String topology for stacks

K. Behrend, G. Ginot, B. Noohi and P. Xu

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Abstract

We establish the general machinery of string topology for differentiable stacks. This machinery allows us to treat on equal footing free loops in stacks and hidden loops.

We construct a bivariant (in the sense of Fulton and MacPherson) theory for topological stacks: it gives us a flexible theory of Gysin maps which are automatically compatible with pullback, pushforward and products. Further we prove an excess formula in this context. We introduce oriented stacks, generalizing oriented manifolds, which are stacks on which we can do string topology. We prove that the homology of the free loop stack of an oriented stack and the homology of hidden loops (sometimes called ghost loops) are a Frobenius algebra which are related by a natural morphism of Frobenius algebras. We also prove that the homology of free loop stack has a natural structure of **BV**-algebra, which together with the Frobenius structure fits into an homological conformal field theories with closed positive boundaries. We also use our constructions to study an analogue of the loop product for stacks of maps of (n-dimensional) spheres to oriented stacks and compatible power maps in their homology. Using our general machinery, we construct an intersection pairing for (non necessarily compact) almost complex orbifolds which is in the same relation to the intersection pairing for manifolds as Chen-Ruan orbifold cup-product is to ordinary cup-product of manifolds. We show that the hidden loop product of almost complex orbifold is isomorphic to the orbifold intersection pairing twisted by a canonical class. Finally we gave some examples including the case of the classifying stacks $[*/G]$ of a compact Lie group.

Contents

Introduction

String topology is a term coined by Chas-Sullivan [15] to describe the rich algebraic structure on the homology of the free loop manifold $\mathrm{L}M$ of an oriented manifold M. The algebraic structure in question is induced by geometric operations on loops such as gluing or pinching of loops. In particular, $H_\bullet(\mathrm{L}M)$ inherits a canonical product and coproduct yielding a structure of Frobenius algebra [15, 22]. Furthermore, the canonical action of S^1 on $\mathrm{L}M$ together with the multiplicative structure make $H_\bullet(\mathrm{L}M)$ into a $\mathbf{BV}$-algebra [15]. These algebraic structures, especially the loop product, are known to be related to many subjects in mathematics and in particular mathematical physics [64, 14, 21, 2, 25].

Many interesting geometric objects in (algebraic or differential) geometry or mathematical physics are *not* manifolds. There are, for instance, orbifolds, classifying spaces of compact Lie groups, or, more generally, global quotients of a manifold by a Lie group. All these examples belong to the realm of (geometric) stacks. A natural generalization of smooth manifolds, including the previous examples, is given by differentiable stacks [8] (on which one can still do differentiable geometry). Roughly speaking, differential stacks are Lie groupoids *up to Morita equivalence.*

One important feature of differentiable stacks is that they are *non-singular*, when viewed as stacks (even though their associated coarse spaces are typically singular). For this reason, differentiable stacks have an intersection product on their homology, and a loop product on the homology of their free loop stacks.

The aim of this paper is to establish the general machinery of string topology for differentiable stacks. This machinery allows us to treat on an equal footing free loops in stacks and *hidden* loops. Roughly speaking, the latter are loops inside the stack, which vanish on the associated coarse space. The stack of hidden loops in the stack $\mathfrak{X}$ is the *inertia stack* of $\mathfrak{X}$, notation $\Lambda\mathfrak{X}$. The inertia stack $\Lambda\mathfrak{X} \to \mathfrak{X}$ is an example of a family of commutative *(sic!)* groups over the stack $\mathfrak{X}$, and the theory of hidden loops generalizes to arbitrary commutative families of groups over stacks.

In the realm of stacks several new difficulties arise whose solutions should be of independent interest.

First, we need a good notion of *free loop stack* $\mathrm{L}\mathfrak{X}$ of a stack $\mathfrak{X}$, and more generally of mapping stack $\mathrm{Map}(Y, \mathfrak{X})$ (the stack of stack morphisms $Y \to \mathfrak{X}$). For the general theory of mapping stacks, we do not need a differentiable structure on $\mathfrak{X}$; we work with topological stacks. This is developed in [56] and is discussed in Section 5.1. The issue here is to obtain a mapping stack with a topological structure which is functorial both in $\mathfrak{X}$ and Y and behaves well enough with respect to pushouts in order to get geometric operations on loops. For instance, a key point in string topology is the identification $\mathrm{Map}(S^1 \vee S^1, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$. Since pushouts are a delicate matter in the realm of stacks, extra care has to be taken in finding the correct class of topological stacks to work with (Section 1.6 and [57]). For this reason, we restrict our attention to the class of *Hurewicz* topological stacks. These are topological stacks which admit an atlas with a certain fibrancy property. Without restricting to this special class of topological stacks, $S^1 \vee S^1$ would not be the pushout of two copies of S^1, in the category of stacks.

A crucial step in usual string topology is the existence of a canonical Gysin homomorphism $H_\bullet(\mathrm{L}M \times \mathrm{L}M) \to H_{\bullet-d}(\mathrm{L}M \times_M \mathrm{L}M)$ when M is a d-dimensional manifold. In fact, the loop product is the composition

$$H_p(\mathrm{L}M) \otimes H_q(\mathrm{L}M) \to \\ \to H_{p+q}(\mathrm{L}M \times \mathrm{L}M) \to H_{p+q-d}(\mathrm{L}M \times_M \mathrm{L}M) \to H_{p+q-d}(\mathrm{L}M)\,, \quad (0.1)$$

where the last map is obtained by gluing two loops at their base point.

Roughly speaking the Gysin map can be obtained as follows. The free loop manifold is equipped with a structure of Banach manifold such that the evaluation map $\mathrm{ev} : \mathrm{L}M \to M$ which maps a loop f to $f(0)$ is a surjective submersion. The pullback along $\mathrm{ev} \times \mathrm{ev}$ of a tubular neighborhood of the diagonal $M \to M \times M$ in $M \times M$ yields a normal bundle of codimension d for the embedding $\mathrm{L}M \times_M \mathrm{L}M \to \mathrm{L}M$. The Gysin map can then be constructed using a standard argument on Thom isomorphism and Thom collapse [23].

This approach does not have a straightforward generalization to stacks. For instance, the free loop stack of a differentiable stack is not a Banach stack in general, and neither is the inertia stack. In order to obtain a flexible theory of Gysin maps, we construct a *bivariant theory* in the sense of Fulton-MacPherson [32] for topological stacks, whose underlying homology theory is singular homology. A bivariant theory is an efficient tool encompassing into a unified framework both homology and cohomology as well as many (co)homological operations, in particular Gysin homomorphisms. The Gysin maps of a bivariant theory are automatically compatible with pullback, pushforward, cup and cap-products (see [32]). (Our bivariant theory is somewhat weaker than that of Fulton-MacPherson, in that products are not always defined.) Our bivariant theory applies in particular to all orbifolds. Moreoverer we gave an excess formula which allows us to compute Gysin maps for relative regular embeddings.

In Section 8.3 we introduce *oriented stacks*. These are the stacks over which we are able to do string topology. Examples of oriented stacks include: oriented manifolds, oriented orbifolds, and quotients of oriented manifolds by compact Lie groups (if the action is orientation preserving and of finite orbit type). A topological stack $\mathfrak{X}$ is *orientable* if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ factors as

$$\mathfrak{X} \xrightarrow{0} \mathfrak{N} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{X} \times \mathfrak{X}\,, \quad (0.2)$$

where $\mathfrak{N}$ and $\mathfrak{E}$ are orientable vector bundles over $\mathfrak{X}$ and $\mathfrak{X} \times \mathfrak{X}$ respectively, and $\mathfrak{N} \to \mathfrak{E}$ is an isomorphism onto an open substack (there is also the technical assumption that $\mathfrak{E}$ is metrizable, and $\mathfrak{X} \to \mathfrak{E}$ factors through the unit disk bundle). The embedding $\mathfrak{N} \to \mathfrak{E}$ plays the role of a tubular neighborhood. The dimension of $\mathfrak{X}$ is $\mathrm{rk}\,\mathfrak{N} - \mathrm{rk}\,\mathfrak{E}$.

The factorization (0.2) gives rise to a bivariant class $\theta \in H(\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X})$, the *orientation* of $\mathfrak{X}$.

Sections 10-15 are devoted to the string topology operations, focusing on the Frobenius, **BV**-algebra and homological conformal field theory structures. The bivariant formalism has the following consequence: if $\mathfrak{X}$ is an oriented stack of dimension d, then any cartesian square

$$\begin{tikzcd} \mathfrak{Y} \arrow[r] \arrow[d] & \mathfrak{Z} \arrow[d] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X}\times\mathfrak{X} \end{tikzcd}$$

defines a canonical Gysin map $\Delta^!\colon H_\bullet(\mathfrak{Z}) \to H_{\bullet-d}(\mathfrak{Y})$. For example, the cartesian square

$$\begin{tikzcd} \mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X} \arrow[r] \arrow[d] & \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[d] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X}\times\mathfrak{X} \end{tikzcd}$$

Gives rise to a Gysin map $\Delta^!\colon H_\bullet(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})$, and we can construct a loop product

$$\star : H_\bullet(L\mathfrak{X})\otimes H_\bullet(L\mathfrak{X}) \to H_{\bullet-d}(L\mathfrak{X}),$$

as in 0.1, or [15, 23, 22].

We also obtain a coproduct

$$\delta\colon H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X})\otimes H_j(\mathrm{L}\mathfrak{X})\,.$$

Furthermore, $\mathrm{L}\mathfrak{X}$ admits a natural S^1-action yielding the operator $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ which is the composition:

$$H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\times\omega} H_{\bullet+1}(\mathrm{L}\mathfrak{X}\times S^1) \longrightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

where $\omega\in H_1(S^1)$ is the fundamental class. Thus we prove that $(H_\bullet(\mathrm{L}\mathfrak{X}),\star,\delta)$ is a Frobenius algebra and that the shifted homology $(H_{\bullet+d}(\mathrm{L}\mathfrak{X}),\star,D)$ is a **BV**-algebra [7]. Using Sullivan's chord diagram [22] and our formalism of Gysin maps given by the bivariant theory, we extend the previous **BV** and Frobenius structure into a homological conformal field theory (in the sense of [26, 61]) with closed positive boundaries (said otherwise non-unital and non-counital) in Section 14. Roughly, this means that to any compact Riemann surface Σ with only closed boundaries (with say n incoming ones and m outgoing ones), and such that any connected component of Σ has a positive number of both incoming and outgoing boundary components, and to any class α in the homology of the mapping class group of Σ, we associate an operation $\mu_\alpha : H(\mathrm{L}\mathfrak{X})^{\otimes n} \to H(\mathrm{L}\mathfrak{X})^{\otimes m}$ compatible with the glueing and disjoint union of surfaces.

Since the inertia stack can be considered as the stack of hidden loops, the general machinery of Gysin maps yields, for any oriented stack $\mathfrak{X}$, a product and a coproduct on the homology $H_\bullet(\Lambda\mathfrak{X})$ of the inertia stack $\Lambda\mathfrak{X}$, making it a Frobenius algebra, too. Moreover in Section 12.4, we construct a natural map $\Phi\colon \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ inducing a morphism of Frobenius algebras in homology.

In Section 15, we explain how to adapt the loop product to the case of spheres spaces $\mathrm{Map}(S^n, \mathfrak{X})$. We obtain an analogue of the loop product, called the brane product, and also study power maps $\lambda^k : H_\bullet(\mathrm{Map}(S^n, \mathfrak{X})) \to H_\bullet(\mathrm{Map}(S^n, \mathfrak{X}))$ induced by the degree k maps $S^n \to S^n$. We show that for $n \geq 2$, the maps λ^k are maps of algebras with respect to the brane product.

In Section 16, we consider almost complex orbifolds (not necessarily compact). Using Gysin maps and the obstruction bundle of Chen-Ruan [19], we construct the *orbifold intersection pairing* on the homology of the inertia stack. It is in the same relation to the intersection pairing on the homology of a manifold as the Chen-Ruan orbifold cup-product [19] is to the ordinary cup product on the cohomology of a manifold.

The orbifold intersection pairing defines a structure of associative, graded commutative algebra on $H_\bullet^{\mathrm{orb}}(\mathfrak{X})$ for any almost complex orbifold $\mathfrak{X}$. As a vector space the orbifold homology $H_\bullet^{\mathrm{orb}}(\mathfrak{X})$ coincides with the homology of the inertia stack $\Lambda\mathfrak{X}$, but the grading is shifted according to the age as in [19, 29].

In the compact case, the orbifold intersection pairing is identified with the Chen-Ruan product, via orbifold Poincaré duality.

We also prove that the loop product, hidden loop product and intersection pairing (for almost complex orbifolds) can be twisted by a cohomology class in $H_\bullet(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$ or $H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, satisfying the 2-cocycle condition (see Propositions 10.11, 11.3, and 16.7). The notion of twisting provides a connection between the orbifold intersection pairing and the hidden loop product. In fact, we associate to an almost complex orbifold $\mathfrak{X}$ a canonical vector bundle $\mathcal{O}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}}$ over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ and prove that the orbifold intersection pairing, twisted by the Euler class of $\mathcal{O}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}}$, is the hidden loop product of $\mathfrak{X}$.

Parallel to our work, the hidden loop product for global quotient orbifolds was studied independently in [50, 37]. Furthermore, a nice interpretation of the hidden loop product in terms of the Chen-Ruan product of the cotangent bundle was given by González et al. [37]. A loop product for global quotients of a manifold by a finite group was studied in [51, 50]. Also purely homotopical techniques to study string topology of classifying spaces of Lie groups have been recently developed in [18].

We close this introduction by remarking that our construction of string operations for stacks can in fact be extended to generalized (co)homology theories other than singular. For instance, in view of the Freed-Hopkins-Teleman's work [31], using K-theory may lead to interesting consequences. In the case of manifolds, Cohen and Godin have already considered such generalization in [22].

The key point in extending our theory to other (co)homology theories is to cast such a (co)homology theory as part of a bivariant theory. Once this is done, the formalism developed in Section 7 applies to produce the desired Gysin maps, and these in turn give rise to string operations. The main input needed to make the construction of the Gysin maps is to produce an orientation class θ in the bivariant cohomology of the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, and this is done by making use of appropriate Thom classes (Definition 4.1) for the given (co)homology theory.

Conventions

Topological spaces

All topological spaces are compactly generated. The category of topological spaces endowed with the Grothendieck topology of open coverings is denoted Top. This is the *site* of topological spaces.

Manifolds

All manifolds are second countable and Hausdorff. In particular they are regular Lindelöf and paracompact.

Groupoids

We will commit the usual abuse of notation and abbreviate a groupoid to $[X_1 \rightrightarrows X_0]$. A *topological groupoid*, is a groupoid $[X_1 \rightrightarrows X_0]$, where X_1 and X_0 are topological spaces, but no further assumptions is made on the source and target maps, except continuity. A topological groupoid is a *Lie groupoid* (or a *differentiable groupoid*) if X_1, X_0 are manifolds, all the structures maps are smooth and, in addition, the source and target maps are surjective submersions.

Stacks

For stacks, we use the words *equivalent* and *isomorphic* interchangeably. We will often omit 2-isomorphisms from the notation. For example, we may call morphisms equal if they are 2-isomorphic. The stack associated to a groupoid $[X_1 \rightrightarrows X_0]$ is denoted by $[X_0/X_1]$, because we think of it as the quotient. Also if G is a topological group acting on a space X, we denote the stack associated to the transformation groupoid $[X \rtimes G \rightrightarrows X]$ by $[X/G]$.

Our terminology is different from that in [57]. The quotient stack $\mathfrak{X}$ of a topological groupoid $[X_0/X_1]$ is called a *topological stack* in this paper, where as in [ibid.] these are called *pretopological stacks.* If the source and target map of $[X_0/X_1]$ are local Hurewicz fibrations, then we say that $\mathfrak{X}$ is a *Hurewicz topological stack*; see Section 1.4.

Warning 0.1 In Sections 10, 12.2, 13, 14 and 15, unless otherwise stated, the (base) stack $\mathfrak{X}$ will always be assumed to be a Hurewicz stack (see Definition 1.6). Note that all differentiable stacks (which are our main interest) are Hurewicz (Example 1.2).

(Co)homology

The coefficients of our (co)homology theories will be taken in a commutative unital ring k. All tensors products are over k unless otherwise specified.

We will write both $H(\mathfrak{X})$, $H_\bullet(\mathfrak{X})$ for the total homology groups $\bigoplus H_n(\mathfrak{X})$. We use the first notation when we deal with ungraded elements and ungraded maps, while we use the second when dealing with homogeneous homology classes and graded maps. Similarly, in Section 7, we use respectively the notations $H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ and $H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ for the total bivariant cohomology groups when we want to deal with ungraded maps or with graded ones.

A 2-commutative (respectively, 2-cartesian) diagram of stacks will simply be referred to as a commutative (respectively, cartesian) diagram.

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1 Topological stacks

We review some basic facts about topological stacks. More details can be found in [57].

1.1 Stacks over Top

Throughout these notes, by a stack we mean a stack over the site Top of compactly generated topological spaces with the standard Grothendieck topology. This means, a stack is a category $\mathfrak{X}$ fibered in groupoids over Top satisfying the descent condition, see Appendix A for more details. Alternatively, we can think of $\mathfrak{X}$ as a presheaf of groupoids over Top which satisfies descent.

We list some basic facts about stacks.

1. Stacks over Top form a 2-category in which 2-morphisms are invertible. Therefore, given two stacks $\mathfrak{X}$ and $\mathfrak{Y}$, we have the *groupoid* $\mathrm{Hom}(\mathfrak{Y},\mathfrak{X})$ of morphisms between them. In the case where the source stack $\mathfrak{Y} = T$ is a topological space, we usually use the alternative notation $\mathfrak{X}(T)$ for the above hom-groupoid. This is sometimes referred to as the groupoid of *T-valued points* of $\mathfrak{X}$.

   Although in practice one may really be interested only in the category of stacks which obtained by identifying 2-isomorphic 1-morphisms, the 2-category structure can not be ignored. For example, when we talk about *fiber products of stacks*, we exclusively mean the 2-fiber product in the 2-category of stacks.

2. The 2-category of stacks has fiber products and inner homs, so it is cartesian closed. The 2-fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is characterized by the property that, for every topological space T, its groupoid of T-valued points is given equivalent to
$$\mathfrak{X}(T) \times_{\mathfrak{Z}(T)} \mathfrak{Y}(T).$$
   Given stacks $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks over Top, the inner hom between them, called the *mapping stack* $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$, is defined by the rule
$$T \in \mathsf{Top} \quad \mapsto \quad \mathrm{Hom}(T \times \mathfrak{Y}, \mathfrak{X}).$$
   Note that we have a natural equivalence of groupoids
$$\mathrm{Map}(\mathfrak{Y},\mathfrak{X})(*) \cong \mathrm{Hom}(\mathfrak{Y},\mathfrak{X}),$$
   where $*$ is a point. The mapping stack has the exponential property. That is, given stacks $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$, we have a natural equivalence of stacks
$$\mathrm{Map}(\mathfrak{Z} \times \mathfrak{Y},\mathfrak{X}) \cong \mathrm{Map}(\mathfrak{Z},\mathrm{Map}(\mathfrak{Y},\mathfrak{X})).$$

3. The category of topological spaces embeds fully faithfully in the 2-category of stacks. This means, given two topological spaces X and Y, viewed as stacks via the functor they represent, the hom-groupoid $\mathrm{Hom}(X,Y)$ is equivalent to a set, and this set is in a natural bijection with the set of continuous functions from X to Y.

   This way, we can think of a topological space as a stack.

   This embedding preserves the closed cartesian structure on Top. This means that fiber products of spaces get sent to 2-fiber products of the corresponding stacks, and the mapping spaces (with the compact-open topology) get sent to mapping stacks.

4. The embedding of the category of topological spaces in the 2-category of stacks admits a left adjoint. That is, to every stack $\mathfrak{X}$ one can associate a topological space, together with a natural map $\pi\colon \mathfrak{X} \to \mathfrak{X}_{mod}$ which is universal among maps from $\mathfrak{X}$ to topological spaces. (That is, every map from $\mathfrak{X}$ to a topological space T factors uniquely through π.) ; See [57], Section 4.3.

   The space $\mathfrak{X}_{mod}$ is called the *coarse moduli space* of $\mathfrak{X}$ and it should be thought of as the "underlying space" of $\mathfrak{X}$.

   In particular, the underlying set of $\mathfrak{X}_{mod}$ is the set of isomorphism classes of the groupoid $\mathfrak{X}(*)$, where $*$ stands for a point. In other words, the points in $\mathfrak{X}_{mod}$ are the 2-isomorphism classes of points of $\mathfrak{X}$, where by a *point* of $\mathfrak{X}$ we mean a morphism $x\colon * \to \mathfrak{X}$.

   The underlying set of the coarse moduli space of the mapping stack $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$ is the set of 2-isomorphism classes of morphisms from $\mathfrak{Y}$ to $\mathfrak{X}$.

5. To a point $x\colon * \to \mathfrak{X}$ of a stack $\mathfrak{X}$ there is associated a group I_x, called the *inertia group* of $\mathfrak{X}$ at x. By definition, I_x is the group of 2-isomorphisms from the point x to itself. An element in I_x is sometimes referred to as a *ghost* or *hidden loop*; see ([57], Section 10) since its image under the map $\mathfrak{X} \xrightarrow{\pi} \mathfrak{X}_{mod}$ is constant.

   The groups I_x assemble into a stack $\Lambda\mathfrak{X} \to \mathfrak{X}$ over $\mathfrak{X}$ called the *inertia stack*. The inertia stack is defined by the following 2-fiber square

$$\begin{tikzcd} \Lambda\mathfrak{X} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\Delta"] \\ \mathfrak{X} \arrow[r, "\Delta"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

   and will be studied in more details in Section 11.

1.2 Morphisms of stacks

A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of stacks is called **representable** if for every map $T \to \mathfrak{Y}$ from a topological space T, the fiber product $T \times_{\mathfrak{Y}} \mathfrak{X}$ is a topological space. This is, roughly speaking, saying that the fibers of f are topological spaces.

Any property $\mathbf{P}$ of morphisms of topological spaces which is invariant under base change can be defined for an arbitrary representable morphism of stacks. More precisely, we say that a representable morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ is $\mathbf{P}$, if for every map $T \to \mathfrak{Y}$ from a topological space T, the base extension $f_T\colon T \times_{\mathfrak{Y}} \mathfrak{X} \to T$ is $\mathbf{P}$ as a map of topological spaces; see ([57], Section 4.1).

This way we can talk about *embeddings (closed, open, locally closed, or arbitrary) of stacks, proper morphisms, finite morphisms*, and so on.

We say that $f\colon \mathfrak{X} \to \mathfrak{Y}$ is an *epimorphism*, if it is an epimorphism in the sheaf theoretic sense. In the case where f is representable, this is equivalent to saying that every base extension f_T of f over a topological space T admits local sections.

1.3 Transformation groupoids

Let G be a topological group acting on a topological space X. We define the *transformation groupoid* $[X \rtimes G \rightrightarrows X]$ of this action as follows. As the notation suggests, the space of objects is X and the space of arrows is $X \times G$. The source map $s\colon X \times G \to X$ is the first projection and the target map is the action $X \times G \to X$. The composition of arrows is induced from the multiplication in G.

More generally, we can associate a transformation groupoid to a groupoid $\Gamma = [\Gamma_1 \rightrightarrows \Gamma_0]$ acting on a space X. Here, X is equipped with a base map $p\colon X \to \Gamma_0$ which we have suppressed from the notation. As the notation suggest, the transformation groupoid $[X \rtimes \Gamma_1 \rightrightarrows X]$ has X as its space of objects. The space of arrows is

$$X \rtimes \Gamma_1 := X \times_{\Gamma_0} \Gamma_1,$$

where $\Gamma_1 \to \Gamma_0$ is the source map. As above, the source map $s\colon X \times_{\Gamma_0} \Gamma_1 \to X$ is the first projection and the target map is the action $X \times_{\Gamma_0} \Gamma_1 \to X$. The composition of arrows is induced from the composition in Γ.

There is a natural map of groupoids

$$[X \rtimes \Gamma_1 \rightrightarrows X] \to [\Gamma_1 \rightrightarrows \Gamma_0].$$

On the level of objects, it is given by $p\colon X \to \Gamma_0$. On the level of arrows it is given by the second projection $X \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$.

1.4 Topological stacks

A **topological** stack ([57], Definition 7.1) is a stack $\mathfrak{X}$ over Top which admits a representable epimorphism $p\colon X \to \mathfrak{X}$ from a topological space X. Equivalently,

$\mathfrak{X}$ is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $[X_1 \rightrightarrows X_0]$. This quotient stack, by definition, is the stack associated to the presheaf of groupoids

$$T \;\mapsto\; [X_1(T) \rightrightarrows X_0(T)].$$

This stack is equivalent to the stack of torsors for the groupoid $[X_1 \rightrightarrows X_0]$; see ([57], Section 12). The groupoid $[X_1 \rightrightarrows X_0]$ is recovered from the atlas $p\colon X \to \mathfrak{X}$ by setting $X_0 := X$ and $X_1 := X \times_{\mathfrak{X}} X$. Under this correspondence between topological stacks and topological groupoids, morphisms of stacks correspond to Hilsum-Skandalis bibundles.

An important example to keep in mind is the case of a topological group G acting on a topological space X. The quotient stack of the associated transformation groupoid $[X \rtimes G \rightrightarrows X]$ is denoted by $[X/G]$. For a topological space T, the groupoid $[X/G](T)$ of T-points of $[X/G]$ is the groupoid of pairs (P, φ), where P is a principal G-bundle over T, and $\varphi\colon P \to X$ is a G-equivariant map. In the case where X is a point, the corresponding quotient stack $[*/G]$ is called the *classifying stack* of G. Its groupoid of T-points is precisely the groupoid of principal G-bundles over T. There is a natural morphism of stack $[X/G] \to [*/G]$.

We can repeat this discussion with G replaced by a topological groupoid Γ. The quotient stack of the associated transformation groupoid $[X \rtimes \Gamma \rightrightarrows X]$ is denoted by $[X/\Gamma]$. It comes with a natural morphism of stacks $[X/\Gamma] \to [\Gamma_0/\Gamma_1]$.

We list some basic facts about topological stacks.

1. Topological stacks form a full sub 2-category of the 2-category of stacks over Top.

2. The 2-category of topological stacks is closed under fiber products. It, however, does not seem to have inner hom objects. That is, it does not seem to be the case in general that the mapping stack $\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})$ of two topological stacks $\mathfrak{X}$ and $\mathfrak{Y}$ is a topological stack. This *is* the case, however, if $\mathfrak{Y}$ is the quotient stack of a groupoid $[Y_1 \rightrightarrows Y_0]$ such that Y_0 and Y_1 are compact topological spaces; see Proposition 5.1.

3. The stack associated to a topological space X is topological. It is, in fact, equivalent to the stack associated to the trivial groupoid $[X \rightrightarrows X]$. Thus, the category of topological spaces is a full subcategory of the 2-category of topological stacks.

4. Let $\mathfrak{X} = [X_0/X_1]$ be the quotient stack of a topological groupoid $[X_1 \rightrightarrows X_0]$. Then, the coarse moduli space $\mathfrak{X}_{mod}$ of $\mathfrak{X}$ is naturally homeomorphic to the course quotient space of the groupoid $[X_1 \rightrightarrows X_0]$. In particular, the coarse moduli space of the quotient stack $[X/G]$ is the orbit space X/G of the action of G on X. The coarse moduli space of the classifying stack $[*/G]$ of G is just a single point.

5. For a point $x\colon * \to \mathfrak{X}$ of a topological stack $\mathfrak{X}$, the inertia group I_x is naturally a topological group. The inertia stack $\Lambda\mathfrak{X}$ is a topological stack, and the natural map $\Lambda\mathfrak{X} \to \mathfrak{X}$ is representable.

6. Every morphism $T \to \mathfrak{X}$ from a topological space T to a topological stack $\mathfrak{X}$ is representable.

1.5 Substacks of a topological stack

Let $\mathfrak{X}$ be a topological stack. A representable morphism $i\colon \mathfrak{Y} \to \mathfrak{X}$ is called an **embedding** if for every map $T \to \mathfrak{X}$, with T a topological space, the base extension $i_T\colon T \times_{\mathfrak{X}} \mathfrak{Y} \to T$ is an embedding of topological spaces (that is, i_T maps $T \times_{\mathfrak{X}} \mathfrak{Y}$ homeomorphically onto a subspace of T). In this case, we say that $\mathfrak{Y}$ is a **substack** of $\mathfrak{X}$. We can similarly define **open**, **closed**, and **locally closed** substacks. With a slight abuse of notation, we often use the notation $\mathfrak{Y} \subseteq \mathfrak{X}$ for a substack.

Let $p\colon X \to \mathfrak{X}$ be an atlas for the topological stack $\mathfrak{X}$, and let $[X_1 \rightrightarrows X]$ be the corresponding groupoid presentation. Then, taking inverse image via p induces a bijection between substacks $\mathfrak{Y} \subseteq \mathfrak{X}$ and invariant subspaces $Y \subset X$. Under this bijection open (respectively, closed, locally closed) substacks of $\mathfrak{X}$ correspond to open (respectively, closed, locally closed) subspaces of X.

Given a family of substacks $\mathfrak{Y}_\alpha$ of $\mathfrak{X}$, we define their **intersection** $\bigcap\limits_{\alpha}\mathfrak{Y}_\alpha$ to be the largest substack of $\mathfrak{X}$ which is a substack of all of $\mathfrak{Y}_\alpha$. The **union** $\bigcup\limits_{\alpha}\mathfrak{Y}_\alpha$ is defined similarly. The intersection and union of substacks always exist. In fact, if $p\colon X \to \mathfrak{X}$ is an atlas for $\mathfrak{X}$, then the intersection $\bigcap\limits_{\alpha}\mathfrak{Y}_\alpha$ corresponds to the invariant subspace $\bigcap\limits_{\alpha} Y_\alpha$ of X. The same goes with the union.

Given a substack $\mathfrak{Y}$ of $\mathfrak{X}$, we define its **closure** $\bar{\mathfrak{Y}}$ to be the smallest closed substack containing $\mathfrak{X}$. The **interior** $\mathfrak{Y}^\circ$ is defined to be the largest open substack of $\mathfrak{X}$ contained in $\mathfrak{Y}$. The **complement** $\mathfrak{Y}^c$ of a substack $\mathfrak{Y} \subseteq \mathfrak{X}$ is the largest substack of $\mathfrak{X}$ whose intersection with $\mathfrak{Y}$ is empty. Given two substacks $\mathfrak{Y}$ and $\mathfrak{Z}$ of $\mathfrak{X}$, we define the **difference** $\mathfrak{Y} - \mathfrak{Z}$ to be the substack $\mathfrak{Y} \cap \mathfrak{Z}^c$. All these exist, are well-defined, and can be constructed by taking the corresponding invariant subspaces of an atlas $p\colon X \to \mathfrak{X}$.

1.6 Hurewicz topological stacks

As we will see in Section 1.7, in order to have nice gluing properties for maps into a stack $\mathfrak{X}$, we need to assume $\mathfrak{X}$ is a Hurewicz stack. This will be needed later on when we work with loop stacks. We recall the definition of a Hurewicz stack[1].

A *Hurewicz fibration* is a map having the homotopy lifting property for all topological spaces. A map $f\colon X \to Y$ of topological spaces is a *local Hurewicz*

[1]We use a different terminology here than [57]. What we call a Hurewicz stack here is called *topological* in [ibid.].

fibration if for every $x \in X$ there are opens $x \in U$ and $f(x) \in V$ such that $f(U) \subseteq V$ and $f|_U \to V$ is a Hurewicz fibration. The most important example for us is the case of a topological submersion: a map $f\colon X \to Y$, such that locally U is homeomorphic to $V \times \mathbb{R}^n$, for some n.

Dually, we have the notion of *local cofibration.* It is known ([62]), that if $A \to Z$ is a closed embedding of topological spaces, it is a local cofibration if and only if there exists and open neighborhood $A \subset U \subset Z$ such that A is a strong deformation retract of U. If $A \to Z$ is a local cofibration, so is $A \times T \to Z \times T$ for every topological space T. Moreover, the following result is essential for our purposes ([63]):

given a commutative diagram, with $A \to Z$ a local cofibration and $X \to Y$ a local fibration

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

then for every point $a \in A$ there exists an open neighborhood Z' of a in Z, such that there exists a lifting (the dotted arrow) giving two commutative triangles

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where $A' = A \cap Z'$.

Definition 1.1 A topological stack $\mathfrak{X}$ is called **Hurewicz** if it is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $[X_1 \rightrightarrows X_0]$ whose source and target maps are local Hurewicz fibrations.

Example 1.2 A topological space is a Hurewicz topological stack. Every substack of a Hurewicz topological stack is a Hurewicz topological stack. The topological stack underlying any differentiable stack is a Hurewicz topological stack. In particular, any global quotient $[M/G]$ of a manifold by a Lie group is a Hurewicz topological stack.

1.7 Pushouts in the category of stacks

The following generalizes ([57], Theorem 16.2).

Proposition 1.3 *Let $A \to Y$ be a closed embedding of Hausdorff spaces, which is a local cofibration. Let $A \to Z$ be a finite proper map of Hausdorff spaces.*

Suppose we are given a pushout diagram in the category of topological spaces

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Z \vee_A Y \end{tikzcd}$$

Then this diagram remains a pushout diagram in the 2-category of Hurewicz topological stacks. In other words, for every Hurewicz topological stack $\mathfrak{X}$, the morphism

$$\mathfrak{X}(Z \vee_A Y) \longrightarrow \mathfrak{X}(Z) \times_{\mathfrak{X}(A)} \mathfrak{X}(Y)$$

is an equivalence of groupoids.

Proof. Let us abbreviate the pushout by $U = Z \vee_A Y$

The fully faithful property only uses that $\mathfrak{X}$ is a topological stack and that U is a pushout. Let us concentrate on essential surjectivity. Because $\mathfrak{X}$ is a stack and we already proved full faithfulness, the question is local in U. Assume given $Z \to \mathfrak{X}$ and $Y \to \mathfrak{X}$, and an isomorphism over A. Let $[X_1 \rightrightarrows X_0]$ be a groupoid presenting $\mathfrak{X}$, whose source and target maps are local fibrations.

Let us remark that both $Z \to U$ and $Y \to U$ are finite proper maps of Hausdorff spaces. Thus we can cover U by open subsets U_i, such that for every i, both $Z_i = U_i \cap Z$ and $Y_i = Y \cap U_i$ admit liftings to X_0 of their morphisms to $\mathfrak{X}$. We thus reduce to the case that we have $Z \to X_0$, $Y \to X_0$, and $A \to X_1$. Next, we need to construct the dotted arrow in

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, dotted] \\ X_1 \arrow[r] & X_0 \end{tikzcd}$$

We can cover Y by opens over which this arrow exists, because $A \to Y$ is a local cofibration and $X_1 \to X_0$ a local fibration. Then for a point $u \in U$ we choose an open neighborhood in U small enough such that the preimage in Y is a disjoint union of sets over wich the dotted arrow exists. Passing to such a neighborhood of u reduces to the case that the dotted arrow exists. Then there is nothing left to prove. □

1.8 Orbifolds as topological stacks

The most familiar examples of topological stacks are the orbifolds. An orbifold, by definition, is a topological stack which can be covered by open substacks of the form $[X/G]$, with G a finite group. Any orbifold is the quotient stack of an étale groupoid $[X_1 \rightrightarrows X_0]$. Recall that being étale means that the source (hence also the target) $X_1 \to X_0$ is a local homeomorphism – in particular, an orbifold is a Hurewicz topological stack. Moreover, it can be shown that the diagonal map $X_1 \to X_0 \times X_0$ is a closed map (with finite fibers). In fact, the converse is also true in the locally connected case. Namely, the quotient stack of a locally connected étale groupoid whose diagonal map is closed with finite fibers is an orbifold (see [57], Propostion 14.9).

We should point out that there is some inconsistency in the literature about terminology: in the definition of orbifold, some authors assume that the action of G on X is generically free. For this reason, and by analogy to their algebraic geometric counterparts, in *loc. cit.* the term *Deligne-Mumford* has been used instead of orbifold. The orbifolds for which the above generic freeness condition is satisfied are sometimes called reduced orbifolds.

Although every orbifold $\mathfrak{X}$ is locally the quotient stack $[X/G]$ of a finite group action, this may not be the case globally, i.e., $\mathfrak{X}$ may not be *good.* (For a characterization of good orbifolds in terms of their fundamental group see [57], Theorem 18.24.) It is known, however, that a reduced differentiable orbifold $\mathfrak{X}$ can always be globally written as a quotient stack $[X/G]$, where G is a Lie group acting with finite stabilizers on a manifold X. This is not known to be the case for general orbifolds though.

Orbifolds clearly form a small subclass of all topological stacks. For instance, every point on an orbifold has finite stabilizer group, and this is not true for an arbitrary topological stack. The simplest example of a topological stack which is not an orbifold is the quotient stack $[*/G]$, where G is any topological group which is not finite.

1.9 Geometric stacks

In this paper we will encounter other types of stacks as well. A **differentiable stack** is a stack on the category of C^∞-manifolds, which is isomorphic to the quotient stack of a Lie groupoid. Every differentiable stack has an underlying topological stack that is Hurewicz. If the Lie groupoid $[X_1 \rightrightarrows X_0]$ presents the differentiable stack $\mathfrak{X}$, the underlying topological groupoid presents the underlying topological stack. Often we will tacitly pass from a differentiable stack to its underlying topological stack. For more on differentiable stacks, see [8].

An **almost complex stack** is a stack on the category of almost complex manifolds, which is isomorphic to the quotient stack of an almost complex Lie groupoid, i.e., a Lie groupoid $[X_1 \rightrightarrows X_0]$, where X_0 and X_1 are almost complex manifolds, and all structure maps respect the almost complex structures. Every almost complex stack has an underlying differentiable stack and hence also an underlying topological stack.

2 Homotopy type of a topological stack

2.1 Classifying space of a topological groupoid

We recall the construction of the (Haefliger-Milnor) classifying space $B\mathbb{X}$ and the **universal bundle** $E\mathbb{X}$ of a topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$ from [58].

