/(u^n v, u^n w)$$

and $H_*^{(0)}(L\mathbb{CP}_n) = k[u]/(u^{n+1})$. As in Example 5.8, we get a Hodge decomposition even if $\mathrm{char}(k) > 0$. In particular, the Harrison cohomology groups are

$$Har^*(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) = \big(k[u]/(u^n)\big)[-2n] \oplus \big(k[u]/(u^n)\big)[1]$$

if $\mathrm{char}(k) = 0$ and

$$Har^*(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) = \big(k[u]/(u^n)\big)\prod_{i\geq 0} k[-2n(pi-i+1)] \oplus k[1-2in(p-1)]$$

if $\mathrm{char}(k) = p$. The Hodge decomposition in Hochschild homology is given by

$$HH_*^{(p)}(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) = k[x]/(x^n)[2np-2n+1] \oplus k[x]/(x^n)[2np+2],$$
$$HH_*^{(0)}(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) = k[x]/(x^{n+1})$$

where the degrees are cohomological degrees. In particular, if $\mathrm{char}(k) = 0$, the only non trivial Harrison homology groups are

$$Har_{2i-1}(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) = k \quad \text{for} \quad 1 \leq i \leq n,$$
$$Har_{2i}(C^*(\mathbb{CP}_n), C^*(\mathbb{CP}_n)) = k \quad \text{for} \quad n+1 \leq i \leq 2n.$$

6. Concluding remarks

- At the same time as a first draft of this paper, Hamilton and Lazarev [**HL**] (also see the recent updated versions [**HL2, HL3, HL4**]) wrote a paper about cohomology of homotopy algebras, using Kontsevich framework of formal non-commutative geometry. In particular they study Harrison and Hochschild cohomology of C_∞-algebras over a field of characteristic zero and give a Hodge decomposition of $HH^*(R,R)$ and $HH^*(R,R^\star)$. Using that the C_∞-structure is determined by maps $D_i : R^{\otimes i} \to R$, it is easy to check that their definitions are dual and equivalent to ours in this special cases. They also prove that the above cohomology theories yields the good obstruction theory. They finally apply it to (a different from our) issue in string topology, namely the homotopy invariance of the Gerstenhaber algebra structure.
- The Connes operator $B : C^*(R,M) \to C^{*-1}(R,M)$ is well defined for C_∞-algebras and commutes with the Hochschild differential, thus one can define cyclic (co)homology of a C_∞-algebra R see [**GJ1, Tr2, HL**]. Furthermore, it is easy to check that the λ-operations and Hodge decomposition passes to the various cyclic homology theories in characteristic zero [**HL**]. In positive characteristic, the λ-operations passes to cyclic (co)homology but not to negative cyclic (co)homology.
- Besides the BV-algebra structure, there are other string topology operations on $\mathbb{H}_*(LM)$ as well as in equivariant homology $H_*^{S^1}(LM)$, which come from an action of Sullivan chord diagram on LM. It seems interesting to obtain compatibility conditions between the λ-operation/Hodge decomposition and the full scope of string topology operation. It might be achieved by combining the techniques of this paper and [**TZ2**].
- There are power maps $\gamma^k : LM \to LM$ which sends a loop $f : S^1 \to M$ to the loop $u \mapsto f(ku)$. It seems reasonable to expect that these power maps coincides with our λ-operation for simply connected spaces.

References

[Ba] J. H. Baues, *The double bar and cobar constructions*, Compos. Math. 43 (1981), 331-341

[BW] N. Bergeron, L. Wolfgang *The decomposition of Hochschild cohomology and Gerstenhaber operations*, J. Pure Appl. Algebra 79 (1995) 109–129

[CS] M. Chas, D. Sullivan *String Topology*, preprint GT/9911159 (1999)

[Co] R. Cohen *Multiplicative properties of Atiyah duality*, Homology, Homotopy, and its Applications, vol 6 no. 1 (2004), 269–281

[CG] R. Cohen, V. Godin *A polarized view to string topology*, Topology, Geometry, and Quantum Field theory, Lond. Math. Soc. lecture notes vol. 308 (2004), 127–154

[CJ] R. Cohen, J.D.S. Jones *A homotopic realization of string topology*, Math. Annalen, vol 324. 773-798 (2002)

[CJY] R. Cohen, J.D.S. Jone, J. Yan *The loop homology algebra of spheres and projective spaces*, in Categorical Decomposition Techniques in Algebraic Topology, Prog. Math. 215 (2004), 77–92

[EKMM] A. Elmendorf, I. Kriz, M. Mandell, J.P.May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, 47. American Mathematical Society, Providence, RI, 1997.

[FTV] Y. Félix, Y. Thomas, M. Vigué *The Hochschild cohomology of a closed manifold*, Publ. IHES, 99, (2004), 235-252

[FTV2] Y. Félix, Y. Thomas, M. Vigué *Rational string topology*, J. Eur. Math. Soc. (JEMS) 9 (2007), no. 1, 123–156.

