String product for inertia stacks

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Abstract

We introduce a Chas-Sullivan type string product on the homology groups of the inertia stack of an oriented differential stack. We prove that its Poincaré dual is isomorphic to a twisted Chen-Ruan product when the stack is a compact complex orbifold.
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Résumé

Produit pour les lacets fantômes d'un champ. On construit, sur l'homologie du champ d'inertie d'un champ orienté, un produit analogue à celui de Chas-Sullivan pour les lacets libres d'une variété. On montre que ce produit est le dual de Poincaré d'un produit de Chen-Ruan tordu lorsque le champ est un orbifold complexe.
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Le produit, défini par Chas-Sullivan [**?**], sur l'homologie de l'espace des lacets libres d'une variété est relié à de nombreux sujets en mathématiques et particulièrement en mathématique physique. Ce produit d'intersection des lacets se décrit simplement comme la composée [**?**] :

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$$H_\bullet(LX)\otimes H_\bullet(LX)\cong H_\bullet(LX\times LX)\xrightarrow{G_\Delta^{\mathrm{ev}_0}} H_{\bullet-d}(LX_X LX)) \xrightarrow{p_*} H_{\bullet-d}(LX). \quad (1)$$

Ici $G_\Delta^{\mathrm{ev}_0}$ est le morphisme de Gysin obtenu à partir du diagramme cartésien

$$\begin{array}{ccc} LX\times_X LX & \longrightarrow & LX\times LX \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_0)} \\ X & \xrightarrow{\Delta} & X\times X \end{array}$$

en appliquant la construction de Thom au tiré en arrière du fibré normal de $X\to X\times X$ (cf. [?]). Le morphisme p_* est l'application naturelle associée à la multiplication de Pontrjagin des lacets $p: LX\times_X LX\to LX$.

Par ailleurs, un ingrédient essentiel de la cohomologie quantique des orbifolds étudiée par Chen-Ruan [?] est une structure d'anneau sur la cohomologie de l'orbifold d'inertie associé à un orbifold compact (quasi)-complexe. Le produit de Chas-Sullivan et celui de Chen-Ruan sont tous deux intimement reliés à la théorie des cordes. Il est donc naturel d'essayer d'établir un lien entre eux. Pour ce faire, il convient d'abord de construire un produit analogue à celui de Chas-Sullivan sur la (co)homologie d'un orbifold d'inertie. Un ingrédient essentiel est alors l'existence de morphismes de Gysin pour les orbifolds. Une approche catégorique très efficace pour étudier et construire des morphismes de Gysin est due à Fulton-Mac Pherson [?]. Dans cette note, nous introduisons des morphismes de Gysin à la Fulton-Mac Pherson associé à tout diagramme cartésien de champs topologiques satisfaisant certaines propriétés d'orientation. Lorsque la diagonale d'un champ $\mathfrak{X}$ vérifie ces conditions d'orientation, nous disons que $\mathfrak{X}$ est orienté.

Il est usuel de considérer les champs d'inertie comme des champs de lacets fantômes [?]. En utilisant les morphismes de Gysin, on adapte alors aisément le diagramme (??) pour définir un produit sur $H_*(\Lambda\mathfrak{X})$.

Théorème 0.1 *Soit $\mathfrak{X}$ un champ différentiel orienté de dimension d. L'application donnée par la composée $H_\bullet(\Lambda\mathfrak{X})\otimes H_\bullet(\Lambda\mathfrak{X})\xrightarrow{G} H_\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})\xrightarrow{m_*} H_\bullet(\Lambda\mathfrak{X})$ définit un produit associatif, commutatif gradué sur $\mathbb{H}_\bullet(\Lambda\mathfrak{X}):=H_{\bullet+d}(\Lambda\mathfrak{X})$, appelé le produit fantôme.*

Afin de relier notre produit fantôme à celui de Chen-Ruan, on introduit, pour toute classe $w\in H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$ satisfaisant l'équation de cocycle Eq. (??), un produit de Chen-Ruan $\cup^w_{\mathrm{orb}}$ tordu par w. On obtient alors

Théorème 0.2 *Le dual de Poincaré du produit fantôme est le produit de Chen-Ruan $\cup^w_{\mathrm{orb}}$, où w est la classe d'Euler d'un fibré explicite sur $\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}$ défini à partir du fibré d'excès et du fibré d'obstruction de Chen-Ruan.*