An element in $E\mathbb{X}$ is a sequence $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)$, where $\alpha_i \in R$ are such that $s(\alpha_i)$ are equal to each other, and $t_i \in [0,1]$ are such that all but finitely many of them are zero and $\sum t_i = 1$. As the notation suggests, we set $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) = (t'_0\alpha'_0, t'_1\alpha'_1, \cdots, t'_n\alpha'_n, \cdots)$ if $t_i = t'_i$ for all i and $\alpha_i = \alpha'_i$ if $t_i \neq 0$.

Let $t_i \colon E\mathbb{X} \to [0,1]$ denote the map $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) \mapsto t_i$, and let $\alpha_i \colon t_i^{-1}(0,1] \to R$ denote the map $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots) \mapsto \alpha_i$. The topology on $E\mathbb{X}$ is the weakest topology in which $t_i^{-1}(0,1]$ are all open and t_i and α_i are all continuous.

The classifying space $B\mathbb{X}$ is defined to be the quotient of $E\mathbb{X}$ under the following equivalence relation: two elements $(t_0\alpha_0, t_1\alpha_1, \cdots, t_n\alpha_n, \cdots)$ and $(t'_0\alpha'_0, t'_1\alpha'_1, \cdots, t'_n\alpha'_n, \cdots)$ of $E\mathbb{X}$ are equivalent, if $t_i = t'_i$ for all i, and if there is an element $\gamma \in X_1$ such that $\alpha_i = \gamma\alpha'_i$. (So, in particular, $t(\alpha_i) = t(\alpha'_i)$ for all i.)

2.2 Classifying space of a topological stack

Many facts about topological stacks can be reduced to the case of topological spaces by virtue of the following.

Theorem 2.1 *For every topological stack $\mathfrak{X}$, there exists a topological space X together with a morphism $\varphi \colon X \to \mathfrak{X}$ which has the property that, for every morphism $T \to \mathfrak{X}$ from a topological space T, the pullback $T \times_{\mathfrak{X}} X \to T$ is a weak homotopy equivalence.*

A topological space X with the above property is called a **classifying space** for $\mathfrak{X}$. A classifying space for $\mathfrak{X}$ can be constructed by taking the classifying space $B\mathbb{X}$ of a groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$ whose quotient stack is $\mathfrak{X}$ (see [58], Theorem 6.3). The above theorem implies that the classifying space of a topological stack is unique up to a unique isomorphism in the weak homotopy category of topological spaces (i.e., the category of topological spaces with weak homotopy equivalences inverted).

In the case where $\mathfrak{X} = [X/G]$ is the quotient stack of a group action, the Borel construction $X \times_G EG$ is a classifying space for $\mathfrak{X}$. Here EG is the total space of the universal principal G-bundle in the sense of Milnor.

2.3 Paracompactness of the classifying space

In many applications, it is important to find a classifying space for $\mathfrak{X}$ which is *paracompact*. There are various conditions on a groupoid $[X_1 \rightrightarrows X_0]$ which guarantee that the fat realization of the nerve of $[X_1 \rightrightarrows X_0]$ is paracompact. The following is one.

Definition 2.2 A topological stack $\mathfrak{X}$ is called **regular Lindelöf** if it is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $[X_1 \rightrightarrows X_0]$ such that X_1, X_0 are regular Lindelöf spaces.

The proof of the following proposition will appear elsewhere.

Proposition 2.3 *If $\mathfrak{X}$ is a regular Lindelöf stack, there exists a classifying space for $\mathfrak{X}$ which is a regular Lindelöf space, in particular paracompact.*

Remark 2.4 Every differentiable stack is regular Lindelöf and hence has a paracompact classifying space.

2.4 (Co)homology theories for topological stacks

Theorem 2.1 allows one to extend every (generalized) (co)homology theory h to the 2-category of topological stacks. For instance, let us explain how to define $h(\mathfrak{X}, \mathfrak{A})$ for a pair $(\mathfrak{X}, \mathfrak{A})$ of topological stacks.

Choose a classifying space $\varphi\colon X \to \mathfrak{X}$, and let $A := \varphi^{-1}\mathfrak{A}$. It follows that the pair (X, A) is well-defined in the weak homotopy category of pairs (i.e., is independent of the choice of a particular classifying space X). So, we can define $h(\mathfrak{X}, \mathfrak{A})$ to be $h(X, A)$. It can be easily verified that this construction is functorial in morphisms of pairs.

The cohomology theory thus defined on topological stacks will maintain all natural properties that it had on spaces. For example, it will be homotopy invariant (in particular, it will not distinguish 2-isomorphic morphisms), it will satisfy excision, it will maintain all the products (cap, cup, etc.) that it had on spaces, and so on.

In the case where $\mathfrak{X} = [X/G]$, we will recover the usual G-equivariant (co)homology of X defined using the Borel construction. That is, $h([X/G]) \cong h(X \times_G EG)$.

Every (co)homology theory for topological stacks which is invariant under weak equivalences is induced from one on topological spaces. This is due to existence of a classifying space $\varphi\colon X \to \mathfrak{X}$ (Theorem 2.1) which forces the (co)homology of $\mathfrak{X}$ to be equal to that of its classifying space X.

2.5 Eilenberg-Steenrod axioms for topological stacks

We recall the Eilenberg-Steenrod axioms for a homology theory, formulated in the context of topological stacks. Let H_n be a sequence of functors from the category of pairs $(\mathfrak{X}, \mathfrak{A})$ of topological stacks to the category of abelian groups. (By a pair $(\mathfrak{X}, \mathfrak{A})$ we mean a topological stack $\mathfrak{X}$ and a substack $\mathfrak{A}$.) This sequence is equipped with natural transformations $\partial\colon H_i(\mathfrak{X}, \mathfrak{A}) \to H_{i-1}(\mathfrak{A})$, called the boundary maps. The Eilenberg-Steenrod axioms are the following:

1. *Homotopy.* If $f, g\colon (\mathfrak{X}, \mathfrak{A}) \to (\mathfrak{Y}, \mathfrak{B})$ are homotopic as morphisms of pairs of stacks (in the sense of [57], Definition 17.2), then they induce the same map on H_n for all n.

2. *Excision.* Let $(\mathfrak{X}, \mathfrak{A})$ be a pair of topological stacks. Let $\mathfrak{U}$ be a substack of $\mathfrak{X}$ such that the closure of $\mathfrak{U}$ is contained in the interior of $\mathfrak{A}$. Then, the inclusion map $(\mathfrak{X} - \mathfrak{U}, \mathfrak{A} - \mathfrak{U}) \hookrightarrow (\mathfrak{X}, \mathfrak{A})$ induces an isomorphism in homology.

3. *Dimension.* $H_n(*) = 0$ for all $n \neq 0$, where $*$ is the one point space.

4. *Additivity.* For any collection $\{\mathfrak{X}_\alpha\}$ of topological stacks, $H_n(\coprod_\alpha \mathfrak{X}_\alpha) \cong \bigoplus_\alpha H_n(\mathfrak{X}_\alpha)$

5. *Exactness.* For every pair $(\mathfrak{X}, \mathfrak{A})$ of topological stacks, the maps $i\colon \mathfrak{A} \to \mathfrak{X}$ and $j\colon (\mathfrak{X}, \emptyset) \to (\mathfrak{X}, \mathfrak{A})$ induce a long exact sequence

$$\cdots \longrightarrow H_n(\mathfrak{A}) \xrightarrow{i_*} H_n(\mathfrak{X}) \xrightarrow{j_*} H_n(\mathfrak{X}, \mathfrak{A}) \xrightarrow{\partial} H_{n-1}(\mathfrak{A}) \longrightarrow \cdots .$$

In the case of singular homology with coefficients in an abelian group A, we have $H_n = 0$ for all $n < 0$ and $H_0(*) = A$.

2.6 Singular homology and cohomology

We will fix once and for all a coefficient ring and drop it from the notation consistently.

Singular homology and cohomology for spaces extend to topological stacks. The singular (co)homology of a topological stack $\mathfrak{X}$ can be defined to be the singular (co)homology of its classifying space, as we saw in Section 2.4. Alternatively, but equivalently, we can define singular (co)homology as follows [6].

Let $\mathbb{X} := [X_1 \rightrightarrows X_0]$ be a topological groupoid presentation of $\mathfrak{X}$. Let $X_p = X_1 \times_{X_0} \ldots \times_{X_0} X_1$ (p-fold) be the space of composable sequences of p arrows in the groupoid $\mathbb{X}$. It yields a simplicial space $X_\bullet$

$$\ldots X_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; X_1 \rightrightarrows X_0 \,. \tag{2.1}$$

The *singular chain complex* of $X_\bullet$ is the total complex of the double complex $C_\bullet(X_\bullet)$, where $C_q(X_p)$ is the linear space generated by the continuous maps $\Delta_q \to X_p$. Its homology groups $H_q(X_\bullet) = H_q\big(C_\bullet(X_\bullet)\big)$ are called the *singular homology groups* of $\mathfrak{X}$. The *singular cochain complex* of $X_\bullet$ is the dual of $C_\bullet(X_\bullet)$, i.e., it is the total complex of the bicomplex $C^p(X_q)$. It gives rise to *singular cohomology groups* of $\mathfrak{X}$.

These groups are Morita invariant (i.e., they only depend on the quotient stack $[X_0/X_1]$). In fact, they are naturally isomorphic to the (co)homology groups of the quotient stack $\mathfrak{X} = [X_0/X_1]$ defined in terms of a classifying space of $\mathfrak{X}$ (see Section 2.4).

The above definition of singular (co)homology extends to pairs $(\mathfrak{X},\mathfrak{A})$ of topological stacks in the obvious way and, again, it coincides with the definition in terms of classifying spaces. In particular, Eilenberg-Steenrod axioms are satisfied and we also have cup products. These (co)homology groups coincide with the usual singular (co)homology when $(\mathfrak{X},\mathfrak{A})$ is a pair of topological spaces. In the case when $\mathfrak{X}=[X/G]$ is the quotient stack of a topological group action, with $\mathfrak{A}\subset\mathfrak{X}$ the substack associated to an invariant subspace $A\subset X$, $H(\mathfrak{X},\mathfrak{A})$ is the G-equivariant (co)homology of the pair $(\mathfrak{X},\mathfrak{A})$.

The Künneth formula also holds for singular (co)homology of topological stacks. We only formulate the cohomology version but the homology one holds as well (with the same proof).

Proposition 2.5 (Künneth formula) *In the case of field coefficients, singular cohomology of topological stacks satisfies Künneth formula. That is, we have an isomorphism of graded groups*

$$H^\bullet(\mathfrak{X},\mathfrak{A})\otimes H^\bullet(\mathfrak{Y},\mathfrak{B})\cong H^\bullet(\mathfrak{X}\times\mathfrak{Y},\mathfrak{X}\times\mathfrak{B}\cup\mathfrak{A}\times\mathfrak{Y}).$$

PROOF. Like other properties of singular cohomology, this is proved by choosing classifying spaces $X\to\mathfrak{X}$ and $Y\to\mathfrak{Y}$ and pulling back everything along $X\times Y\to\mathfrak{X}\times\mathfrak{Y}$. $\square$

Remark 2.6 When the coefficient is only a ring, there are still natural cross-product homomorphisms $H^\bullet(\mathfrak{X},\mathfrak{A})\otimes H^\bullet(\mathfrak{Y},\mathfrak{B})\to H^\bullet(\mathfrak{X}\times\mathfrak{Y},\mathfrak{X}\times\mathfrak{B}\cup\mathfrak{A}\times\mathfrak{Y})$ in cohomology and in homology as well $H_\bullet(\mathfrak{X},\mathfrak{A})\otimes H_\bullet(\mathfrak{Y},\mathfrak{B})\to H_\bullet(\mathfrak{X}\times\mathfrak{Y},\mathfrak{X}\times\mathfrak{B}\cup\mathfrak{A}\times\mathfrak{Y})$ (the later being further a monomorphism). As for the proof of Proposition 2.5, this can be seen by choosing classifying space or, alternatively, by working directly with the singular (co)chain complexes of associated groupoid presentations.

Proposition 2.7 *Let $\mathfrak{X}$ be a topological stack and $\mathfrak{A},\mathfrak{B}\subseteq\mathfrak{X}$ substacks. Then, we have a cohomology long exact sequence*

$$\cdots\to H^{n-1}(\mathfrak{A},\mathfrak{A}\cap\mathfrak{B})\to H^n(\mathfrak{X},\mathfrak{A}\cup\mathfrak{B})\to H^n(\mathfrak{X},\mathfrak{B})\to H^n(\mathfrak{A},\mathfrak{A}\cap\mathfrak{B})\to H^{n+1}(\mathfrak{X},\mathfrak{A}\cup\mathfrak{B})\cdots.$$

PROOF. By Excision $H^n(\mathfrak{A},\mathfrak{A}\cap\mathfrak{B})\cong H^n(\mathfrak{A}\cup\mathfrak{B},\mathfrak{B})$. The result follows from the long exact cohomology sequence for the triple $(\mathfrak{X},\mathfrak{A}\cup\mathfrak{B},\mathfrak{B})$. $\square$

The following less standard fact is also true about H.

Proposition 2.8 *Let $\mathfrak{A}\hookrightarrow\mathfrak{B}\hookrightarrow\mathfrak{X}$ be closed embeddings of topological stacks. Suppose that $\mathfrak{X}$ is regular Lindelöf . Then, there is a natural product*

$$H^m(\mathfrak{X},\mathfrak{X}-\mathfrak{B})\otimes H^n(\mathfrak{B},\mathfrak{B}-\mathfrak{A})\to H^{m+n}(\mathfrak{X},\mathfrak{X}-\mathfrak{A})$$

which coincides with the cup product if $\mathfrak{B}=\mathfrak{X}$.

Proof. One uses the fact that the classifying space is paracompact (Proposition 2.3). It is a general fact (for instance see [46]) that if F is a sheaf over a paracompact space X and $Z \subset X$ is closed, then $\varinjlim_{U \supset Z} \Gamma(U, F) \xrightarrow{\sim} \Gamma(Z, F)$, where U is open. Then the result follows from the same argument as for topological spaces in [32], Section 3. $\square$

3 Vector bundles on stacks

We begin with the definition of a (representable) vector bundle on a stack.

Definition 3.1 Let $\mathfrak{X}$ be a (topological) stack. A real **vector bundle** on $\mathfrak{X}$ is a representable morphism of stacks $\mathfrak{E} \to \mathfrak{X}$ which makes $\mathfrak{E}$ a vector space object relative to $\mathfrak{X}$. That is, we have an addition morphism $\mathfrak{E} \times_{\mathfrak{X}} \mathfrak{E} \to \mathfrak{E}$ and an $\mathbb{R}$-action $\mathbb{R} \times \mathfrak{E} \to \mathfrak{E}$, both relative to $\mathfrak{X}$, which satisfy the usual axioms. A complex vector bundle is defined analogously.

A linear map between two vector bundles is defined in the obvious way. Vector bundles on $\mathfrak{X}$ and linear maps between them form a category. (Notice that since a vector bundle $\mathfrak{E}$ is representable over its base $\mathfrak{X}$, we only get a category and not a 2-category.)

There are alternative ways of defining vector bundles over a stack $\mathfrak{X}$ as we will see in the next proposition. All definitions are equivalent to the one given above.

Proposition 3.2 *The following three definitions for a vector bundle on a stack $\mathfrak{X}$ are equivalent to the one given in Definition 3.1, in the sense that the corresponding categories of vector bundles over a given stack $\mathfrak{X}$ are naturally equivalent as linear categories:*

1. *A vector bundle on $\mathfrak{X}$ is a representable morphism of stacks $\mathfrak{E} \to \mathfrak{X}$ such that, for every $f\colon U \to \mathfrak{X}$ with U a topological space, the pullback $E_U \to U$ is endowed with the structure of a vector bundle. Here, $E_U := f^*\mathfrak{E} = U \times_{\mathfrak{X}} \mathfrak{E}$.*

$$\begin{array}{ccc} E_U & \longrightarrow & \mathfrak{E} \\ \downarrow & & \downarrow \\ U & \xrightarrow[f]{} & \mathfrak{X} \end{array}$$

We also require, for every $a\colon V \to U$, that the natural isomorphism

$$\varphi_a\colon (f \circ a)^*\mathfrak{E} \to a^*(f^*\mathfrak{E})$$

be a bundle map.

2. *A vector bundle on $\mathfrak{X}$ is assignment of a vector bundle $E_U \to U$ to every morphism $f\colon U \to \mathfrak{X}$ from a topological space U, together with isomorphisms $\varphi_a\colon a^*E_U \to E_V$ of vector bundles, for every 2-commutative triangle*

$$\begin{tikzcd} V \arrow[rr, "a"] \arrow[dr, "g"'] & \arrow[d, phantom, "\Uparrow\alpha"] & U \arrow[dl, "f"] \\ & \mathfrak{X} & \end{tikzcd}$$

We require the isomorphisms φ to satisfy the cocycle condition

$$\varphi_{a\circ b} = \varphi_b \circ (b^*\varphi_a)$$

for every pair of composable triangles a and b. (Note the abuse of notation: the vector bundle E_U also depends on f, and the isomorphism φ_a also depends on the 2-morphism α.)

3. *Let $\mathbb{X} = [s,t\colon X_1 \rightrightarrows X_0]$ be a groupoid presentation for $\mathfrak{X}$. Then, a vector bundle on $\mathfrak{X}$ is an $\mathbb{X}$-equivariant vector bundle. Recall that an $\mathbb{X}$-equivariant vector bundle consists of a vector bundle E over X_0, and an isomorphism $\psi\colon s^*E \to t^*E$ of vector bundles over X_1 such that the three restrictions of ψ to $X_1 \times_{X_0} X_1$ satisfy the cocycle condition.*

PROOF. We briefly explain how to go from one definition to the other.

Let $\mathfrak{E} \to \mathfrak{X}$ be a vector bundle in the sense of Definition 3.1. It is clear that the pullback vector bundles E_U satisfy the conditions of (1).

To go from (1) to (2) is obvious.

Given a vector bundle in the sense of (2), we obtain a vector bundle E_{X_0} on X_0 corresponding to the quotient map $p\colon X_0 \to \mathfrak{X}$. It follows from the cocycle condition on (2) that this is an $\mathbb{X}$-equivariant bundle.

Finally, given an $\mathbb{X}$-equivariant vector bundle E, we define $\mathfrak{E}$ to be the quotient stack of the groupoid $[E_1 \rightrightarrows E_0]$, where $E_0 := E$ and $E_1 := s^*E = X_1 \times_{X_0} E$. The source map $E_1 \to E_0$ is the projection map $\mathrm{pr}_2\colon X_1 \times_{X_0} E \to E_0$. The target map is $\mathrm{pr}_2 \circ \psi$. It is easy to verify that $\mathfrak{E}$ is a vector bundle over $\mathfrak{X}$ in the sense of Definition 3.1. $\square$

3.1 Operations on vector bundles

The standard operations on vector bundles on spaces (e.g., direct sum, tensor product, exterior powers, and so on) can be carried out on vector bundles on stacks *mutatis mutandis*. This is more easily seen if we think of a vector bundle as in Proposition 3.2 (2). In this case, we simply perform the desired operation simultaneously on the E_U, for varying U, and the resulting family of vector bundles, say F_U, will give rise to a vector bundle $\mathfrak{F}$ on $\mathfrak{X}$.

In view of Proposition 3.2 (3), operations on vector bundles on $\mathfrak{X}$ correspond to operations on $\mathbb{X}$-equivariant vector bundles.

Similarly, we can define a **metric** on a vector bundle. More precisely, a metric on $\mathfrak{E}$ is the same thing as a compatible family of metrics on E_U, for varying U. Given a presentation $\mathbb{X} = [X_1 \rightrightarrows X_0]$ for $\mathfrak{X}$, a metric on $\mathfrak{E}$ is the same thing as an invariant metric on the $\mathbb{X}$-equivariant vector bundle E. (The latter simply means a metric on the vector bundle E over X_0 such that the isomorphism $\psi \colon s^*E \to t^*E$ is an isometry.)

Example 3.3 Let X be a paracompact topological space (say, a manifold) and G a compact Lie group acting on it. Set $\mathfrak{X} := [X/G]$. Then every vector bundle $\mathfrak{E}$ on $\mathfrak{X}$ admits a metric. In fact, metrics on $\mathfrak{E}$ are in bijection with G-invariant metrics on the vector bundle $E := p^*\mathfrak{E}$ over X. (Here $p \colon X \to \mathfrak{X}$ is the quotient map.)

3.2 Tangent, normal, and excess bundles

The main examples of vector bundles we encounter in this paper are tangent and normal bundles. In this section we explain how they are defined.

Tangent bundle

Let $\mathfrak{X}$ be a differentiable stack and choose a differentiable groupoid $[X_1 \rightrightarrows X_0]$ presenting it. Taking tangent bundles gives rise to a new differentiable groupoid $[TX_1 \rightrightarrows TX_0]$. The quotient stack $[TX_0/TX_1]$ is denoted by $\mathfrak{T}\mathfrak{X}$ and is the *tangent bundle* of $\mathfrak{X}$. The base maps induce a groupoid morphism $[TX_1 \rightrightarrows TX_0] \to [X_1 \rightrightarrows X_0]$. After passing to quotient stacks, this induces a morphism of stacks $\mathfrak{T}\mathfrak{X} \to \mathfrak{X}$ which we regard as the base map of $\mathfrak{T}\mathfrak{X}$. It is not hard to see that, up to isomorphism of stacks over $\mathfrak{X}$, $\mathfrak{T}\mathfrak{X}$ is independent of the choice of the groupoid presentation. The tangent bundle $\mathfrak{T}\mathfrak{X}$ is functorial in the obvious sense.

Example 3.4 Let $\mathfrak{X}$ be a differentiable orbifold. Choose a presentation for it by a smooth étale groupoid $\mathbb{X} = [s, t \colon X_1 \rightrightarrows X_0]$. The tangent bundle TX_0 of X_0 is naturally $\mathbb{X}$-equivariant because the two pullbacks $s^*(TX_0)$ and $t^*(TX_0)$ are both naturally isomorphic to TX_1. The corresponding vector bundle on $\mathfrak{X}$ is the tangent bundle of $\mathfrak{X}$.

Warning 3.5 The above example shows that, when $\mathfrak{X}$ is a differentiable orbifold, the tangent bundle $\mathfrak{T}\mathfrak{X}$ is indeed a vector bundle in the sense of Definition 3.1. This, however, is not the case for arbitrary differentiable stacks. This is seen by observing that the fiber $\mathfrak{T}_x\mathfrak{X}$ of the map $\mathfrak{T}\mathfrak{X} \to \mathfrak{X}$ over a point x in $\mathfrak{X}$ is not a vector space in general. In fact, the map $\mathfrak{T}\mathfrak{X} \to \mathfrak{X}$ may not even be representable.

Notation: when $\mathfrak{X}$ is an orbifold we use the notation $T\mathfrak{X}$ for the tangent bundle.

Example 3.6 Let G be a Lie group, and let $\mathfrak{X} = BG = [*/G]$ be its classifying stack. Let $\mathfrak{g}$ be the Lie algebra of G. Then, $\mathfrak{T}BG = B(\mathfrak{g} \rtimes G)$, where G is acting on $\mathfrak{g}$ by the adjoint action. The fiber of the base map $\mathfrak{T}BG \to BG$ over the point $* \to BG$ is $B\mathfrak{g} = [*/\mathfrak{g}]$. (Here, $\mathfrak{g}$ is regarded as a group via its vector space addition. The stack $B\mathfrak{g} = [*/\mathfrak{g}]$ is a simple example of a "2-vector space.")

As indicated in Example 3.6, the definition of the tangent bundle for a general differentiable stack $\mathfrak{X}$ requires a more general class of vector bundles which are quite a bit subtler. These are what some authors call 'stacky vector bundles' or '2-vector bundles' and have the property that their fibers are, in general, 2-vector spaces. In particular, the structure map $\mathfrak{E} \to \mathfrak{X}$ of a stacky vector bundle is no longer representable.

Locally, the tangent 2-vector bundle of $\mathfrak{X}$ can be presented by a length 2 complex of vector bundles ("the tangent complex"). A suitable model for this is the complex $E \xrightarrow{\rho} TX$, where ρ is the anchor map of the Lie algebroid associated to a Lie groupoid presentation $[X_1 \rightrightarrows X]$ for $\mathfrak{X}$. Here E is the normal bundle of the unit map $\eta\colon X \to X_1$. It is naturally identified with the relative tangent bundle T_t of the target map $t : X_1 \to X$ and the anchor map ρ is the composition $\rho : E \cong \eta^*(T_t) \hookrightarrow \eta^*(T_t) \oplus TX \cong \eta^*(TX_1) \cong \eta^*(T_s) \oplus TX \to TX$ where T_s is the relative tangent bundle of the source map $s : X_1 \to X$. In Example 3.6, the tangent bundle TBG can be represented by the complex of vector spaces $\mathfrak{g} \to 0$ (viewed as a complex of vector bundles on a point).

In the rest of the paper, the only instances where we encounter tangent bundles are when $\mathfrak{X}$ is an orbifold.

Normal bundle

Let $\mathfrak{Y}$ be a differentiable stack and $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ a differentiable substack. We would like to define the normal bundle of $\mathfrak{X}$ in $\mathfrak{Y}$. When X and Y are smooth manifolds, one defines the normal bundle either as the quotient $(TY|_X)/TX$, or as the orthogonal complement to TX in $TY|_X$ (upon fixing a metric on the latter).

None of these definitions are available to us in the context of stacks (except when $\mathfrak{Y}$ is an orbifold). Nevertheless, it *is* possible to define the normal bundle as a vector bundle on $\mathfrak{X}$. To do so, pick an atlas $Y \to \mathfrak{Y}$ and let $X \subset Y$ be the invariant submanifold corresponding to $\mathfrak{X}$. Let $N_{X/Y} = (TY|_X)/TX$ be the normal bundle of X in Y. This is an equivariant vector bundle with respect to the induced groupoid structure on X, hence, after passing to the quotient, gives rise to a vector bundle on $\mathfrak{X}$, which we denote by $\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}$ and call the **normal bundle** to $\mathfrak{X}$ in $\mathfrak{Y}$.

Example 3.7 Let $\mathfrak{Y}$ be a differentiable orbifold. Then, we have $\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} = (T\mathfrak{Y}|_{\mathfrak{X}})/T\mathfrak{X}$. In fact, since we can always choose a metric on $T\mathfrak{Y}$ (because $\mathfrak{X}$ is paracompact), we have a direct sum decomposition $T\mathfrak{Y}|_{\mathfrak{X}} = \mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} \oplus T\mathfrak{X}$.

Excess bundle and transversality

Consider a 2-cartesian diagram of differentiable stacks

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathfrak{Y}' \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, hook, "i"'] & \mathfrak{Y} \end{tikzcd}$$

in which the horizontal morphisms are embeddings. (Note that if q is a submersion, i being an embedding implies that j is an embedding. When q is representable, this is seen by pulling back everything along an atlas $Y \to \mathfrak{Y}$. The general case reduces to the representable case by pulling back j along an atlas $Y' \to \mathfrak{Y}'$.) The bundle $\mathfrak{N}_{\mathfrak{X}'/\mathfrak{Y}'}$ is naturally a subbundle of $p^*\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}$. (This is seen using the same pullback argument we just gave to prove that j is an embedding.) We call the quotient bundle $\mathfrak{E} := (p^*\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}})/(\mathfrak{N}_{\mathfrak{X}'/\mathfrak{Y}'})$, which is a vector bundle on $\mathfrak{X}'$, the **excess normal bundle** of the diagram. We say that q is **transversal** to f if $\mathfrak{E}$ is trivial.

Example 3.8 Let $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ and $j\colon \mathfrak{Y} \hookrightarrow \mathfrak{Z}$ be embeddings of differentiable stacks. Consider the 2-cartesian diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook, "\mathrm{id}"] \arrow[d, hook, "i"'] & \mathfrak{X} \arrow[d, hook, "j\circ i"] \\ \mathfrak{Y} \arrow[r, hook, "j"'] & \mathfrak{Z} \end{tikzcd}$$

The excess normal bundle for this diagram is $i^*\mathfrak{N}_{\mathfrak{Y}/\mathfrak{Z}}$. The excess normal bundle for the transpose diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook, "i"] \arrow[d, hook, "\mathrm{id}"'] & \mathfrak{Y} \arrow[d, hook, "j"] \\ \mathfrak{X} \arrow[r, hook, "j\circ i"'] & \mathfrak{Z} \end{tikzcd}$$

is also $i^*\mathfrak{N}_{\mathfrak{Y}/\mathfrak{Z}}$, because we have a short exact sequence

$$0 \to \mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} \to \mathfrak{N}_{\mathfrak{X}/\mathfrak{Z}} \to i^*\mathfrak{N}_{\mathfrak{Y}/\mathfrak{Z}} \to 0.$$

This can be checked by choosing an atlas for $\mathfrak{Z}$.

4 Thom isomorphism

Definition 4.1 We say a vector bundle $p\colon \mathfrak{E} \to \mathfrak{X}$ of rank n on a topological stack $\mathfrak{X}$ is **orientable**, if there is a class $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ such that the map

$$\begin{array}{ccc} H^i(\mathfrak{X}) & \xrightarrow{\tau} & H^{i+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \\ c & \mapsto & p^*(c) \cup \mu \end{array}$$

is an isomorphism for all $i \in \mathbb{Z}$. The class μ is called a **Thom class**, or an **orientation**, for $p\colon \mathfrak{E} \to \mathfrak{X}$.

Lemma 4.2 *Let $\mathfrak{E} \to \mathfrak{X}$ be an oriented vector bundle and $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ a Thom class for it. Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a morphisms of stacks. Then $f^*\mathfrak{E} \to \mathfrak{Y}$ is an oriented vector bundle and $f^*(\mu)$ is a Thom class for it.*

Lemma 4.3 *Let $\mathfrak{E} \to \mathfrak{X}$ be a vector bundle. Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a trivial fibration of topological stacks, and let ν be a Thom class for the vector bundle $f^*\mathfrak{E} \to \mathfrak{Y}$. Then, there is a unique Thom class μ for $\mathfrak{E}$ such that $f^*(\mu) = \nu$.*

Proposition 4.4 *Let $p\colon \mathfrak{E} \to \mathfrak{X}$ be an orientable vector bundle of rank n, and let $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ be a Thom class for it. Let $\mathfrak{K} \subset \mathfrak{X}$ be a closed substack. Then, the homomorphism*

$$\begin{array}{ccc} H^\bullet(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \xrightarrow{\tau} & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \\ c & \mapsto & p^*(c) \cup \mu \end{array}$$

is an isomorphism. Here, we have identified $\mathfrak{K}$ with a closed substack of $\mathfrak{E}$ via the zero section of $\mathfrak{E} \to \mathfrak{X}$.

Proof. Let $\mathfrak{U} = \mathfrak{X} - \mathfrak{K}$. The map $c \mapsto p^*(c) \cup \mu$ induces a map between long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^{\bullet+n}(\mathfrak{E}|_{\mathfrak{U}}, \mathfrak{E}|_{\mathfrak{U}} - \mathfrak{U}) \arrow[r] & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \arrow[r] & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \arrow[r] & \cdots \\ \cdots \arrow[r] & H^\bullet(\mathfrak{U}) \arrow[r] \arrow[u, "\cong"] & H^\bullet(\mathfrak{X}) \arrow[r] \arrow[u, "\cong"] & H^{\bullet+n}(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) \arrow[r] \arrow[u] & \cdots . \end{tikzcd}$$

(The top sequence is long exact by Proposition 2.7.) The claim follows from 5-lemma. $\square$

Proposition 4.5 *In Proposition 4.4, identify $\mathfrak{X}$ with a closed substack of $\mathfrak{E}$ via the zero section. Then, for every $c \in H^*(\mathfrak{X}, \mathfrak{X} - \mathfrak{K})$, we have $\tau(c) = c \cdot \mu$, where $\cdot$ is the product of Proposition 2.8.*

Proposition 4.6 *In Proposition 4.4, assume that $\mathfrak{E}$ is metrized, and let $\mathfrak{D}$ denote its disc bundle of radius r. Set $\mathcal{L} = p^{-1}(\mathfrak{K}) \cap \mathfrak{D}$. and let $\rho\colon H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathcal{L})$ be the restriction homomorphism. Then the homomorphism*

$$\begin{array}{ccc} H^\bullet(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \xrightarrow{\tau} & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathcal{L}) \\ c & \mapsto & \rho(p^*(c) \cup \mu) \end{array}$$

is an isomorphism. In particular, the map ρ is an isomorphism.

Proof. Let $\mathfrak{U} = \mathfrak{X} - \mathfrak{K}$. In the case where $\mathfrak{K} = \mathfrak{X}$, a standard deformation retraction argument shows that ρ is an isomorphism, so the result follows from Proposition 4.4. The general case reduces to this case by considering the map of long exact sequences induced by $c \mapsto \rho(p^*(c) \cup \mu)$,

$$\begin{array}{ccccccccc}
\cdots & \to & H^{\bullet+n}(\mathfrak{E}|_{\mathfrak{U}}, \mathfrak{E}|_{\mathfrak{U}} - \mathfrak{D}|_{\mathfrak{U}}) & \to & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{D}) & \to & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathcal{L}) & \to & \cdots \\
 & & \cong\uparrow & & \cong\uparrow & & \uparrow & & \\
\cdots & \longrightarrow & H^{\bullet}(\mathfrak{U}) & \longrightarrow & H^{\bullet}(\mathfrak{X}) & \longrightarrow & H^{\bullet+n}(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \to & \cdots,
\end{array}$$

and applying 5-lemma. □

The following lemma strengthens Proposition 4.4.

Lemma 4.7 *Let $p \colon \mathfrak{E} \to \mathfrak{X}$ be an orientable vector bundle of rank n, and let $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ be a Thom class for it. Let $\mathfrak{K} \subset \mathfrak{X}$ be a closed substack, and $\mathfrak{K}' \subset \mathfrak{E}$ a closed substack of $\mathfrak{E}$ mapping isomorphically to $\mathfrak{K}$ under p. Then, we have a natural isomorphism $H^{\bullet}(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) \cong H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}')$.*

Lemma 4.8 *Let $p \colon \mathfrak{E} \to \mathfrak{X}$ and $q \colon \mathfrak{F} \to \mathfrak{X}$ be vector bundles over $\mathfrak{X}$, and assume that $\mathfrak{E}$ is oriented. Then, an orientation for $\mathfrak{F}$ determines an orientation for $\mathfrak{E} \oplus \mathfrak{F}$, and vice versa. Indeed, if μ is an orientation for $\mathfrak{E}$, and ν an orientation for $\mathfrak{F}$, then $\mu \cdot p^*(\nu) = \nu \cdot q^*(\mu)$ is an orientation for $\mathfrak{E} \oplus \mathfrak{F}$. Here, $\cdot$ is the product of Proposition 2.8.*

Proof. We only prove one of the statements, namely, the case where $\mathfrak{E}$ and $\mathfrak{E} \oplus \mathfrak{F}$ are oriented. We show that $\mathfrak{F}$ is also oriented. Assume $\mathfrak{E}$ and $\mathfrak{F}$ have rank m and n, respectively, and let $\mu \in H^m(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ and $\nu \in H^{m+n}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X})$ be orientations for $\mathfrak{E}$ and $\mathfrak{E} \oplus \mathfrak{F}$. The class $q^*(\mu) \in H^m(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{F})$ is an orientation for the pullback bundle $q^*(\mathfrak{E}) \cong \mathfrak{E} \oplus \mathfrak{F}$ over $\mathfrak{F}$; note that the bundle map $q^*(\mathfrak{E}) \to \mathfrak{F}$ can be naturally identified with the second projection map $\pi \colon \mathfrak{E} \oplus \mathfrak{F} \to \mathfrak{F}$. By Proposition 4.4, applied to the vector bundle $\pi \colon \mathfrak{E} \oplus \mathfrak{F} \to \mathfrak{F}$, we have an isomorphism

$$\begin{array}{rcl}
H^n(\mathfrak{F}, \mathfrak{F} - \mathfrak{X}) & \to & H^{n+m}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X}) \\
c & \mapsto & \pi^*(c) \cup q^*(\mu).
\end{array}$$

The inverse image of ν under this isomorphism is the desired orientation class in $H^n(\mathfrak{F}, \mathfrak{F} - \mathfrak{X})$. □

In Lemma 4.8, we call the orientation on $\mathfrak{E} \oplus \mathfrak{F}$ the **sum** of the orientations of $\mathfrak{E}$ and $\mathfrak{F}$, and the orientation on $\mathfrak{F}$ the **difference** of the orientations on $\mathfrak{E} \oplus \mathfrak{F}$ and $\mathfrak{E}$.

Lemma 4.9 *Let $0 \to \mathfrak{E} \to \mathfrak{M} \to \mathfrak{F} \to 0$ be a short exact sequence of vector bundles over a topological stack $\mathfrak{X}$. Then, the choice of orientations on two of* *the three vector bundles uniquely determines an orientation on the third one. Moreover, we have the following relation between the orientation classes:*

$$\mu_{\mathfrak{E}} \cdot p^*(\mu_{\mathfrak{F}}) = \mu_{\mathfrak{M}}.$$

Here, p stands for the morphism of pairs $(\mathfrak{M}, \mathfrak{M} - \mathfrak{E}) \to (\mathfrak{F}, \mathfrak{F} - \mathfrak{X})$, and $\cdot$ is the product of Proposition 2.8.

Proof. Apply Lemma 4.3 to the trivial fibration $f\colon \mathfrak{M} \to \mathfrak{X}$ to reduce the problem to the split case and then apply Lemma 4.8. $\square$

Lemma 4.10 *In Lemma 4.7, assume we are given another oriented vector bundle $\mathfrak{F} \to \mathfrak{X}$ of rank m, and endow $\mathfrak{E} \oplus \mathfrak{F}$ with the sum orientation. Let $\mathfrak{K}'' \subset \mathfrak{E} \oplus \mathfrak{F}$ be a closed substack mapping isomorphically to $\mathfrak{K}'$ under the projection $\mathfrak{E} \oplus \mathfrak{F} \to \mathfrak{E}$. Then, the diagram*

$$\begin{tikzcd}
H^\bullet(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}') \arrow[dl, "\cong"] \\
& H^{\bullet+n+m}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{K}'') &
\end{tikzcd}$$

commutes. (All the isomorphisms in this diagram are the ones of Lemma 4.7. So, in the case where $\mathfrak{K} = \mathfrak{K}' = \mathfrak{K}''$, the isomorphisms are simply the Thom isomorphisms of Proposition 4.4.)