[Fr1] B. Fresse, *Homologie de Quillen pour les algèbres de Poisson*, C.R. Acad. Sci. Paris Sér. I Math. 326(9) (1998), 1053–1058

[Fr2] B. Fresse, *Théorie des opérades de Koszul et homologie des algèbres de Poisson*, preprint

[Ge] M. Gerstenhaber, *The Cohomology Structure Of An Associative ring* Ann. Maths. 78(2) (1963)

[GS1] M. Gerstenhaber, S. Schack, *A Hodge-type decomposition for commutative algebra cohomology* J. Pure Appl. Algebra 48 (1987), no. 3, 229–247

[GS2] M. Gerstenhaber, S. Schack, *The shuffle bialgebra and the cohomology of commutative algebras* J. Pure Appl. Algebra 70 (1991), 263–272

[GV] M. Gerstenhaber, A. Voronov, *Homotopy G-algebras and moduli space operad*, Internat. Math. Res. Notices (1995), no. 3, 141–153

[GJ1] E. Getzler, J.D.S. Jones, *A_∞-algebras and the cyclic bar complex*, Illinois J. Math. 34 (1990) 12–159

[GJ2] E. Getzler, J.D.S. Jones *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint hep-th/9403055 (1994)

[Gi] G. Ginot, *Homologie et modèle minimal des algèbres de Gerstenhaber*, Ann. Math. Blaise Pascal 11 (2004), no. 1, 95–127

[GH1] G. Ginot, G. Halbout *A formality theorem for Poisson manifold*, Let. Math. Phys. 66 (2003) 37–64

[GH2] G. Ginot, G. Halbout *Lifts of G_∞-morphism to C_∞ and L_∞-morphisms*, Proc. Amer. Math. Soc. 134 (2006) 621–630.

[GK] V. Ginzburg, M. Kapranov, *Koszul duality for operads*, Duke Math. J. 76 (1994), No 1, 203–272

[HL] A. Hamilton, A. Lazarev *Homotopy algebras and noncommutative geometry*, preprint, math.QA/0410621

[HL2] A. Hamilton, A. Lazarev *Cohomology theories for homotopy algebras and noncommutative geometry*, preprint, math.QA/0707.2311

[HL3] A. Hamilton, A. Lazarev *Symplectic C_∞-algebras*, preprint, math.QA/0707.3951.

[HL4] A. Hamilton, A. Lazarev *Symplectic A_∞-algebras and string topology operations*, preprint, math.QA/0707.4003.

[Hi] H. Hiller, *λ-rings and algebraic K-theory.* J. Pure Appl. Algebra 20 (1981), no. 3, 241–266.

[HS] J. Huebschmann, J. Stasheff *Formal solution of the master equation via HPT and deformation theory*, Forum. Math. 14 (2002), no. 6, 847–868

[Jo] J.D.S. Jones *Cyclic homology and equivariant homology*, Inv. Math. 87, no.2 (1987), 403–423

[Ka] T. Kadeishvili *On the homology theory of fiber spaces*, Russian Mathematics Surveys 6 (1980), 231–238.

[KST] T. Kimura, J. Stasheff, A. Voronov *Homology of moduli spaces of curves and commutative homotopy algebras*, Comm. Math. Phys. 171 (1995), 1–25

[Lo1] J.-L. Loday, *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. Math. 96 (1989), No. 1, 205–230

[Lo2] J.-L. Loday, *Cyclic homology*, Springer Verlag (1993)

[Lo3] J.-L. Loday, *Série de Hausdorff, idempotents Eulériens et algèbres de Hopf*, Expo. Math. 12 (1994), 165–178

[Ma] Y. Manin, *Frobenius Manifolds, Quantum Cohomology and Moduli Spaces*, Colloquium publications 47 (1991), American Mathematical Society

[MMSS] M. Mandell, J.P. May, S. Schwede, B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) 82 (2001), no. 2, 441–512.

[Me] L. Menichi *Batalin-Vilkovisky algebras and cyclic cohomology of Hopf algebras*, K-Theory 32 (2004), 231–251.

[Mer] S. Merkulov, *De Rham model for string topology*, Int. Math. Res. Not. 2004, no. 55, 2955–2981.

[Pa] F. Patras, *La décomposition en poids des algèbres de Hopf*, Ann. Inst. Fourier 43 (1993), No. 4, 1067–1087

[St] J.D. Stasheff, *Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. 108 (1963), 275–292

[Sm] V. Smirnov, *Simplicial and operad methods in algebraic topology*, Transl. Math. Monographs 198, American Mathematical Society (2001)

[Su] D. Sullivan, *Appendix to Infinity structure of Poincaré duality spaces*, Algebr. Geom. Topol. 7 (2007), 233–260.

[Ta] D. Tamarkin, *Another proof of M. Kontsevich's formality theorem*, math.QA/9803025

[Tr1] T. Tradler, *Infinity-inner-products on a A-infinity-algebras*, preprint arXiv AT:0108027

[Tr2] T. Tradler, *The BV Algebra on Hochschild Cohomology Induced by Infinity Inner Products*, preprint arXiv QA:0210150

[TZ] T. Tradler, M. Zeinalian *Infinity structure of Poincaré duality spaces*, Algebr. Geom. Topol. 7 (2007), 233–260.

[TZ2] T. Tradler, M. Zeinalian *On the cyclic Deligne conjecture*, J. Pure Appl. Algebra 204 (2006), no. 2, 280–299

[Vi] M. Vigué *Décompositions de l'homologie cyclique des algèbres différentielles graduées*, *K*-theory 4 (1991) -399–410

[WGS] J. Wu, M. Gerstenhaber, J. Stasheff, *On the Hodge decomposition of differential graded bi-algebras* J. Pure Appl. Algebra 162 (2001), no. 1, 103–125

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