1. Introduction

The string product on homology groups of loop manifolds as introduced by Chas-Sullivan [?] has been known to be related to many subjects in mathematics and in particular mathematical physics. A simple way to obtain this string product can be described as the composition of the following maps [?]:

$$H_\bullet(LX)\otimes H_\bullet(LX)\cong H_\bullet(LX\times LX)\xrightarrow{G_\Delta^{\mathrm{ev}_0}} H_{\bullet-d}(LX_X LX)) \xrightarrow{p_*} H_{\bullet-d}(LX). \quad (2)$$

Here $G_\Delta^{\mathrm{ev}_0}$ is the Gysin map obtained by applying Thom collapse construction to the pullback of the normal bundle of the diagonal map $X\to X\times X$ [?]. The pullback is taken along the evaluation map $\mathrm{ev}_0: LX\to X$, mapping a loop f to $f(0)$. The Pontrjagin multiplication of free loops with the same basepoint yields a map $p: LX\times_X LX\to LX$, which induces the map p_* in homology.

On the other hand, Chen-Ruan [?] introduced a ring structure on the cohomology groups of the inertia orbifold of a compact (almost) complex orbifold in connection with the study of quantum cohomology. Since both Chas-Sullivan string product and Chen-Ruan product are closely related to string theory, it is natural to expect that there is a close connection between these two products. The purpose of this Note is to study such a connection. In other words, we introduce a Chas-Sullivan type product on the homology groups of inertia orbifolds and also for more general differential stacks. For this purpose, a crucial step is the existence of a Gysin map for stacks. An efficient categorical way to study Gysin maps is to use Fulton-MacPherson theory. Indeed this is the approach we take in this Note. More precisely, we introduce a Fulton-MacPherson type Gysin map for cartesian diagram of topological stacks satisfying some orientation property.

It is a folksome that inertia stacks $\Lambda\mathfrak{X}$ can be considered as "ghost" loop stacks [?]. Using such a viewpoint and with the help of Gysin maps, we adapt diagram (**??**) to define an associative product on $H_*(\Lambda\mathfrak{X})$, called the string product.

To reveal the connection between our string product and Chen-Ruan product, we also introduce a twisted Chen-Ruan product $\cup^w_{\mathrm{orb}}$ for any class $w \in H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ satisfying the 2-cocycle condition Eq. (**??**). As a result, we prove that the Poincaré dual of the string product is isomorphic to the twisted Chen-Ruan product $\cup^w_{\mathrm{orb}}$, where w is the Euler class of some vector bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

We note that recently Lupercio et. al. [?] obtained similar result independently in the particular case of symmetric product orbifold.

2. Gysin maps for topological stacks

By a *stack* we mean a stack over the site **Top** of compactly generated topological spaces. A stack is called *topological* if it is equivalent to the quotient stack $[\Gamma_0/\Gamma_1]$ of a topological groupoid $\Gamma_1 \rightrightarrows \Gamma_0$. If $\Gamma_1 \rightrightarrows \Gamma_0$ is a Lie groupoid, the associated quotient stack is called a differential stack [?]. The quotient stack depends only on the Morita equivalence class of the groupoid Γ.

Let $\Gamma_p = \Gamma_1 \times_{\Gamma_0} \ldots \times_{\Gamma_0} \Gamma_1$ (p-fold) be the space of composable sequences of p arrows in the groupoid $\Gamma_1 \rightrightarrows \Gamma_0$. It yields a simplicial space. Recall that the *singular chain complex* of $\Gamma_\bullet$ is the total complex associated to the double complex $C_\bullet(\Gamma_\bullet)$ [?]. Here $C_q(\Gamma_p)$ is the linear space generated by the continuous maps $\Delta_q \to \Gamma_p$. Its homology groups $H_q(\Gamma_\bullet) = H_q\big(C_\bullet(\Gamma_\bullet)\big)$ are called the *homology groups* of Γ. These groups are Morita invariant (i.e. only depend on the quotient stack $[\Gamma_0/\Gamma_1]$).