Finally, we prove a lemma about compatibility of Thom isomorphism with excision.

Lemma 4.11 *Let X be a manifold, and let $E \to X$ and $N \to X$ be vector bundles of rank n. Assume that E is oriented. Let $i\colon N \to E$ be an open embedding which sends the zero section of N to the zero section of E. (Note that N is naturally isomorphic to E, hence oriented, via the isomorphisms $TX \oplus N \cong TE \cong TX \oplus E$.) Then, the following diagram commutes:*

$$\begin{tikzcd}
H^{\bullet+n}(N, N - X) \arrow[rr, "excision", "\cong"'] & & H^{\bullet+n}(E, E - X) \\
& H^\bullet(X) \arrow[ul, "\cong", "\tau_N"'] \arrow[ur, "\cong"', "\tau_E"] &
\end{tikzcd}$$

5 Loop stacks

5.1 Mapping stacks and the free loop stack

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks over Top. Recall from Section 1.1 that the mapping stack $\mathrm{Map}(\mathfrak{Y}, \mathfrak{X})$ is defined by the rule

$$T \in \mathsf{Top} \quad \mapsto \quad \mathrm{Hom}(T \times \mathfrak{Y}, \mathfrak{X}),$$

where Hom denotes the groupoid of stack morphisms. The mapping stack $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$ is functorial in $\mathfrak{X}$ and $\mathfrak{Y}$ and satisfies the exponential law:

$$\mathrm{Map}(\mathfrak{Z}\times\mathfrak{Y},\mathfrak{X})\cong\mathrm{Map}(\mathfrak{Z},\mathrm{Map}(\mathfrak{Y},\mathfrak{X})).$$

It follows from the exponential law for mapping spaces [68] that when X and Y are spaces, with Y Hausdorff, then $\mathrm{Map}(Y,X)$ is representable by the usual mapping space from Y to X (endowed with the compact-open topology).

Proposition 5.1 *Let $\mathfrak{X}$ be a topological stack and A a compact topological space. Then* $\mathrm{Map}(A,\mathfrak{X})$ *is a topological stack.*

Proof. This follows from Theorem 1.1 of [56]. $\square$

Let $\mathfrak{X}$ be a topological stack. Then $\mathrm{L}\mathfrak{X}=\mathrm{Map}(S^1,\mathfrak{X})$ is also a topological stack. It is called the **loop stack** of $\mathfrak{X}$. By functoriality of mapping stacks, for every $t\in S^1$ we have the corresponding evaluation map $\mathrm{ev}_t\colon \mathrm{L}\mathfrak{X}\to\mathfrak{X}$. In particular, denoting by $0\in S^1$ the standard choice of a base point, there is an evaluation map

$$\mathrm{ev}_0\colon \mathrm{L}\mathfrak{X}\to\mathfrak{X}. \tag{5.1}$$

Similarly, the **path stack** of $\mathfrak{X}$, which is defined to be $\mathrm{Map}(I,\mathfrak{X})$, is a topological stack.

For the next result, we need to assume that $\mathfrak{X}$ is a Hurewicz topological stack.

Lemma 5.2 *Let A, Y, and Z be as in Proposition 1.3. Let $\mathfrak{X}$ be a Hurewicz topological stack. Then the diagram*

$$\begin{tikzcd}
\mathrm{Map}(Z\vee_A Y,\mathfrak{X}) \arrow[r] \arrow[d] & \mathrm{Map}(Y,\mathfrak{X}) \arrow[d] \\
\mathrm{Map}(Z,\mathfrak{X}) \arrow[r] & \mathrm{Map}(A,\mathfrak{X})
\end{tikzcd}$$

is a 2-cartesian diagram of topological stacks.

Proof. We have to verify that for every topological space T the T-points of the above mapping stacks form a 2-cartesian diagram of groupoids. This follows from Proposition 1.3 applied to $A\times T$, $Y\times T$, and $Z\times T$. $\square$

We denote by '8' the wedge $S^1\vee S^1$ of two circles.

Corollary 5.3 *Let $\mathfrak{X}$ be a Hurewicz topological stack, and let $L\mathfrak{X}$ be its loop stack. Then, the diagram*

$$\begin{tikzcd}
\mathrm{Map}(8, \mathfrak{X}) \arrow[r] \arrow[d] & L\mathfrak{X} \arrow[d] \\
L\mathfrak{X} \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

is 2-cartesian.

5.2 Groupoid presentation

Let us now describe a particular groupoid presentation of the loop stack. For this we will assume that $\mathfrak{X}$ is a *Hausdorff Hurewicz topological stack*. Thus $\mathfrak{X}$ admits a groupoid presentation $\mathbb{X}\colon = [X_1 \rightrightarrows X_0]$, where X_0 and X_1 are Hausdorff topological spaces, $X_1 \to X_0 \times X_0$ is proper, and source and target maps are local fibrations. We will fix the groupoid $\mathbb{X}$.

We will construct a groupoid $\mathrm{L}\mathbb{X}\colon = [\mathrm{L}_1\mathbb{X} \rightrightarrows \mathrm{L}_0\mathbb{X}]$ out of $\mathbb{X}$ which presents $\mathrm{L}\mathfrak{X}$. This groupoid presentation is useful in computations (see Section 12). Our construction resembles the construction of the fundamental groupoid of a groupoid [55].

Let $M\mathbb{X} = [M_1\mathbb{X} \rightrightarrows M_0\mathbb{X}]$ be the morphism groupoid of $\mathbb{X}$. Its object set is $M_0\mathbb{X} = X_1$ and its morphism set $M_1\mathbb{X}$ is the set of commutative squares in the underlying category of $\mathbb{X}$:

$$\begin{tikzcd}
t(h) & t(k) \arrow[l, "g"'] \\
s(h) \arrow[u, "h"] & s(k) \arrow[l, "h^{-1}gk"'] \arrow[u, "k"']
\end{tikzcd} \tag{5.2}$$

The source and target maps are the horizontal arrows in square (5.2). The groupoid multiplication is by (vertical) superposition of such squares. Thus we have $M_1\mathbb{X} \cong X_3 = X_1 \times_{X_0} X_1 \times_{X_0} X_1$. The groupoid $M\mathbb{X}$ is another presentation of the stack $\mathfrak{X}$ and is Morita equivalent to $\mathbb{X}$.

Let $P \subset S^1$ be a finite subset of S^1 which contains the base point of S^1. The points of P are labeled according to increasing angle as $P_0, P_1, \dots, P_n$ in such a way that $P_0 = P_n$ is the base point of S^1. Write I_i for the closed interval $[P_{i-1}, P_i]$. Let S_0^P be the disjoint union $S_0^P = \coprod_{i=1}^n I_i$. There is a canonical map $S_0^P \to S^1$. Let S_1^P be the fiber product $S_1^P = S_0^P \times_{S^1} S_0^P$. There is an obvious topological groupoid structure $[S_1^P \rightrightarrows S_0^P]$. The compact-open topology induces a topological groupoid structure on $\mathrm{L}^P\mathbb{X} = [\mathrm{L}_1^P\mathbb{X} \rightrightarrows \mathrm{L}_0^P\mathbb{X}]$, where $\mathrm{L}_0^P\mathbb{X}$ is the set of continuous strict groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \to [X_1 \rightrightarrows X_0]$ and $\mathrm{L}_1^P\mathbb{X}$ is the set of strict continuous groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \to [M_1\mathbb{X} \rightrightarrows M_0\mathbb{X}]$.

The finite subsets of S^1 including the base point are ordered by inclusion. The ordering is *directed*. For $P \leq Q$ there is a canonical morphism of groupoids $\mathrm{L}^P\mathbb{X} \longrightarrow \mathrm{L}^Q\mathbb{X}$. Using the fact that X_0 and X_1 are Hausdorff, it is not difficult to prove that $\mathrm{L}^P\mathbb{X} \to \mathrm{L}^Q\mathbb{X}$ is an isomorphism onto an open subgroupoid. Define the topological groupoid

$$\mathrm{L}\mathbb{X} = \varinjlim_{P \subset S^1} \mathrm{L}^P\mathbb{X} = \bigcup_{P \subset S^1} \mathrm{L}^P\mathbb{X}\,.$$

Proposition 5.4 *The groupoid* $\mathrm{L}\mathbb{X}$ *presents the loop stack* $\mathrm{L}\mathfrak{X}$.

PROOF. First, we need to construct a morphism $\mathrm{L}_0^P\mathbb{X} \to \mathrm{L}\mathfrak{X}$, for every P. The presentation $\mathrm{L}_0\mathbb{X} \to \mathrm{L}\mathfrak{X}$ will then be obtained by gluing these morphisms using the stack property of $\mathrm{L}\mathfrak{X}$ and the fact the $\mathrm{L}_0^P\mathbb{X}$ form an open covering of the topological space $\mathrm{L}_0\mathbb{X}$.

The structure map $\mathrm{L}_0^P \times S_0^P \to X_0$ gives rise to a morphism $\mathrm{L}_0^P \times S_0^P \to \mathfrak{X}$. This morphism descends to $\mathrm{L}_0^P \times S^1 \to \mathfrak{X}$, by Proposition 1.3, because S^1 is obtained from S_0^P as a pushout covered by that proposition. By adjunction, we obtain the required morphism $\mathrm{L}_0^P \to \mathrm{L}\mathfrak{X}$.

The fact that $\bigcup_P \mathrm{L}_0^P\mathbb{X} \to \mathrm{L}\mathfrak{X}$ is an epimorphism of stacks, follows as in Proposition 5.1.

The fact that $\mathrm{L}_1\mathbb{X}$ is the fibered product of $\mathrm{L}_0\mathbb{X}$ with itself over $\mathrm{L}\mathfrak{X}$ reduces immediately to the case of $\mathrm{L}_1^P\mathbb{X}$, for which it is immediate. $\square$

It is easy to represent evaluation map and functorial properties of the free loop stack at the groupoid level with this model.

Remark 5.5 In particular, there is an equivalence of the underlying categories between $\mathrm{L}\mathbb{X}$ and the groupoid whose objects are the set of generalized morphisms from the space S^1 to $\mathbb{X}$ and has equivalences of such as arrows.

Corollary 5.6 *If* $\mathfrak{X}$ *is a differentiable stack then* $\mathrm{L}\mathfrak{X}$ *is regular Lindelöf .*

Target connected groupoid

Assume the groupoid $\mathbb{X}$ is target connected. This means that if T is a topological space, and $\phi \colon T \to X_1$ a continuous map, then for every point of T there exist an open neighborhood $T' \subset T$ and a homotopy $\Phi \colon T' \times I \to X_1$, such that $\Phi_0 = \phi$ and $\Phi_1 = t \circ \phi$, where $t : X_1 \to X_0$ is the target map. For example, any transformation groupoid with connected Lie group is target connected.

For every finite subset $P \subset S^1$ and $x \in \mathrm{L}_0^P\mathbb{X}$, there are arrows $g_i \in X_1$ with $t(g_i) = P_i \in I_i$ and $s(g_i) = P_i \in I_{i+1}$ (or $P_0 \in I_1$ if $i = n$). These arrows can be continuously deformed to the identity point $P_n \in I_n$. Thus there is an element $\widetilde{x} \in \mathrm{Map}(S^1, X_0) \subset \mathrm{L}^{\{0\}}X_0$ and an arrow $\gamma \in \mathrm{L}_1^P\mathbb{X}$ with $s(\gamma) = \widetilde{x}$ and $t(\gamma) = x$. From this observation, we deduce:

Proposition 5.7 *If $\mathbb{X}$ is target connected, then the groupoid $[\mathrm{L}X_1 \rightrightarrows \mathrm{L}X_0]$ with pointwise source map, target map and multiplication presents the loop stack $\mathrm{L}\mathfrak{X}$. Here $\mathrm{L}X_i$ is the usual free loop space of X_i endowed with the compact-open topology.*

In particular, $\mathrm{L}\mathbb{X}$ is Morita equivalent to the groupoid $[\mathrm{L}X_1 \rightrightarrows \mathrm{L}X_0]$.

Example 5.8 If G is a connected Lie group acting on a manifold M, then Proposition 5.7 implies that $\mathrm{L}[M/G] \cong [\mathrm{L}M/\mathrm{L}G]$.

Discrete group action

To the contrary, if G is a discrete group acting on a space M one can form the global quotient $[M/G]$ which is represented by the transformation groupoid $\mathbb{X} := [M \times G \rightrightarrows M]$. For any $x \in \mathrm{L}^P X_0$ one can easily find an arrow $\gamma \in \mathrm{L}^P X_1$ such that $s(\gamma) = x$ and $t(\gamma) \in \mathrm{L}^{\{0\}} X_0$. Furthermore, since G is discrete, an element of $\mathrm{L}^P X_1$ is described by its source and one element $g_i \in G$ for $i = 0, \ldots, |P|$. From these two observations one proves easily:

Proposition 5.9 *Let G be a discrete group acting on a space M. Then $\mathrm{L}[M/G]$ is presented by the transformation groupoid*

$$\left(\coprod_{g \in G} \mathcal{P}_g M\right) \times G \rightrightarrows \coprod_{g \in G} \mathcal{P}_g M$$

where $\mathcal{P}_g M = \{f \colon [0,1] \to M \text{ such that } f(0) = f(1) \cdot g\}$ and G acts by pointwise conjugation.

Note that if G is finite, one recovers the loop orbifold of [49].

Remark 5.10 An element $f \in \mathcal{P}_g M$ has a canonical extension into a map $f : \mathbb{R} \to M$ satisfying $f(x+k) \cdot g^k = f(x)$.

6 Bounded proper morphisms of topological stacks

Definition 6.1 Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a morphisms of topological stacks and $\mathfrak{E}$ a metrizable vector bundle over $\mathfrak{Y}$. A lifting $i \colon \mathfrak{X} \to \mathfrak{E}$ of f,

$$\begin{tikzcd} & \mathfrak{E} \arrow[d] \\ \mathfrak{X} \arrow[ur, hook, dotted, "i"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

is called **bounded** if there is a choice of metric on $\mathfrak{E}$ such that i factors through the unit disk bundle of $\mathfrak{E}$. A morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is called

bounded proper if there exists a metrizable orientable vector bundle $\mathfrak{E}$ on $\mathfrak{Y}$ and a bounded lifting i as above such that i is a closed embedding.

Definition 6.2 A bounded proper morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ is called **strongly proper** if every orientable metrizable vector bundle $\mathfrak{E}$ on $\mathfrak{X}$ is a direct summand of $f^*(\mathfrak{E}')$ for some orientable metrizable vector bundle $\mathfrak{E}'$ on $\mathfrak{Y}$. (Note that, possibly after multiplying by a positive $\mathbb{R}$-valued function on $\mathfrak{Y}$, we can arrange the inclusion $\mathfrak{E} \hookrightarrow f^*(\mathfrak{E}')$ to be contractive, i.e., have norm at most one.)

Example 6.3

1. Every bounded proper map $f\colon X \to Y$ of a topological spaces with Y compact is strongly proper. In that case, one can use the fact that every vector bundle on a compact space is a subbundle of a trivial bundle.

2. Let $\mathfrak{X}$ be a topological stack such that $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is bounded proper. Then Δ is strongly proper. This follows from the fact that every vector bundle on $\mathfrak{X}$ can be naturally extended to $\mathfrak{X} \times \mathfrak{X}$. Similarly, the iterated diagonal $\Delta^{(n)}\colon \mathfrak{X} \to \mathfrak{X}^n$ is strongly proper.

3. Let X, Y be compact G-manifolds (with G compact) and $f\colon X \to Y$ be a G-equivariant map. Then the induced map of stacks $[f/G]\colon [X/G] \to [Y/G]$ is strongly proper.

It does not seem to be true in general that two bounded proper maps compose to a bounded proper map, but we have the following.

Lemma 6.4 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be strongly proper morphisms. Then $g \circ f\colon \mathfrak{X} \to \mathfrak{Z}$ is strongly proper.*

Proof. It is trivial that every orientable metrizable bundle on $\mathfrak{X}$ is a direct summand of one coming from $\mathfrak{Z}$. Let us now prove that $g \circ f$ is proper. Suppose given factorizations

$$\begin{tikzcd} & \mathfrak{E} \arrow[d] \\ \mathfrak{X} \arrow[ur, hook, dotted, "i"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd} \qquad \begin{tikzcd} & \mathfrak{F} \arrow[d] \\ \mathfrak{Y} \arrow[ur, hook, dotted, "j"] \arrow[r, "g"'] & \mathfrak{Z} \end{tikzcd}$$

for f and g. By enlarging $\mathfrak{E}$, and using that g is strongly proper , we may assume that $\mathfrak{E} = g^*(\mathfrak{E}')$, for some oriented metrized vector bundle $\mathfrak{E}'$ on $\mathfrak{Z}$. Let $i'\colon \mathfrak{X} \to \mathfrak{E}'$ be the composition $\mathrm{pr} \circ i$ where $\mathrm{pr}\colon \mathfrak{E} \to \mathfrak{E}'$ is the projection map. The following diagram shows that $g \circ f$ is proper:

$$\begin{tikzcd} & \mathfrak{E}' \oplus \mathfrak{F} \arrow[d] \\ \mathfrak{X} \arrow[ur, hook, dotted, "{(i', jf)}"] \arrow[r, "g \circ f"'] & \mathfrak{Z} \end{tikzcd}$$

$\square$

6.1 Some technical lemmas

In this section we prove a few technical lemmas that will be needed in Section 7 to define bivariant groups.

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of topological stack that admits a factorization

$$\begin{tikzcd} & \mathfrak{E} \arrow[d] \\ \mathfrak{X} \arrow[ur, hook, dotted, "i"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

For example, every bounded proper f has this property (Definition 6.1). The following series lemmas investigate certain properties of the relative cohomology groups $H^{\bullet}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$.

Lemma 6.5 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of topological stacks, and assume we are given two different factorizations $(i, \mathfrak{E})$ and $(i', \mathfrak{E}')$ for it. Then, there is a canonical isomorphism $H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \cong H^{\bullet+\mathrm{rk}\,\mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X})$.*

Proof. Embed $\mathfrak{X}$ in $\mathfrak{E} \oplus \mathfrak{E}'$ via $(i, i')\colon \mathfrak{X} \to \mathfrak{E} \oplus \mathfrak{E}'$. Consider the diagram

$$(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X}) \leftarrow (\mathfrak{E}' \oplus \mathfrak{E}, \mathfrak{E}' \oplus \mathfrak{E} - \mathfrak{X}) \to (\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$$

of pairs of topological stacks. It follows from Proposition 4.4 that we have natural isomorphisms

$$H^{\bullet+\mathrm{rk}\,\mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X}) \xleftarrow{\sim} H^{\bullet+\mathrm{rk}\,\mathfrak{F}+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}' \oplus \mathfrak{E}, \mathfrak{E}' \oplus \mathfrak{E} - \mathfrak{X}) \xrightarrow{\sim} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}).$$

We can now apply Lemma 4.7. $\square$

Using a triple direct sum argument, it can be shown that given three factorizations $(i, \mathfrak{E})$, $(i', \mathfrak{E}')$, and $(i'', \mathfrak{E}'')$ for f, the corresponding isomorphisms defined in the above lemma are compatible. Also, if we switch the order of $(i, \mathfrak{E})$ and $(i', \mathfrak{E}')$ we get the inverse isomorphism. Finally, when $(i, \mathfrak{E})$ and $(i', \mathfrak{E}')$ are equal we get the identity isomorphism. Therefore, the group $H^{\bullet}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ only depends on the morphism f.

Lemma 6.6 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of topological stacks, and let $\varphi := f \circ \mathrm{pr}\colon \mathfrak{X} \times I \to \mathfrak{Y}$, where I is the unit interval and* pr *stands for projection. Suppose we are give a factorization*

$$\begin{tikzcd} & \mathfrak{E} \arrow[d] \\ \mathfrak{X} \times I \arrow[ur, hook, dotted, "\iota"] \arrow[r, "\varphi"'] & \mathfrak{Y} \end{tikzcd}$$

for φ. Let $0 \leq a \leq 1$, and define $\iota_a \colon \mathfrak{X} \to \mathfrak{E}$ to be the restriction of ι to $\mathfrak{X} = \mathfrak{X} \times \{a\}$. Then, the natural map $\phi_a \colon H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota_a(\mathfrak{X})) \to H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota(\mathfrak{X} \times I))$ induced by the map of pairs $(\mathfrak{E}, \mathfrak{E} - \iota(\mathfrak{X} \times I)) \to (\mathfrak{E}, \mathfrak{E} - \iota_a(\mathfrak{X}))$ is an isomorphism and it is independent of a.

PROOF. We may assume that the image of ι does not intersect the zero section of $\mathfrak{E}$. (For example, we lift everything to $\mathfrak{E} \oplus \mathbb{R}$ via $(\iota, 1) \colon \mathfrak{X} \to \mathfrak{E} \oplus \mathbb{R}$ and apply Proposition 4.7 to the vector bundle $\mathfrak{E} \oplus \mathbb{R} \to \mathfrak{E}$.).

Let $\mathfrak{E}' = \mathfrak{E} \oplus \mathbb{R}$ and define $\beta \colon \mathfrak{X} \times I \hookrightarrow \mathfrak{E}'$ by $\beta(x, t) = (\iota(x, 0), t)$. This is a closed embedding, so by Lemma 6.5, we have a commutative diagram

$$\begin{array}{ccc}
H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota(\mathfrak{X} \times I)) & \xrightarrow{\cong} & H^\bullet(\mathfrak{E}', \mathfrak{E}' - \beta(\mathfrak{X} \times I)) \\
{\scriptstyle \phi_a}\uparrow & & \uparrow{\scriptstyle \phi'_a} \\
H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota_a(\mathfrak{X})) & \xrightarrow{\cong} & H^\bullet(\mathfrak{E}', \mathfrak{E}' - \beta_a(\mathfrak{X}))
\end{array}$$

This reduces the problem to the case where our map is β instead of ι, in which case the result is obvious. $\square$

7 Bivariant theory for topological stacks

We define a bivariant cohomology theory [32] on the category of topological stacks whose associated covariant and contravariant theories are singular homology and cohomology, respectively. Our bivariant theory satisfies weaker axioms than those of [32] in that products are not always defined. We show, however, that there are enough products to enable us to define Gysin morphisms as in [32].

The underlying category of our bivariant theory is the category TopSt of topological stacks. The confined morphisms are all maps and independent squares are 2-cartesian diagrams.

7.1 Bivariant groups

To a morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks, we associate a category $\mathsf{C}(f)$ as follows. The objects of $\mathsf{C}(f)$ are morphisms $a \colon \mathfrak{K} \to \mathfrak{X}$ such that $fa \colon \mathfrak{K} \to \mathfrak{Y}$ is bounded proper (Definition 6.1). A morphism in $\mathsf{C}(f)$ between $a \colon \mathfrak{K} \to \mathfrak{X}$ and $b \colon \mathfrak{L} \to \mathfrak{X}$ is a homotopy class (relative to $\mathfrak{X}$) of morphisms $g \colon \mathfrak{K} \to \mathfrak{L}$ over $\mathfrak{X}$.

Lemma 7.1 *The category $\mathsf{C}(f)$ is cofiltered.*

Once and for all, we choose, for each object $a \colon \mathfrak{K} \to \mathfrak{X}$, a vector bundle $\mathfrak{E} \to \mathfrak{Y}$ through which fa factors, as in Definition 6.2.

We define the **bivariant singular homology** of an arbitrary morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ to be the $\mathbb{Z}$-graded abelian group

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) = \varinjlim_{\mathsf{C}(f)} H^{\bullet + \mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

The homomorphisms in this direct limit are defined as follows. Consider a morphism $\varphi\colon \mathfrak{K} \to \mathfrak{K}'$ in $\mathsf{C}(f)$. From this we will construct a natural graded pushforward homomorphism $\varphi_*\colon H^{\bullet+m}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to H^{\bullet+n}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}')$, where $m = \operatorname{rk} \mathfrak{E}$ and $n = \operatorname{rk} \mathfrak{E}'$.

Let $\mathfrak{F} = \mathfrak{E} \oplus \mathfrak{E}'$ with the sum orientation. Let $p\colon \mathfrak{E}' \to \mathfrak{Y}$ be the projection map. Then, $p^*(\mathfrak{E})$ is an oriented vector bundle over $\mathfrak{E}'$. Note that the projection map $\pi\colon p^*(\mathfrak{E}) \to \mathfrak{E}'$ is naturally isomorphic to the second projection map $\mathfrak{F} = \mathfrak{E} \oplus \mathfrak{E}' \to \mathfrak{E}'$; this allows us to view $\mathfrak{F}$ as an oriented vector bundle of rank m over $\mathfrak{E}'$. Let $\mathfrak{D} \subseteq \mathfrak{F}$ be the unit disc bundle. It follows from the assumptions that $\mathfrak{K} \subseteq \mathfrak{D}$, hence also $\mathfrak{K} \subseteq \mathfrak{L} := \pi^{-1}(\mathfrak{K}') \cap \mathfrak{D}$. The restriction homomorphism

$$\varphi_*\colon H^{\bullet+m+n}(\mathfrak{F}, \mathfrak{F} - \mathfrak{K}) \to H^{\bullet+m+n}(\mathfrak{F}, \mathfrak{F} - \mathfrak{L}) \cong H^{\bullet+n}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}'),$$

induced by the inclusion of pairs $(\mathfrak{F}, \mathfrak{F} - \mathfrak{L}) \to (\mathfrak{F}, \mathfrak{F} - \mathfrak{K})$ is the desired pushforward homomorphism; here, we have used the isomorphism of Proposition 4.6.

Next we have to show that the map φ_* is independent of the homotopy class of φ. Consider $a \circ \mathrm{pr}\colon \mathfrak{K} \times I \to \mathfrak{X}$, and let $\rho_0, \rho_1\colon \mathfrak{K} \to \mathfrak{K} \times I$ be the times 0 and time 1 maps. Note that $a \circ \mathrm{pr}\colon \mathfrak{K} \times I \to \mathfrak{X}$ is an object of $\mathsf{C}(f)$. Since every homotopy (relative to $\mathfrak{X}$) between maps with domain $\mathfrak{K}$ factors through $\mathfrak{K} \times I$, it is enough to show that $\rho_{0,*} = \rho_{1,*}$. This follows from Lemma 6.6.

Remark 7.2 The objects of the category $\mathsf{C}(f)$ should be regarded as *supports* of our theory, so by taking the colimit in the definition of the bivariant groups we are ensuring that the (homology theory) associated to our bivariant theory is compactly supported, which is what is expected from singular homology. If we did not do this we would end up with a Borel-Moore homology theory.

Remark 7.3 Let $\mathfrak{K} \to \mathfrak{Y}$ be a bounded proper morphism. It follows from Lemma 6.5, that the cohomology $H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$ is independent of choice of the vector bundle $\mathfrak{E}$ and the embedding $i\colon \mathfrak{K} \hookrightarrow \mathfrak{E}$, up to a canonical isomorphism. Furthermore, the pushforward maps constructed above are compatible with these canonical isomorphisms. So, $H^\bullet(f)$ is independent of all choices involved in its definition.

Lemma 7.4 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a bounded proper morphism and $\mathfrak{X} \xrightarrow{i} \mathfrak{E} \to \mathfrak{Y}$ a factorization for f, where i is a closed embedding (but $\mathfrak{E}$ is not necessarily metrizable). Then we have a natural isomorphism*

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \cong H^{\bullet+\operatorname{rk}\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}).$$

In particular, when $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a closed embedding, then the bivariant group

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \cong H^\bullet(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$$

coincides with relative cohomology.

Proof. Follows from Lemma 6.5. $\square$

7.2 Independent pullbacks

Consider a cartesian diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d] & \mathfrak{Y}' \arrow[d, "h"] \\ \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

We define the pullback $h^*\colon H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \to H(\mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}')$ as follows.

Pullback along h induces a functor $h^*\colon \mathsf{C}(f) \to \mathsf{C}(f')$, $\mathfrak{K} \mapsto h^*\mathfrak{K} := \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{K}$. Furthermore, we have a natural homomorphism

$$H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(h^*\mathfrak{E}, h^*\mathfrak{E} - h^*\mathfrak{K})$$

induced by the map of pairs $(h^*\mathfrak{E}, h^*\mathfrak{E} - h^*\mathfrak{K}) \to (\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$. Using Lemma 6.5, this induces the desired homomorphism of colimits

$$h^*\colon \varinjlim_{\mathsf{C}(f)} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to \varinjlim_{\mathsf{C}(f')} H^{\bullet+\mathrm{rk}\,\mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}').$$

7.3 Confined pushforwards

Let $h\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of topological stacks (Definition 6.1) fitting in a commutative triangle

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathfrak{Y} \arrow[dl, "g"] \\ & \mathfrak{Z} & \end{tikzcd}$$

We define the pushforward homomorphism $h_*\colon H(\mathfrak{X} \xrightarrow{f} \mathfrak{Z}) \to H(\mathfrak{Y} \xrightarrow{g} \mathfrak{Z})$ as follows.

There is a natural functor $\mathsf{C}(f) \to \mathsf{C}(g)$, which sends $a\colon \mathfrak{K} \to \mathfrak{X}$ to $ha\colon \mathfrak{K} \to \mathfrak{Y}$. A factorization for fa gives a factorization for gha in a trivial manner:

$$\begin{tikzcd} \mathfrak{K} \arrow[r, hook, "i"] \arrow[d, "a"'] & \mathfrak{E} \arrow[d] \\ \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Z} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathfrak{K} \arrow[r, hook, "i"] \arrow[d, "ha"'] & \mathfrak{E} \arrow[d] \\ \mathfrak{Y} \arrow[r, "g"'] & \mathfrak{Z} \end{tikzcd}$$

Using Lemma 6.5, this induces the desired homomorphism

$$h_*\colon \varinjlim_{\mathsf{C}(f)} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to \varinjlim_{\mathsf{C}(g)} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

7.4 Products

Unfortunately, we are not able to define product for arbitrary pairs of composable morphisms f and g. However, under an extra assumption on g this will be possible.

Definition 7.5 A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is called **adequate** if in the cofiltered category $\mathsf{C}(f)$ the subcategory consisting of $a\colon \mathfrak{K} \to \mathfrak{X}$ such that $fa\colon \mathfrak{K} \to \mathfrak{Y}$ is strongly proper is cofinal.

Example 7.6

1. Every strongly proper morphism is adequate. (Because in this case $\mathsf{C}(f)$ has a final object that is strongly proper over $\mathfrak{Y}$.)

2. A morphism $f\colon \mathfrak{X} \to Y$ in which Y is a paracompact topological space is adequate. (In this case every object in $\mathsf{C}(f)$ is strongly proper over Y; see Example 6.3)

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of topological stacks, and assume g is adequate. Then we can define products of any two classes $\alpha \in H(f)$ and $\beta \in H(g)$. The construction of the product is as follows. Consider objects $(\mathfrak{K}, a) \in \mathsf{C}(f)$ and $(\mathfrak{L}, b) \in \mathsf{C}(g)$, and choose factorizations

$$\begin{tikzcd} \mathfrak{K} \arrow[r, hook, "i"] \arrow[d, "a"'] & \mathfrak{E} \arrow[d] \\ \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{L} \arrow[r, hook, "j"] \arrow[d, "b"'] & \mathfrak{F} \arrow[d] \\ \mathfrak{Y} \arrow[r, "g"'] & \mathfrak{Z} \end{tikzcd}$$

We may assume $gb\colon \mathfrak{L} \to \mathfrak{Z}$ is strongly proper. There exists a metrizable oriented vector bundle $\mathfrak{E}'$ over $\mathfrak{Z}$ such that $b^*\mathfrak{E}$ is isomorphic to a subbundle of $(gb)^*(\mathfrak{E}')$ as vector bundles over $\mathfrak{L}$. Note that, possibly after multiplying by a positive $\mathbb{R}$-valued function on $\mathfrak{Z}$, we can arrange the inclusion $b^*\mathfrak{E} \hookrightarrow (gb)^*(\mathfrak{E}')$ to be contractive (i.e., have norm at most one). Let us denote $b^*\mathfrak{E}$ by $\mathfrak{E}_0$, $(gb)^*(\mathfrak{E}')$ by $\mathfrak{E}_1$, and the codimension of $\mathfrak{E}_0$ in $\mathfrak{E}_1$ by c.

We define the product

$$H^r(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \otimes H^s(\mathfrak{F}, \mathfrak{F} - \mathfrak{L}) \to H^{r+s+c}(\mathfrak{E}' \oplus \mathfrak{F}, \mathfrak{E}' \oplus \mathfrak{F} - \mathfrak{K} \times_{\mathfrak{Y}} \mathfrak{L}).$$

as follows. (Note that $(\mathfrak{K} \times_{\mathfrak{Y}} \mathfrak{L}, a \circ \mathrm{pr})$ belongs to $C(g \circ f)$ and we have a factorization

$$\begin{tikzcd} \mathfrak{K} \times_{\mathfrak{Y}} \mathfrak{L} \arrow[r, hook, "{(i,j)}"] \arrow[d] & \mathfrak{E}' \oplus \mathfrak{F} \arrow[d] \\ \mathfrak{X} \arrow[r, "g \circ f"'] & \mathfrak{Z} \end{tikzcd}$$

for it. We explain this in more detail shortly.) By pulling back the map i along $\varpi\colon \mathfrak{E}_0 \to \mathfrak{E}$, we obtain a closed embedding $\mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} \hookrightarrow \mathfrak{E}_0$. On the other hand, we have a closed embedding $\mathfrak{E}_1 \hookrightarrow \mathfrak{E}' \oplus \mathfrak{F}$; this is simply the pullback of j along the projection map $\pi\colon \mathfrak{E}' \oplus \mathfrak{F} \to \mathfrak{F}$. Using the inclusion $\mathfrak{E}_0 \hookrightarrow \mathfrak{E}_1$, we find a factorization

$$\begin{tikzcd}
\mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} \arrow[r, hook] \arrow[rrr, bend left, "{(i,j)}"] & \mathfrak{E}_0 \arrow[r, hook] & \mathfrak{E}_1 \arrow[r, hook] & \mathfrak{E}' \oplus \mathfrak{F}.
\end{tikzcd}$$

Now, let $\alpha \in H^r(\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$ and $\beta \in H^s(\mathfrak{F}, \mathfrak{F} - \mathcal{L})$ be two cohomology classes. We define $\alpha \cdot \beta \in H^{r+s+c}(\mathfrak{E}' \oplus \mathfrak{F}, \mathfrak{E}' \oplus \mathfrak{F} - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L})$ to be $\tau(\varpi^*(\alpha)) \cdot \pi^*(\beta)$, where the latter $\cdot$ is the product of Proposition 2.8. In more detail, we have $\pi^*(\beta) \in H^s\big(\mathfrak{E}' \oplus \mathfrak{F}, \mathfrak{E}' \oplus \mathfrak{F} - \mathfrak{E}_1\big)$, $\varpi^*(\alpha) \in H^r(\mathfrak{E}_0, \mathfrak{E}_0 - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L})$, and $\tau\colon H^r(\mathfrak{E}_0, \mathfrak{E}_0 - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L}) \to H^{r+c}\big(\mathfrak{E}_1, \mathfrak{E}_1 - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L}\big)$ is the Thom isomorphism of Proposition 4.4 for the vector bundle $\mathfrak{E}_1$ over $\mathfrak{E}_0$; to obtain this Thom isomorphism, we have used that, since the bundles are metrizable, $\mathfrak{E}_0$ is a direct summand of $\mathfrak{E}_1$ and its complement is oriented (Lemma 4.8). Finally, our $\cdot$ is the one in Proposition 2.8 with the inclusions $\mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} \hookrightarrow \mathfrak{E}_1 \hookrightarrow \mathfrak{E}' \oplus \mathfrak{F}$, $n = r + c$ and $m = s$.

7.5 Künneth formula

For bivariant singular cohomology with field coefficients we have the following Künneth formula.

Proposition 7.7 (Künneth formula) *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $f'\colon \mathfrak{X}' \to \mathfrak{Y}'$ be morphisms of topological stacks. Then, we have a natural isomorphism of graded groups*

$$H^\bullet(f \times f') \cong H^\bullet(f) \otimes H^\bullet(f'),$$

where $f \times f'\colon \mathfrak{X} \times \mathfrak{X}' \to \mathfrak{Y} \times \mathfrak{Y}'$ is the product map.

PROOF.

First, suppose that f and f' are bounded proper, and choose factorizations as in Definition 6.2. We obtain a factorization

$$\begin{tikzcd}
 & \mathfrak{E} \boxtimes \mathfrak{E}' \arrow[d] \\
\mathfrak{X} \times \mathfrak{X}' \arrow[ur, hook, dotted, "{(i,i')}"] \arrow[r, "{(f,f')}"'] & \mathfrak{Y} \times \mathfrak{Y}'
\end{tikzcd}$$

for $f \times f'$. Note that the total space of the vector bundle $\mathfrak{E} \boxtimes \mathfrak{E}'$ is $\mathfrak{E} \times \mathfrak{E}'$. Let $\mathfrak{U} = \mathfrak{E} - i(\mathfrak{X})$ and $\mathfrak{U}' = \mathfrak{E}' - i(\mathfrak{X}')$. Then, $\mathfrak{E} \times \mathfrak{E}' - (i, i')(\mathfrak{X} \times \mathfrak{X}') = \mathfrak{E} \times \mathfrak{U}' \cup \mathfrak{U} \times \mathfrak{E}'$. So, by Proposition 2.5, we have

$$H^\bullet(f \times f') \cong H^{\bullet+n+n'}(\mathfrak{E} \times \mathfrak{E}' - (i, i')(\mathfrak{X} \times \mathfrak{X}')) \cong H^{\bullet+n}(\mathfrak{E}, \mathfrak{U}) \otimes H^{\bullet+n'}(\mathfrak{E}', \mathfrak{U}'), \quad (7.1)$$

and the latter is equal to $\cong H^\bullet(f) \otimes H^\bullet(f')$.

