# String topology for loop stacks

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**Abstract**

We prove that the homology groups of the free loop stack of an oriented stack are equipped with a canonical loop product and coproduct, which makes it into a Frobenius algebra. Moreover, the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ admits a BV algebra structure.

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**Résumé**

**Topologie des cordes pour les lacets libres d'un champ.** On munit les groupes d'homologie du champ des lacets libres d'un champ orienté d'un produit et d'un coproduit induisant une structure d'algèbre de Frobenius. De plus, l'homologie en degrés décalés $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ est une algèbre BV.

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## Version française abrégée

Dans cette note on généralise le produit de Chas et Sullivan [3], défini sur les groupes d'homologie de l'espace des lacets libres d'une variété orientée, aux champs. On généralise également deux autres constructions de la topologie des cordes : l'opérateur BV [3] et le coproduit [4]. Pour ce faire, la bonne notion de champ est celle de champ différentiel muni d'une diagonale normalement non-singulière orientée

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(*cf.* [1]). Nous dirons d'un tel champ qu'il est *orienté*. Rappelons qu'un champ est dit topologique s'il est représentable par un groupoide topologique et que les champs différentiels sont ceux représentables par un groupoide de Lie. A un champ topologique $\mathfrak{X}$, on peut associer fonctoriellement un champ $\mathrm{L}\mathfrak{X} = \mathrm{hom}(S^1, \mathfrak{X})$ qui est topologique. Ici, hom désigne le champ des morphismes de champ [9]. De fait, pour toute présentation $\Gamma$ de $\mathfrak{X}$, on donne une construction naturelle d'un groupoide $\mathrm{L}\Gamma$ représentant $\mathrm{L}\mathfrak{X}$. Cette construction s'obtient comme la limite, sur tous les sous-ensembles finis $P = \{0 < p_1 < \cdots < p_{n-1}\}$ de $S^1 = [0,1]/\{0 \sim 1\}$ ($n \geq 1$), du groupoide des morphismes de groupoides stricts $S^P \to M\Gamma$. Le groupoide $M\Gamma$ est le groupoide des carrés commutatifs dans la catégorie $\Gamma$ et $S^P$ est le groupoide $S_0^P \times_{S^1} S_0^P \rightrightarrows S_0^P$ où, en notant $p_n = 1 = 0$, $S_0^P$ est la réunion disjointe $\coprod_{i=1}^n [p_{i-1}, p_i]$ .

De manière similaire à [1], [2], pour tout champ différentiel orienté $\mathfrak{X}$ de dimension $d$, on construit un produit

$$\star : H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X}).$$

et un coproduit

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}).$$

Les propriétés de fonctorialité de $\mathrm{L}\mathfrak{X}$ lui confèrent une action de $S^1$. On obtient alors un opérateur $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ qui est l'application composée :

$$H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\times\omega} H_{\bullet+1}(\mathrm{L}\mathfrak{X} \times S^1) \longrightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

où $\omega \in H_1(S^1)$ est la classe fondamentale de $S^1$. La dernière flèche est induite par l'action de $S^1$ sur $\mathrm{L}\mathfrak{X}$. Les propriétés de fonctorialité et de naturalité des morphismes de Gysin (cf. [1] Proposition 2.2) autorisent l'utilisation des méthodes opéradiques de Cohen, Jones et Voronov [5], [6] pour montrer que $D$ est un opérateur $BV$. Les résultats principaux de cette note sont résumés dans le théorème suivant :

**Théorème 0.1** *Soit $\mathfrak{X}$ un champ orienté de dimension d.*

*(i) L'homologie en degrés décalés $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$, munie du produit $\star : \mathbb{H}_*(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_*(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_*(\mathrm{L}\mathfrak{X})$ et de l'opérateur $D : \mathbb{H}_*(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{*+1}(\mathrm{L}\mathfrak{X})$, est une algèbre $BV$.*

*(ii) De plus l'homologie $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ est une algèbre de Frobenius non nécessairement (co)unitaire.*

L'homologie du champ d'inertie $\Lambda\mathfrak{X}$ associé à $\mathfrak{X}$ est aussi munie d'une structure d'algèbre de Frobenius [2]. Par ailleurs, il existe un morphisme naturel de champs $\Phi : \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.