Definition 2.1 *We say that a representable morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is* normally nonsingular*, if there exist vector bundles $\mathfrak{F}$ and $\mathfrak{E}$ over the stacks $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, and a commutative diagram*

$$\begin{array}{ccc} \mathfrak{F} & \xhookrightarrow{\;i\;} & \mathfrak{E} \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle p} \\ \mathfrak{X} & \xrightarrow{\;f\;} & \mathfrak{Y} \end{array}$$

where s is the zero section of the vector bundle $\mathfrak{F} \to \mathfrak{X}$, and i is an open immersion. A normally nonsingular f is called oriented *if $\mathfrak{F}$ and $\mathfrak{E}$ are oriented vector bundles (i.e. have Thom classes). The integer* $c = \mathrm{rk}\,\mathfrak{E} - \mathrm{rk}\,\mathfrak{F}$ *depends only on f and is called the* codimension *of f.*

If $\mathfrak{X}$ and $\mathfrak{Y}$ are differentiable stacks that are equivalent to quotient stacks of compact Lie group action on manifolds, then every representable $f : \mathfrak{X} \to \mathfrak{Y}$ is automatically normally nonsingular. In particular, any smooth map of manifolds is normally nonsingular. The projection map $\mathfrak{E} \to \mathfrak{X}$ of a vector bundle $\mathfrak{X}$ is also normally nonsingular.

An efficient categorical way to deal with Gysin maps is to develop the Fulton-MacPherson theory [?] for topological stacks. In this way, we obtain a bivariant theory h for the category of topological stacks whose corresponding covariant theory is the singular homology while its contravariant theory is the singular cohomology. Moreover, every *proper* oriented normally nonsingular morphism $f : \mathfrak{X} \to \mathfrak{X}'$ of codimension c carries a canonical *strong orientation class* $\theta \in h^c(f)$ in the sense of Fulton-MacPherson [?] and thus gives rise to wrong way Gysin maps $f^! : H_\bullet(\mathfrak{X}') \to H_{\bullet-c}(\mathfrak{X})$, and $f_! : H^\bullet(\mathfrak{X}) \to H^{\bullet+c}(\mathfrak{X}')$. In fact, we have

Proposition 2.2 *Let $f : \mathfrak{X} \to \mathfrak{X}'$ be a proper oriented normally nonsingular morphism of codimension c. Every cartesian diagram of topological stacks*

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\widetilde{f}} & \mathfrak{Y}' \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{X}' \end{array} \tag{3}$$

gives rise to a Gysin map $G_f^p : H_\bullet(\mathfrak{Y}') \to H_{\bullet-c}(\mathfrak{Y})$, which satisfies the following properties:

(i) $G_f^{id} = f^!$;

(ii) Given a commutative diagram of cartesian squares

$$\begin{array}{ccccc} \mathfrak{Z} & \longrightarrow & \mathfrak{Z}' & & \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} & & \\ \mathfrak{Y} & \longrightarrow & \mathfrak{Y}' & \longrightarrow & \mathfrak{Y}'' \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{X}' & \xrightarrow{g} & \mathfrak{X}'' \end{array} \tag{4}$$

then the induced Gysin morphisms satisfy

$$G^q_{g\circ f} = G^p_f \circ G^q_g, \qquad \textit{and} \qquad r_* \circ G_f^{\widetilde{p\circ p}} = G^p_f \circ \widetilde{p}_*.$$

In the case of smooth manifolds, these Gysin maps coincide with the classical ones [?].

3. String product on inertia stacks

A topological stack $\mathfrak{X}$ is said to be *oriented* if the diagonal $\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is an oriented normally nonsingular morphism. In this case, we define $\operatorname{codim}\Delta$ as the dimension of $\mathfrak{X}$. When $\mathfrak{X}$ is a differential stack, this definition coincides with the usual dimension of $\mathfrak{X}$, i.e. $\dim \mathfrak{X} = 2\dim \Gamma_0 - \dim \Gamma_1$ for any Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ representing $\mathfrak{X}$.