To prove the isomorphism for general f and f', consider the functor $P\colon \mathsf{C}(f) \times \mathsf{C}(f') \to \mathsf{C}(f \times f')$ which sends a pair $(a, a') \in \mathsf{C}(f) \times \mathsf{C}(f')$, with $a\colon \mathfrak{K} \to \mathfrak{X}$ and $a'\colon \mathfrak{K}' \to \mathfrak{X}'$, to $a \times a'\colon \mathfrak{K} \times \mathfrak{K}' \to \mathfrak{X} \times \mathfrak{X}'$. Since we know the result for bounded proper morphisms, to prove the Künneth isomorphism for f and f' we observe that, in general, for every directed system indexed by $\mathsf{C}(f \times f')$, the induced directed system (via P) indexed by $\mathsf{C}(f) \times \mathsf{C}(f')$ has the same colimit. This is due to the fact that P has a left adjoint $Q\colon \mathsf{C}(f \times f') \to \mathsf{C}(f) \times \mathsf{C}(f')$, defined by sending $a\colon \mathfrak{K} \to \mathfrak{X} \times \mathfrak{X}'$ to $(\mathrm{pr}_1 \circ a, \mathrm{pr}_2 \circ a) \in \mathsf{C}(f) \times \mathsf{C}(f')$. $\square$

When the coefficient is only a ring, the cohomology cross-product (see Remark 2.6) yields a bivariant cross-product.

Proposition 7.8 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $f'\colon \mathfrak{X}' \to \mathfrak{Y}'$ be morphisms of topological stacks. Then, we have a natural homomorphism of graded groups*

$$H^\bullet(f) \otimes H^\bullet(f') \to H^\bullet(f \times f')$$

where $f \times f'\colon \mathfrak{X} \times \mathfrak{X}' \to \mathfrak{Y} \times \mathfrak{Y}'$ is the product map.

Proof. The proof of Proposition 7.7 applies with the only difference that the last isomorphism in the sequence (7.1) of isomorphisms is replaced by the cross product $H^{\bullet+n+n'}(\mathfrak{E} \times \mathfrak{E}' - (i, i')(\mathfrak{X} \times \mathfrak{X}')) \leftarrow H^{\bullet+n}(\mathfrak{E}, \mathfrak{U}) \otimes H^{\bullet+n'}(\mathfrak{E}', \mathfrak{U}')$ homomorphism (Remark 2.6). $\square$

7.6 Associated covariant and contravariant theories

By definition, the n^{th} graded piece of the contravariant theory associated to the bivariant theory H is give by

$$H^n(\mathfrak{X}) = H^n(\mathfrak{X} \xrightarrow{\mathrm{id}} \mathfrak{X}) = \varinjlim_{\mathsf{C}(\mathrm{id}_{\mathfrak{X}})} H^{n+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

The category $\mathsf{C}(\mathrm{id}_{\mathfrak{X}})$ has a final object $(\mathfrak{X}, \mathfrak{X})$, so the above colimit is isomorphic to $H^n(\mathfrak{X}, \mathfrak{X} - \mathfrak{X}) = H^n(\mathfrak{X})$, the usual singular cohomology.

The n^{th} graded piece of the covariant theory associated to H is defined to be

$$H_n(\mathfrak{X}) = H^{-n}(\mathfrak{X} \to pt) = \varinjlim_{\mathsf{C}(\mathfrak{X})} H^{e-n}(E, E - K) \cong \varinjlim_{K \to \mathfrak{X}} H_n(K).$$

Here, $\mathsf{C}(\mathfrak{X})$ is the category whose object are pairs (E, K) where E is a Euclidean space of dimension e and K is a compact subspace of E together with a map $K \to \mathfrak{X}$. In the latter colimit, we have used the Spanier-Whitehead duality $H_n(K) \cong H^{e-n}(E, E - K)$, and the limit is taken over the category of all maps $K \to \mathfrak{X}$ with K a compact topological space that is embeddable in some Euclidean space. By the following proposition, the latter colimit is, indeed, isomorphic to the singular cohomology $H_n(\mathfrak{X})$.

Proposition 7.9 *Let $\mathfrak{X}$ be a topological stack. Then, we have a natural isomorphism*

$$\varinjlim_{K \to \mathfrak{X}} H_n(K) \cong H_n(\mathfrak{X}),$$

where the limit is taken over the category of all maps $K \to \mathfrak{X}$ with K a compact topological space that is embeddable in some Euclidean space.

It is possible to generalize the axiomatic framework for (skew-symmetric) bivariant theories [32] to include the present case, where products are only defined for a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ if $Y \xrightarrow{g} Z$ belongs to a subclass of morphisms called adequate. See Appendix B for the axioms. Details will appear elsewhere.

8 Regular embeddings, submersions, and normally nonsingular morphisms

8.1 Submersions

Definition 8.1 Let $p\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differentiable stacks. For p representable, we say it is a **submersion** if its base extension along any differentiable map $Y \to \mathfrak{Y}$ from a manifold Y is a submersion of manifolds. (It is enough to check this for one atlas $Y \to \mathfrak{Y}$.) If p is not necessarily representable, we say that p is a submersion if for some (hence every) atlas $q\colon X \to \mathfrak{X}$, the composition $p \circ q\colon X \to \mathfrak{Y}$ is a submersion.

Example 8.2 The following are some simple examples of submersions.

1. A differentiable map $p\colon X \to Y$ of manifolds is a submersion in the above sense if and only if it is a submersion in the usual sense.

2. Any projection $\mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X}$ is a submersion.

3. Let $\mathfrak{E}$ be a vector bundle over $\mathfrak{X}$. Then, the base map $p\colon \mathfrak{E} \to \mathfrak{X}$ is a submersion. More generally, if p is an affine bundle (for example, a surjection $\mathfrak{E} \to \mathfrak{F}$ of vector bundles over a base $\mathfrak{X}$), then p is a submersion.

4. Let $p\colon X \to \mathfrak{X}$ be an atlas for the differentiable stack $\mathfrak{X}$. Then p is a submersion. In other words, if $\mathfrak{X} = [X_0/X_1]$ is the quotient stack of a differentiable groupoid $[X_1 \rightrightarrows X_0]$, then the quotient map $p\colon X_0 \to \mathfrak{X}$ is a submersion.

Lemma 8.3 *Let $p\colon \mathfrak{X} \to \mathfrak{Y}$ and $q\colon \mathfrak{Y} \to \mathfrak{Z}$ be submersions. Then, we have the following.*

1. *The composition $q \circ p\colon \mathfrak{X} \to \mathfrak{Z}$ is a submersion.*

2. *For an arbitrary morphism $\mathfrak{Y}' \to \mathfrak{Y}$ of differentiable stacks, the base extension $p'\colon \mathfrak{X}' \to \mathfrak{Y}'$ of p is a submersion.*

Lemma 8.4 *Consider the 2-cartesian diagram of differentiable stacks*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathfrak{Y}' \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, hook, "i"'] & \mathfrak{Y} \end{tikzcd}$$

in which the horizontal morphisms are embeddings and q is a submersion. Then, q is transversal to i. That is, $p^*\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} = \mathfrak{N}_{\mathfrak{X}'/\mathfrak{Y}'}$ (equivalently, the excess bundle $\mathfrak{E}$ is trivial).

Proof. Precomposing q with an atlas $Y' \to \mathfrak{Y}'$, we are reduced to the case where q is representable. By making a base change along an atlas $Y \to \mathfrak{Y}$, we reduce further to the case where we have a diagram of smooth manifolds, in which case the result is clear. $\square$

8.2 Regular embeddings

In differential topology existence of a tubular neighborhood for a submanifold is a strong tool which allows one to linearize the situation by passing to the normal bundle of the submanifold. Unfortunately, this tool is not always available in the world of differentiable stacks, as a substack may not necessarily admit a tubular neighborhood. To our knowledge, the only situation where existence of tubular neighborhoods is guaranteed is when the ambient differentiable stack is the quotient stack of a compact Lie group action on a smooth manifold (Example 8.7).

In this section, we introduce a class of embeddings of differentiable stacks, called *regular embeddings*, which behave as if they have tubular neighborhoods. We begin with a preliminary definition.

Definition 8.5 We say that an embedding $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ of topological stacks admits a **tubular neighborhood** if there is a vector bundle $\mathfrak{N}$ over $\mathfrak{X}$ and a factorization

$$\mathfrak{X} \overset{s}{\hookrightarrow} \mathfrak{N} \overset{j}{\hookrightarrow} \mathfrak{Y}$$

for i, where s is the zero section of $\mathfrak{N}$ and j is an open embedding. The bundle $\mathfrak{N}$ is called a tubular neighborhood of $\mathfrak{X}$ in $\mathfrak{Y}$. (Note that the vector bundle $\mathfrak{N}$ is canonically isomorphic to the normal bundle $N_{\mathfrak{X}/\mathfrak{Y}}$.)

Example 8.6 Let $\mathfrak{X}$ be a topological stack and $\mathfrak{E}$ a vector bundle over $\mathfrak{X}$. Let $s\colon \mathfrak{X} \to \mathfrak{E}$ be the zero section. Then s admits a tubular neighborhood. The normal bundle and the tubular neighborhood of $\mathfrak{X}$ in $\mathfrak{E}$ are both $\mathfrak{E}$ itself.

Example 8.7 Let $\mathfrak{Y} = [Y/G]$ be the quotient stack $[Y/G]$ of a topological group G action on a topological space Y. Let $\mathfrak{X} \subseteq \mathfrak{Y}$ be a closed substack of $\mathfrak{Y}$, and let

$X \subseteq Y$ be the corresponding invariant subspace. Then, tubular neighborhoods of $\mathfrak{X}$ in $\mathfrak{Y}$ are in bijection with G-equivariant tubular neighborhoods of X in Y. In particular, if $\mathfrak{Y}$ is a differentiable stack which is isomorphic to the quotient stack $[Y/G]$ of a compact Lie group G action on a smooth manifold Y, then any differentiable substack $\mathfrak{X}$ of $\mathfrak{Y}$ admits a tubular neighborhood. This follows from the G-equivariant tubular neighborhood theorem; see ([11], Section VI, Theorem 2.2) and also the proof of Proposition 8.18.

Embeddings which admit tubular neighborhoods have the expected nice properties, but they are not flexible enough for our purposes. For instance, composition of two such embeddings does not appear to admit a tubular neighborhood in general. Also, pullback of such an embedding $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ along a submersion onto $\mathfrak{Y}$ (even along the base map of vector bundle $\mathfrak{E} \to \mathfrak{Y}$) does not seem to always admit a tubular neighborhood. Definition 8.11 is devised to fix these deficiencies.

Definition 8.8 Let $\mathfrak{Y}$ be a differentiable stack. Let $i \colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ a differentiable substack with normal bundle $\mathfrak{N} = N_{\mathfrak{X}/\mathfrak{Y}}$. Let $c \in H^k(\mathfrak{N}, \mathfrak{N} - \mathfrak{X})$ be a cohomology class. We say that a class $\bar{c} \in H^k(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ is **compatible** with c if for every differentiable atlas $q \colon Y \to \mathfrak{Y}$, the class $q^*(c) \in H^k(N, N - X)$ corresponds to the class $q^*(\bar{c}) \in H^k(Y, Y - X)$ under the isomorphism $H^k(N, N - X) \cong H^k(Y, Y - X)$ obtained by identifying $N := p^*\mathfrak{N} \cong N_{X/Y}$ with a tubular neighborhood of $X := p^{-1}\mathfrak{X}$ in Y (and applying excision).

$$\begin{tikzcd} N \arrow[r, dash] & X \arrow[r, hook] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ \mathfrak{N} \arrow[r, dash] & \mathfrak{X} \arrow[r, hook, "i"'] & \mathfrak{Y} \end{tikzcd}$$

Since tubular neighborhoods of submanifolds are unique up to isotopy, the isomorphism $H^k(N, N - X) \cong H^k(Y, Y - X)$ in the above definition is independent of the choice of the tubular neighborhood.

Lemma 8.10 justifies the above definition. Before proving it we quote a useful Lemma from [69].

Lemma 8.9 *Let $\mathfrak{X}$ be a differentiable stack and M a smooth manifold. Let $f \colon M \to \mathfrak{X}$ be a continuous map (i.e., a morphism of underlying topological stacks). Then, there exits a differentiable atlas $q \colon X \to \mathfrak{X}$ such that f lifts to a continuous map $\widetilde{f} \colon M \to Y$. If f is differentiable (i.e., a morphism of differentiable stacks), then $\widetilde{f}$ can also be taken to be differentiable.*

PROOF. The case where f is differentiable is Lemma 3.10 of [69]. The case where f is only continuous is proved using the same argument given in *loc. cit.* $\square$

Lemma 8.10 *Notation being as in Definition 8.8, the class $\bar{c} \in H^k(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ is unique (if it exists).*

PROOF. The statement is true when $\mathfrak{Y}$ is a manifold. So, it follows that if $\bar{c}_1, \bar{c}_2 \in H^k(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ are two such classes, the difference $d := \bar{c}_1 - \bar{c}_2$ has the property that $q^*(d) \in H^k(Y, Y - X)$ is zero for every differentiable atlas $q\colon Y \to \mathfrak{Y}$. We claim that this can only happen if $d = 0$.

Suppose that $d \in H^k(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ is a nonzero cohomology class. Choose a classifying space $\varphi\colon Y_0 \to \mathfrak{Y}$ for $\mathfrak{Y}$ (see Section 2.2). Then $\varphi^*(d) \in H^k(Y_0, Y_0 - X_0)$ is nonzero, where $X_0 = \varphi^{-1}(\mathfrak{X}) \subseteq Y_0$. We can find a finite simplicial complex K and a map $f\colon K \to Y_0$ such that $f^*\varphi^*(d) \in H^k(K, K - L)$ is nonzero, where $L = f^{-1}\varphi^{-1}(\mathfrak{X}) \subseteq K$. By embedding K in some Euclidean space and choosing a small tubular neighborhood M of K which retracts to K, we obtain a manifold M and a continuous map $g\colon M \to Y_0$ such that $g^*\varphi^*(d)$ is nonzero. Therefore, we have succeeded in finding a manifold M and a morphism (of topological stacks) $\varphi \circ g\colon M \to \mathfrak{Y}$ which sees d. It follows from Lemma 8.9 that the map $\varphi \circ g\colon M \to \mathfrak{Y}$ factors through a differentiable atlas $q\colon Y \to \mathfrak{Y}$. So, q also sees d, that is $q^*(d) \in H^k(Y, Y - X)$ is nonzero, which is what we wanted to prove. $\square$

Definition 8.11 We say that an embedding $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ of differentiable stacks is a **regular embedding** if for every orientation class $\mu \in H^n(\mathfrak{N}, \mathfrak{N} - \mathfrak{X})$ (i.e., a Thom class as in Definition 4.1), there is a class $\bar{\mu} \in H^n(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ compatible with it in the sense of Definition 8.8. Here, $\mathfrak{N} = \mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}$ is the normal bundle to $\mathfrak{X}$ in $\mathfrak{Y}$ and n is its rank.

Lemma 8.12 *Regular embeddings enjoy the following properties:*

1. *Any embedding of smooth manifolds is a regular embedding.*

2. *Any embedding $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ which admits a tubular neighborhood (Definition 8.5) is a regular embedding.*

PROOF. By excision, $H^\bullet(\mathfrak{N}, \mathfrak{N} - \mathfrak{X}) \cong H^\bullet(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$. This implies (2), and (2) implies (1). $\square$

Lemma 8.13 (Composition) *If $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ and $j\colon \mathfrak{Y} \hookrightarrow \mathfrak{Z}$ are regular embeddings, then $j \circ i\colon \mathfrak{X} \hookrightarrow \mathfrak{Z}$ is also a regular embedding. Moreover, if $\mu_i \in H^m(\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}, \mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} - \mathfrak{X})$ and $\mu_j \in H^n(\mathfrak{N}_{\mathfrak{Y}/\mathfrak{Z}}, \mathfrak{N}_{\mathfrak{Y}/\mathfrak{Z}} - \mathfrak{Y})$ are orientations (Definition 4.1) and $\overline{\mu_i} \in H^m(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ and $\overline{\mu_j} \in H^n(\mathfrak{Z}, \mathfrak{Z} - \mathfrak{Y})$ are the corresponding compatible classes, then the induced orientation $\mu_{j\circ i} \in H^{m+n}(\mathfrak{N}_{\mathfrak{X}/\mathfrak{Z}}, \mathfrak{N}_{\mathfrak{X}/\mathfrak{Z}} - \mathfrak{X})$ (see Lemma 4.9 and Example 3.8) is compatible with $\overline{\mu_i} \cdot \overline{\mu_j}$, where the latter product is the one of Proposition 2.8. In other words, $\overline{\mu_{j\circ i}} = \overline{\mu_i} \cdot \overline{\mu_j}$.*

PROOF. As we saw in Example 3.8, we have a short exact sequence

$$0 \to N_{\mathfrak{X}/\mathfrak{Y}} \to N_{\mathfrak{X}/\mathfrak{Z}} \to i^*N_{\mathfrak{Y}/\mathfrak{Z}} \to 0$$

of vector bundles over $\mathfrak{X}$. By Lemma 4.9, the orientation classes μ_i and $i^*\mu_j$ induce an orientation class $\mu_{j\circ i} := \mu_i \cdot p^* i^*(\mu_j)$ on $N_{\mathfrak{X}/\mathfrak{Z}}$, where p stands for the map of pairs $(N_{\mathfrak{X}/\mathfrak{Z}}, N_{\mathfrak{X}/\mathfrak{Z}} - N_{\mathfrak{X}/\mathfrak{Y}}) \to (i^*N_{\mathfrak{Y}/\mathfrak{Z}}, i^*N_{\mathfrak{Y}/\mathfrak{Z}} - \mathfrak{X})$. To prove the lemma, we have to show that $\overline{\mu_i} \cdot \overline{\mu_j}$ is compatible with $\mu_{j\circ i}$. By definition of compatibility, it is enough to verify this after pulling back everything along an atlas $Z \to \mathfrak{Z}$. That is, it is enough to prove the lemma in the case of smooth manifolds. By choosing a suitable tubular neighborhood for the embedding $X \hookrightarrow Z$, we can further reduce to the case where the given embeddings are zero sections of vector bundles, in which case the result is trivial. $\square$

Lemma 8.14 (Pullback) *Consider the 2-cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, hook, "i'"] \arrow[d, "p"'] & \mathfrak{Y}' \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, hook, "i"'] & \mathfrak{Y} \end{tikzcd}$$

Suppose that i is a regular embedding and q is a submersion (Definition 8.1). Then, i' is a regular embedding. Furthermore, if $\mu \in H^n(\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}, \mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}} - \mathfrak{X})$ is an orientation class and $\bar{\mu} \in H^n(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ is compatible with μ, then $q^*(\bar{\mu})$ is compatible with the pullback orientation on $\mathfrak{N}_{\mathfrak{X}'/\mathfrak{Y}'} = p^*\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}$. Here, $q^*\colon H^n(\mathfrak{Y}, \mathfrak{Y}-\mathfrak{X}) \to H^n(\mathfrak{Y}', \mathfrak{Y}'-\mathfrak{X}')$ is the induced map on relative cohomology.

Proof. It follows from the definition of compatibility (Definition 8.8) that an embedding $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ is a regular embedding if and only if its base extension along every atlas $Y \to \mathfrak{Y}$ is. This, combined with Lemmas 8.3.1 and 8.4, proves the lemma. $\square$

In the above lemma, the case we are particularly interested in is where q is an affine bundle (e.g, when q is the base map of a vector bundle, see Example 8.2.3).

8.3 Normally nonsingular morphisms of stacks and oriented stacks

Definition 8.15 We say that a representable morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of stacks is **normally nonsingular**, (**nns** for short), if there exist vector bundles $\mathfrak{N}$ and $\mathfrak{E}$ over the stacks $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, and a commutative diagram

$$\begin{tikzcd} \mathfrak{N} \arrow[r, hook, "i"] & \mathfrak{E} \arrow[d, "p"] \\ \mathfrak{X} \arrow[u, "s"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

where s is the zero section of $\mathfrak{N}$, i is an open embedding, and $\mathfrak{E}$ is oriented. When $\mathfrak{N}$ is also oriented, we say that the diagram is **oriented**. (For the definition of on orientation on a morphism f see Definition 8.21 below.) The integer $c = \mathrm{rk}\,\mathfrak{N} - \mathrm{rk}\,\mathfrak{E}$ depends only on f and is called the **codimension** of f. (Note that in the case where $\mathfrak{E}$ is of rank zero this coincides with Definition 8.5, so f admits a tubular neighborhood.)

A diagram as above is called a *normally nonsingular diagram* for f. The vector bundle $\mathfrak{N}$ is a tubular neighborhood of $\mathfrak{X}$ in $\mathfrak{E}$ in the sense of Definition 8.5.

The following are two extreme examples of nns morphisms.

Example 8.16 Let $\mathfrak{X}$ be a topological stack and $\mathfrak{E}$ a vector bundle over $\mathfrak{X}$. Let $s\colon \mathfrak{X} \to \mathfrak{E}$ be the zero section. Then, the diagram

$$\begin{tikzcd} \mathfrak{E} \arrow[r, hook, "\mathrm{id}"] & \mathfrak{E} \arrow[d, "\mathrm{id}"] \\ \mathfrak{X} \arrow[u, "s"] \arrow[r, "s"'] & \mathfrak{E} \end{tikzcd}$$

is an nns diagram for s. Here, we are regarding $\mathfrak{E}$ as a rank zero vector bundle over itself. The diagram is oriented if and only if $\mathfrak{E}$ is. The codimension of s is equal to $\mathrm{rk}\,\mathfrak{E}$. (For future use, let us also record the fact that s is a strongly proper morphism.)

Example 8.17 Let $\mathfrak{X}$ be a topological stack and $\mathfrak{E}$ an oriented vector bundle over $\mathfrak{X}$. Let $p\colon \mathfrak{E} \to \mathfrak{X}$ be the base map. Then, the diagram

$$\begin{tikzcd} \mathfrak{E} \arrow[r, hook, "\mathrm{id}"] & \mathfrak{E} \arrow[d, "p"] \\ \mathfrak{E} \arrow[u, "\mathrm{id}"] \arrow[r, "p"'] & \mathfrak{X} \end{tikzcd}$$

is an oriented nns diagram for p. Here, we are regarding $\mathfrak{E}$ as a rank zero vector bundle over itself. The codimension of p is equal to $-\,\mathrm{rk}\,\mathfrak{E}$. The normal bundle and the tubular neighborhood of $\mathfrak{E}$ in $\mathfrak{E}$ are both $\mathfrak{E}$ itself.

Proposition 8.18 *Let G be a compact Lie group, and X and Y smooth G-manifolds, with $\mathfrak{X} = [X/G]$ and $\mathfrak{Y} = [Y/G]$ the corresponding quotient stacks. Assume further that X is of finite orbit type. Then, for every G-equivariant smooth map $X \to Y$, the induced morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of quotient stacks is normally nonsingular.*

Proof. First we claim that, there is a vector bundle $\mathfrak{V} \to BG$ and a smooth embedding $j \colon \mathfrak{X} \to \mathfrak{V}$, as in the following commutative diagram:

$$\begin{tikzcd} & & \mathfrak{V} \arrow[d, "p"] \\ \mathfrak{X} \arrow[r, "f"'] \arrow[urr, dotted, "j"] & \mathfrak{Y} \arrow[r, "\pi_{\mathfrak{Y}}"'] & BG \end{tikzcd}$$

This statement is equivalent to the fact that every G-manifold X of finite orbit type embeds G-equivariantly into a linear G-representation V ([11], Section II, Theorem 10.1). We can arrange for the G-action on V to be orientation preserving by simply replacing V with $V \oplus V$.

Let $\mathfrak{E} := \mathfrak{Y} \times_{BG} \mathfrak{V}$ be the pullback of $\mathfrak{V}$ over $\mathfrak{Y}$. We obtain the following commutative diagram

$$\begin{tikzcd} & \mathfrak{E} \arrow[d, "p"] \\ \mathfrak{X} \arrow[r, "f"'] \arrow[ur, dotted, "{(f,j)}"] & \mathfrak{Y} \end{tikzcd}$$

Observe that (f, j) is a smooth closed embedding (this can be checked by pulling back the whole picture along a chart, say $* \to BG$, for BG). Let $\mathfrak{N}$ be the normal bundle of $(f, j)(\mathfrak{X})$ in $\mathfrak{E}$. By the existence of G-equivariant tubular neighborhoods ([11], Section VI, Theorem 2.2), we find a vector bundle $\mathfrak{N}$ over $\mathfrak{X}$ and an open embedding $i \colon \mathfrak{N} \to \mathfrak{E}$ making the following diagram commutative:

$$\begin{tikzcd} \mathfrak{N} \arrow[r, hook, "i"] & \mathfrak{E} \arrow[d, "p"] \\ \mathfrak{X} \arrow[u, "s"] \arrow[r, "f"'] \arrow[ur, dotted, "{(f,j)}"] & \mathfrak{Y} \end{tikzcd}$$

This is exactly what we were looking for. $\square$

Example 8.19 The action of a finite group on a manifold has finite orbit type. More interestingly, the action of a compact Lie group on a manifold whose $\mathbb{Z}$-coefficient homology groups are finitely generated has finite orbit type. This is Mann's Theorem, see [11], Section IV.10.

Definition 8.20 Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a strongly proper morphism. A bivariant class $\theta \in H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$, not necessarily homogeneous, is called a **strong orientation** if for every $g \colon \mathfrak{Z} \to \mathfrak{X}$, multiplication by θ induces an isomorphism $H(\mathfrak{Z} \xrightarrow{g} \mathfrak{X}) \xrightarrow{\sim} H(\mathfrak{Z} \xrightarrow{f \circ g} \mathfrak{Y})$.

The above definition can be made for every adequate morphism in a (generalized) bivariant theory. However we do not need this generality.

Definition 8.21 A strongly proper morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is called **strongly oriented**, if it is normally nonsingular and it is endowed with a strong orientation $\theta_f \in H^c(f)$, where $c = \operatorname{codim} f$; see Definition 8.15. A topological stack $\mathfrak{X}$ is called strongly oriented if the diagonal $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is strongly oriented. In this case, we define $\dim \mathfrak{X} := \operatorname{codim} \Delta$.

Remark 8.22 As we have avoided the discussion of 2-vector bundles in this paper, we will not give an intrinsic definition of orientation in terms of the tangent 2-vector bundle of a differentiable stack $\mathfrak{X}$. However, we point out that an orientation for $\mathfrak{X}$ (in the bivariant sense) amounts to an orientation for the "tangent complex" of $\mathfrak{X}$, by which we (rather imprecisely) mean the anchor map $E \overset{\rho}{\to} TX$ of the Lie algebroid associated to a Lie groupoid presentation $[X_1 \rightrightarrows X]$ for $\mathfrak{X}$; see Example 3.4. (By an orientation on a complex $V_1 \to V_0$ of vector bundles on a manifold we simply mean an orientation on $V_1 \oplus V_0$. Note that this sloppy definition works because we are using singular (co)homology.) Conversely, assuming that $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ has a normally nonsingular diagram as in Definition 8.15, an orientation for $E \overset{\rho}{\to} TX$ gives rise to an orientation for Δ (hence, by definition, for $\mathfrak{X}$).

Lemma 8.23 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be strongly proper morphisms, and let $\theta \in H(f)$ and $\psi \in H(g)$ be strong orientation classes. Then, $\theta \cdot \psi$ is a strong orientation class for $g \circ f\colon \mathfrak{X} \to \mathfrak{Z}$. (Note that $g \circ f$ is strongly proper by Lemma 6.4.)*

Lemma 8.24 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a strongly proper map and $\theta \in H(f)$ a strong orientation class for it. Then multiplication by θ induces an isomorphism $H(\mathfrak{X}) \xrightarrow{\sim} H(f)$. If $\theta' \in H(f)$ is another orientation class for f, then there is a unique unit $u \in H(\mathfrak{X})$ such that $\theta' = u \cdot \theta$.*

The following result states that an oriented normally nonsingular diagram gives rise a canonical strong orientation.

Proposition 8.25 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a strongly proper morphism of topological stacks equipped with an oriented normally nonsingular diagram. Then, f has a canonical strong orientation class $\theta_f \in H^c(f)$, where $c = \operatorname{codim} f$.*

PROOF. The proof is essentially the same as the one given in [32]. (The 'strongly proper' assumption is a technical condition we need to impose on f in order to be able to multiply bivariant classes. This does not come up in [32] as they only use trivial vector bundles when defining bivariant classes and the decent condition of Definition 6.2 is automatic in this case.) $\square$

Example 8.26 (Euler class) Let $\mathfrak{X}$ be a topological stack and $\mathfrak{E}$ an oriented vector bundle over $\mathfrak{X}$. Let $s\colon \mathfrak{X} \to \mathfrak{E}$ be the zero section. As we saw in Example 8.16, s is a strongly proper morphism equipped with a natural nns diagram.

It follows from Proposition 8.25 that s has a canonical strong orientation class $\theta \in H^n(s)$, where $n = \operatorname{rk} \mathfrak{E}$. Consider the following 2-cartesian diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathfrak{X} \arrow[d, "s"] \\ \mathfrak{X} \arrow[r, "s"'] & \mathfrak{E} \end{tikzcd}$$

The pullback $s^*(\theta) \in H^n(\mathrm{id}_{\mathfrak{X}}) = H^n(\mathfrak{X})$ is the *Euler class* of $\mathfrak{E}$.

Lemma 8.27 *Let $i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}$ be an embedding of codimension n of differentiable stacks. If i is nns then it is a regular embedding (Definition 8.11). In fact, for any choice of orientation $\mu \in H^n(\mathfrak{N}, \mathfrak{N} - \mathfrak{X})$ for the normal bundle $\mathfrak{N} = \mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}$, the canonical strong orientation class $\theta_i \in H^n(i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}) \cong H^n(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ (Proposition 8.25) coincides with the compatible class $\bar{\mu} \in H^n(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ (Definition 8.8).*

Proof. Pick an nns diagram

$$\begin{tikzcd} \mathfrak{N}' \arrow[r, hook, "j"] & \mathfrak{E} \arrow[d, "p"] \\ \mathfrak{X} \arrow[u, "s"] \arrow[r, "i"'] & \mathfrak{Y} \end{tikzcd}$$

for i. We have a natural isomorphism $\mathfrak{N}' \cong \mathfrak{N}_{\mathfrak{X}/\mathfrak{E}}$. This gives rise to a split short exact sequence

$$0 \to \mathfrak{N} \to \mathfrak{N}' \to \mathfrak{E}|_{\mathfrak{X}} \to 0$$

of vector bundles over $\mathfrak{X}$. Recall that, by definition, $\mathfrak{E}$ is oriented. By regarding $\mathfrak{N}'$ as a vector bundle over $\mathfrak{N}$ which is the pullback of $\mathfrak{E}|_{\mathfrak{X}}$ along the base map $\mathfrak{N} \to \mathfrak{X}$, we obtain isomorphisms

$$H^\bullet(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X}) \cong H^{\bullet+r}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \cong H^{\bullet+r}(\mathfrak{E}, \mathfrak{E} - js(\mathfrak{X})),$$

where $r = \operatorname{rk} E$. In the last equality we have used Lemma 4.7. Also, we have isomorphisms

$$H^\bullet(\mathfrak{N}, \mathfrak{N} - \mathfrak{X}) \cong H^{\bullet+r}(\mathfrak{N}', \mathfrak{N}' - \mathfrak{X}) \cong H^{\bullet+r}(\mathfrak{E}, \mathfrak{E} - js(\mathfrak{X})),$$

where in the last equality we have used excision because $\mathfrak{N}'$ can be identified with an open substack of $\mathfrak{E}$ via j. Under the above identifications, the orientation class $\mu \in H^n(\mathfrak{N}, \mathfrak{N} - \mathfrak{X})$ corresponds to its compatible class $\bar{\mu} \in H^n(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$ and the strong orientation class $\theta_i \in H^n(i\colon \mathfrak{X} \hookrightarrow \mathfrak{Y}) = H^{n+r}(\mathfrak{E}, \mathfrak{E} - js(\mathfrak{X}))$. (This latter equality is the very definition of bivariant cohomology.) $\square$

The following proposition shows that any morphism between strongly oriented topological stacks has a natural strong orientation. Proposition 8.30 shows that this class is multiplicative.

Proposition 8.28 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a strongly proper normally nonsingular morphism of topological stacks, and assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are both strongly oriented (Definition 8.21). Let $d = \dim \mathfrak{X}$ and $c = \dim \mathfrak{Y} - \dim \mathfrak{X}$. Then, there is a unique strong orientation class $\theta_f \in H^c(f)$ which satisfies the equality $\theta_f \cdot \theta_{\mathfrak{Y}} = (-1)^{cd}\theta_{\mathfrak{X}} \cdot (\theta_f \times \theta_f)$, as in the diagram*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\theta_f", "f"'] \arrow[d, "\theta_{\mathfrak{X}}"', "\Delta"] & \mathfrak{Y} \arrow[d, "\Delta"', "\theta_{\mathfrak{Y}}"] \\
\mathfrak{X} \times \mathfrak{X} \arrow[r, "f \times f", "\theta_f \times \theta_f"'] & \mathfrak{Y} \times \mathfrak{Y}
\end{tikzcd}$$

PROOF. By Proposition 8.25, there exists a strong orientation θ for f. It is easy to see that $\theta \times \theta$ is a strong orientation for $f \times f\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y} \times \mathfrak{Y}$. By Lemma 8.23, $\theta \cdot \theta_{\mathfrak{Y}}$ and $\theta_{\mathfrak{X}} \cdot (\theta \times \theta)$ are both strong orientation classes for $\mathfrak{X} \to \mathfrak{Y} \times \mathfrak{Y}$. Therefore, by Lemma 8.24, there is a unit $u \in H^0(\mathfrak{X})$ such that $\theta \cdot \theta_{\mathfrak{Y}} = u \cdot \theta_{\mathfrak{X}} \cdot (\theta \times \theta)$. It follows that $\theta_f \colon = (-1)^{cd} u \cdot \theta$ has the desired property; see Lemma 8.29 below. $\square$

Lemma 8.29 *Let $\mathfrak{X}$ be a topological stack and $\theta \in H(\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X})$. Let $u, v \in H^0(\mathfrak{X})$, and let $u \times v \in H^0(\mathfrak{X}) \times H^0(\mathfrak{X})$ be their exterior product. Then, $\theta \cdot (u \times v) = u \cdot v \cdot \theta$, as classes in $H(\Delta)$.*

PROOF. Since $u \times v = (u \times 1) \cdot (1 \times v)$, it is enough to prove the statement in the case where $v = 1$. Recall that $u \times 1$ is defined via independent pullback in the righthand square in the diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\theta", "\Delta"'] \arrow[d, "u"', "\mathrm{id}"] & \mathfrak{X} \times \mathfrak{X} \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "u \times 1"', "\mathrm{id}"] & \mathfrak{X} \arrow[d, "\mathrm{id}"', "u"] \\
\mathfrak{X} \arrow[r, "\Delta", "\theta"'] & \mathfrak{X} \times \mathfrak{X} \arrow[r, "\mathrm{pr}_2"] & \mathfrak{X}
\end{tikzcd}$$

The equality follows from the skew commutativity of the bivariant theory applied to the lefthand square. $\square$

Proposition 8.30 *Assume $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ are strongly proper normally nonsingular morphisms of strongly oriented topological stacks. Let $\theta_f \in H^c(f)$, $c = \operatorname{codim} f$, and $\theta_g \in H^d(g)$, $d = \operatorname{codim} g$, be the strong orientations constructed in Proposition 8.28. Then, $g \circ f$ is a strongly proper normally nonsingular. Furthermore, $\theta_f \cdot \theta_g = \theta_{g \circ f}$.*

Proof. By Lemma 6.4, $g \circ f$ is strongly proper. Consider the normally nonsingular diagrams for f and g

$$\begin{tikzcd} \mathfrak{N} \arrow[r, hook, "i"] & \mathfrak{E} \arrow[d] & \mathfrak{M} \arrow[r, hook, "j"] & \mathfrak{F} \arrow[d] \\ \mathfrak{X} \arrow[u] \arrow[r, "f"'] & \mathfrak{Y} & \mathfrak{Y} \arrow[u] \arrow[r, "g"'] & \mathfrak{Z} \end{tikzcd}$$

By adding a vector bundle to $\mathfrak{E}$, we may assume that $\mathfrak{E} = g^*(\mathfrak{E}')$ for some orientable vector bundle $\mathfrak{E}'$ over $\mathfrak{Z}$. The following is a normally nonsingular diagram for $g \circ f$

$$\begin{tikzcd} f^*\mathfrak{M} \oplus \mathfrak{N} \arrow[r, hook, "k"] & \mathfrak{F} \oplus \mathfrak{E}' \arrow[d] \\ \mathfrak{X} \arrow[u] \arrow[r, "g \circ f"'] & \mathfrak{Z} \end{tikzcd}$$

where k is the composite

$$f^*\mathfrak{M} \oplus \mathfrak{N} \xhookrightarrow{(\mathrm{pr}, i)} \mathfrak{M} \oplus \mathfrak{E} \xhookrightarrow{(j, \mathrm{pr})} \mathfrak{F} \oplus \mathfrak{E}'.$$

This proves that $g \circ f$ is normally nonsingular.

The equality $\theta_f \cdot \theta_g = \theta_{g \circ f}$ follows from the identity in Proposition 8.28. $\square$

If $\mathfrak{X}$ is strongly oriented (Definition 8.21), its iterated diagonals $\Delta^{(n)} \colon \mathfrak{X} \to \mathfrak{X}^n$ are strongly proper (Example 6.3).

Corollary 8.31 *Let $\mathfrak{X}$ be a oriented stack. Then the diagonals $\Delta^{(n)} \colon \mathfrak{X} \to \mathfrak{X}^n$ are canonically strongly oriented.*

Proposition 8.32 *Notation being as in Proposition 8.18, assume further that X and Y are oriented and that the G-actions are orientation preserving. Then, every normally nonsingular diagram for $f \colon \mathfrak{X} \to \mathfrak{Y}$ is naturally oriented. In particular, when f is strongly proper, we have a strong orientation class $\theta_f \in H^c(f)$, $c = \dim Y - \dim X$. Furthermore, this class is independent of the choice of the normally nonsingular diagram.*

Proof. Let us first fix a notation: given a manifold X with an action of G, we denote $[TX/G]$ by $T\mathfrak{X}$. (So, $T\mathfrak{X}$ does depend on X, and not just on $\mathfrak{X}$. Since in what follows all stacks are quotients of a G-action on a given manifold, this should not cause confusion.)