**Theorem 0.1** *L'application induite $\Phi_* : H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$ est un morphsime d'algèbres de Frobenius.*

## 1. Introduction

In this Note, we generalize the loop product [3] and coproduct [4], as well as the BV-operator [3], for loop homology of manifolds to stacks. The relevant notion is oriented differential stacks [1], which we simply call oriented stacks in the Note. In fact, if $\mathfrak{X}$ is a topological stack, there is a functorial construction of the free loop stack $\mathrm{L}\mathfrak{X}$ which is a topological stack. Indeed for any presentation $\Gamma$ of the stack $\mathfrak{X}$, we present a natural construction of a topological groupoid, which represents the free loop stack $\mathrm{L}\mathfrak{X}$. Similar to the constructions in [1], [2], for any oriented differential stack of dimension $d$, we construct a product

$$\star : H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X}),$$

and a coproduct

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X})$$

which makes $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ into a Frobenius algebra. Furthermore, we give a natural map $\Phi : \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$, where $\Lambda\mathfrak{X}$ is the inertia stack of $\mathfrak{X}$, which induces a nontrivial morphism of Frobenius algebras in homology.

Due to its functorial property, $\mathrm{L}\mathfrak{X}$ admits a natural $S^1$-action which induces a square zero operator $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$. We prove that the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ together with the string product $\star : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ and the operator $D : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$ becomes a $BV$-algebra.

## 2. Free loop stack

We use the conventions and notations from [1], [2]. Let $\mathfrak{X}$ be a topological stack and $A$ a compactly generated topological space. We define the stack $\mathrm{Map}(A, \mathfrak{X})$, called the **mapping stack** from $A$ to $\mathfrak{X}$, by the rule

$$T \in \mathbf{Top} \quad \mapsto \quad \hom(T \times A, \mathfrak{X}).$$

Here, the right hand side stands for the groupoid of stack morphisms from $T \times A$ to $\mathfrak{X}$. The mapping stack $\mathrm{Map}(A, \mathfrak{X})$ is functorial in $A$ and $\mathfrak{X}$.

**Lemma 2.1** *If $A$ is compact, then* $\mathrm{Map}(A, \mathfrak{X})$ *is a topological stack.*

In the case where $\mathfrak{X}$ is a manifold, $\mathrm{Map}(A, \mathfrak{X})$ represents the usual mapping space with the compact-open topology.

When $A = S^1$ is the unit circle, we denote $\mathrm{Map}(S^1, \mathfrak{X})$ by $\mathrm{L}\mathfrak{X}$ and call it the **free loop stack** of $\mathfrak{X}$. By functoriality of mapping stacks, for every $t \in S^1$ we have the corresponding evaluation map $\mathrm{ev}_t : \mathrm{L}\mathfrak{X} \to \mathfrak{X}$.

Let us now describe, for any presentation $\Gamma : \Gamma_1 \rightrightarrows \Gamma_0$ of a topological stack $\mathfrak{X}$, a natural useful construction of a groupoid which represents the free loop stack $\mathrm{L}\mathfrak{X}$. We use this presentation at the end of Section 5. Note that our construction is somehow similar to the construction of the fundamental groupoid of a groupoid [8]. Let $P \subset S^1$ be a finite subset of $S^1$ which contains the base point $0 \sim 1 \in S^1$. The points of $P$ are labeled according to increasing angle as $P_0, P_1, \ldots, P_n$ in such a way that $P_0 = P_n$ is the base point of $S^1$. Write $I_i$ for the closed interval $[P_{i-1}, P_i]$. Let $S_0^P$ be the disjoint union $S_0^P = \coprod_{i=1}^n I_i$. There is a canonical map $S_0^P \to S^1$. Let $S_1^P$ be the fiber product $S_1^P = S_0^P \times_{S^1} S_0^P$. There is an obvious topological groupoid structure $S_1^P \rightrightarrows S_0^P$. The compact-open topology induces a topological groupoid structure on $\mathrm{L}^P\Gamma : \mathrm{L}_1^P\Gamma \rightrightarrows \mathrm{L}_0^P\Gamma$, where $\mathrm{L}_0^P\Gamma$ is the set of continuous strict groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \to [\Gamma_1 \rightrightarrows \Gamma_0]$ and $\mathrm{L}_1^P\Gamma$ is the set of strict continuous groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \to [M_1\Gamma \rightrightarrows M_0\Gamma]$. Here $M\Gamma = [M_1\Gamma \rightrightarrows M_0\Gamma]$ is the morphism groupoid of $\Gamma$. Recall that the groupoid $M\Gamma$ is the groupoid where $M_1\Gamma$ is the set of commutative squares

$$\begin{array}{ccc} t(h) & \xleftarrow{\;g\;} & t(k) \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle k} \\ s(h) & \xleftarrow{\;h^{-1}gk\;} & s(k) \end{array} \qquad (1)$$

in the underlying category of $\Gamma$. The source and target maps are the horizontal arrows as in square (1) and the groupoid multiplication is by superposition of squares. Thus we have $M_0\Gamma \cong \Gamma_1$ and $M_1\Gamma \cong \Gamma_3 = \Gamma_1 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Gamma_1$.