From now on, we assume that $\mathfrak{X}$ is an oriented differential stack of dimension d and Γ a Lie groupoid representing $\mathfrak{X}$. Let $S\Gamma = \{g \in \Gamma_1 |\ s(g) = t(g)\}$ be the space of closed loops. There is a natural action of Γ on $S\Gamma$ by conjugation. Thus one forms the transformation groupoid $\Lambda\Gamma : S\Gamma \rtimes \Gamma \rightrightarrows S\Gamma$, which is always a topological groupoid, called the *inertia groupoid*. Its corresponding topological stack is called the *inertia stack*. Indeed one obtains the following double groupoid

$$\begin{array}{ccc} \Lambda\Gamma & \rightrightarrows & S\Gamma \\ \downdownarrows & & \downdownarrows \\ \Gamma_1 & \rightrightarrows & \Gamma_0. \end{array} \tag{5}$$

It is easy to see that one has the following cartesian square

$$\begin{array}{ccc} \Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X}\times\Lambda\mathfrak{X} \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle \iota\times\iota} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X}\times\mathfrak{X} \end{array} \tag{6}$$

where $\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}$ is the stack corresponding to the transformation groupoid $\big(S\Gamma\times_{\Gamma_0}S\Gamma\big)\rtimes_{\Gamma_0}\Gamma_1\rightrightarrows S\Gamma\times_{\Gamma_0}S\Gamma$. The groupoid multiplication of the left vertical groupoid in square (**??**) yields a groupoid morphism $\Lambda\Gamma\times_{\Gamma}\Lambda\Gamma\to\Lambda\Gamma$ and thus a "Pontrjagin" multiplication map $m:\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}\to\Lambda\mathfrak{X}$. By Proposition **??**, there is a Gysin map $G_{\Delta}^{\iota\times\iota}$, whose composition with the cross product gives rise to a degree d map $G: H_i(\Lambda\mathfrak{X})\otimes H_j(\Lambda\mathfrak{X})\longrightarrow H_{i+j-d}(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$.

Theorem 3.1 *Let $\mathfrak{X}$ be an oriented differential stack of dimension d. The composition*

$$H_i(\Lambda\mathfrak{X})\otimes H_j(\Lambda\mathfrak{X})\xrightarrow{G}H_{i+j-d}(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})\xrightarrow{m_*}H_{i+j-d}(\Lambda\mathfrak{X})$$

defines an associative graded commutative product on $\mathbb{H}_\bullet(\Lambda\mathfrak{X}):=H_{\bullet+d}(\Lambda\mathfrak{X})$, called the string product.

When $\mathfrak{X}$ is a manifold M, then $\Lambda\mathfrak{X}=M$ and the string product simply reduces to the intersection product. If $\mathfrak{X}$ is the zero dimensional stack $\mathfrak{X}=[*/G]$ with G being a finite group, then $\Lambda\mathfrak{X}=[G/G]$ where G acts on itself by conjugation. Then $(\mathbb{H}_\bullet(\Lambda\mathfrak{X}),\bullet)$ is isomorphic as an algebra to the center $Z(\mathbb{C}(G))$ of the group algebra of G, which also follows easily from Proposition **??** below.

4. String product for orbifolds

The goal of this section is to study the relation between the cohomology ring of Chen-Ruan [**?**] for complex orbifolds and the string product of Section **??**. We consider global quotient complex orbifold $\mathfrak{X}=[Y/G]$, and all (co)homology groups are taken with coefficients in $\mathbb{C}$. As a vector space Chen-Ruan cohomology ring of $[Y/G]$ is the cohomology of the inertia groupoid $\Lambda\Gamma$, where Γ is the transformation groupoid $Y\rtimes G\rightrightarrows Y$. That is $H^\bullet(\Lambda\mathfrak{X})\cong H^\bullet(\Lambda\Gamma)\cong\big(\bigoplus_{g\in G}H^\bullet(Y^g)\big)^G$. For any $g,h\in G$, over the fixed point space $Y^{g,h}=Y^g\cap Y^h$, there is a canonical vector bundle $F(g,h)$, which is equivariant under the conjugacy action of G [**?**]. It yields a bundle over $\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}$, called the *obstruction bundle*. Let $f(g,h)$ be its Euler class. Also let $i_g: Y^{g,h}\to Y^g$, $i_h: Y^{g,h}\to Y^h$ and $m_{gh}: Y^{g,h}\to Y^{gh}$ be the natural inclusions. We denote by $m_{gh}{}^!: H(Y^g\times Y^h)\to H(Y^{g,h})$ the cohomology Gysin map (induced by the oriented manifolds structures). It is known [**?**] that the pairing defined, for $\alpha\in H^\bullet(Y^g)$ and $\beta\in H^\bullet(Y^h)$, by

$$\alpha\cup_{\mathrm{orb}}\beta:=m_{gh}{}^!\big(i_g^*(\alpha)\cup i_h^*(\beta)\cup f(g,h)\big)$$

induces the Chen-Ruan product on $H^\bullet(\Lambda\mathfrak{X})$.