Consider a normally nonsingular diagram

$$\begin{tikzcd} \mathfrak{N} \arrow[r, hook, "i"] & \mathfrak{E} \arrow[d, "p"] \\ \mathfrak{X} \arrow[u, "s"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

as in the proof of Proposition 8.18. We show that $\mathfrak{N}$ is naturally oriented. By Lemma 4.9, there is a natural orientation on $T\mathfrak{E}$, because it fits in the following short exact sequence

$$0 \to p^*\mathfrak{E} \to T\mathfrak{E} \to p^*T\mathfrak{X} \to 0.$$

In particular, we have an orientation on $f^*(T\mathfrak{E})$. We have an isomorphism of vector bundles over $\mathfrak{X}$

$$T\mathfrak{X} \oplus \mathfrak{N} \cong f^*(T\mathfrak{E}).$$

It now follows from Lemma 4.8 that $\mathfrak{N}$ also carries a natural orientation. This proves the first part of the proposition. In particular, when f is proper, we obtain a class $\theta_f \in H^c(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ as in Proposition 8.25.

Now, we show that the class θ_f is independent of the normally nonsingular diagram above. Consider another oriented normally nonsingular diagram for f

$$\begin{tikzcd} \mathfrak{M} \arrow[r, hook, "j"] & \mathfrak{F} \arrow[d, "q"] \\ \mathfrak{X} \arrow[u, "t"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

We have to show that the following diagram commutes

$$\begin{tikzcd} H^\bullet(\mathfrak{X}) \arrow[r, "\cong"] \arrow[d, "="'] & H^{\bullet+\operatorname{rk}\mathfrak{N}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \arrow[d, "\cong"] \\ H^\bullet(\mathfrak{X}) \arrow[r, "\cong"] & H^{\bullet+\operatorname{rk}\mathfrak{M}}(\mathfrak{F}, \mathfrak{F} - \mathfrak{X}) \end{tikzcd}$$

where the horizontal isomorphisms are the one of Proposition 8.25, and the vertical isomorphism is the one of Lemma 6.5. First we prove a special case.

Special case. Assume $\mathfrak{E} = \mathfrak{F}$, and $is = tj$. In this case, we can choose a third vector bundle $\mathfrak{L} \to \mathfrak{X}$ and an open embedding $k\colon \mathfrak{L} \hookrightarrow \mathfrak{E}$ that factors through both $\mathfrak{N}$ and $\mathfrak{M}$. The two orientations induced on $\mathfrak{L}$ from $\mathfrak{M}$ and $\mathfrak{N}$, as in Lemma 4.11, are the same (because they are equal to the orientation induced from $\mathfrak{E}$, as described above). The claim now follows from the commutative diagram of Lemma 4.11 (applied once to the open embedding $\mathfrak{L} \hookrightarrow \mathfrak{N}$ and once to the open embedding $\mathfrak{L} \hookrightarrow \mathfrak{M}$).

General case. To prove the general case, we make use of the following auxiliary oriented nonsingular diagrams:

$$\begin{tikzcd} \mathfrak{N} \oplus f^*\mathfrak{F} \arrow[r, hook, "{(i,\mathrm{pr})}"] & \mathfrak{E} \oplus \mathfrak{F} \arrow[d] \\ \mathfrak{X} \arrow[u, "{(s,jt)}"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{M} \oplus f^*\mathfrak{E} \arrow[r, hook, "{(j,\mathrm{pr})}"] & \mathfrak{F} \oplus \mathfrak{E} \arrow[d] \\ \mathfrak{X} \arrow[u, "{(t,is)}"] \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

Here, the two maps pr stand for the projection maps $f^*\mathfrak{F} = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{F} \to \mathfrak{F}$ and $f^*\mathfrak{E} = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{E} \to \mathfrak{E}$. Let us denote the ranks of $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{N}$, and $\mathfrak{M}$ by e, f, n, and m. (Hopefully, presence of two different f in the notation will not cause confusion!) The first normally nonsingular diagram gives rise to the following commutative diagram of isomorphisms:

$$\begin{tikzcd}
H^\bullet(\mathfrak{X}) \arrow[r, "\cong"] \arrow[rr, bend left, "\varphi"] \arrow[d, "="'] & H^{\bullet+n+f}(\mathfrak{N} \oplus f^*\mathfrak{F}, \mathfrak{N} \oplus f^*\mathfrak{F} - \mathfrak{X}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H^{\bullet+n+f}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X}) \arrow[d, "\cong"] \\
H^\bullet(\mathfrak{X}) \arrow[r, "\cong"] & H^{\bullet+n}(\mathfrak{N}, \mathfrak{N} - \mathfrak{X}) \arrow[r, "\cong"] & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})
\end{tikzcd}$$

The commutativity of the left square is because of Lemma 4.10, and the commutativity of the right square is because Thom isomorphism (vertical) commutes with excision (horizontal).

Similarly, the second normally nonsingular diagram gives rise to the following commutative diagram of isomorphisms

$$\begin{tikzcd}
H^\bullet(\mathfrak{X}) \arrow[r, "\cong"] \arrow[rr, bend left, "\psi"] \arrow[d, "="'] & H^{\bullet+m+e}(\mathfrak{M} \oplus f^*\mathfrak{E}, \mathfrak{M} \oplus f^*\mathfrak{E} - \mathfrak{X}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H^{\bullet+m+e}(\mathfrak{F} \oplus \mathfrak{E}, \mathfrak{F} \oplus \mathfrak{E} - \mathfrak{X}) \arrow[d, "\cong"] \\
H^\bullet(\mathfrak{X}) \arrow[r, "\cong"] & H^{\bullet+m}(\mathfrak{M}, \mathfrak{M} - \mathfrak{X}) \arrow[r, "\cong"] & H^{\bullet+m}(\mathfrak{F}, \mathfrak{F} - \mathfrak{X})
\end{tikzcd}$$

On the other hand, using the special case that we just proved, the two normally nonsingular diagrams give rise to the following commutative diagram:

$$\begin{tikzcd}
H^\bullet(\mathfrak{X}) \arrow[r, "\varphi", "\cong"'] \arrow[d, "="'] & H^{\bullet+n+f}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X}) \arrow[d, "="] \\
H^\bullet(\mathfrak{X}) \arrow[r, "\psi", "\cong"'] & H^{\bullet+m+e}(\mathfrak{F} \oplus \mathfrak{E}, \mathfrak{F} \oplus \mathfrak{E} - \mathfrak{X})
\end{tikzcd}$$

The general case now follows from combining this diagram with (the other rectangles) of the previous two diagrams. $\square$

Corollary 8.33 *Let $\mathfrak{X}$ be a stack that is equivalent to the quotient stack $[X/G]$ of smooth orientation preserving action of a compact Lie group G on a smooth oriented manifold X having finitely generated homology groups. Then, the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is naturally oriented. In particular, the diagonal of the classifying stack BG of a compact Lie group G is naturally oriented.*

Remark 8.34 Let $\mathfrak{X}$, $\mathfrak{Y}$ and f be as in Proposition 8.32. There are two ways of giving a strong orientation to f. Either we can use Proposition 8.32 directly, or we first apply Corollary 8.33 to endow $\mathfrak{X}$ and $\mathfrak{Y}$ with a strong orientation, and then apply Proposition 8.28. The orientations we get are the same for f. We denote θ_f this strong orientation.

Proposition 8.35 *Let $\mathfrak{X}$ be a paracompact orbifold whose tangent bundle (Example 3.4) is oriented. Then the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is strongly oriented and in particular, $\mathfrak{X}$ is naturally oriented.*

Proof. Locally, we can find a tubular neighborhood for the diagonal. The result follows using partition of unity. $\square$

9 Gysin maps

As in [17, 23, 25], the main step in our construction of the **BV**-structure on the homology of the loop stack is the systematic development of Gysin maps for oriented morphisms of stacks. To this end, we use (a slightly generalized version of) Fulton-MacPherson's bivariant theory. This is in spirit very close to Chataur's bordism approach which relies on Jakob's bivariant theory for differentiable manifolds [17], although bivariant theories are not explicitly mentioned in [17].

9.1 Construction of the Gysin maps

We recall the construction of Gysin homomorphisms associated to a bivariant class [32].

Fix an element $\theta \in H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$. Let $u : \mathfrak{Y}' \to \mathfrak{Y}$ be an arbitrary morphism of topological stacks and $\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ the base change given by the cartesian square:

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow u \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array} \tag{9.1}$$

Then θ determines **Gysin homomorphisms**

$$\theta^! \colon H_j(\mathfrak{Y}') \to H_{j-i}(\mathfrak{X}')$$

and

$$\theta_! \colon H^j(\mathfrak{X}') \to H^{j+i}(\mathfrak{Y}').$$

For the cohomology Gysin map, we need to assume that f' is adequate. These homomorphisms are defined by

$$\theta^!(a) = \big(u^*(\theta)\big) \cdot a, \quad \text{for} \;\; a \in H_j(\mathfrak{Y}') = H^{-j}(\mathfrak{Y}' \to pt),$$

and

$$\theta_!(b) = f'_*\big(b \cdot u^*(\theta)\big), \quad \text{for } \ b \in H^j(\mathfrak{X}') = H^j(\mathfrak{X}' \xrightarrow{\text{id}} \mathfrak{X}').$$

The homology Gysin map is defined because the map $\mathfrak{X}' \to *$ is adequate (see Example 7.6).

9.2 Standard Properties of Gysin maps

By Proposition 8.25, when the map $f\colon \mathfrak{X} \to \mathfrak{Y}$ in Diagram (9.1) is strongly oriented, it has a canonical strong orientation θ_f. In this case, we have a canonical Gysin morphism

$$f^! := (\theta_f)^!\colon H_\bullet(\mathfrak{Y}') \to H_{\bullet-c}(\mathfrak{X}'),$$

where c is the codimension of f. In this subsection we collect some of the standard properties of these Gysin morphisms.

1. *Functoriality.* Assume given a commutative diagram of cartesian squares

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{Y}' \arrow[r] \arrow[d] & \mathfrak{Z}' \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \arrow[r, "g"] & \mathfrak{Z} \end{tikzcd} \tag{9.2}$$

   with $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ strongly oriented of codimensions c and d, respectively. Then, the induced Gysin morphisms $f^!\colon H_\bullet(\mathfrak{Y}') \to H_{\bullet-c}(\mathfrak{X}')$ and $g^!\colon H_\bullet(\mathfrak{Z}') \to H_{\bullet-d}(\mathfrak{Y}')$ satisfy the functoriality identity

$$(g \circ f)^! = f^! \circ g^!.$$

2. *Naturality.* Assume given a commutative diagram of cartesian squares

$$\begin{tikzcd} \mathfrak{X}'' \arrow[r] \arrow[d, "v"'] & \mathfrak{Y}'' \arrow[d, "u"] \\ \mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{Y}' \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd} \tag{9.3}$$

   with f strongly oriented. Then, the induced Gysin morphisms satisfy

$$v_* \circ f^! = f^! \circ u_*.$$

3. *Commutation with cross product.* Given two cartesian squares

$$\begin{array}{ccc} \mathfrak{X}'_1 & \longrightarrow & \mathfrak{Y}'_1 \\ \downarrow & & \downarrow{\scriptstyle u_1} \\ \mathfrak{X}_1 & \xrightarrow{f_1} & \mathfrak{Y}_1 \end{array} \qquad \begin{array}{ccc} \mathfrak{X}'_2 & \longrightarrow & \mathfrak{Y}'_2 \\ \downarrow & & \downarrow{\scriptstyle u_2} \\ \mathfrak{X}_2 & \xrightarrow{f_2} & \mathfrak{Y}_2 \end{array}$$

consider the induced product square

$$\begin{array}{ccc} \mathfrak{X}'_1 \times \mathfrak{X}'_2 & \longrightarrow & \mathfrak{Y}'_1 \times \mathfrak{Y}'_2 \\ \downarrow & & \downarrow{\scriptstyle u_1 \times u_2} \\ \mathfrak{X}_1 \times \mathfrak{X}_2 & \xrightarrow{f_1 \times f_2} & \mathfrak{Y}_1 \times \mathfrak{Y}_2 \end{array} \tag{9.4}$$

If f_1 and f_2 are strongly oriented, then so is $f_1 \times f_2$. Moreover, the three Gysin morphisms satisfy the equation

$$(f_1 \times f_2)^!(- \times -) = {f_1}^!(-) \times {f_2}^!(-).$$

4. *Commutation with pullback.* Given a cartesian square

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow{\scriptstyle u} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array} \tag{9.5}$$

with f strongly oriented and $y' \in H^\bullet(\mathfrak{Y}')$, we have

$$f^!(y' \cap -) = (-1)^{\deg(y')\,\mathrm{codim}(f)} f'^*(y') \cap f^!(-).$$

Proof of Properties 1,2,3 and 4

Everything follows from the axioms of a bivariant theory. By Proposition 8.30, the products of two strongly oriented maps is canonically strongly oriented. Thus Property 1 follows from Axiom **A13**. Property 2 is Axiom **A3** followed by Axiom **A123**. Taking direct products of vector bundles shows that strongly proper and normally nonsingular morphisms are stable by products. Hence Property 3 follows from the definition and naturality of the cross product (Proposition 7.8). Property 4 is a consequence of the skew-commutativity.

Remark 9.1 (Cohomology Gysin maps) When in Diagram (9.1) f' is adequate, there is an induced cohomology Gysin map $f^!\colon H^\bullet(\mathfrak{X}') \to H^{\bullet+c}(\mathfrak{Y}')$. Properties 1,2,3 and 4 above have obvious analogs in cohomology when all the relevant maps involved are adequate. Recall that a strongly oriented map is strongly proper hence adequate.

Remark 9.2 We have emphasized the case of strongly oriented maps for simplicity and because it is sufficient for our purpose. Nevertheless, by pullback axiom, any bivariant class $\theta \in H^r(f)$ yields a bivariant class $u^*(\theta)\colon H^r(f')$ and thus a Gysin map $H_\bullet(\mathfrak{Y}') \to H_{\bullet-r}(\mathfrak{X}')$. Properties 1,2,3 and 4 above will hold true in this more general setting.

9.3 A special case: G-equivariant Gysin maps

Let M, N be oriented compact manifolds and G a Lie group acting on M, N by orientation preserving diffeomorphisms. By Proposition 8.32, if $f\colon M \to N$ is a G-equivariant map, then f is canonically strongly oriented. Gysin maps for equivariant (co)homology were already considered, for example, by Atiyah and Bott [4].

Proposition 9.3 *The Gysin maps $f_!, f^!$ associated to f in (co)homology coincide with the equivariant Gysin maps in the sense of Atiyah and Bott [4].*

Proof. Gysin map in [4] are obtained by the use of fiber integration and Thom classes over the spaces $M_G = M \times_G EG$ and $N_G = N \times_G EG$. These spaces are respectively classifying spaces of the stacks $[M/G]$ and $[N/G]$ and thus are respectively the pullbacks $[M/G] \times_{[*/G]} BG$, $[N/G] \times_{[*/G]} BG$. The pullback of the normally nonsingular diagram of Proposition 8.18 along the natural maps $[M/G] \times_{[*/G]} BG \to [M/G]$ and $[N/G] \times_{[*/G]} BG \to [N/G]$ yields a bundle $\mathfrak{N}_G = \mathfrak{N} \times_{[*/G]} BG$ over M_G and a bundle $\mathfrak{E}_G = \mathfrak{E} \times_{[*/G]} BG$ over N_G. This defines a nonsingular diagram for the induced map $f_G\colon M_G \to N_G$. Unfolding the definition of bivariant classes, it is straightforward to check that the Gysin map associated to the strong orientation class of Proposition 8.32 is induced by the Thom isomorphism associated to the bundle $\mathfrak{N}_G$ over M_G. $\square$

Let G be a subgroup of a finite (discrete) group H. Let Y be a manifold endowed with a (right) H-action (and thus a G-action). Consider the quotient stacks $[Y/G]$ and $[Y/H]$. There are well known "transfer maps" $\mathrm{tr}_H^G\colon H_\bullet^H(Y) \to H_\bullet^G(Y)$ (see [9])

Lemma 9.4 *When G is a finite group, the Gysin map associated to the cartesian square*

$$\begin{array}{ccc} [Y/G] & \longrightarrow & [Y/H] \\ \downarrow & & \downarrow \\ [*/G] & \longrightarrow & [*/H] \end{array}$$

where the lower map is induced by the inclusion $G \hookrightarrow H$, is the usual "transfer map" $H_\bullet^H(Y) \to H_\bullet^G(Y)$ in equivariant homology.

Proof. The space $Y \times H$ is endowed with a natural right H-action given by $(y,h).k = (y.k, k^{-1}h)$ as well as a right G-action $(y,h).g = (y,hg)$. These two actions commutes hence we can form the quotient stack $[Y \times H/H \times G] \cong [Y \times (H/G)/H]$. Clearly the map $(y,h) \mapsto yh$ is equivariant with respect to the G action on the target and $H \times G$-action on the source. One easily checks that this map induces an equivalence $[Y \times (H/G)/H] \cong [Y/G]$. We are thus left to study the Gysin map of an equivariant covering with fibers the set H/G. The argument of Proposition 9.3 easily shows that it coincides with the usual transfer maps for coverings by a finite group and thus with the transfer. $\square$

Assuming we take coefficient in a field of characteristic coprime with $|H|$ for the singular homology, we have

$$H_\bullet([Y/H]) \cong H_\bullet^H(Y) \cong \big(H_\bullet(Y)\big)_H.$$

In that case, the map $\mathrm{tr}_H^G \colon \big(H_\bullet(Y)\big)_H \to \big(H_\bullet(Y)\big)_G$ is explicitly given by

$$\mathrm{tr}_H^G(x) \quad = \sum_{h \in H/G} h.x. \tag{9.6}$$

9.4 The excess formula

The main result of this subsection is the following.

Proposition 9.5 (Excess formula) *Consider the 2-cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathfrak{Y}' \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, hook, "i"'] & \mathfrak{Y} \end{tikzcd}$$

in which i and j are regular embeddings with normal bundles $\mathfrak{N}$, $\mathfrak{N}'$, respectively. Let $\mathfrak{E} = p^*(\mathfrak{N})/\mathfrak{N}'$ be the excess bundle (see Section 3.2). Fix orientations on $\mathfrak{N}$ and $\mathfrak{N}'$ and endow $\mathfrak{E}$ with the induced orientation as in Lemma 4.9. (In the case where $\mathfrak{N}$ and $\mathfrak{N}'$ have equal ranks the orientation on $\mathfrak{N}'$ is uniquely determined by the one on $\mathfrak{N}$.) Let $\theta_i \in H^n(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X}) = H^n(i)$ and $\theta_j \in H^n(\mathfrak{Y}', \mathfrak{Y}' - \mathfrak{X}') = H^{n'}(j)$ be the classes compatible with the orientations on $\mathfrak{N}$ and $\mathfrak{N}'$, respectively (Definition 8.8). Then,

$$q^*(\theta_i) = e(\mathfrak{E}) \cdot \theta_j, \tag{9.7}$$

where $e(\mathfrak{E}) \in H^\bullet(\mathfrak{X}')$ is the Euler class of $\mathfrak{E}$ (see Example 8.26).

Proof. In the case where q is a submersion, the proposition follows from Lemmas 8.4 and 8.14. We use this to reduce the problem to the case of manifolds.

By Lemmas 8.4, 8.10, and 8.14, and the fact that bivariant product commutes with pullback, it is enough to prove the formula after passing to an arbitrary atlas $Y' \to \mathfrak{Y}'$. So, we may assume that $\mathfrak{Y}' =: Y'$ and $\mathfrak{X}' =: X'$ are smooth manifolds. By Lemma 8.9, we can find an atlas $Y \to \mathfrak{Y}$ through which $q\colon Y' \to \mathfrak{Y}$ factors. Since the atlas $Y \to \mathfrak{Y}$ is a submersion and the proposition is true for submersions, we may assume, after pulling back everything along $Y \to \mathfrak{Y}$, that $\mathfrak{Y} =: Y$ and $\mathfrak{X} =: X$ are also smooth manifolds. From now on, we use the notation N, N' and E instead of $\mathfrak{N}$, $\mathfrak{N}'$ and $\mathfrak{E}$.

We are reduced to proving the result in the case of manifolds. Since $q\colon Y' \to Y$ factors as the composition

$$Y' \xrightarrow{(1,q)} Y' \times Y \longrightarrow Y$$

of an embedding and a submersion, it is enough, by functoriality of pullbacks, to consider the cases where q is a submersion and q is an embedding separately. The former case is easy, as $\mathfrak{E}$ is the zero bundle and $q^*(\theta_i) = \theta_j$ by Lemma 8.14.

It remains to prove the proposition in the case where q is an embedding. By choosing appropriate tubular neighborhoods, we reduce to the case where $Y = F$ is a vector bundle over X and $i\colon X \to F$ is the zero section. We may also assume that X' is a submanifold of X and $Y' = N'$ is a vector bundle over X' which is a subbundle of $F|_{X'}$, having zero section $j\colon X' \to N'$. Moreover, after choosing a metric on F, we may write $F|_{X'} = E \oplus N'$. Finally, replacing F with $F|_{X'}$, we may assume that $X = X'$. Summarizing all the reduction we have made, we are in a situation where we have a manifold X, with vector bundles N' and E on it, so that the 2-cartesian square of the proposition has the form

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "p=\mathrm{id}"'] & N' \arrow[d, "q"] \\ X \arrow[r, hook, "i"'] & E \oplus N' \end{tikzcd}$$

Here, the horizontal maps are zero sections and q is the inclusion of the summand N'. We expand this square to the 2-cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, hook, "j"] \arrow[d, "s"'] & N' \arrow[d, "q"] \\ X \arrow[r, hook, "s"] \arrow[rr, bend right, "i"'] & E \arrow[r, hook] & E \oplus N' \end{tikzcd}$$

where s stands for the zero section. Let θ_E be the strong orientation class of $X \hookrightarrow E$, and θ' the strong orientation class of $E \hookrightarrow E \oplus N'$. By Lemma 8.13, we have $\theta_i = \theta_E \cdot \theta'$. Then, since pullback respects bivariant product, and $s^*(\theta_E) = e(E)$ (Example 8.26), we find

$$q^*(\theta_i) = e(E) \cdot q^*(\theta').$$

Making the rightmost square in the above diagram upside down and using the obvious projection maps, as in the diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & E \oplus N' \arrow[d, "\pi"] \\ X \arrow[r, hook, "j"] & N' \end{tikzcd}$$

we see that $E \hookrightarrow E \oplus N'$ has $\pi^*(\theta_j)$ as its canonical strong orientation, that is $\theta' = \pi^*(\theta_j)$. Hence, $q^*(\theta') = \theta_j$, and the above displayed formula becomes $q^*(\theta_i) = e(E) \cdot \theta_j$, which is the desired excess formula. $\square$

It is worth noticing that when i and j are nns, then, by Lemma 8.27, the classes $\theta_i \in H^n(i\colon \mathfrak{X} \to \mathfrak{Y})$ and $\theta_j \in H^n(j\colon \mathfrak{X}' \to \mathfrak{Y}')$ are precisely the canonical strong orientations constructed in Proposition 8.25.

The following immediate corollary of Proposition 9.5 is useful in computing Gysin maps.

Corollary 9.6 *Consider the 2-cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}'' \arrow[r] \arrow[d, "u"'] & \mathfrak{Y}'' \arrow[d] \\ \mathfrak{X}' \arrow[r, "j"] \arrow[d, "p"'] & \mathfrak{Y}' \arrow[d, "q"] \\ \mathfrak{X} \arrow[r, "i"'] & \mathfrak{Y} \end{tikzcd}$$

in which the lower square is as in Proposition 9.5. Let n and n' be the ranks of $\mathfrak{N}$ and $\mathfrak{N}'$ respectively. Let

$$i^!\colon H_\bullet(\mathfrak{Y}'') \to H_{\bullet-n}(\mathfrak{X}''),$$

$$j^!\colon H_\bullet(\mathfrak{Y}'') \to H_{\bullet-n'}(\mathfrak{X}'')$$

be the corresponding Gysin maps. Then, for any $c \in H_\bullet(\mathfrak{Y}'')$ we have the equality

$$i^!(c) = u^* e(\mathfrak{E}) \cdot j^!(c).$$

In particular, if q is transversal to i (e.g., when q is a submersion), then $i^! = j^!$.

10 The loop product

In this section we consider (Hurewicz) *strongly oriented* stacks (Definition 8.21). We obtain a loop product on the homology of the free loop stack of an oriented stack which generalizes Chas-Sullivan product for the homology of a loop manifolds [15]. Recall that a stack $\mathfrak{X}$ is called strongly oriented if the diagonal $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ has a strong orientation class (Definition 8.20). For instance, oriented manifolds and oriented orbifolds are oriented stacks. More generally, the quotient stack of a compact Lie group acting by orientation preserving automorphisms on an oriented manifold is an oriented stack.

Note that it is possible to have two different group actions, say a Lie group G acting on a manifold X and another Lie group H acting on another manifold Y, which give rise to the same quotient stacks, i.e., $[X/G] \cong [Y/H]$. By definition, our notion of orientation, as well as our construction of the loop product (and all other string operations that we construct), are independent of the choice of the presentation and only depend on the resulting quotient stack. Put differently, and slightly more generally, what we do is we use a Morita invariant notion of orientation for Lie groupoids, and for such oriented Lie groupoids we construct Morita invariant string operations.

10.1 Construction of the loop product

Let $\mathfrak{X}$ be a Hurewicz oriented stack of finite dimension d. The construction of the loop product

$$H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$$

is divided into 3 steps.

Step 1 There is a well-known external product (called the "cross product")

$$H_p(\mathrm{L}\mathfrak{X}) \otimes H_q(\mathrm{L}\mathfrak{X}) \xrightarrow{S} H_{p+q}(\mathrm{L}\mathfrak{X}).$$

Step 2 The diagonal $\Delta\colon \mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ and the evaluation map $\mathrm{ev}_0 : \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ (5.1) yield the cartesian square

$$\begin{array}{ccc} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_0)} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}. \end{array} \tag{10.1}$$

We will usually denote by $e : \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ the map $(\mathrm{ev}_0, \mathrm{ev}_0)$. Since $\mathfrak{X}$ is Hurewicz, Corollary 5.3 implies that there is a natural equivalence of stacks

$$\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \cong \mathrm{Map}(8, \mathfrak{X}),$$

where the figure "8" stands for the topological stack associated to the topological space $S^1 \vee S^1$. The wedge $S^1 \vee S^1$ is taken with respect to the basepoint 0 of S^1. Since $\mathfrak{X}$ is oriented, its diagonal $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is oriented normally nonsingular and according to Section 9.1, there is a Gysin map

$$\Delta^!\colon H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \cong H_{\bullet-d}(\mathrm{Map}(8, \mathfrak{X})).$$

Step 3 The map $S^1 \to S^1 \vee S^1$ that pinches $\frac{1}{2}$ to 0, induces a natural map of stacks $m : \mathrm{Map}(8, \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$, called the *Pontrjagin multiplication.* Hence we have an induced map on homology

$$m_*\colon H_\bullet(\mathrm{Map}(8, \mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X}).$$

We define the *loop product* to be the following composition

$$H_p(\mathrm{L}\mathfrak{X}) \otimes H_q(\mathrm{L}\mathfrak{X}) \overset{S}{\to} H_{p+q}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \overset{\Delta^!}{\to} H_{p+q-d}(\mathrm{Map}(8, \mathfrak{X})) \overset{m_*}{\to} H_{p+q-d}(\mathrm{L}\mathfrak{X}). \tag{10.2}$$

Theorem 10.1 *Let $\mathfrak{X}$ be an oriented (Hurewicz[2]) stack of dimension d. The loop product induces a structure of associative and graded commutative algebra for the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) := H_{\bullet+d}(\mathrm{L}\mathfrak{X})$.*

The loop product is of degree $d = \dim(\mathfrak{X})$ because the Gysin map involved in Step 2 is of degree d. If we denote $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) := H_{\bullet+\dim(\mathfrak{X})}(\mathrm{L}\mathfrak{X})$ the shifted homology groups, then the loop product induces a degree 0 multiplication $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$.

Indeed one can introduce a "twisted" version of loop product. Let α be a class in $\bigoplus_{r\geq 0} H^r(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$. The *twisted loop product*

$$\star_\alpha\colon H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$$

is defined, for all $x, y \in H_\bullet(\mathrm{L}\mathfrak{X})$,

$$x \star_\alpha y = m_*\big(\Delta^!(x \times y) \cap \alpha\big).$$

Remark 10.2 The twisted product $\star_\alpha$ is not graded since we do not assume α to be homogeneous. However, if $\alpha \in H^r(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$ is homogeneous of degree r, then $\star_e\colon H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d-r}(\mathrm{L}\mathfrak{X})$ is of degree $r + \dim(\mathfrak{X})$.

Let us introduce some notation. We denote, respectively,

$$p_{12}, p_{23}\colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$$

the projections on the first two and the last two factors. Also let

$$(m \times 1)\colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$$

and

$$(1 \times m)\colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$$

be the Pontrjagin multiplication of the two first factors and two last factors respectively. Furthermore, there are flip maps

$$\sigma\colon \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X},$$

$$\widetilde{\sigma}\colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$$

permuting the two factors of $\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}$.

[2]Recall that every differentiable stack is Hurewicz.

Theorem 10.3 *Let α be a class in $\bigoplus_{r\geq 0} H^r(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})$.*

1. *If α satisfies the 2-cocycle condition*

$$p_{12}^*(x)\cup(m\times 1)^*(\alpha) \quad = \quad p_{23}^*(\alpha)\cup(1\times m)^*(\alpha) \tag{10.3}$$

   in $H^\bullet(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})$, then $\star_e : H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X})\to H_\bullet(\mathrm{L}\mathfrak{X})$ is associative.

2. *If α satisfies the flip condition $\widetilde{\sigma}^*(\alpha)=\alpha$, then the twisted Loop product $\star_\alpha\colon \mathbb{H}(\mathrm{L}\mathfrak{X})\otimes\mathbb{H}(\mathrm{L}\mathfrak{X})\to\mathbb{H}(\mathrm{L}\mathfrak{X})$ is graded commutative.*

Example 10.4 If E is an oriented vector bundle over a stack $\mathfrak{X}$ it has a Euler class $e(E)$. Note that the rank may vary on different connected components of $\mathfrak{X}$. In particular, any vector bundle E over $\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}$ defines a twisted loop product $\star_E:=\star_{e(E)}\colon H(\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X})\to H(\mathrm{L}\mathfrak{X})$. Moreover, $\widetilde{\sigma}^*(e(E))\cong e(E)$ whenever $\widetilde{\sigma}^*E\cong E$. Since identities between Euler classes are equivalent to identities in K-theory we have:

Corollary 10.5 *Let $\mathfrak{X}$ be an oriented (Hurewicz) stack and E a vector bundle over $\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}$.*

1. *If E satisfies the cocycle condition*

$$p_{12}^*(E)+(m\times 1)^*(E)=p_{23}^*(E)+(1\times m)^*(E)$$

   in K-theory, then $\star_E$ is associative.

2. *If $\widetilde{\sigma}^*E\cong E$, then the twisted Loop product $\star_E : H(\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X})\to H(\mathrm{L}\mathfrak{X})$ is graded commutative.*

Remark 10.6 Let M be an oriented manifold and G a finite group acting on M by orientation preserving diffeomorphisms and $\mathfrak{X}=[M/G]$ be the associated global quotient orbifold. Using Proposition 5.9, Proposition 9.3 and the argument of the proof of Proposition 17.10 below to identify evaluation maps and Pontrjagin map, it is straightforward to prove the Loop product $\star : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})\otimes\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})\to\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ coincides with the one introduced in [50].

10.2 Proof of Theorems

The Pontrjagin multiplication $m:\mathrm{Map}(8,\mathfrak{X})\to\mathrm{L}\mathfrak{X}$ is induced by the pinch map $S^1\to S^1\vee S^1$. The latter is homotopy coassociative, thus there is a chain homotopy equivalence between

$$m(m\times\mathrm{id})\colon C_\bullet(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})\to C_\bullet(\mathrm{L}\mathfrak{X})$$

and $m(\mathrm{id}\times m)$. This proves the next lemma:

Lemma 10.7 *The Pontrjagin multiplication satisfies*

$$m_*\big((\mathrm{id}\times m)_*\big) = m_*\big((m\times \mathrm{id})_*\big).$$

Proposition 10.8 *The loop product $H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\bullet} H_{\bullet-d}(\mathrm{L}\mathfrak{X})$ is associative.*

Proof. It is well known that the cross product is associative so that

$$S^{(2)}\colon H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X})$$

is equal to both $S(S\times 1)$ and $S(1\times S)$. We write $m^{(2)}$ for the iterated map

$$m_*(m\times 1)_* = m_*(1\times m)_*$$

as in Lemma 10.7 and $\Delta^{(2)}$ the iterated diagonal

$$\Delta(\Delta\times 1) = \Delta(1\times\Delta)\colon \mathfrak{X}\to\mathfrak{X}^{\times 3}.$$

Also let $e^{(2)}\colon \mathrm{L}\mathfrak{X}^{\times 3}\to\mathfrak{X}^{\times 3}$ denote the product $\mathrm{ev}_0\times\mathrm{ev}_0\times\mathrm{ev}_0$ of the evaluation map on each component. It is enough to prove that, for all $x,y,z\in H_\bullet(\mathrm{L}\mathfrak{X})$,

$$(x\bullet y)\bullet z = m^{(2)}\big({\Delta^{(2)}}^!(x\times y\times z)\big) = x\bullet(y\bullet z).$$

The first equality is given by the commutativity of the following diagram:

$$\begin{tikzcd}
H(\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X}) \arrow[d, "S\otimes 1"'] \arrow[dr, "S^{(2)}"] & & & \\
H(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X}) \arrow[r, "S"] \arrow[d, "\Delta^!\otimes 1"'] \arrow[dr, phantom, "(3)"] & H(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}) \arrow[r, "{\Delta^{(2)}}^!"] \arrow[d, "\Delta\times 1^!"] \arrow[dr, phantom, "(1)"] & H(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}) \arrow[d, equal] & \\
H(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X}) \arrow[r, "S"] \arrow[d, "m_*\otimes 1"'] & H(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}) \arrow[r, "\Delta^!"] \arrow[d, "(m\times 1)_*"'] \arrow[dr, phantom, "(2)"] & H(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}) \arrow[d, "\widetilde{m\times 1}"] \arrow[dr, "p^{(2)}"] & \\
H(\mathrm{L}\mathfrak{X})\otimes H(\mathrm{L}\mathfrak{X}) \arrow[r, "S"] & H(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}) \arrow[r, "\Delta^!"] & H(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}) \arrow[r, "m_*"] & H(\mathrm{L}\mathfrak{X}).
\end{tikzcd} \tag{10.4}$$

The commutativity of the bottom left square follows from the naturality of the cross product, and the bottom right triangle from the associativity of m_* according to Lemma 10.7. The three remaining squares commutes thanks to the following reasons:

Square (1) There is a diagram of cartesian squares

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X} \arrow[r] \arrow[d] & \mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{\mathrm{ev}_0}\times\mathrm{ev}_0"] & \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[d, "e^{(2)}"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X}\times\mathfrak{X} \arrow[r, "\Delta\times 1"] & \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}
\end{tikzcd}.$$

Thus the commutativity follows from the functoriality of Gysin maps.

Square (2) Note that the map $\widetilde{\mathrm{ev}_0}$ in square (1) is equal to $\mathrm{ev}_0 \circ m$. The commutativity follows, by naturality of Gysin maps, from the tower of cartesian diagrams:

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{m \times 1}"'] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "m \times 1"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{\mathrm{ev}_0}"'] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "e"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd}$$

Square (3) It is commutative by compatibility of Gysin maps with the cross product.

Hence it follows that, for all $x, y, z \in H(\mathrm{L}\mathfrak{X})$, one has $(x \bullet y) \bullet z = m^{(2)}(\Delta^{(2)^!}(x \times y \times z))$. One proves in a similar way the identity $m^{(2)}(\Delta^{(2)^!}(x \times y \times z)) = x \bullet (y \bullet z)$ from which the equation $(x \bullet y) \bullet z = \bullet(y \bullet z)$ follows. $\square$

Proposition 10.9 *The loop product* $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\star} \mathbb{H}_*(\mathrm{L}\mathfrak{X})$ *is graded commutative.*

Proof. Essentially, this result follows from the homotopy commutativity of the Pontrjagin map $m : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} = \mathrm{Map}(8, \mathfrak{X}) \to \mathfrak{X}$. More precisely we need to prove that, for $x \in \mathbb{H}_p(\mathrm{L}\mathfrak{X}), y \in \mathbb{H}_q(\mathrm{L}\mathfrak{X})$, we have

$$m_*\big(\Delta^!(x \times y)\big) = (-1)^{pq}\big(m_*\big(\Delta^!(y \times x)\big).$$

The tower of pullback squares

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{\sigma}"'] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\sigma"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "e"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

implies that

$$\widetilde{\sigma}_* \circ \Delta^!(x \times y) = (-1)^{pq}\Delta^!(y \times x).$$

Here $\widetilde{\sigma} \colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ and $\sigma \colon \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}$ are flip maps. Hence the result follows from $m_* \circ \widetilde{\sigma}_* = m_*$ in homology. The latter is an immediate consequence of the existence of a homotopy between the pinch map $p \colon S^1 \to S^1 \vee S^1$ and $\sigma \circ p \colon S^1 \to S^1 \vee S^1$ obtained by making the base point $0 \in S^1$ goes to $\frac{1}{2} \in S^1$. Passing to the mapping stack functor $\mathrm{Map}(-, \mathfrak{X})$ yields a homotopy equivalence between $m \circ \widetilde{\sigma}$ and m. $\square$

Remark 10.10 Note that the homotopy between the two pinch maps does not preserve the canonical basepoints. Hence it is crucial to work with the free loop stack (in other words with non pointed mapping stack functors) in this proof.