*Remark 1 Note that this topological groupoid structure is analogous to the one used in Section 2 of [2].*

**Lemma 2.2** *Let $\Gamma$ be a groupoid representing a topological stack $\mathfrak{X}$. The limit*

$$\mathrm{L}\Gamma = \varinjlim_{P \subset S^1} \mathrm{L}^P\Gamma$$

*represents the free loop stack* $\mathrm{L}\mathfrak{X}$.

It is easy to represent evaluation map and functorial properties of the free loop stack at the groupoid level with this model.

## 3. Loop product

In this section we consider *oriented* stacks. Recall that a differential stack $\mathfrak{X}$ is called oriented if the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is an oriented normally nonsingular morphism [1]. For instance, oriented manifolds and oriented orbifolds are oriented stacks. More generally, the quotient stack of a compact Lie group acting by orientation preserving automorphisms on an oriented manifold is an oriented stack.

Consider the cartesian diagram

$$\begin{array}{ccc} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_0)} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

The fact that $\mathfrak{X}$ is topological implies ([9], Proposition 16.1) that there is a natural equivalence of stacks

$$\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \cong \mathrm{Map}(8, \mathfrak{X}).$$

The map $S^1 \to S^1 \vee S^1$ that pinches $\frac{1}{2}$ to 0, induces a natural map $m : \mathrm{Map}(8, \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$, called the *Pontrjagin mltiplication*. Putting these together, we have the following augmented cartesian square:

$$\begin{array}{ccccc} \mathrm{L}\mathfrak{X} & \xleftarrow{m} & \mathrm{Map}(8, \mathfrak{X}) & \longrightarrow & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \\ & & \downarrow & & \downarrow{\scriptstyle e=(\mathrm{ev}_0, \mathrm{ev}_0)} \\ & & \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array} \tag{2}$$

By Proposition 2.2 in [1], since $\Delta$ is an oriented normally nonsingular morphism of codimension $d$, we have a Gysin map $G^e_\Delta : H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{Map}(8, \mathfrak{X}))$. We define the *loop product* to be the following composition

$$H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \cong H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \xrightarrow{G^e_\Delta} H_{\bullet-d}(\mathrm{Map}(8, \mathfrak{X})) \xrightarrow{m_*} H_{\bullet-d}(\mathrm{L}\mathfrak{X}).$$

**Theorem 3.1** *Let $\mathfrak{X}$ be an oriented stack of dimension $d$. The loop product induces a structure of associative and graded commutative algebra for the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) := H_{\bullet+d}(\mathrm{L}\mathfrak{X})$.*

Similar to the string product of inertia stacks, there is also a "twisted" version of loop product. Let $\alpha$ be a class in $H^r(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$. The *twisted loop product* $\star_\alpha : H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d-r}(\mathrm{L}\mathfrak{X})$ is defined, for all $x, y \in H_\bullet(\mathrm{L}\mathfrak{X})$, by

$$x \star_\alpha y = m_*\big(G^e_\Delta(x \times y) \cap \alpha\big).$$

Recall some notations from [1]: let $p_{12}, p_{23} : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ be, respectively, the projections on the first two and the last two factors. Also $(m \times 1)$ and $(1 \times m)$: $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ denote the Pontrjagin multiplication of the two first factors and two last factors respectively.

**Theorem 3.2** *Let $\alpha$ be a class in $H^r(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$. If $\alpha$ satisfies the 2-cocycle condition*

$$p_{12}^*(x) \cup (m \times 1)^*(\alpha) = p_{23}^*(\alpha) \cup (1 \times m)^*(\alpha)$$

*in $H^\bullet(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$, then $\star_e : H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}) \to H_{i+j-d-r}(\mathrm{L}\mathfrak{X})$ is associative.*

## 4. *BV*-structure

In this section, we assume that singular homology is taken with coefficients in a field of characteristic different from 2. In this case, the homology of the free loop space of a manifold is known to be a *BV*-

algebra [3]. The operadic approach of Cohen, Jones and Voronov [5],[6] for constructing the $BV$-structure relies on the existence of evaluation maps, existence of Gysin maps and their functoriality and naturality properties. Thanks to Proposition 2.2 in [1], one can adapt their approach to stacks.