Remark 1 Note that the Chen-Ruan product does not respect the usual grading of the cohomology groups. In order to have a graded product one needs to use a rational degree shifting described in [?] and [?].

We now introduce a twisted version of Chen-Ruan product. Let $e\in H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$ be a cohomology class, not necessarily homogeneous. The Chen-Ruan pairing twisted by the class e is defined, for all $\alpha,\beta\in H^\bullet(\Lambda\mathfrak{X})$, by

$$\alpha\cup_{\mathrm{orb}}^{e}\beta:=m^!\big(i^*(\alpha)\cup i^*(\beta)\cup f(g,h)\cup e(g,h)\big).$$

It is not graded in general (see Remark **??** above).

Remark 2 If E is a vector bundle over $\mathfrak{X}$, let $e(E)\in H^{2\mathrm{rk}(E)}(\mathfrak{X})$ be its Euler class. Hence, any vector bundle E over $\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}$ defines a twisted Chen-Ruan product $\cup_{\mathrm{orb}}^{E}:=\cup_{\mathrm{orb}}^{e(E)}$.

We now describe the condition ensuring the associativity of the twisted product $\cup_{\text{orb}}^e$. First of all, we need to fix some notations. Let $p_{12}, p_{23} : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ be, respectively, the projections on the first two and the last two factors. Also $(m \times 1) : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ denotes the Pontrjagin multiplication of the two first factors and $(1 \times m)$ the one on the two last factors. We write $\sigma : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ for the flip map.

Proposition 4.1 *Let e be a class in $H^r(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$.*

*(i) If e satisfies the 2-cocycle condition with respect to the left vertical groupoid structure in square (**??**), i.e. the identity*

$$p_{12}^*(e) \cup (m \times 1)^*(e) = p_{23}^*(e) \cup (1 \times m)^*(e) \tag{7}$$

holds in $H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, then the twisted pairing $\cup_{\text{orb}}^e$ is associative.

(ii) If $\sigma^*(e) = e$, the twisted pairing $\cup_{\text{orb}}^e$ is graded commutative.

*Remark 3 One can also define twisted string product by a class $e \in H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ via the formula $x \bullet_e y = m_*\big(G(x \otimes y) \cap e\big)$. Proposition **??** holds as well in this case.*

There is a Poincaré duality for compact orbifolds [**?**]. In particular the string product of Section **??** has a Poincaré dual string product $H^i(\Lambda\mathfrak{X}) \otimes H^j(\Lambda\mathfrak{X}) \xrightarrow{\bullet} H^{i+j+d}(\Lambda\mathfrak{X})$ in cohomology.

The inclusions $i_g : Y^{g,h} \to Y^g$, $i_h : Y^{g,h} \to Y^h$ yield another canonical bundle $Ex(g,h)$ over $Y^{g,h}$, i.e. the excess bundle $Ex(Y, Y^g, Y^h)$. It is just $T_Y - T_{Y^g} - T_{Y^h} + T_{Y^{g,h}}$ (over $Y^{g,h}$) in K-theory. Write $ex(g,h)$ for its Euler class.

Proposition 4.2 *The Poincaré dual of the string product is induced in cohomology by the pairing*

$$\alpha \bullet \beta := {m_{gh}}^!\big(i_g^*(\alpha) \cup i_h^*(\beta) \cup ex(g,h)\big).$$

The Whitney sum $W(g,h)$ of the normal bundle of $Y^{g,h}$ over Y^{gh} with $F(h^{-1}, g^{-1})$ yields a third canonical equivariant bundle over $Y^{g,h}$. We denote by W the corresponding bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

Lemma 4.3 *$F(g,h) + W(g,h) = Ex(g,h)$ as elements of the K-theory group $K^0(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$.*

Theorem 4.4 *The Poincaré dual of the string product is isomorphic to $\cup_{\text{orb}}^{e(W)}$, the twisted Chen-Ruan product by the bundle W.*

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