Proposition 10.11 *If* $\alpha \in H^\bullet(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$ *satisfies the cocycle equation* (10.3), *then the twisted loop product* $\star_\alpha : H(\mathrm{L}\mathfrak{X}) \otimes H(\mathrm{L}\mathfrak{X}) \to H(\mathrm{L}\mathfrak{X})$ *is associative.*

Proof. We write

$$f_1 : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X},$$

$$f_3 : \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$$

for the canonical projections. Also we have canonical maps

$$j_3 : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \hookrightarrow \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X},$$

$$j_1 : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \hookrightarrow \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}.$$

Using the naturality of cup product and cross product, we can write an associativity diagram similar to (10.4) for $\star_\alpha$, for which the only non obviously commuting square is the one labeled by (1) which becomes

$$\begin{tikzcd}
H(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[r, "1\times\Delta^!"] \arrow[d, "\Delta\times 1^!"] & H(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[r, "\cap f_3^*(\alpha)"] & H(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[d, "\Delta^!"] \\
H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[d, "\cap f_1^*(\alpha)"'] & & H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[d, "\cap p_{23}^*(\alpha)"] \\
H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[r, "\Delta^!"] & H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[r, "\cap p_{12}^*(\alpha)"] & H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}).
\end{tikzcd}$$

Since Gysin maps commute with pullback, for any $y \in H_\bullet(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X})$,

$$\begin{aligned}
\Delta^!(y \cap f_1^*(\alpha)) &= \Delta^!(y) \cap (f_1 \circ j_3)^*(\alpha) \\
&= \Delta^!(y) \cap (m \times 1)^*(\alpha).
\end{aligned}$$

Similarly, $\Delta^!(y \cap f_3^*(\alpha)) = \Delta^!(y) \cap (1 \times m)^*(\alpha)$. From square (1) of diagram 10.4 we deduce that the commutativity of the square is equivalent to the identity

$$\left(\Delta^! \circ \Delta \times 1^!\right) \cap (m \times 1)^*(\alpha) \cap p_{12}^*(\alpha) = \left(\Delta^! \circ 1 \times \Delta^!\right) \cap (1 \times m)^*(\alpha) \cap p_{23}^*(\alpha)$$

$$\iff \Delta^{(2)^!} \cap \left((m \times 1)^*(\alpha) \cup p_{12}^*(\alpha)\right) = \Delta^{(2)^!} \cap \left((1 \times m)^*(\alpha) \cup p_{23}^*(\alpha)\right).$$

The last equality follows immediately from the 2-cocycle condition (10.3). □

Proposition 10.12 *If* $\widetilde{\sigma}^*(\alpha) = \alpha$, *then the twisted loop product* $\star_\alpha : \mathbb{H}(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}(\mathrm{L}\mathfrak{X}) \to \mathbb{H}(\mathrm{L}\mathfrak{X})$ *is commutative.*

Proof. The proof of Propositions 10.9 applies verbatim. □

11 Hidden loop product for family of groups over a stack

The Chas-Sullivan product generalizes the intersection product for a manifold M. Indeed, the embedding of M as the space of constant loop in LM makes $H_\bullet(M)$ a subalgebra of the loop homology and the restriction of the loop product to this subalgebra is the intersection product [15].

In the context of stacks, there are more interesting "constant" loops, namely, loops which, roughly, are constant on the coarse moduli space, but not necessarily on the stack itself. In mathematical physics such loops are sometimes called *ghost loops*. The ghost loops form a stack, called the *inertia stack*.

11.1 Hidden loop product

In this section we construct a hidden loop product for the inertia stack. From the categorical point of view the **inertia stack** $\Lambda\mathfrak{X}$ of a stack $\mathfrak{X}$ is the stack of pairs (x, φ) where x is an object of $\mathfrak{X}$ and φ an automorphism of x. If $\mathfrak{X}$ is a Hurewicz topological stack then so is $\Lambda\mathfrak{X}$. However, if $\mathfrak{X}$ is differentiable, $\Lambda\mathfrak{X}$ is not necessarily differentiable. Let $\mathbb{X}$ be a topological groupoid presenting $\mathfrak{X}$. Let $S\mathbb{X} = \{g \in X_1 |\ s(g) = t(g)\}$ be the space of closed loops. There is a natural action of $\mathbb{X}$ on $S\mathbb{X}$ by conjugation. The associated transformation groupoid $\Lambda\mathbb{X} = [S\mathbb{X} \rtimes X_1 \rightrightarrows S\mathbb{X}]$ is called the *inertia groupoid*. It presents the inertia stack $\Lambda\mathfrak{X}$. We have a morphism of groupoids $\mathrm{ev}_0 \colon \Lambda\mathbb{X} \to \mathbb{X}$,

$$\mathrm{ev}_0 \colon [S\mathbb{X} \rtimes X_1 \rightrightarrows S\mathbb{X}] \to [X_1 \rightrightarrows X_0] \tag{11.1}$$

which on the level of objects sends a closed loop g to its base point $s(g) = t(g)$. On the level of arrows, we have $\mathrm{ev}_0(g, \gamma) = \gamma$. The groupoid morphism $\mathrm{ev}_0 : \Lambda\mathbb{X} \to \mathbb{X}$ induces the evaluation map

$$\mathrm{ev}_0 : \Lambda\mathfrak{X} \to \mathfrak{X} \tag{11.2}$$

on the corresponding stacks.

The construction of the hidden loop product can be made in 3 steps.

Step 1 The external product induces a map:

$$H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}) \xrightarrow{S} H_{p+q}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}).$$

Step 2 We can form the pullback of the evaluation map $\mathrm{ev}_0 : \Lambda\mathfrak{X} \to \mathfrak{X}$ along the diagonal $\Delta \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, thus obtaining the cartesian square

$$\begin{tikzcd} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r] \arrow[d] & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_0)"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}. \end{tikzcd} \tag{11.3}$$

Again we denote by $e : \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ the map $(\mathrm{ev}_0, \mathrm{ev}_0)$. Since $\mathfrak{X}$ is strongly oriented, so is its diagonal $\Delta \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. Hence we have a Gysin map:

$$\Delta^! \colon H_\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \to H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}).$$

Step 3 The stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is known as the *double inertia stack*. Its objects are triples (x, φ, ψ) where x is an object of $\mathfrak{X}$ and φ and ψ are automorphisms of x. On the groupoid level the stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is presented by the transformation groupoid

$$[(S\mathbb{X} \times_{X_0} S\mathbb{X}) \rtimes X_1 \rightrightarrows S\mathbb{X} \times_{X_0} S\mathbb{X}]$$

where $\mathbb{X}$ acts on $S\mathbb{X} \times_{X_0} S\mathbb{X}$ by conjugation diagonally. The double inertia stack is endowed with a "Pontrjagin" multiplication map $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ given by $m(x, \varphi, \psi) = (x, \varphi\psi)$. It induces a morphism on homology

$$m_* \colon H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \to H_\bullet(\Lambda\mathfrak{X}).$$

Composing the three maps in the above steps one obtains a product

$$\star : H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}) \to H_{p+q-d}(\Lambda\mathfrak{X}),$$

called the hidden loop product:

$$H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}) \xrightarrow{S} H_{p+q}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{\Delta^!} H_{p+q-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m_*} H_{p+q-d}(\Lambda\mathfrak{X}). \quad (11.4)$$

As for the loop product, the hidden loop product is a degree 0 multiplication on the shifted homology groups: $\mathbb{H}_\bullet(\Lambda\mathfrak{X}) = H_{\bullet+d}(\Lambda\mathfrak{X})$.

Theorem 11.1 *Let $\mathfrak{X}$ be an oriented stack of dimension d. The shifted homology $\mathbb{H}_\bullet(\Lambda\mathfrak{X})$ of the inertia stack is an associative graded commutative algebra.*

Before proving Theorem 11.1, let us remark that the "Pontrjagin" map $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ corresponds to the multiplication is associative. Thus, passing to homology one has the following lemma.

Lemma 11.2 $m_* \colon H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \to H_\bullet(\Lambda\mathfrak{X})$ *satisfies the associativity condition:*

$$m_*((\mathrm{id} \times m)_*) = m_*((m \times \mathrm{id})_*).$$

Less obvious is that it is also commutative: indeed there is a 2-arrow α

$$\begin{tikzcd} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r, "m"] \arrow[d, "\mathrm{flip}"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \Lambda\mathfrak{X} \\ \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "m"'] & \end{tikzcd} \quad (11.5)$$

which associates to (x, φ, ψ) in the double inertia the isomorphism φ^{-1}

$$\begin{tikzcd} x \arrow[r, "\varphi\psi"] \arrow[d, "\varphi^{-1}"'] \arrow[dr, phantom, "\cong"] & x \arrow[d, "\varphi^{-1}"] \\ x \arrow[r, "\psi\varphi"] & x \end{tikzcd} \tag{11.6}$$

between $(x, \varphi\psi)$ and $(x, \psi\varphi)$ in $\Lambda\mathfrak{X}$.

PROOF OF THEOREM 11.1. Associativity follows *mutatis mutandis* from the proof of Theorem 10.8, substituting $\mathrm{L}\mathfrak{X}$ with $\Lambda\mathfrak{X}$ in the argument. Similarly, the proof of Theorem 10.9 leaves us to proving that the induced map $m_* \circ \widetilde{\sigma} \colon H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \to H_\bullet(\Lambda\mathfrak{X})$ in homology is equal to m_*. Here again $\widetilde{\sigma}$ is the flip map. Passing to any groupoid $\mathbb{X}$ representing $\mathfrak{X}$ and denoting $\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X} = [\big(S\mathbb{X} \times_{X_0} S\mathbb{X}\big) \rtimes X_1 \rightrightarrows \big(S\mathbb{X} \times_{X_0} S\mathbb{X}\big)]$, it is enough to check that the induced map

$$m_* \circ \widetilde{\sigma}_* \colon H_\bullet\left(\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X}\right) \to H_\bullet(\Lambda\mathbb{X})$$

in groupoid homology is equal to m_*. At the level of groupoids, the 2-arrow α of diagram (11.5) yields the identity

$$\begin{aligned} m_*(\sigma(n_1, n_2)) &= \mu(n_2, n_1) \\ &= \big(\mu(n_1, n_2)\big)^{n_2^{-1}} \end{aligned}$$

for all $x = (n_1, n_2, \gamma) \in \big(S\mathbb{X} \times_{X_0} S\mathbb{X}\big) \rtimes X_1$. Here $\mu : S\mathbb{X} \times_{X_0} S\mathbb{X} \to S\mathbb{X}$ is the restriction of the groupoid multiplication of $\mathbb{X}$. Thus $m_*(\sigma(n_1, n_2))$ is canonically conjugate to $m_*(n_1, n_2)$ and in a equivariant way. It follows that after passing to groupoid homology, one has $m_* = m_* \circ \widetilde{\sigma}$. An explicit homotopy $h : C_n(\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X}) \to C_{n+1}(\Lambda\mathbb{X})$ between m_* and $m_* \circ \widetilde{\sigma}$ at the chain level is given by $h = \sum_{i=0}^{n} (-1)^i h_i$ where

$$\begin{aligned} h_i((n_1, n_2), g_1, \ldots, g_n) = \Big(&\big(\mu(n_1, n_2)\big)^{n_2^{-1}}, g_1, \ldots \\ &\ldots, g_i, (g_1 \ldots g_i)^{-1} n_2 (g_1 \ldots g_i), g_{i+1}, \ldots, g_n \Big) \end{aligned}$$

for $i > 0$ and $h_0((n_1, n_2), g_1, \ldots, g_n) = \left(\big(\mu(n_1, n_2)\big)^{n_2^{-1}}, n_2, g_1, \ldots, g_n \right)$. $\square$

If α is a cohomology class in $\bigoplus_{r \geq 0} H^r(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, one defines the twisted hidden loop product

$$\star_\alpha \colon H_\bullet(\Lambda\mathfrak{X}) \otimes H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\Lambda\mathfrak{X})$$

as follows. For any $x, y \in H_\bullet(\Lambda\mathfrak{X})$,

$$x \star_\alpha y = m_*\big(\Delta^!(x \times y) \cap \alpha\big).$$

We use similar notations as for Theorem 10.3: denote

$$p_{12}, p_{23} \colon \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$$

the projections on the first two and the last two factors

Proposition 11.3 *Let α be a class in $\bigoplus_{r\geq 0} H^r(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$.*

1. *If α satisfies the cocycle condition:*

$$p_{12}^*(\alpha) \cup (m \times 1)^*(\alpha) \quad = \quad p_{23}^*(\alpha) \cup (1 \times m)^*(\alpha) \tag{11.7}$$

   in $H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, then $\star_\alpha \colon H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X})$ is associative.

2. *If α satisfies the flip condition $\widetilde{\sigma}^*(\alpha) = \alpha$, then the twisted hidden loop product $\star_\alpha \colon \mathbb{H}(\Lambda\mathfrak{X}) \otimes \mathbb{H}(\Lambda\mathfrak{X}) \to \mathbb{H}(\Lambda\mathfrak{X})$ is graded commutative.*

Proof. The argument of Proposition 10.11 and Proposition 10.12 applies. □

Corollary 10.5 has an obvious counterpart for inertia stack.

Corollary 11.4 *Let $\mathfrak{X}$ be an oriented stack and E a vector bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.*

1. *If E satisfies the cocycle condition*

$$p_{12}^*(E) + (m \times 1)^*(E) = p_{23}^*(E) + (1 \times m)^*(E)$$

   in K-theory, then $\star_E$ is associative.

2. *If $\widetilde{\sigma}^* E \cong E$, then the twisted Loop product $\star_E : H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X})$ is graded commutative.*

11.2 Family of commutative groups and crossed modules

The hidden loop product can be defined for more general "ghost loops" stacks than the mere inertia stack. In fact, we can replace the commutative family $\Lambda\mathfrak{X} \to \mathfrak{X}$ by an arbitrary commutative family of groups.

A **family of groups** over a (topological) stack $\mathfrak{X}$ is a (topological) stack $\mathfrak{G}$ together with a morphism of (topological) stacks $\mathrm{ev} : \mathfrak{G} \to \mathfrak{X}$ and an associative multiplication $m : \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \to \mathfrak{G}$. A family of groups $\mathfrak{G} \to \mathfrak{X}$ (over $\mathfrak{X}$) is said to be a **commutative family of groups** (over $\mathfrak{X}$) if there exists an invertible 2-arrow α making the following diagram

$$\begin{tikzcd} \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[r, "m"] \arrow[d, "\mathrm{flip}"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathfrak{G} \\ \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \arrow[ur, "m"'] & \end{tikzcd} \tag{11.8}$$

commutative. Clearly, the inertia stack is a commutative family of groups (see Equation 11.6).

In the groupoid language, a nice class of commutative family of groups can be represented by crossed modules as follows. A *crossed module* of (topological) groupoids is a morphism of groupoids

$$\begin{array}{ccc} N_1 & \xrightarrow{i} & \Gamma_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ N_0 & \xrightarrow{=} & \Gamma_0 \end{array}$$

which is the identity on the base spaces (in particular $N_0 = \Gamma_0$) and where $[N_1 \rightrightarrows N_0]$ is a family of groups (i.e. source and target are equal), together with a right action $(\gamma, n) \to n^\gamma$ of Γ on N by automorphisms satisfying:

1. For all $(n, \gamma) \in N \rtimes \Gamma_1$, $i(n^\gamma) = \gamma^{-1} i(n) \gamma$;
2. For all $(x, y) \in N \times_{\Gamma_0} N$, $x^{i(y)} = y^{-1} x y$.

Note that the equalities in (1) and (2) make sense because N is a family of groups. We use the short notation $[N \xrightarrow{i} \Gamma]$ for a crossed module.

Remark 11.5 In the literature, groupoids for which source equals target are sometimes called bundle of groups. Since we do not assume the source to be locally trivial, we prefer the terminology family of groups.

Since a crossed module $[N \xrightarrow{i} \Gamma]$ comes with an action of Γ on N, one can form the transformation groupoid $\Lambda[N \xrightarrow{i} \Gamma] := [N_1 \rtimes \Gamma_1 \rightrightarrows N_1]$, which is a topological groupoid. Furthermore, the projection $N_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$ on the second factor induces a (topological) groupoid morphism $\mathrm{ev} : \Lambda[N \xrightarrow{i} \Gamma] \to \Gamma$. Let $\mathfrak{G}$ and $\mathfrak{X}$ be the quotient stack $[N_1/N_1 \rtimes \Gamma_1]$ and $[\Gamma_0/\Gamma_1]$ respectively. Then $\mathrm{ev} : \mathfrak{G} \to \mathfrak{X}$ is a commutative family of groups over $\mathfrak{X}$.

We say that a commutative family of groups is a **strong commutative family of groups** if it can be presented by a crossed module as above.

Clearly, the inertia stack $\Lambda\mathfrak{X}$ corresponds to the crossed module $[S\Gamma \hookrightarrow \Gamma]$ for any groupoid presentation Γ of $\mathfrak{X}$. Obviously $\Lambda[S\Gamma \hookrightarrow \Gamma]$ is the inertia groupoid $\Lambda\Gamma$. The inertia stack is universal among commutative family of groups over $\mathfrak{X}$:

Lemma 11.6 *Let* $\mathrm{ev} : \mathfrak{G} \to \mathfrak{X}$ *be a strong commutative family of groups over* $\mathfrak{X}$*. There exists a unique factorization*

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "e"] \arrow[dr, "\mathrm{ev}"'] & \Lambda\mathfrak{X} \arrow[d, "\mathrm{ev}_0"] \\ & \mathfrak{X}. \end{tikzcd}$$

In fact, for any crossed module $[N \xrightarrow{i} \Gamma]$*, there is a unique map* $e : \Lambda[N \xrightarrow{i} \Gamma] \to \Lambda\Gamma$ *making the following diagram commutative:*

$$\begin{tikzcd} \Lambda[N \xrightarrow{i} \Gamma] \arrow[r, "e"] \arrow[dr, "\mathrm{ev}"'] & \Lambda\Gamma \arrow[d, "\mathrm{ev}_0"] \\ & \Gamma. \end{tikzcd}$$

Example 11.7 Let $\mathfrak{X}$ be an abelian orbifold, that is an orbifold which can be locally represented by quotients $[X/G]$ where G is (finite) abelian. Then the k-twisted sectors of [19] carries a natural crossed module structure $[S_\Gamma^k \xrightarrow{\mu} \Gamma]$ where μ is the $k-1$-fold multiplication $S_\Gamma^k \to S_\Gamma$ followed by the inclusion ι. Of course, for $k = 1$, it is well-known that the induced stack is the inertia stack and that the abelian hypothesis can be dropped. The associated commutative family of groups is $\Lambda_k\mathfrak{X} \to \mathfrak{X}$ where $\Lambda_k\mathfrak{X} = \Lambda\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is the k^{th}-inertia stack.

Let $\mathfrak{G} \to \mathfrak{X}$ be a commutative family of groups over a stack $\mathfrak{X}$. If $\mathfrak{X}$ is strongly oriented, Section 9.1 yields a canonical Gysin map

$$\Delta^! \colon H_\bullet(\mathfrak{G} \times \mathfrak{G}) \to H_{\bullet-d}(\mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G}).$$

Thus one can form the composition

$$\star : H_p(\mathfrak{G}) \otimes H_q(\mathfrak{G}) \xrightarrow{S} H_{p+q}(\mathfrak{G} \times \mathfrak{G}) \xrightarrow{\Delta^!} H_{p+q-d}(\mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G}) \xrightarrow{m_*} H_{p+q-d}(\mathfrak{G}) \quad (11.9)$$

Since $m : \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \to \mathfrak{G}$ is associative and commutative as for the inertia stack in Section 11.1, Step 3, the argument of Theorem 11.1 yields easily

Proposition 11.8 *Let* $\mathfrak{G}$ *be a commutative family of groups over an oriented stack* $\mathfrak{X}$ *(with* $\dim(\mathfrak{X}) = d$*). The multiplication* $\star$ *(see Equation* (11.9)*) endows the shifted homology groups* $\mathbb{H}_\bullet(\mathfrak{G}) \cong H_{\bullet+d}(\mathfrak{G})$ *with a structure of associative, graded commutative algebra.*

Remark 11.9 It is easy to define twisted ring structures on $\mathbb{H}_\bullet(\mathfrak{G})$ along the lines of Theorem 11.3. Details are left to the reader.

Remark 11.10 If $\mathfrak{X}$ is a oriented stack and if $\mathfrak{G} \to \mathfrak{X}$ is a family of groups which is not supposed to be commutative, the product $\star$ (Equation (11.9)) is still defined. Moreover the proof of Theorem 11.1 shows that $(\mathbb{H}_\bullet(\mathfrak{G}), \star)$ is an associative algebra. The k^{th}-inertia stack $\Lambda_k\mathfrak{X} = \Lambda\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is an example of non (necessarily) commutative family of groups.

Remark 11.11 Unlike for free loop stacks in Section 10, we do not need to assume $\mathfrak{X}$ to be Hurewicz in this Section. However, we do not know any interesting example in which it is not the case.

12 Frobenius algebra structures

The loop homology (with coefficients in a field) of a manifold carries a rich algebraic structure besides the loop product. It is known [22] that there exists also a coproduct, which makes it a Frobenius algebra (without counit).

It is natural to expect that such a structure also exists on $H_\bullet(\mathrm{L}\mathfrak{X})$ for an oriented stack $\mathfrak{X}$. In Section 12.2 we show that this is indeed the case. We also prove a similar statement for the homology of inertia stacks.

In this section we assume that our coefficient ring k is a field, since we will use the Künneth formula $H_\bullet(\mathfrak{X} \otimes \mathfrak{Y}) \xrightarrow{\sim} H_\bullet(\mathfrak{X}) \otimes H_\bullet(\mathfrak{Y})$ (Proposition 2.5).

12.1 Quick review on Frobenius algebras

Let k be a field and A a k-vector space. Recall that A is said to be a *Frobenius algebra* if there is an associative commutative multiplication $\mu : A^{\otimes 2} \to A$ and a coassociative cocommutative comultiplication $\delta\colon A \to A^{\otimes 2}$ satisfying the following compatibility condition

$$\delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \delta) = (1 \otimes \mu) \circ (\delta \circ 1) \tag{12.1}$$

in $\mathrm{Hom}(A^{\otimes 2}, A^{\otimes 2})$. Here we do not require the existence of a unit nor a counit. Also we allow A to be graded and the maps μ and δ to be graded as well. When both maps are of the same degree d, we say that A is a Frobenius algebra of *degree* d. Note that this grading is adding signs in an usual way to the story so that the multiplication becomes associative and graded commutative after shifting A to $A[d]$ and similarly for the comultiplication if one shifts A to $A[-d]$. The precise relationship between the involved signs are given by the structure of a d-dimensional homological conformal theory as in Section 14; also see [27, 34]. The tensor product of two Frobenius algebras A and B is naturally a Frobenius algebra with the multiplication $(\mu \otimes \mu) \circ (\tau_{23})$ and comultiplication $\tau_{23}^{-1} \circ (\delta \otimes \delta)$ where $\tau_{23}\colon A \otimes B \otimes A \otimes B \to A^{\otimes 2} \otimes B^{\otimes 2}$ is the map permuting the two middle components.

Warning 12.1 We need a few words of caution concerning our definition of Frobenius algebras. In the literature, one often encounters (commutative) Frobenius algebras which are both unital and counital such that, if $c : A \to k$ is the counit, then $c \circ \mu : A \otimes A \to k$ is a nondegenerate pairing.

Remark 12.2 It is well-known [60, 3] that a structure of $1+1$-dimensional Topological Quantum Field Theory on A is equivalent to a structure of unital and counital Frobenius algebra on A such that the pairing $c \circ \mu : A \otimes A \to k$, where c is the counit and μ the multiplication, is non-degenerate. Theorem 12.5 and Theorem 12.3 below imply that $H_\bullet(\mathrm{L}\mathfrak{X})$ and $H_\bullet(\Lambda\mathfrak{X})$ have the structure of $1+1$-positive boundary TQFT in the sense of [22]. Positive boundary TQFT are obtained by considering only cobordism Σ with boundary $\partial\Sigma = -S_1 \coprod S_2$ such that $S_1, S_2 \neq \emptyset$ (see [22] for details).

Further, in the case of the free loop stack, we will see in section 14.2 that the TQFT structure on $H_\bullet(\mathrm{L}\mathfrak{X})$ can be extended to a whole homological conformal field theory with positive closed boundary (Theorem 14.2).

12.2 Frobenius algebra structure for loop stacks

In this subsection we prove the existence of a Frobenius algebra structure on the homology of the free loop stack of an oriented (Hurewicz) stack. Let $\mathrm{ev}_0, \mathrm{ev}_{1/2} \colon \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ be the evaluation maps defined as in Equation (5.1), where $\mathfrak{X}$ is a Hurewicz topological stack. To simplify the notations, let $\breve{e}$ be the evaluation map $(\mathrm{ev}_0, \mathrm{ev}_{1/2}) \colon \mathrm{L}\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$.

Lemma 12.3 *The stack* $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ *fits into a cartesian square*

$$\begin{tikzcd} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "m"] \arrow[d] & \mathrm{L}\mathfrak{X} \arrow[d, "\breve{e}"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd} \tag{12.2}$$

where $\Delta \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ *is the diagonal.*

Proof. Since S^1 is compact and $\mathfrak{X}$ is a Hurewicz topological stack, Lemma 5.2 ensures that the pushout diagram of topological spaces

$$\begin{tikzcd} \mathrm{pt} \coprod \mathrm{pt} \arrow[r, "0 \coprod \frac{1}{2}"] \arrow[d] & S^1 \arrow[d] \\ \mathrm{pt} \arrow[r] & S^1 \vee S^1 \end{tikzcd}$$

becomes a pullback diagram after applying the mapping stack functor $\mathrm{Map}(-, \mathfrak{X})$. This is precisely diagram (12.2). $\square$

Remark 12.4 The argument of Lemma 12.3 can be applied to iterated diagonals as well. In particular, $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ (with n-terms) is the mapping stack $\mathrm{Map}(S^1 \vee \cdots \vee S^1, \mathfrak{X})$ (with n copies of S^1) and moreover there is a cartesian square

$$\begin{tikzcd} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d] & \mathrm{L}\mathfrak{X} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_{1/n}, \ldots, \mathrm{ev}_{n-1/n})"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \cdots \times \mathfrak{X}. \end{tikzcd} \tag{12.3}$$

Now assume further that $\mathfrak{X}$ is oriented of dimension d. According to Section 9.1, the cartesian square (12.2) yields a Gysin map

$$\Delta^!\colon H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}).$$

By diagram (10.1), there is a canonical map $\mathrm{Map}(8, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \xrightarrow{j} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}$. Thus we obtain a degree d map

$$\delta\colon H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\Delta^!} H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \xrightarrow{j_*} H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}).$$

Theorem 12.5 *Let $\mathfrak{X}$ be an oriented Hurewicz stack of dimension d. Then $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ is a Frobenius algebra, where both operations $\star$ and δ are of degree d.*

Proof. It remains to prove the coassociativity, cocommutativity of the coproduct and the Frobenius compatibility relation. Denote by $\delta_S : H_\bullet(\mathfrak{X} \times \mathfrak{Y}) \to H_\bullet(\mathfrak{X}) \otimes H_\bullet(\mathfrak{Y})$ the inverse of the cross product induced by the Künneth isomorphism and $\delta_S^{(n)}$ for its iteration.

i) Coassociativity Let $\breve{e}^{(2)}\colon \mathrm{L}\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ be the iterated evaluation map $(\mathrm{ev}_0, \mathrm{ev}_{1/3}, \mathrm{ev}_{2/3})$. According to Corollary 8.31, the iterated diagonal $\Delta^{(2)}\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ is naturally normally nonsingular oriented. Thus, Remark 12.4 implies that there is a Gysin map ${\Delta^{(2)}}^!$. Similarly there is a canonical map

$$j^{(2)}\colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \cong \mathrm{Map}(S^1 \vee S^1 \vee S^1, \mathfrak{X}) \to \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}.$$

The argument of the proof of Theorem 10.8 shows that it is sufficient to prove that the following diagram is commutative (which is, in a certain sense, is the dual of diagram (10.4)).

$$\begin{tikzcd}
H(\mathrm{L}\mathfrak{X}) \otimes H(\mathrm{L}\mathfrak{X}) \otimes H(\mathrm{L}\mathfrak{X}) & & & \\
H(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \otimes H(\mathrm{L}\mathfrak{X}) \arrow[u, "\delta_S \otimes 1"] \arrow[dr, phantom, "(5)"] & H(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[l, "\delta_S"'] \arrow[ul, "\delta_S^{(2)}"'] \arrow[dr, phantom, "(1)"] & H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[l, "j_*^{(2)}"'] & \\
H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \otimes H(\mathrm{L}\mathfrak{X}) \arrow[u, "j_* \otimes 1"] \arrow[dr, phantom, "(3)"] & H((\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \times \mathrm{L}\mathfrak{X}) \arrow[l, "1 \times \delta_S"'] \arrow[u, "(j \times 1)_*"'] \arrow[dr, phantom, "(2)"] & H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[l, "(1 \times j)_*"'] \arrow[u, equal] & \\
H(\mathrm{L}\mathfrak{X}) \otimes H(\mathrm{L}\mathfrak{X}) \arrow[u, "\Delta^! \otimes 1"] & H(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[l, "\delta_S"] \arrow[u, "\Delta^!"] & H(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[l, "j_*"] \arrow[u, "\Delta^!"] & H(\mathrm{L}\mathfrak{X}) \arrow[l, "\Delta^!", "(4)"'] \arrow[ul, "{\Delta^{(2)}}^!"']
\end{tikzcd} \tag{12.4}$$

where p and $\widetilde{p}$ denote, respectively, the projections $\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ and $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ on the first factor. Square (5) is commutative by naturality of the cross coproduct δ_S and the upper left triangle by its coassociativity. We are left to study the three remaining squares (1), (2), (3) and triangle (4).

Square (1) The square commutes in view of the identity $j^{(2)} = (j \times 1) \circ (1 \times j)$ which follows from the natural isomorphism

$$(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \times \mathrm{L}\mathfrak{X} \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} (\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}).$$

Here the map $\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ is the composition $\mathrm{ev}_0 \circ p$. In the sequel, we use this isomorphism without further notice.

Square (2) Since $\breve{e} \circ \widetilde{p} = \big(\breve{e} \circ p\big) \circ j : \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, the commutativity of square (2) follows immediately, by naturailty of Gysin maps, from the tower of cartesian diagrams

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d, "1\times j"'] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "j"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[r, "m\times 1"] \arrow[d] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\breve{e}\circ p"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd}$$

Square (3) The square commutes by the same argument as for square (3) in diagram (10.4).

Triangle (4) The sequence of cartesian diagrams

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "m\times 1"] \arrow[d] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "m"] \arrow[d, "\breve{e}\circ\widetilde{p}"] & \mathrm{L}\mathfrak{X} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}}, \mathrm{ev}_{\frac{1}{4}})"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \arrow[r, "\Delta\times 1"] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd} \tag{12.5}$$

implies, by naturality of Gysin maps, that

$$\Delta^{!} \circ (\Delta \times 1)^{!} \quad = \quad {\Delta^{(2)}}^{!}. \tag{12.6}$$

There is an homeomorphism $h : S^1 \xrightarrow{\sim} S^1$ which, together with the flip map σ, induces a commutative diagram

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[rr, bend left] \arrow[d] & \mathrm{L}\mathfrak{X} \arrow[r, "h^*"'] \arrow[d, "\breve{e}^{(2)}"'] & \mathrm{L}\mathfrak{X} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}}, \mathrm{ev}_{\frac{1}{4}})"] \\
\mathfrak{X} \arrow[r, "\Delta^{(2)}"] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \arrow[r, "1\times\sigma"] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}
\end{tikzcd} \tag{12.7}$$

As $h^* = \mathrm{Map}(-, \mathfrak{X})(h)$ is a homeomorphism and $(1 \times \sigma) \circ \Delta^{(2)} = \Delta^{(2)}$, diagram (12.7) identifies ${\Delta^{(2)}}^{!}$ with the Gysin map (denoted ${\Delta^{(2)}}^{!}$ by abuse of notation) associated to Diagram (12.5). Since $(\Delta \times 1)^{!} = \Delta^{!}$ the commutativity of Triangle (4) follows from Equation (12.6).

ii) Let's turn to the point of cocommutativity. It is sufficient to prove that

$$\Delta^! \quad = \quad \widetilde{\sigma}_* \circ \Delta^!, \tag{12.8}$$

where $\widetilde{\sigma}\colon \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ is the flip map. There is a natural homotopy $F\colon I \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ between $m \circ \widetilde{\sigma}$ and m (see the proof of Theorem 10.9). Equation (12.8) follows easily by naturality of Gysin maps applied to the cartesian squares below (where $t \in I$)

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "{F(t,-)}"] \arrow[d, "{(t,1)}"'] & \mathrm{L}\mathfrak{X} \arrow[d, "{(t,1)}"] \\
I \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "{(1\circ p,F)}"] \arrow[d] & I \times \mathrm{L}\mathfrak{X} \arrow[d, "\breve{e}"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd}$$

The map $(t,1)\colon \mathrm{L}\mathfrak{X} \to I \times \mathrm{L}\mathfrak{X}$ is the map $\mathrm{L}\mathfrak{X} \xrightarrow{\sim} \{t\} \times \mathrm{L}\mathfrak{X} \to I \times \mathrm{L}\mathfrak{X}$. The left upper vertical map is similar. The maps $(t,1)$ are homotopy equivalences inverting the canonical projections $I \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$, $I \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$.

iii) It remains to prove the Frobenius relation (12.1). To avoid confusion between different Gysin maps, we now denote $m^!\colon = \Delta^!\colon H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$ and $j^! := \Delta^!\colon H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$ the Gysin maps inducing the product and coproduct. The cartesian squares

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "1\times m"] \arrow[d] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "\breve{e}\circ\widetilde{p}"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "\widetilde{j}"] \arrow[d] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "(\mathrm{ev}_0 \times \mathrm{ev}_0)\circ(1\times\widetilde{p})"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

give rise to Gysin maps (see Section 9.1)

$$(1 \times m)^!\colon H_\bullet(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$$

and

$$\widetilde{j}^!\colon H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}).$$

There is a canonical map $\widetilde{\widetilde{j}}$ sitting in the pullback diagram

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "\widetilde{\widetilde{j}}"] \arrow[d] & (\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \times \mathrm{L}\mathfrak{X} \arrow[d, "\widetilde{\mathrm{ev}_0} \times \mathrm{ev}_0"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

Consider the following diagram

$$\begin{tikzcd}
H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[r, "(1\times m)^!"] \arrow[d, "j^!"'] \arrow[dr, "m^!_{23}"] \arrow[d, phantom, shift left=6ex, "(a)"] & H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[r, "(1\times j)_*"] \arrow[d, "\widetilde{j}^!"] \arrow[d, phantom, shift right=6ex, "(a')"] \arrow[dr, phantom, "(b')"] & H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[d, "(j\times 1)^!"] \\
H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[r, "1\times_{\mathfrak{X}} m^!"'] \arrow[d, "m_*"'] \arrow[dr, phantom, "(b)"] & H_{\bullet-2d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[r, "\widetilde{\widetilde{j}}_*"] \arrow[d, "m_* \times_{\mathfrak{X}} 1"] \arrow[dr, phantom, "(c)"] & H_{\bullet-2d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \arrow[d, "m_* \times 1"] \\
H_{\bullet-d}(\mathrm{L}\mathfrak{X}) \arrow[r, "m^!"'] & H_{\bullet-2d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \arrow[r, "j_*"'] & H_{\bullet-2d}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X})
\end{tikzcd} \tag{12.9}$$

where $m^!_{23}\colon H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \to H_{\bullet-2d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$ is the Gysin map determined by the cartesian square (applying Corollary 8.31)

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "j\circ(1\times_{\mathfrak{X}} m)"] \arrow[d] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_0 \times \check{e}"] \\
\mathfrak{X} \arrow[r, "\Delta^{(2)}"] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd}$$

The triangle (a) in diagram (12.9) is commutative because we have a sequence of cartesian squares

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "1\times_{\mathfrak{X}} m"] \arrow[d] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "j"] \arrow[d, "\check{e}\circ\widetilde{p}"] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_0 \times \check{e}"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \arrow[r, "\Delta\times 1"] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd} \tag{12.10}$$

Similarly, triangle (a') is commutative, i.e., $\widetilde{j}^! \circ (1 \times m)^! = m^!_{23}$. By naturality of Gysin maps, the towers of cartesian squares

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "1\times_{\mathfrak{X}} m"] \arrow[d, "1\times_{\mathfrak{X}} m"'] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "m"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "m"] \arrow[d] & \mathrm{L}\mathfrak{X} \arrow[d] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{\widetilde{j}}"'] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "1\times j"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[r, "j\times 1"] \arrow[d, "\widetilde{\mathrm{ev}_0}\circ(1\times p)"'] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "(\mathrm{ev}_0 \times \mathrm{ev}_0)\circ(1\times p)"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

give the commutativity of squares (b) and (b') in diagram (12.9). The commutativity of Square (c) is trivial. Thus diagram (12.9) is commutative. Up to the identification $H_\bullet(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}) \cong H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X})$, the composition of the bottom horizontal map and the left vertical one in diagram (12.9) is the composition $\delta(-\star-)$. The composition of the right vertical map with the upper arrow is $(a\star b^{(1)})\otimes b^{(2)}$. Finally commutation of the Gysin maps with the cross product yields the identity

$$\delta(a\star b) = a\star b^{(1)}\otimes b^{(2)}.$$

The proof of identity $\delta(a\star b) = a^{(1)}\otimes a^{(2)}\star b$ is similar. $\square$

12.3 Frobenius algebra structure for inertia stacks

In this section we show that the homology of the inertia stack is also a Frobenius algebra, similarly to Theorem 12.5.

Let $\mathfrak{X}$ be a topological stack of dimension d and $\mathbb{X}$ a topological groupoid representing $\mathfrak{X}$. Thus its inertia stack $\Lambda\mathfrak{X}$ is the stack associated to the inertia groupoid $\Lambda\mathbb{X} = [S\mathbb{X}\rtimes X_1 \rightrightarrows S\mathbb{X}]$, where $S\mathbb{X}$ is the space of closed loops. Any loop $S^1\to X$ on a topological space X can be evaluated in 0 but also in 1/2. It is folklore to think of $\Lambda\mathfrak{X}$ as a ghost loop stack. Hence evaluation map at 0 and 1/2 should make sense as well. We first construct these evaluation maps for the inertia stack which leads to the construction of the Frobenius structure on $H_\bullet(\Lambda\mathfrak{X})$ when $\mathfrak{X}$ is oriented.