By the functorial properties of Lemma 2.1, the free loop stack $\mathrm{L}\mathfrak{X}$ inherits an $S^1$-action. Introduce an operator $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ by the composition

$$H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\times\omega} H_{\bullet+1}(\mathrm{L}\mathfrak{X} \times S^1) \longrightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

where $\omega \in H_1(S^1)$ is the fundamental class and the last arrow is induced by the action. It is not hard to check that $D^2 = 0$.

**Theorem 4.1** *Let $\mathfrak{X}$ be an oriented stack of dimension $d$. The shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ admits a $BV$-algebra structure given by the loop product $\star : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ and the operator $D : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$.*

*Example 1 When $\mathfrak{X}$ is an oriented manifold $M$, then $\mathrm{L}\mathfrak{X}$ is the free loop space of $M$. Then the $BV$-structure on $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ coincides with the one of Chas and Sullivan [3],[5].*

*If $\mathfrak{X}$ is a global quotient orbifold which is oriented, then the $BV$-structure on $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ coincides with the one introduced (in characteristic zero) in [7].*

## 5. Frobenius structure and inertia stack

It is known [4] that there is also a coproduct on the homology of a free loop manifold which induces a Frobenius algebra structure. Also, in [2] it is shown that the homology of the inertia stack of an oriented stack $\mathfrak{X}$ admits a Frobenius algebra structure. Thus it is reasonable to expect that such a structure exist on $H_\bullet(\mathrm{L}\mathfrak{X})$ as well. We show that this is indeed the case. Let $\mathrm{ev}_0, \mathrm{ev}_{1/2} : \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ be the evaluation maps defined in Section 2.

**Lemma 5.1** *The stack $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ fits into a cartesian square*

$$\begin{array}{ccc} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_{1/2})} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array} \tag{3}$$

*where $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is the diagonal.*

According to Proposition 2.2 of [1], if $\mathfrak{X}$ is an oriented differential stack of dimension $d$, the cartesian square (3) yields a Gysin map

$$G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})} : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}).$$

By diagram (2), there is a map $\mathrm{Map}(8, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \xrightarrow{\rho} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}$. Thus we obtain a degree $d$ map

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})}} H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \xrightarrow{\rho_*} H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}).$$

**Theorem 5.2** *Let $\mathfrak{X}$ be an oriented stack of dimension $d$. Then $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ is a Frobenius algebra, where both operations $\star$ and $\delta$ are of degree $d$.*

We now introduce a morphism of stacks $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$. Let $\Gamma$ be a groupoid representing $\mathfrak{X}$ and $\Lambda\Gamma$ its inertia groupoid representing $\Lambda\mathfrak{X}$. We refer to [1] [2] for details. Following the notations as in Lemma 2.2, we take $P = \{1\} \subset S^1$ as a trivial subset of $S^1$. Any element $(g, h) \in S\Gamma \rtimes \Gamma$ (i.e. $g \in \Gamma_1$ with $s(g) = t(g)$) determines a commutative diagram $D(g, h)$ in the category $\Gamma$

$$
[\mathtt{D(g,h)}]: \quad \begin{array}{ccc} t(h) & \xleftarrow{\;g\;} & t(h) \\ {\scriptstyle h}\big\uparrow & & \big\uparrow{\scriptstyle h} \\ s(h) & \xleftarrow{\;h^{-1}gh\;} & s(h) \end{array},
$$

thus an element of $M_1\Gamma$. In particular it induces a (constant) groupoid morphism $[S_1^P \rightrightarrows S_0^P] \to [M_1\Gamma \rightrightarrows M_0\Gamma]$. The map $(g,h) \mapsto D(g,h)$ is easily seen to be a groupoid morphism. We denote by $\Phi : \Lambda\Gamma \to \mathrm{L}\Gamma$ its composition with the inclusion $\mathrm{L}^P\Gamma \to \mathrm{L}\Gamma$.

**Lemma 5.3** *The map $\Phi : \Lambda\Gamma \to \mathrm{L}\Gamma$ induces a map of stacks $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.*

Thus there is an induced map $\Phi_\star : H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$.

**Theorem 5.4** *The map $\Phi_\star : H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$ is a morphism of Frobenius algebras.*

If $\mathfrak{X} = [*/G]$ with $G$ being a compact Lie group, then $\mathrm{L}[*/G]$ is homotopy equivalent to $\Lambda[*/G]$ and the map $\Phi : H_\bullet(\Lambda[*/G]) \to H_\bullet(\mathrm{L}[*/G])$ is an isomorphism of Frobenius algebras. This Frobenius structure is studied (with real coefficients) in [2]. In this case, the $BV$-operator is trivial.


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