First of all, let us introduce another groupoid $\widetilde{\Lambda\mathbb{X}}$ which is Morita equivalent to $\Lambda\mathbb{X}$. Objects of $\widetilde{\Lambda\mathbb{X}}$ consist of all diagrams

$$\begin{tikzcd} x & y \arrow[l, bend right, "g_1"'] & x \arrow[l, bend right, "g_2"'] \end{tikzcd} \tag{12.11}$$

in $\mathbb{X}$. Note that the composition g_1g_2 is a loop over x. Arrows of $\widetilde{\Lambda\mathbb{X}}$ consist of commutative diagrams

$$\begin{tikzcd} x & y \arrow[l, bend right, "g_1"'] & x \arrow[l, bend right, "g_2"'] \\ x' \arrow[u, "h_0"] & y' \arrow[l, bend left] \arrow[u, "h_{1/2}"] & x' \arrow[l, bend left] \arrow[u, "h_0"'] \end{tikzcd}$$

Note that the left and right vertical arrows are the same. The target map is the top row

$$\begin{tikzcd} x & y \arrow[l, bend right, "g_1"'] & x \arrow[l, bend right, "g_2"'] \end{tikzcd}$$

while the source map is the bottom row

$$\begin{tikzcd} x' & y' \arrow[l, bend right, "h_0^{-1}g_1h_{1/2}"'] & x' \arrow[l, bend right, "h_{1/2}^{-1}g_2h_0"'] \end{tikzcd}.$$

The unit map is obtained by taking identities as vertical arrows. The composition is obtained by superposing two diagrams and deleting the middle row of the diagram, i.e.

$$\begin{tikzcd} x & y \arrow[l, bend right, "g_1"'] & x \arrow[l, bend right, "g_2"'] \\ x' \arrow[u, "h_0"] & y' \arrow[l, bend left] \arrow[u, "h_{1/2}"] & x' \arrow[l, bend left] \arrow[u, "h_0"'] \end{tikzcd} * \begin{tikzcd} x' & y' \arrow[l, bend right] & x' \arrow[l, bend right] \\ x" \arrow[u, "h'_0"] & y" \arrow[l, bend left] \arrow[u, "h'_{1/2}"] & x" \arrow[l, bend left] \arrow[u, "h'_0"'] \end{tikzcd}$$

is mapped to

$$\begin{tikzcd} x & y \arrow[l, bend right, "g_1"'] & x \arrow[l, bend right, "g_2"'] \\ x" \arrow[u, "h_0 h'_0"] & y" \arrow[l, bend left] \arrow[u, "h_{1/2}h'_{1/2}"] & x" \arrow[l, bend left] \arrow[u, "h_0 h'_0"'] \end{tikzcd} .$$

In other words, $\widetilde{\Lambda\mathbb{X}}$ is the transformation groupoid $\widetilde{S\mathbb{X}} \rtimes (X_1 \times X_1)$, where $\widetilde{S\mathbb{X}} = \{(g_1, g_2) \in X_2 \,|\, t(g_1) = s(g_2)\}$, the momentum map $\widetilde{S\mathbb{X}} \to X_0 \times X_0$ is (t, t), and the action is given, for all compatible $(h_0, h_{1/2}) \in X_1 \times X_1$, $(g_1, g_2) \in \widetilde{S\mathbb{X}}$, by

$$(g_1, g_2) \cdot (h_0, h_{1/2}) = (h_0^{-1} g_1 h_{1/2}, h_{1/2}^{-1} g_2 h_0).$$

One defines evaluation maps taking by the vertical arrows of $\widetilde{S\mathbb{X}}$, i.e. $\forall (g_1, g_2, h_0, h_{1/2}) \in \widetilde{\Lambda\mathbb{X}}_1$ define

$$\mathrm{ev}_0 : (g_1, g_2, h_0, h_{1/2}) \mapsto h_0, \quad \mathrm{ev}_{1/2} \colon (g_1, g_2, h_0, h_{1/2}) \mapsto h_{1/2}.$$

It is simple to prove

Lemma 12.6 *Both evaluation maps* $\mathrm{ev}_0 : \widetilde{\Lambda\mathbb{X}} \to \mathbb{X}$ *and* $\mathrm{ev}_{1/2} \colon \widetilde{\Lambda\mathbb{X}} \to \mathbb{X}$ *are groupoid morphisms.*

There is a map

$$p \colon \widetilde{\Lambda\mathbb{X}} \to \Lambda\mathbb{X} \tag{12.12}$$

obtained by sending a diagram in $\widetilde{\Lambda\mathbb{X}}_1 = \widetilde{S\mathbb{X}} \rtimes (X_1 \times X_1)$ to the composition of the horizontal arrows, i.e.,

$$\begin{tikzcd} x & y \arrow[l, bend right, "g_1"'] & x \arrow[l, bend right, "g_2"'] \\ x' \arrow[u, "h_0"] & y' \arrow[l, bend left] \arrow[u, "h_{1/2}"] & x' \arrow[l, bend left] \arrow[u, "h_0"'] \end{tikzcd} \quad \text{is mapped to} \quad \begin{tikzcd} x & x \arrow[l, bend right, "g_1 g_2"'] \\ x' \arrow[u, "h_0"] & x' \arrow[l, bend left] \arrow[u, "h_0"'] \end{tikzcd}$$

In other words $p(g_1, g_2, h_0, h_{1/2}) = (g_1 g_2, h_0)$.

Lemma 12.7 *The map* $p \colon \widetilde{\Lambda\mathbb{X}} \to \Lambda\mathbb{X}$ *is a Morita morphism.*

Proof. The map $p_0 : \widetilde{\Lambda\mathbb{X}}_0 \to \Lambda\mathbb{X}_0$ is a surjective submersion with a section given by $g \mapsto (g, 1_{s(g)})$ for $g \in S\mathbb{X}$. Let $g, g' \in S\mathbb{X}$. Assume given $(g_1, g_2) \in X_2$ with $g_1 g_2 = g$ and $(g'_1, g'_2) \in X_2$ with $g'_1 g'_2 = g'$. Then any arrow in $\widetilde{\Lambda\mathbb{X}}$ from $x \xleftarrow{g_1} y \xleftarrow{g_2} x$ to $x \xleftarrow{g'_1} y \xleftarrow{g'_2} x$ is uniquely determined by $h_0 \in X_1$ satisfying $h_0^{-1} g_1 g_2 h_0 = g'_1 g'_2$. Indeed, $h_{1/2}$ is given by $h_{1/2} = g_2 h_0 {g'_2}^{-1}$. □

As a consequence the groupoid $\widetilde{\Lambda\mathbb{X}}$ also presents the inertia stack $\Lambda\mathfrak{X}$, and Lemma 12.6 implies that there are two stack maps $\mathrm{ev}_0, \mathrm{ev}_{1/2} \colon \Lambda\mathfrak{X} \to \mathfrak{X}$.

We now proceed to construct the hidden loop coproduct. As in Section 11.1 above, $\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X}$ is the transformation groupoid

$$[\big(S\mathbb{X} \times_{X_0} S\mathbb{X}\big) \rtimes X_1 \rightrightarrows S\mathbb{X} \times_{X_0} S\mathbb{X}],$$

where $\mathbb{X}$ acts on $S\mathbb{X} \times_{X_0} S\mathbb{X}$ by conjugations diagonally. Its corresponding stack is $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

Lemma 12.8 *The stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ fits into the cartesian square*

$$\begin{tikzcd} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r, "m"] \arrow[d] & \Lambda\mathfrak{X} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_{1/2})"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd} .$$

As in Section 12.5, we denote $\breve{e} := (\mathrm{ev}_0, \mathrm{ev}_{1/2}) \colon \Lambda\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ the right vertical map in the diagram of lemma 12.8.

PROOF.We use $\widetilde{\Lambda\mathbb{X}}$ as a groupoid representative of $\Lambda\mathfrak{X}$. By the definition of the evaluation maps, the fiber product

$$\begin{tikzcd} \mathbb{X} \times_{\mathbb{X}\times\mathbb{X}} \widetilde{\Lambda\mathbb{X}} \arrow[r] \arrow[d] & \widetilde{\Lambda\mathbb{X}} \arrow[d, "(ev_0, ev_{1/2})"] \\ \mathbb{X} \arrow[r, "\Delta"] & \mathbb{X} \times \mathbb{X} \end{tikzcd}$$

can be identified with the subgroupoid of $\widetilde{\Lambda\mathbb{X}}$, which consists of $(g_1, g_2, h_0, h_{1/2})$ such that $h_0 = h_{1/2}$. The latter is simply the transformation groupoid

$$[(S\mathbb{X} \times_{X_0} S\mathbb{X}) \rtimes X_1 \rightrightarrows S\mathbb{X} \times_{X_0} S\mathbb{X}]$$

which is precisely $\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X}$. Moreover the composition

$$\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X} \cong \mathbb{X} \times_{\mathbb{X}\times\mathbb{X}} \widetilde{\Lambda\mathbb{X}} \to \widetilde{\Lambda\mathbb{X}} \xrightarrow{p} \Lambda\mathbb{X},$$

where p is defined by equation (12.12), is precisely the "Pontrjagin" map m : $\Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X} \to \Lambda\mathbb{X}$ in Section 11.1. $\Box$

Remark 12.9 It is not hard to generalize the above construction to any finite number of evaluation maps and obtain the following cartesian square (see the proof of Theorem 12.3 below)

$$\begin{tikzcd} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r] \arrow[d] & \Lambda\mathfrak{X} \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{X} \times \cdots \times \mathfrak{X}. \end{tikzcd}$$

If $\mathfrak{X}$ is oriented of dimension d, the cartesian square of Lemma 12.8 yields a Gysin map (Section 9.1)

$$\Delta^!\colon H_\bullet(\Lambda\mathfrak{X}) \longrightarrow H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}).$$

As shown in Section 11.1, there is also a canonical map $j : \Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X} \to \Lambda\mathfrak{X}\times\Lambda\mathfrak{X}$.

Theorem 12.10 *Assume $\mathfrak{X}$ is an oriented stack of dimension d. The composition*

$$H_n(\Lambda\mathfrak{X}) \xrightarrow{\Delta^!} H_{n-d}(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) \xrightarrow{j} H_{n-d}(\Lambda\mathfrak{X}\times\Lambda\mathfrak{X}) \cong \bigoplus_{i+j=n-d} H_i(\Lambda\mathfrak{X})\otimes H_j(\Lambda\mathfrak{X})$$

yields a coproduct $\delta\colon H_\bullet(\Lambda\mathfrak{X}) \to \bigoplus_{i+j=\bullet-d} H_i(\Lambda\mathfrak{X})\otimes H_j(\Lambda\mathfrak{X})$ which is a coassociative and graded cocommutative coproduct on the shifted homology $\mathbb{H}_\bullet(\Lambda\mathfrak{X}) := H_{\bullet+d}(\Lambda\mathfrak{X})$, called the hidden loop coproduct of $\Lambda\mathfrak{X}$.

Proof. The proof is very similar to that of Theorem 12.5. We only explain the difference.

i) First we introduce a third evaluation map $\mathrm{ev}_{2/3}\colon \Lambda\mathfrak{X} \to \mathfrak{X}$ similar to $\mathrm{ev}_{1/2}$. Taking a representative $\mathbb{X}$ of $\mathfrak{X}$, the idea is to replace the Lie groupoid $\Lambda\mathbb{X}$ presenting $\Lambda\mathfrak{X}$ by another groupoid $\widetilde{\widetilde{\widetilde{\Lambda\mathbb{X}}}}$, where $\widetilde{\widetilde{\widetilde{\Lambda\mathbb{X}}}}_1$consists of commutative diagrams:

$$\begin{tikzcd}
x & y \arrow[l, bend right, "g_1"'] & z \arrow[l, bend right, "g_2"'] & x \arrow[l, bend right, "g_3"'] \\
x' \arrow[u, "h_0"] & y' \arrow[l, bend left] \arrow[u, "h_{1/2}"'] & z' \arrow[l, bend left] \arrow[u, "h_{2/3}"'] & x' \arrow[l, bend left] \arrow[u, "h_0"']
\end{tikzcd}$$

The source and target maps are, respectively, given by the bottom and upper lines. The multiplication is by superposition of diagrams. There are evaluation maps $\mathrm{ev}_0, \mathrm{ev}_{1/2}, \mathrm{ev}_{2/3}\colon \widetilde{\widetilde{\widetilde{\Lambda\mathbb{X}}}} \to \mathbb{X}$, respectively, given by $h_0, h_{1/2}, g_{2/3}$. A proof similar to those of Lemmas 12.7 and Lemmas 12.8 gives the following facts:

1. the groupoid $\widetilde{\widetilde{\widetilde{\Lambda\mathbb{X}}}}$ is Morita equivalent to $\Lambda\mathbb{X}$. Hence it also presents the stack $\Lambda\mathfrak{X}$.

2. The evaluation maps induce a cartesian square

$$\begin{tikzcd}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r] \arrow[d] & \Lambda\mathfrak{X} \arrow[d, "\breve{e}^{(2)}"] \\
\mathfrak{X} \arrow[r, "\Delta^{(2)}"] & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

   which yields a Gysin map ${\Delta^{(2)}}^!\colon H_\bullet(\Lambda\mathfrak{X}) \to H_{\bullet-2d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$.

It follows that one can form a diagram similar to (12.4) for $\Lambda\mathfrak{X}$ and prove that all its squares (1), (2), (3), (5) are commutative *mutatis mutandis*. The proof of the commutativity of triangle (4) is even easier: it follows immediately from the sequence of cartesian square

$$\begin{array}{ccccc}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \Lambda\mathfrak{X} \\
\downarrow & & \downarrow{\scriptstyle \breve{e}\circ\widetilde{p}} & & \downarrow{\scriptstyle \breve{e}^{(2)}} \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} & \xrightarrow{\Delta\times 1} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}.
\end{array}$$

ii) Since $p \circ \widetilde{\sigma}$ is conjugate to p, the proof of the cocommutativity of δ is similar to the proof of Proposition 12.5 and of Proposition 11.1. □

Theorem 12.11 *The homology groups $(H_\bullet(\Lambda\mathfrak{X}), \bullet, \delta)$ form a (non unital, non counital) Frobenius algebra of degree d.*

Proof. According to Theorems 11.1, 12.3 it suffices to prove the compatibility condition between the hidden loop product and hidden loop coproduct. The argument of the proof of Theorem 12.5.iii) applies. □

Remark 12.12 If $\mathfrak{X}$ has finitely generated homology groups in each degree, then by universal coefficient theorem, $H^\bullet(\Lambda\mathfrak{X})$ inherits a Frobenius coalgebra structure which is unital iff $(H_\bullet(\Lambda\mathfrak{X}), \delta)$ is counital.

12.4 The canonical morphism $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$

There is a morphism of stacks $\Phi\colon \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ generalizing the canonical inclusion of a space into its loop space (as a constant loop).

Remark 12.13 Objects of $\Lambda\mathfrak{X}$ are pairs (x, φ) where x is an object of $\mathfrak{X}$ and φ an automorphism of x. The morphism Φ may be thought of as maps $(x, \varphi) \in \Lambda\mathfrak{X}$ to the isotrivial family over S^1, which is obtained from the constant family X_I over the interval by identifying the two endpoints via φ.

We show in this Section that Φ induces a morphism of Frobenius algebras in homology.

Let $\mathbb{X}$ be a groupoid representing the oriented stack $\mathfrak{X}$ (of dimension d) and $\Lambda\mathbb{X}$ its inertia groupoid representing $\Lambda\mathfrak{X}$. Proposition 5.4 gives a groupoid $\mathrm{L}\mathbb{X}$ representing the free loop stack $\mathrm{L}\mathfrak{X}$. We use the notations of Section 5.2. Recall that the topological groupoid $\mathrm{L}\mathbb{X}$ is a limit of topological groupoids $L^P\mathbb{X}$ where P is a finite subset of S^1 which can be described as an increasing (with respect to a cyclic ordering on S^1) sequence $\{P_0, P_1, \dots, P_{n-1}\}$. We take $n = 1$ and $\{P_0\} = \{1\} \subset S^1$ the trivial subset of S^1. We will construct a morphism of groupoids $\Lambda\mathbb{X} \to L^P\mathbb{X}$ inducing the map $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.

Any $(g,h) \in S\mathbb{X} \rtimes X_1 = \Lambda\mathbb{X}_1$ (i.e. $g \in X_1$ with $s(g) = t(g)$ determines a commutative diagram $\Phi(g,h)$ in the underlying category of the groupoid $\mathbb{X}$:

$$\begin{tikzcd} t(h) & t(h) \arrow[l, "g"'] \\ s(h) \arrow[u, "h"] & s(h) \arrow[l, "h^{-1}gh"] \arrow[u, "h"'] \end{tikzcd} \quad . \tag{12.13}$$

The square $\Phi(g,h)$ (defined by diagram (12.13)) being commutative, it is an element of $M_1\mathbb{X}$. Since P is a trivial subset of S^1, a morphism $[S_1^P \rightrightarrows S_0^P] \to [M_1\mathbb{X} \rightrightarrows M_0\mathbb{X}]$ is given by a path $f\colon [0,1] \to X_1$ and elements $k, k' \in X_1$ such that the diagram

$$\begin{tikzcd} t(f(0)) & t(f(1)) \arrow[l, "k"'] \\ s(f(0)) \arrow[u, "f(0)"] & s(f(1))) \arrow[l, "k'"] \arrow[u, "f(1)"'] \end{tikzcd}$$

commutes. In particular, the diagram $\Phi(g,h) \in M_1\mathbb{X}$ yields a (constant) groupoid morphism $[S_1^P \rightrightarrows S_0^P] \to [M_1\mathbb{X} \rightrightarrows M_0\mathbb{X}]$ defined by $t \mapsto f(t) = h$. The map $(g,h) \mapsto \Phi(g,h)$ is easily seen to be a groupoid morphism. We denote by $\Phi\colon \Lambda\mathbb{X} \to \mathrm{L}\mathbb{X}$ its composition with the inclusion $\mathrm{L}^P\mathbb{X} \to \mathrm{L}\mathbb{X}$. It is still a morphism of groupoids. Hence we have the following

Lemma 12.14 *The map $\Phi\colon \Lambda\mathbb{X} \to \mathrm{L}\mathbb{X}$ induces a functorial map of stacks $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.*

In particular there is an induced map $\Phi_*\colon H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$ in homology.

Theorem 12.15 *Let $\mathfrak{X}$ be an oriented Hurewicz stack. The map $\Phi_*\colon (H_\bullet(\Lambda\mathfrak{X}), \bullet, \delta) \to (H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ is a morphism of Frobenius algebras.*

PROOF. Let $\mathbb{X}$ be a groupoid representing $\mathfrak{X}$. For any $(g,h) \in S\mathbb{X} \rtimes X_1 = \Lambda\mathbb{X}_1$, one has

$$\mathrm{ev}_0\big(\Phi(g,h)\big) = h = \mathrm{ev}_0(g,h)$$

where ev_0 stands for both evaluation maps $\mathrm{L}\mathbb{X} \to \mathbb{X}$, $\Lambda\mathbb{X} \to \mathbb{X}$. Thus the cartesian square of Step (2) in the construction of the hidden loop product factors through the one of the loop product and we have a tower of cartesian squares:

$$\begin{tikzcd}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{\Phi}"'] & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \arrow[d, "\Phi\times\Phi"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r] \arrow[d] & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_0"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd} \tag{12.14}$$

where $\widetilde{\Phi}$ is induced by $\Phi \times \Phi$. The square (12.14) shows that

$$\Delta^! \circ \big(\Phi \times \Phi\big)_* \quad = \quad \widetilde{\Phi}_* \circ \Delta^!. \tag{12.15}$$

Since $\mathrm{L}\mathbb{X}$ is a presentation of $\mathrm{L}\mathfrak{X}$, the cartesian square $\mathrm{L}\mathbb{X} \times_{\mathbb{X}} \mathrm{L}\mathbb{X}$ presents the stack $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$. Given any (g_1, g_2, h) in $\big(S\mathbb{X} \times_{X_0} S\mathbb{X}\big) \rtimes X_1 = \Lambda\mathbb{X} \times_{\mathbb{X}} \Lambda\mathbb{X}$, one can form a commutative diagram $\widetilde{\Phi}(g_1, g_2, h)$:

$$\begin{tikzcd}
t(h) & t(h) \arrow[l, bend right, "g_1"'] & t(h) \arrow[l, bend right, "g_2"'] \\
s(h) \arrow[u, "h"] & s(h) \arrow[u, "h"'] \arrow[l, bend left, "h^{-1}g_1h"] & s(h), \arrow[u, "h"'] \arrow[l, bend left, "h^{-1}g_2h"]
\end{tikzcd}$$

which induces canonically an arrow of $\mathrm{L}\mathbb{X} \times_{\mathbb{X}} \mathrm{L}\mathbb{X}$ as in the construction of Φ. The map $(g_1, g_2, h) \mapsto \widetilde{\Phi}(g_1, g_2, h)$ presents the stack morphism $\widetilde{\Phi}$. Since

$$m\big(\widetilde{\Phi}(g_1, g_2, h)\big) \quad = \quad \Phi(g_1 g_2, h)$$

the diagram

$$\begin{tikzcd}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r] \arrow[d, "\widetilde{\Phi}"'] & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "m"] \\
\Lambda\mathfrak{X} \arrow[r, "\Phi"] & \mathrm{L}\mathfrak{X}
\end{tikzcd} \tag{12.16}$$

is commutative. Hence, diagram (12.16) and Equation (12.15) implies that Φ_* is an algebra morphism. Similarly Φ_* is a coalgebra morphism since the diagram

$$\begin{tikzcd}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[r, "m"] \arrow[d, "\widetilde{\Phi}"'] & \Lambda\mathfrak{X} \arrow[d, "\Phi"] \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[r, "m"] \arrow[d] & \mathrm{L}\mathfrak{X} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_{1/2})"] \\
X \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd}$$

is commutative. □

Remark 12.16 If the stack $\mathfrak{X}$ is actually a manifold X, then its inertia stack is X itself and $\mathrm{L}\mathfrak{X} = LX$ the free loop space of X. It is clear that the map Φ becomes the usual inclusion $X \hookrightarrow LX$ identifying X with constant loops. For manifolds, the map Φ_* is injective but not surjective (except in trivial cases). However, for general stacks, Φ_* is not necessary injective nor surjective. See Section 17.4.

13 The BV-algebra on the homology of free loop stack

13.1 BV-structure

In this section we construct a **BV**-algebra structure on the homology of the loop stack. First we recall the definition of a **BV**-algebra.

A **Batalin-Vilkovisky algebra** (**BV-algebra** for short) is a graded commutative associative (non necessarily unital) algebra with a degree 1 operator D such that $D(1) = 0$, $D^2 = 0$, and the following identity is satisfied:

$$D(abc) - D(ab)c - (-1)^{|a|}aD(bc) - (-1)^{(|a|+1)|b|}bD(ac) + \\ + D(a)bc + (-1)^{|a|}aD(b)c + (-1)^{|a|+|b|}abD(c) = 0. \quad (13.1)$$

In other words, D is a second-order differential operator.

Now, let $\mathfrak{X}$ be a topological stack and $\mathrm{L}\mathfrak{X}$ its loop stack. The circle S^1 acts on itself by left multiplication. By functoriality of the mapping stack, this S^1-action confers an S^1-action to $\mathrm{L}\mathfrak{X}$ for any topological stack $\mathfrak{X}$. This action endows $H_\bullet(\mathrm{L}\mathfrak{X})$ with a degree one operator D as follows. Let $[S^1] \in H_1(S^1)$ be the fundamental class. Then a linear map $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ is defined by the composition

$$H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\times[S^1]} H_{\bullet+1}(\mathrm{L}\mathfrak{X} \times S^1) \xrightarrow{\rho_*} H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

where the last arrow is induced by the action $\rho : S^1 \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.

Lemma 13.1 *The operator D satisfies $D^2 = 0$, i.e. is a differential.*

Proof. Write $m : S^1 \times S^1 \to S^1$ for the group multiplication on S^1. The naturality of the cross product implies, for any $x \in \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$, that

$$D^2(x) = \rho_*\big(m_*([S^1] \times [S^1]) \times x)\big) = 0$$

since $m_*([S^1] \times [S^1]) \in H_2(S^1) = 0$. □

Theorem 13.2 *Let $\mathfrak{X}$ be an oriented (Hurewicz[3].) stack of dimension d. Then the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ admits a* **BV***-algebra structure given by the loop product $\star$: $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ and the operator $D : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$.*

Remark 13.3 We will give a proof of Theorem 13.2 using conformal field theory in Section 14.2. However, here we wish to give a direct proof, hoping this proof can also be applied to some family of commutative groups as introduced in Section 11.

[3] Recall that any differentiable stack is Hurewicz

13.2 Gerstenhaber bracket and proof of Theorem 13.2

We start by some well-known facts on **BV**-algebras. Let $(A, \cdot, D)$ be a **BV**-algebra. We can define a degree 1 binary operator $\{\,;\,\}$ by the following formula:

$$\{a;b\} \quad = \quad (-1)^{|a|}D(a\cdot b) - (-1)^{|a|}D(a)\cdot b - a\cdot D(b) \tag{13.2}$$

The **BV**-identity (13.1) and commutativity of the product implies that $\{\,;\,\}$ is a derivation of each variable (and anti-symmetric with respect to the degree shifted down by 1). Further the relation $D^2 = 0$ implies the (graded) Jacobi identity for $\{\,;\,\}$. In other words, $(A, \cdot, \{\,;\,\})$ is a *Gerstenhaber* algebra, that is, a commutative graded algebra equipped with a bracket $\{\,;\,\}$ that makes $A[1]$ a graded Lie algebra and satisfying a graded Leibniz rule [23].

Indeed it is standard (see [33]) that a graded commutative algebra $(A, \cdot)$ equipped with a degree 1 operator D, such that $D^2 = 0$, is a **BV**-algebra if and only if the operator $\{\,;\,\}$ defined by the formula (13.2) is a derivation of the second variable, that is

$$\{a;bc\} = \{a;b\}\cdot c + (-1)^{|b|(|a|+1)}b\cdot\{a;c\}. \tag{13.3}$$

By Theorem 10.1 and Lemma 13.1, the shifted homology $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, D)$, equipped with the loop product and operator D induced by the circle action on $\mathrm{L}\mathfrak{X}$, is a graded commutative algebra and $D^2 = 0$. In order to prove Theorem 13.2, we will thus prove the identity (13.3). First, we identify the bracket $\{\,;\,\}$ given by formula (13.2). We need to introduce some notations to do so.

Let

$$\mathrm{ev}_\rho : S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$$

be the (twisted by ρ) evaluation map defined by

$$\mathrm{ev}_\rho(t,\gamma,\beta) = \begin{cases} \mathrm{ev}_0(\gamma)\times \mathrm{ev}_{2t}(\beta) & \text{if } \ 0\le t\le 1/2 \\ \mathrm{ev}_{2t-1}(\gamma)\times \mathrm{ev}_1(\beta) & \text{if } \ 1/2\le t\le 1 \end{cases}$$

where $\mathrm{ev}_x : \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ is the evaluation map defined in Section 5.1. We let $L_\rho(\mathfrak{X}\times\mathfrak{X})$ be the pullback stack of the diagonal along ev_ρ:

$$\begin{tikzcd} L_\rho(\mathfrak{X}\times\mathfrak{X}) \arrow[r, "i_\rho"] \arrow[d] & S^1\times \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_\rho"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X}\times\mathfrak{X} \end{tikzcd} \tag{13.4}$$

Note that $L_\rho(\mathfrak{X}\times\mathfrak{X}) \cong S^1 \times \mathrm{Map}(S^1\vee S^1, \mathfrak{X})$ (by Lemma 5.2) and that, under this identification, i_ρ becomes the map

$$(t,\gamma) \mapsto \begin{cases} \big(t, \gamma^{(1)}, \rho(2t)(\gamma^{(2)})\big) & \text{if } \ 0\le t\le 1/2 \\ \big(t, \rho(2t-1)(\gamma^{(1)}), \gamma^{(2)}\big) & \text{if } \ 1/2\le t\le 1 \end{cases}$$

where $\gamma \mapsto (\gamma^{(1)}, \gamma^{(2)})$ is the map

$$\mathrm{Map}(S^1 \vee S^1, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}.$$

For any $0 \leq t \leq 1/2$, we have a pinching map $p_t : S^1 \to S^1 \vee S^1$ defined by

$$p_t(u) = \begin{cases} (0, 2u - 2t) & \text{if } \ 0 \leq u \leq t \\ (2u - 2t, 0) & \text{if } \ t \leq u \leq t + 1/2 \\ (0, 2u - 1 - 2t) & \text{if } \ t + 1/2 \leq u \end{cases}$$

Here, we have identified $S^1 \vee S^1$ with the union of two basic circles of $S^1 \times S^1$. For $1/2 \leq t \leq 1$, we similarly define

$$p_t(u) = \begin{cases} (2u - 2t + 1, 0) & \text{if } \ u \leq t - 1/2 \\ (0, 2u - 2t + 1) & \text{if } \ t - 1/2 \leq u \leq t \\ (2u - 2t, 0) & \text{if } \ u \geq t \end{cases}$$

Note that $p_0 = p_1$ is the pinching map of Section 10. We take $m_\rho : L_\rho(\mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ to be the map $(t, \gamma) \mapsto \gamma \circ p_t$ induced by the above pinching maps.

Remark 13.4 Informally, the map m_ρ can be described as follow. An element in $L_\rho(\mathfrak{X} \times \mathfrak{X})$ is given, for each $1/2 \geq t \in S^1$, by two loops $a, b \in \mathrm{L}\mathfrak{X}$ such that $\mathrm{ev}_0(a) = \mathrm{ev}_{2t}(b)$. Then $m_\rho(t, a, b)$ is the loop starting at $\mathrm{ev}_0(b)$, describing b until it reaches $\mathrm{ev}_{2t}(b) = \mathrm{ev}_0(a)$ where it follows the loop a and then follows b back to $\mathrm{ev}_0(b)$. There is a similar picture for $t \geq 1/2$. Note that, in the proof of Theorem 13.2, we will use several times an informal description similar to the one of m_ρ to describe various maps that are rigorously defined using a parametrized pinching procedure as above.

When $\mathfrak{X}$ is an oriented stack of dimension d, then the pullback diagram (13.4) induces a Gysin map $\Delta_\rho^! : H_\bullet\big(S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}\big) \to H_{\bullet+d}\big(L_\rho(\mathfrak{X} \times \mathfrak{X})\big)$.

Lemma 13.5 *If $\mathfrak{X}$ is an oriented stack and $a, b \in \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$, one has*

$$\{a; b\} = m_{\rho_*} \circ \Delta_\rho^!\Big([S^1] \times a \times b\Big).$$

Proof. The proof is the stack analogue of a result of Tamanoi [65, Theorem 5.4] and [65, Definition 3.1]. To apply the proof of [65], we only need to use the evaluation maps as we did to define $L_\rho(\mathfrak{X} \times \mathfrak{X})$ and Gysin maps induced by the pullback along the diagonal $\Delta \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ which is strongly oriented by assumption. Then all identities involving Gysin maps in [65] follow using the Gysin map given by the bivariant theory (see Section 9.1, and the techniques of the proofs of Theorem 10.1 and Theorem 12.5) so that the proof of the Lemma for manifolds [65] goes through the category of oriented stacks. $\square$

Proof of Theorem 13.2. We already have proved that $(\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}), \star)$ is a graded commutative algebra (see Theorem 10.1) and that the operator $D\colon \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$ squares to zero: $D^2 = 0$ (Lemma 13.1). Thus, we only need to prove identity (13.3) in order to prove Theorem 13.2.

By Lemma 13.5, given $a, b, c \in H(\mathrm{L}\mathfrak{X})$, the left hand side of identity (13.3) is $f\big([S^1] \times a \times b \times c\big)$ where $f : H(S^1 \times \big(\mathrm{L}\mathfrak{X}\big)^3) \to H(\mathrm{L}\mathfrak{X})$ is the composition

$$H(S^1 \times \big(\mathrm{L}\mathfrak{X}\big)^3) \overset{\mathrm{id}\times\Delta^!}{\longrightarrow} H(S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$$
$$\overset{\mathrm{id}\times m_*}{\longrightarrow} H(S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \overset{\Delta_\rho^!}{\longrightarrow} H(L_\rho(\mathfrak{X}\times\mathfrak{X})) \overset{m_{\rho_*}}{\longrightarrow} H(\mathrm{L}\mathfrak{X}).$$

We denote $L_{\rho,1}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X})$ the pullback

$$\begin{tikzcd}
L_{\rho,1}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[r] \arrow[d, "\mu"'] & S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{id}\times m"] \\
L_\rho(\mathfrak{X}\times\mathfrak{X}) \arrow[r, "i_\rho"] & S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}
\end{tikzcd}$$

of $S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ along i_ρ. We also denote $m_{\rho,1} : L_{\rho,1}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ the composition $m_\rho \circ \mu$. Consider the following tower of pullback squares:

$$\begin{tikzcd}
L_{\rho,1}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[r] \arrow[d, "\mu"'] & S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{id}\times m"] \\
L_\rho(\mathfrak{X}\times\mathfrak{X}) \arrow[r, "i_\rho"] \arrow[d] & S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_\rho"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X}\times\mathfrak{X}
\end{tikzcd} \tag{13.5}$$

The naturality of Gysin maps with respect to this tower yields that

$$\{a; b\star c\} \quad = \quad m_{\rho,1_*} \circ \Delta^!_{\rho,1} \circ (\mathrm{id}\times\Delta^!)\big([S^1] \times a \times b \times c\big) \tag{13.6}$$

where $\Delta^!_{\rho,1} : H(S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \to H(L_{\rho,1}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}))$ is the Gysin map obtained by pulling-back the Gysin map of the strongly oriented diagonal $\Delta : \mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ along $\mathrm{ev}_\rho \circ(\mathrm{id}\times m)$.

Now the main point is to analyze the composition $\Delta^!_{\rho,1} \circ (\mathrm{id}\times\Delta^!)$. By Section 9.1 each Gysin map is obtained by pulling back the normally non-singular diagram of the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ (see Definition 8.21) along, respectively, $\mathrm{ev}_\rho \circ(\mathrm{id}\times m)$ and $\mathrm{id}\times \mathrm{ev}_0 \times \mathrm{ev}_0$ and taking the Thom class of the associated diagram.

Recall that each normally non-singular (nns for short) diagram yields a tubular neighborhood in the sense of Definition 8.5 (after replacing the target by a fiber bundle) and similarly after taking pullbacks. Then the composition $\Delta^!_{\rho,1} \circ (\mathrm{id} \times \Delta^!)$ is the product in the bivariant theory (see Section 7) of these Thom classes. It is essentially obtained by considering fiber products (over $S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$) of the above pulled-back normally non singular diagrams; more precisely by taking suitable pullbacks of the (pulled-back along $\mathrm{ev}_\rho \circ (\mathrm{id} \times m)$ and $\mathrm{id} \times \mathrm{ev}_0 \times \mathrm{ev}_0$) tubular neighborhoods (as in Section 7.4) and composing them in a way similar to the proof of Lemma 8.13. As in Definition 8.21, we let $\theta_\Delta \in H^d(\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X})$ be the strong orientation class of $\mathfrak{X}$ and $\theta_{\Delta^{(2)}} \in H^{2d}(\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X})$ be the strong orientation class of the iterated diagonal (see Corollary 8.31).

In order to carry on the analysis, we divide the circle S^1 into the joint of 4-intervals I_i ($i = 1, \dots, 4$) corresponding to $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ (here we identify $S^1 = [0,1]/(0 \sim 1)$) with the obvious identifications. Note that the regular embeddings (induced by the nns diagram of the diagonals) inducing the Thom classes can be obtained by gluing together the regular embeddings restricted over each $I_i \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$, that is by taking the fiber product (over the regular embeddings obtained by restricting to $\{i/4\} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$) of the regular embeddings over each $I_i \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$. We first consider a restriction of $S^1 \times (\mathrm{L}\mathfrak{X})^3$ to $[1/2, 1] \times (\mathrm{L}\mathfrak{X})^3$. It yields a commutative diagram of pullback squares:

$$\begin{tikzcd}
P_{\rho,1}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \arrow[r] \arrow[d, hook] & {[\tfrac{1}{2}, 1] \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}} \arrow[r] \arrow[d, hook] & {[\tfrac{1}{2}, 1] \times (\mathrm{L}\mathfrak{X})^3} \arrow[dd, "\mathrm{ev}_{\rho \times \rho}"] \\
L_{\rho,1}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \arrow[r, "i_\rho"] \arrow[d] & S^1 \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_\rho^{(2)}"] & \\
\mathfrak{X}^2 \arrow[r, "\Delta \times \mathrm{id}"'] & \mathfrak{X}^3 \arrow[r, "\tau_{34} \circ (\mathrm{id} \times \Delta \times \mathrm{id})"'] & \mathfrak{X}^4
\end{tikzcd} \tag{13.7}$$

where $\tau_{3,4} : \mathfrak{X}^4 \to \mathfrak{X}^4$ switches the last two factors. The vertical map $\mathrm{ev}_\rho^{(2)}$ is the composition $\Big(\mathrm{ev}_\rho \circ (\mathrm{id} \times m), \pi_1 \circ \mathrm{ev}_\rho\Big)$ (where π_1 is the projection on the first component), that is the map defined by

$$\mathrm{ev}_\rho^{(2)}(t, \gamma, \beta, \eta) = \begin{cases} \big(\mathrm{ev}_0(\gamma), \mathrm{ev}_{2t}(\beta), \mathrm{ev}_0(\gamma)\big) & \text{if } \ 0 \le t \le 1/2 \\ \big(\mathrm{ev}_{2t-1}(\gamma), \mathrm{ev}_1(\beta), \mathrm{ev}_{2t-1}(\gamma)\big) & \text{if } \ 1/2 \le t \le 1 \end{cases}$$

Similarly, the map $\mathrm{ev}_{\rho \times \rho}$ is the map defined by

$$\mathrm{ev}_{\rho \times \rho}(t, \gamma, \beta, \eta) = \big(\mathrm{ev}_{2t-1}(\gamma), \mathrm{ev}_1(\beta), \mathrm{ev}_{2t-1}(\gamma), \mathrm{ev}_1(\eta)\big)$$

for $1/2 \le t \le 1$. By diagram (13.7), the (restriction to $[1/2, 1] \times \big(\mathrm{L}\mathfrak{X}\big)^3$ of the) product ${\mathrm{ev}_\rho^{(2)}}^*(\theta_\Delta) \cdot \big(\mathrm{id}\,\mathrm{ev}_0 \times_e v_0\big)^*(\theta_\Delta)$ is computed by ${\mathrm{ev}_{\rho \times \rho}}^*(\theta_{\Delta^{(2)}})$.

Let Δ_2 be the standard two dimensional simplex

$$\Delta_2 := \{(u, s),\ 0 \le u \le s \le 1\}$$

and consider the following diagram (in which $P_H(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X})$ is defined so that the square is cartesian)

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} & P_H(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[l, "m_H"'] \arrow[r] \arrow[d] & \Delta_2\times\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_H"] \\
& \mathfrak{X}\times\mathfrak{X} \arrow[r, "\Delta\times\Delta"] & \mathfrak{X}^4
\end{tikzcd}$$

Here the vertical (evaluation) map ev_H is defined by

$$\mathrm{ev}_H\left((u,s),a,b,c\right) = \left(\mathrm{ev}_u(a),\mathrm{ev}_0(b),\mathrm{ev}_s(a),\mathrm{ev}_0(c)\right)$$

and $m_H\colon P_H(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X})\to\mathrm{L}\mathfrak{X}$ is defined in a similar way as the map $m_\rho\colon L_\rho(\mathfrak{X}\to\mathfrak{X})\to\mathrm{L}\mathfrak{X}$; that is, $m_H\big((u,s),a,b,c\big)$ (with u,s,a,b,c satisfying the pullback relation) is the loop starting at $\mathrm{ev}_0(a)$, describing a until it reaches $\mathrm{ev}_u(a)$, then describing b back to $\mathrm{ev}_0(b)=\mathrm{ev}_u(a)$, following a again until it reaches $\mathrm{ev}_s(a)$, then following c until reaches $\mathrm{ev}_1(c)=\mathrm{ev}_s(a)$ again and finishes to follow a back to $\mathrm{ev}_1(a)=\mathrm{ev}_0(a)$. Here we use an informal description of the map m_H that can be obtained by a pinching procedure as explained before Lemma 13.5 above.

Let $\iota:[1/2,1]\to\Delta_2$ be the map $\iota(t)=(2t,2t)$. The diagram

$$\begin{tikzcd}
& P_{\rho,1}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[dl, "m_{\rho,1}"'] \arrow[r] \arrow[d] & {[\tfrac{1}{2},1]\times(\mathrm{L}\mathfrak{X})^3} \arrow[d, "\iota\times\mathrm{id}"'] \arrow[dd, bend left=60, "\mathrm{ev}_{\rho\times\rho}"] \\
\mathrm{L}\mathfrak{X} & P_H(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[l, "m_H"'] \arrow[r] \arrow[d] & \Delta_2\times(\mathrm{L}\mathfrak{X})^3 \arrow[d, "\mathrm{ev}_H"'] \\
& \mathfrak{X}\times\mathfrak{X} \arrow[r, "\Delta\times\Delta"] & \mathfrak{X}^4
\end{tikzcd} \tag{13.8}$$

is commutative. Since the subset $\{(u,s),\ u=s\}\subset\Delta_2$ is a boundary component of the dimension 2 simplex, it follows from Lemma 6.6 and diagram (13.8) that, to compute the Thom class of the composition of the top lines in diagram (13.7), we can replace the top line of diagram (13.7) by the restriction to the subset $\{u=0 \text{ or } s=1\}\subset\Delta_2$ (that is the complementary boundary of Δ_2) of the top line of diagram (13.8).

Now we analyze the restriction of diagram (13.8) to $I_4\times(\mathrm{L}\mathfrak{X})^3=[3/4,1]\times(\mathrm{L}\mathfrak{X})^3$ which reduces to the analysis of the pullback square

$$\begin{tikzcd}
P_{4\rho}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[r] \arrow[d] & \left[\tfrac{3}{4},1\right]\times(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X})\times\mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_\rho\times\mathrm{ev}_0"] \\
\mathfrak{X}^2 \arrow[r, "\Delta\times\Delta"] & \mathfrak{X}^4
\end{tikzcd} \tag{13.9}$$

and its induced tubular neighborhood (given by the pullback along $\mathrm{ev}_\rho \times \mathrm{ev}_0$ of the nns diagram of the diagonal). Here the vertical map $\mathrm{ev}_\rho \times \mathrm{ev}_0$ is the map

$$\mathrm{ev}_\rho \times \mathrm{ev}_0(t,a,b,c) = \big(\mathrm{ev}_{4t-3}(a), \mathrm{ev}_0(b), \mathrm{ev}_0(a), \mathrm{ev}_0(c)\big)$$

(and $P_{4\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X})$ is defined by the pullback property). Furthermore, the restriction of the map $m_H \colon P_H(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ (considered above) to this boundary component identifies with the map $m_{4,\rho} : P_{4\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ which maps (t,a,b,c) to the loop starting at $\mathrm{ev}_0(a)$, following a until it reaches $\mathrm{ev}_{4t-3}(a) = \mathrm{ev}_0(b)$ then follows b until it gets back to $\mathrm{ev}_1(b) = \mathrm{ev}_0(b)$, follows a back to $\mathrm{ev}_1(a) = \mathrm{ev}_0(c)$ and then goes through c. Similarly, the restriction of diagram (13.8) to $I_3 \times (\mathrm{L}\mathfrak{X})^3 = [1/2, 3/4] \times (\mathrm{L}\mathfrak{X})^3$ yields a pullback $P_{3\rho}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X})$ similar to the pullback (13.9) as well as a map $m_{3,\rho} \colon P_{3\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$.

Furthermore, restricting diagram (13.5) to $[0,1/4] \times (\mathrm{L}\mathfrak{X})^3$ yields a cartesian square

$$\begin{tikzcd} P_{1\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \arrow[r] \arrow[d] & {[0,\frac{1}{4}]} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_\rho \times \mathrm{ev}_0"] \\ \mathfrak{X}^2 \arrow[r, "\Delta\times\Delta"] & \mathfrak{X}^4 \end{tikzcd} \tag{13.10}$$

where $\mathrm{ev}_\rho \times \mathrm{ev}_0(t,a,b,c) = \big(\mathrm{ev}_0(a), \mathrm{ev}_{4t}(b), \mathrm{ev}_0(b), \mathrm{ev}_0(c)\big)$ and, similary, restricting $[1/4, 1/2] \times (\mathrm{L}\mathfrak{X})^3$, a cartesian diagram

$$\begin{tikzcd} P_{2\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \arrow[r] \arrow[d] & {[\frac{1}{4},\frac{1}{2}]} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \arrow[d, "\mathrm{ev}_\rho \times \mathrm{ev}_0"] \\ \mathfrak{X}^2 \arrow[r, "\Delta\times\Delta"] & \mathfrak{X}^4 \end{tikzcd} \tag{13.11}$$

where $\mathrm{ev}_\rho \times \mathrm{ev}_0(t,a,b,c) = \big(\mathrm{ev}_0(a), \mathrm{ev}_{4t-1}(c), \mathrm{ev}_0(b), \mathrm{ev}_0(c)\big)$. Note also that restricting the map $m_{\rho,1} : L_{\rho,1}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ to $I_1 = [0,1/4]$ and $I_2 = [1/4, 1/2]$ gives rise to two maps $m_{1,\rho} : P_{1\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$ and $m_{2,\rho} : P_{2\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$.

Now, to compute $m_{\rho,1_*} \circ \Delta^!_{\rho,1} \circ (\mathrm{id} \times \Delta^!)$, *i.e.*, the Thom classes

$$(\mathrm{ev}_\rho \times \mathrm{id}) \circ (\mathrm{id} \times m)^*(\theta_\Delta) \cdot (\mathrm{id} \times \mathrm{ev}_0 \times \mathrm{ev}_0)(\theta_\Delta)$$

we are left to study the pullbacks of the tubular neighborhood induced by the normally non-singular diagrams of the iterated diagonal $\Delta \times \Delta$ along the 4 various $\mathrm{ev}_\rho \times \mathrm{ev}_0$-maps (corresponding to the spaces $P_{i,\rho}(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X})$) and to join them (as noted above). First, we remark that the restrictions to the points $\{1/4\}$ and $\{3/4\}$ of the cartesian squares (13.10), (13.9), (13.11) and of the maps $m_{i,\rho}$ are identical. Thus it is enough to study first the joint

$$S^1_+ := [3/4, 1] \cup [0, 1/4]/(3/4 \sim 1/4)$$

of I_1 and I_4 with the boundary points identified and the Thom class induced by the cartesian squares (13.9), (13.10). Note that diagram (13.9) factors into the following diagram whose vertical squares and top horizontal square are cartesian:

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} & & \mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X} \arrow[ll, "m_\rho"'] \arrow[rrr, hook] \arrow[ddd] & & & \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[ddd] & \\
P_{4\rho}(\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}) \arrow[u, "m_{4,\rho}"] \arrow[urr, "m_{4,\rho}"] \arrow[rrr, hook] \arrow[ddd] & & & P_{4\rho}(\mathfrak{X}\times\mathfrak{X})\times\mathrm{L}\mathfrak{X} \arrow[urr, "m_\rho^+\times\mathrm{id}"] \arrow[rrr, hook] \arrow[ddd] & & & {[\frac{3}{4},1]}\times(\mathrm{L}\mathfrak{X})^3 \arrow[ddd, "\mathrm{ev}_\rho\times\mathrm{ev}_0"] \\
& & & & & & \\
& & \mathfrak{X} \arrow[rrr, "\Delta"'] & & & \mathfrak{X}\times\mathfrak{X} & \\
\mathfrak{X}^2 \arrow[urr, "\pi_2"'] \arrow[rrr, "\mathrm{id}\times\Delta"'] & & & \mathfrak{X}\times\mathfrak{X}^2 \arrow[urr, "\pi_{2,3}"'] \arrow[rrr, "\Delta\times\mathrm{id}"'] & & & \mathfrak{X}^4
\end{tikzcd} \tag{13.12}$$

Here π_2 and $\pi_{2,3}$ denote the projection on the last factors. By functoriality and naturality of the construction of Gysin maps, we get that

$$m_{4,\rho_*}\big((\mathrm{ev}_\rho\times\mathrm{ev}_0)^*(\theta_{\Delta\times\Delta})\big) = m_{\rho_*}\big((\mathrm{ev}_0\times\mathrm{ev}_0)^*(\theta_\Delta)\big)\cdot(m_\rho^+\times\mathrm{id})_*\big((\mathrm{ev}_\rho\times\mathrm{ev}_0)^*(\theta_\Delta\big)$$

There is a diagram similar to (13.12) associated to the cartesian square (13.10). Joining these two diagrams, we get:

$$\begin{tikzcd}
\mathrm{L}\mathfrak{X} & \mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X} \arrow[l, "m_\rho"'] \arrow[rr, hook] \arrow[dd] & & \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \arrow[dd, "\mathrm{ev}_0"] & \\
& & L_{1,\rho}(\mathfrak{X}\times\mathfrak{X})\times\mathrm{L}\mathfrak{X} \arrow[ur, "m_\rho^+\times\mathrm{id}"] \arrow[rr, hook] \arrow[dd] & & S^1_+\times(\mathrm{L}\mathfrak{X})^3 \arrow[dd, "\mathrm{ev}_\rho"] \\
& \mathfrak{X} \arrow[rr, "\Delta"'] & & \mathfrak{X}\times\mathfrak{X} & \\
& & \mathfrak{X} \arrow[rr, "\Delta"'] & & \mathfrak{X}^2
\end{tikzcd} \tag{13.13}$$

which exhibits the restriction to $S^1_+\times\big(\mathrm{L}\mathfrak{X}\big)^3$ of $m_{1,\rho_*}\big(\Delta^!_{\rho,1}\circ(\mathrm{id}\times\Delta^!)\big)$ as $m_{\rho_*}\Big(\mathrm{ev}_0^*(\theta_\Delta)\cdot(m_\rho^+\times\mathrm{id})_*\big(\mathrm{ev}_0^*(\theta_\Delta)\big)\Big)$. Note that the above diagram (13.13) is precisely the diagram defining $\{a,b\}\star c$; in other words,

$$\begin{aligned} & m_{\rho_*}\Big(\mathrm{ev}_0^*(\theta_\Delta)\cdot(m_\rho^+\times\mathrm{id})_*\big(\mathrm{ev}_0^*(\theta_\Delta)\big)\Big)\big([S^1_+]\times a\times b\times c\big) \\ &\qquad = m_{\rho_*}\circ\Delta^!\circ(m_\rho^+\times\mathrm{id})_*\circ\Delta^!_\rho\big([S^1_+]\times a\times b\times c\big) \;=\; \{a,b\}\star c \end{aligned}$$

The above arguments for S^1_+ apply similarly to study the joint

$$S^1_- := [3/4,1]\cup[0,1/4]/(3/4\sim 1/4)$$

of I_1 and I_4 with the boundary points identified. It yields that

$$m_{\rho_*}\Big(\mathrm{ev}_0^*(\theta_\Delta)\cdot(m_\rho^-\times\mathrm{id})_*\big(\mathrm{ev}_0^*(\theta_\Delta)\big)\Big)\big([S^1_+]\times a\times b\times c\big) \;=\; (-1)^{|b||c|}\{a,c\}\star b$$

where the sign comes from the fact that one has to exchange b and c in that case (with a transposition similar to the one appearing in the bottom line of diagram (13.7)). Recall from above that the restrictions to the points $\{1/4\}$ and $\{3/4\}$ of the cartesian squares (13.10), (13.9), (13.11) and of the maps $m_{i,\rho}$ are identical. It follows that the computation of the Gysin maps for $S^1 \times \big(\mathrm{L}\mathfrak{X}\big)^3$ factors through the one of $(S^1_+ \vee S^1_-) \times \big(\mathrm{L}\mathfrak{X}\big)^3$. Thus, we deduce from the above computations for S^1_+ and S^1_- and identity (13.6) that

$$\{a, b \star c\} \;=\; \{a,b\} \star c + (-1)^{|b||c|}\{a,c\} \star b$$

that is, identity (13.3) holds, by graded commutativity of the loop product. □

14 Homological conformal field theory and free loop stacks

In this section, we extend the **BV**-structure of Theorem 13.2 and the Frobenius structure of Theorem 12.5 into the whole structure of an *homological conformal field theory* (with positive boundaries) following ideas of Cohen-Godin [22, 34] for manifolds and Chataur-Menichi [18] for classifying spaces of groups. As in Section 12 (and for the same reasons), we assume in this section that our ground ring k is a field. Note that, unlike in Godin's paper [34], we can only allow *closed* boundaries and a positive number of *both* incoming and outgoing boundaries components.

14.1 Quick review on Homological Conformal Field theory with positive boundaries

We start by recalling some definitions of Homological Conformal Field theories. We will make strong restrictions on the type of boundary we consider (which simplify greatly the theory). We follow [26, 27, 34, 61].

A (closed)[4] homological conformal field theory is an algebra over the PROP of the homology of the stack (or moduli space) of compact oriented Riemann surfaces or, equivalently a symmetric monoidal functor from the (homology of the) Segal category of Riemann surfaces [61] to the category of graded vector spaces. Let us start by giving more details on what this definition means, following [25, 26, 27, 33].

We first recall that, a *complex cobordism* from a family $\coprod_{i=1}^n S^1$ of circles to another family $\coprod_{i=1}^m S^1$ of circles is a closed (non-necessarily connected) Riemann surface Σ equipped with two holomorphic embeddings (with disjoint images) $\rho_{in}\colon \coprod_{i=1}^n D^2 \hookrightarrow \Sigma$ and $\rho_{out}\colon \coprod_{i=1}^M D^2$ of closed disks. The image of ρ_{in} is called the *incoming* boundary and the image of ρ_{out} the outgoing boundary. Two complex cobordism Σ_1 and Σ_2 (from $\coprod_{i=1}^n S^1$ to $\coprod_{i=1}^m S^1$) are equivalent

[4] unless otherwise stated, we only consider closed cobordism in this paper

if there exists a biholomorphism $h\colon \Sigma_1 \overset{\sim}{\to} \Sigma_2$ which fixes the boundary (*i.e.* commutes with ρ_{in} and ρ_{out}).

We denote $\mathfrak{M}_{n,m}$ the moduli space of equivalence classes of complex cobordism from $\coprod_{i=1}^n S^1$ to $\coprod_{i=1}^m S^1$, that is the (coarse moduli space of the) differentiable stack $[\mathfrak{S}_{n,m}/Bihol]$ obtained as the quotient of the space $\mathfrak{S}_{n,m}$ of holomorphic embeddings of disks inside compact Riemann surfaces by the group of biholomorphism fixing the boundary. Note that, there is an isomorphism of stacks

$$[\mathfrak{S}_{n,m}/Bihol] \cong \coprod_{[\Sigma]} [*/\Gamma_{n,m}(\Sigma)]$$

where the union is over a set of representatives of the isomorphism classes of cobordisms (with n incoming and m outgoing closed boundary components) and

$$\Gamma_{n,m}(\Sigma) \colon = \pi_0\Big(\underset{n,m}{\overset{+}{Diff}}(\Sigma)\Big)$$

is the isotopy classes of the group $Diff^+_{n,m}(\Sigma)$ of oriented diffeomorphisms preserving the boundaries pointwise of a surface Σ with n incoming and m outgoing closed boundary components.

The disjoint union of surfaces yields a canonical morphism

$$\mathfrak{M}(n,m) \times \mathfrak{M}(n',m') \to \mathfrak{M}(n+n', m+m').$$

Further, given $\Sigma_1 \in \mathfrak{M}(\ell, n)$ and $\Sigma_2 \in \mathfrak{M}(n,m)$, using the embeddings of disks

$$\Sigma_1 \hookleftarrow \coprod_{i=1}^n D^2 \hookrightarrow \Sigma_2,$$

we can glue Σ_2 on Σ_1 along their common boundary. We denote $\Sigma_2 \circ \Sigma_1 \in \mathfrak{M}_{\ell,m}$ the Riemann surface thus obtained. Applying the singular homology functor to the above operations yields linear maps

$$H_\bullet(\mathfrak{M}_{n,m})) \otimes H_\bullet(\mathfrak{M}_{n',m'}) \overset{H_\bullet(\coprod)}{\longrightarrow} H_\bullet(\mathfrak{M}_{n+n',m+m'})$$

and

$$H_\bullet(\mathfrak{M}_{\ell,n}) \otimes H_\bullet(\mathfrak{M}_{n,m})) \overset{H_\bullet(\circ)}{\longrightarrow} H_\bullet(\mathfrak{M}_{\ell,m})$$

that satisfy natural associativity and compatibility relations. It follows that the collection $\big(H_\bullet(\mathfrak{M}_{n,m})\big)_{n,m\geq 0}$ are the morphisms of a graded linear symmetric monoidal category $\mathcal{C}_{\mathfrak{M}}$ whose objects are the nonnegative integers $n \in \mathbb{N}$ and the monoidal structure is induced by $k \otimes \ell = k + \ell$ on the objects and disjoint union of surfaces on morphisms.

A *homological conformal field theory* is a symmetric monoidal functor from the category $\mathcal{C}_{\mathfrak{M}}$ to the symmetric monoidal category of graded vector spaces (equipped with the usual graded tensor product). Informally, this definition simply means that an homological conformal field theory is a graded vector space

A with an operation $\mu(c)\colon A^{\otimes n} \to A^{\otimes m}$ for any homology class $c \in H_\bullet(\mathfrak{M}_{n,m})$ such that $\mu(c \circ d) = \mu(c) \circ \mu(d)$ and $\mu(c \coprod d) = \mu(c) \otimes \mu(d)$.

Unlike oriented closed manifolds, oriented stacks do not have unit for the loop product (nor counit) in general (see Section 17.4 for instance). This forces us to consider *non-unital and non-counital* homological conformal field theory which are symmetric monoidal functor from the category $\mathcal{C}_{\mathfrak{M}}^{nu,nc}$ to the category of graded vector spaces , where $\mathcal{C}_{\mathfrak{M}}^{nu,nc} \subset \mathcal{C}_{\mathfrak{M}}$ is the (monoidal) subcategory obtained by considering only cobordisms in $\mathfrak{M}_{n,m}$ for which every connected component has at least one ingoing *and* one outgoing boundary component. We may also refer to such algebraic structure as an homological conformal field theory with positive (closed) boundaries.

We wish to make the homology $H_\bullet(\mathrm{L}\mathfrak{X})$ of the free loop stack of an oriented stack an homological conformal field theory (without (co)units). However, since the basic operations we consider are non-trivially graded (for instance the loop product is of degree $\dim(\mathfrak{X})$), we need to plug in a notion of dimension in the definition of conformal field theories to take care of this phenomenon and encode the sign issues. We follow the ideas and presentation of Costello [27] and Godin [34], where the grading is taken into account by a local coefficient system $\det^{\otimes d}$ on the moduli spaces $\mathfrak{M}_{n,m}$.

The *local coefficient system* det is a graded invertible locally constant sheaf (*i.e.* a graded k-linear locally constant sheaf of dimension 1). To a closed Riemann surface $\Sigma \in \mathfrak{M}_{n,m}$, we associate a compact Riemann surface Σ^{bd} with boundary by removing from Σ the interior of (the images of) the closed disks $\rho_{in}\colon \coprod_{i=1}^n D^2 \hookrightarrow \Sigma$ and $\rho_{out}\colon \coprod_{i=1}^m D^2 \hookrightarrow \Sigma$. Restricting ρ_{in} to the boundary $\coprod_{i=1}^n S^1$ of the disks, we get a diffeomorphism from $\coprod_{i=1}^n S^1$ onto the incoming boundary of Σ^{bd}. Following [27] and [34, Section 4.1], we define the fibre of the local coefficient system det at a surface $\Sigma \in \mathfrak{M}_{n,m}$ to be

$$\det(\Sigma)\colon = \det\Big(H^0\big(\Sigma^{bd}, \rho_{in}\big(\coprod_{i=1}^n S^1\big)\big)\Big) \otimes \det\Big(H_1\big(\Sigma^{bd}, \rho_{in}\big(\coprod_{i=1}^n S^1\big)\big)\Big).$$

Here, given a finite dimensional k-vector space V, det denotes the determinant, that is $\det(V) = \bigwedge^{\dim(V)} V$ is the top exterior power of V, and we consider the (relative) homology groups of a pair. This defines the local coefficient system det on $(\mathfrak{M}_{n,m})_{n,m\geq 0}$ and similarly, for an integer d, the local coefficient system $\det^{\otimes d}$ obtained by tensoring det with itself d-times.

It is proved in [27, 34, 18] that the composition of surface induces a natural isomorphism $\det(\Sigma_2)\otimes\det(\Sigma_1) \to \det(\Sigma_2 \circ \Sigma_1)$ which is associative and compatible with the canonical isomorphism $\det(\Sigma_1' \coprod \Sigma_2') \cong \det(\Sigma_1') \otimes \det(\Sigma_2')$. This allows us to see the collection of homology groups $\Big(H_\bullet\big(\mathfrak{M}_{n,m}; \det^{\otimes d}\big)\Big)_{n,m\geq 0}$ with value in the local coefficient $\det^{\otimes d}$ as the morphism of a graded linear symmetric monoidal category $\mathcal{C}_{\mathfrak{M},\det^{\otimes d}}$, and, as above, we also get a graded linear symmetric monoidal category $\mathcal{C}_{\mathfrak{M},\det^{\otimes d}}^{nu,nc}$ by restricting to cobordism with at least one incoming and one outgoing boundary on each connected component.

According to [27, 34], we have the following

Definition 14.1 A (non-unital, non-counital) *d-dimensional homological conformal field theory* is a symmetric monoidal functor from the category $\mathcal{C}^{nu,nc}_{\mathfrak{M},\det^{\otimes d}}$ to the category of graded vector spaces.

14.2 The Homological Conformal Field Theory with positive closed boundaries associated to free loop stacks

It is known (see [33, 25, 53]) that an (non-unital) homological conformal field theory (HCFT for short) carries the structure of a (non-unital) **BV**-algebra as well as that of a Frobenius algebra (without (co)unit). For instance the associative and commutative operation of the **BV** or Frobenius structure is induced by the pair of pants surface lying in $\mathfrak{M}_{2,1}$. The main result of this Section enriches the above structures already obtained for loop stacks into an HCFT over $\mathcal{C}^{nu,nc}_{\mathfrak{M},\det^{\otimes d}}$.

Theorem 14.2 *Let $\mathfrak{X}$ be an oriented (Hurewicz[5]) stack of dimension d. There is a d-dimensional non-unital, non-counital homological conformal field theory on the homology $H_\bullet(\mathrm{L}\mathfrak{X})$ of the free loop stack which induces the* **BV***-algebra and Frobenius structure on the homology $H_\bullet(\mathrm{L}\mathfrak{X})$ given by Theorem 13.2 and Theorem 12.5.*

Remark 14.3 The proof of Theorem 14.2 actually implies the ones of Theorems 13.2 and Theorem 12 as well. However, this proof does *not* apply to prove similar statements (for instance Theorem 12.3) for inertia stack (and thus to define the intersection pairing as in Section 16) or any other family of groups over a stack considered in Section 11. Further, it is not obvious that this proof will also apply to the twisted versions of the loop product studied in Section 10 and aforementioned in Section 16.2.

To prove the above Theorem 14.2, we follow the approach of [18, 34, 27], using chord diagrams/ribbon graphs, but using a stack point of view (instead of a purely homotopical one) and the benefits of the bivariant theory of Section 7. We will first determine the value on a particular cobordism $\Sigma_{g,n,m}$ of the HCFT, which will be given by the linear map (14.9) below.

First, we need to recall some preliminaries on Sullivan's chord diagrams and fat graphs which are taken from [42, 22, 26]. By a graph, we mean a pair $G = (V, H)$ consisting of a finite set of vertices V, of half-edges (which can be thought as oriented edges) H equipped with a map $s : H \to V$ and an involutive map with no-fixed points $e \mapsto \overline{e}$ on the set of half-edges. A fat graph is a graph equipped, at each vertex v, with a cyclic ordering of the half-edges emanating from v. The geometric realization of a fat graph is thus a 1 dimensional cell complex plus extra data. It is well-known that the classifying space of fat graphs

[5]Recall that any differentiable stack is Hurewicz

is equivalent to the moduli space of Riemann surfaces (see [42, 26, 27, 66, 34] for much more precised statements). In particular, every (isomorphism class of a) Riemann surface $\Sigma \in \mathfrak{M}_{n,m}$ is a deformation retract of (the geometric realization of) a fat graph with n-incoming boundary cycles and m outgoing ones (we refer to [42, 22] for the definition of these boundary cycles).

A chord diagram (see [22]) is a special kind of (geometric realization) of a fat graph. A *chord diagram* of type (g, n, m) is a union of n disjoint circles with a disjoint unions of trees whose endpoints are glued on the circles (on distinct points), and such that the induced cell complex is (the geometric realization of) a fat graph representing a surface of genus g with $n+m$ boundary components. The first circles are refered to as the incoming circles and the set of (necessarily path-connected) trees will be denoted $\mathcal{T}(c_{g,n,m})$. The set of points in the circle in which the endpoints of the trees lies is denoted $\mathcal{V}(c_{g,n,m})$ (and called the set of circular vertices).

Let $c_{g,n,m}$ be a chord diagram of type (g, n, m) and $\Sigma_{g,n,m}$ be the surface represented by $c_{g,n,m}$. Here g is the genus of $\Sigma_{g,n,m}$, that is the sum of the genera of its components. In particular, the Euler characteristic of $\Sigma_{g,n,m}$ is given by

$$\chi(\Sigma_{g,n,m}) = 2\#\Sigma_{g,m,n} - 2g - n - m$$

where $\#\Sigma_{g,n,m}$ is the number of (arcwise) connected components of $\Sigma_{g,n,m}$. Then $c_{g,n,m}$ is a deformation retract of $\Sigma_{g,n,m}$. We let $r_{g,n,m} : \Sigma_{g,n,m} \to c_{g,n,m}$ be the retraction and $\iota_{g,n,m} : c_{g,n,m} \to \Sigma_{g,n,m}$ be the inclusion. Note that since every connected component is assumed to have positive incoming and outgoing boundary component, $\chi(\Sigma_{g,n,m})$ is always non-positive in our case.

Given a tree $t \in \mathcal{T}(c_{g,n,m})$, we can associate the subset $V(t) \subset \mathcal{V}(c_{g,n,m})$ of circular vertices given by the endpoints of t and we get a canonical inclusion map $\coprod_{v\in\mathcal{V}(c_{g,n,m})}\{pt\} \to \coprod_{t\in\mathcal{T}(c_{g,n,m})} t$. Applying the mapping stack functor, we get a map

$$d_{c_{g,n,m}} : \prod_{t\in\mathcal{T}(c_{g,n,m})} \mathrm{Map}(t, \mathfrak{X}) \longrightarrow \mathfrak{X}^{\mathcal{V}(c_{g,n,m})}$$

which was already considered in [18].

Lemma 14.4 *Let $\mathfrak{X}$ be an oriented stack of dimension d. Then there is an orientation class*

$$\theta_{d_{c_{g,n,m}}} \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\prod_{t\in\mathcal{T}(c_{g,n,m})} \mathrm{Map}(t, \mathfrak{X}) \stackrel{d_{c_{g,n,m}}}{\longrightarrow} \mathfrak{X}^{\mathcal{V}(c_{g,n,m})}\Big).$$

Proof. Each tree is a deformation retract onto any of its vertex, hence we have deformation retract $t \underset{r_t}{\overset{\iota_t}{\leftrightarrows}} pt$ for each $t \in \mathcal{T}(c_{g,n,m})$ and a factorization

$$\begin{tikzcd} \mathfrak{X}^{\mathcal{T}(c_{g,n,m})} \arrow[r, "\prod r_t^*"] \arrow[rr, bend right, "\Delta^{(\#\mathcal{V}(c_{g,n,m})-\#\mathcal{T}(c_{g,n,m}))}"'] & \prod_{t\in\mathcal{T}(c_{g,n,m})} \mathrm{Map}(t, \mathfrak{X}) \arrow[r, "d_{c_{g,n,m}}"] & \mathfrak{X}^{\mathcal{V}(c_{g,n,m})} \end{tikzcd}$$

The bottom line is an iterated diagonal, hence strongly oriented by Corollary 8.31. Then we take $\theta_{d_{c_{g,n,m}}}$ to be the pushforward $\left(\prod r_t^*\right)_*(\theta)$ of the orientation class $\theta \in H^{d(\#\mathcal{V}(c_{g,n,m})-\#\mathcal{T}(c_{g,n,m}))}(\mathfrak{X}^{\mathcal{T}(c_{g,n,m})} \to \mathfrak{X}^{\mathcal{V}(c_{g,n,m})})$ of this iterated diagonal. Since the Euler characteristic of the chord diagram agrees with the one of the surface it represents, we get $\#\mathcal{V}(c_{g,n,m}) - \#\mathcal{T}(c_{g,n,m}) = -\chi(\Sigma_{g,n,m})$ as in [22] and the lemma follows. $\square$

14.3 Construction of the operations

We now define the operations associated to (the isomorphism class of) a Riemann surface $\Sigma_{g,n,m} \in \mathfrak{M}_{n,m}$ (not necessarily connected). We will first define certain quotient stacks of mapping stacks by diffeomorphism groups. Let $c_{g,n,m}$ be a chord diagram representing $\Sigma_{g,n,m}$. We still denote $r_{c_{g,n,m}} : \Sigma_{g,n,m} \to c_{g,n,m}$ the retraction and $\iota_{c_{g,n,m}} : c_{g,n,m} \to \Sigma_{g,n,m}$ the inclusion, which yield an homotopy equivalence $r_{g,n,m}\colon \mathrm{Map}(c_{g,n,m}, \mathfrak{X}) \xrightarrow{\sim} \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})$ (and $\iota_{g,n,m}\colon \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X}) \xrightarrow{\sim} \mathrm{Map}(c_{g,n,m}, \mathfrak{X})$).

The circular vertices, which are the points where the trees in $\mathcal{T}(c_{g,n,m})$ are glued to the n disjoint circles, yields pushout diagram

$$\begin{tikzcd}
\coprod_{v\in\mathcal{V}(c_{g,n,m})}\{v\} \arrow[r, "i_{\mathcal{V}}"] \arrow[d] & \coprod_{i=1}^n S^1 \arrow[d] \\
\coprod_{t\in\mathcal{T}(c_{g,n,m})} t \arrow[r] & c_{g,n,m}
\end{tikzcd}$$

which, by Lemma 5.2, induces a pullback of stacks

$$\begin{tikzcd}
\mathrm{Map}(c_{g,n,m}, \mathfrak{X}) \arrow[r] \arrow[d] & (\mathrm{L}\mathfrak{X})^n \arrow[d, "\mathrm{ev}_{\mathcal{V}}"] \\
\prod_{t\in\mathcal{T}(c_{g,n,m})} \mathrm{Map}(t, \mathfrak{X}) \arrow[r, "d_{c_{g,n,m}}"] & \mathfrak{X}^{\mathcal{V}(c_{g,n,m})}
\end{tikzcd} \tag{14.1}$$

where $\mathrm{ev}_{\mathcal{V}}\colon \left(\mathrm{L}\mathfrak{X}\right)^n \to \mathfrak{X}^{\mathcal{V}(c_{g,n,m})}$ is the evaluation map induced by the inclusions $i_{\mathcal{V}}\colon \coprod_{v\in\mathcal{V}(c_{g,n,m})}\{v\} \to \coprod_{i=1}^n S^1$. We denote

$$\theta^{\mathfrak{X}}_{g,n,m} = \mathrm{ev}_{\mathcal{V}}^*(\theta_{d_{c_{g,n,m}}}) \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\mathrm{Map}(c_{g,n,m}, \mathfrak{X}) \longrightarrow \left(\mathrm{L}\mathfrak{X}\right)^n\Big) \tag{14.2}$$

the bivariant class induced by the pullback diagram (14.1) and the orientation class $\theta_{d_{c_{g,n,m}}}$ of Lemma 14.4.

The retraction $r_{g,n,m}\colon \mathrm{Map}(c_{g,n,m}, \mathfrak{X}) \xrightarrow{\sim} \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})$ sits inside a commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \arrow[dl, bend right, "\iota_{g,n,m}"'] \arrow[dr, "\rho_{in}"] & \\
\mathrm{Map}(c_{g,n,m},\mathfrak{X}) \arrow[ur, "r_{g,n,m}"'] \arrow[rr] & & (\mathrm{L}\mathfrak{X})^n
\end{tikzcd} \tag{14.3}$$

where the map $\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \to (\mathrm{L}\mathfrak{X})^n$ is induced by the inclusion of the incoming boundary in $\Sigma_{g,n,m}$. Applying the pushforward map along $r_{g,n,m}$ given by the above diagram (14.3) to the bivariant class $\theta^{\mathfrak{X}}_{g,n,m}$ (see (14.2)) gives us a bivariant class

$$r_{g,n,m_*}(\theta^{\mathfrak{X}}_{g,n,m}) \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \longrightarrow \big(\mathrm{L}\mathfrak{X}\big)^n\Big). \tag{14.4}$$

Quotienting by diffeomorphisms To shorten notations, we write G_Σ for the group $Diff^+_{n,m}(\Sigma_{g,n,m})$ of oriented diffeomorphisms of $\Sigma_{g,n,m}$ preserving the boundaries pointwise. The group $G_\Sigma = Diff^+_{n,m}(\Sigma_{g,n,m})$ acts on $\Sigma_{g,n,m}$ and thus on $\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})$ (by functoriality, see Section 5.1) and the restriction map $\rho_{in}\colon \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \longrightarrow (\mathrm{L}\mathfrak{X})^n$ is equivariant (where the action on $(\mathrm{L}\mathfrak{X})^n$ is trivial since the boundary is pointwisely fixed). Similarly to the construction of the transformation groupoid of a topological group acting on a space (see Section 1.4), we can pass to the quotient of the above stacks by the G_Σ-action:

Lemma 14.5 *The action of G_Σ on $\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})$ gives rise to a quotient topological stack $\big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big]$ together with an natural topological stack epimorphism $p_{G_\Sigma}\colon \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \to \big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big]$ and, similarly, a quotient stack $\big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]$ with an natural topological stack epimorphism $p_{G_\Sigma}\colon (\mathrm{L}\mathfrak{X})^n) \to \big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]$ such that*

1. *there is a cartesian square*

$$\begin{tikzcd}
G_\Sigma \times \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \arrow[r] \arrow[d, "pr_2"'] & \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \arrow[d, "p_{G_\Sigma}"] \\
\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \arrow[r, "p_{G_\Sigma}"] & \big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big]
\end{tikzcd}$$

   where the top arrow is given by the G_Σ-action as well as a similar cartesian square with $(\mathrm{L}\mathfrak{X})^n$ instead of $\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})$;

2. *The map $p_{G_\Sigma}\colon \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \to \big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big]$ makes $\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})$ a G_Σ-torsor and similarly for $(\mathrm{L}\mathfrak{X})^n$;*

3. There is a topological stack isomorphism $\left[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\right] \cong (\mathrm{L}\mathfrak{X})^n \times \left[\,*\,/G_\Sigma\right]$ and the following diagram is commutative

$$\begin{tikzcd} \left[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\right] \arrow[r, "\cong"] & (\mathrm{L}\mathfrak{X})^n \times \left[\,*\,/G_\Sigma\right] \\ (\mathrm{L}\mathfrak{X})^n \arrow[u, "p_{G_\Sigma}"] \arrow[ur, "\mathrm{id}\times q_{G_\Sigma}"'] & \end{tikzcd}$$

where $q_{G_\Sigma}\colon * \to \left[\,*\,/G_\Sigma\right]$ is the canonical map.

Proof. We know from Section 5.1, that the stack $\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})$ is the fibered groupoid over Top given by the rule $T \in \mathsf{Top} \mapsto \mathrm{Hom}(T \times \Sigma_{g,n,m}, \mathfrak{X})$, where Hom is the groupoid of (stack) morphisms. Then we can define another fibered groupoid $T \mapsto \mathfrak{M}_{\mathfrak{X},G_\Sigma}(T)$ where the set of objects of $\mathfrak{M}_{\mathfrak{X},G_\Sigma}(T)$ is the set of stack morphisms $hom(T\times\Sigma_{g,n,m}, \mathfrak{X})$ (here $hom(\mathfrak{X}, \mathfrak{Y})$ denotes the set of objects of the groupoid of stack morphisms $\mathrm{Hom}(\mathfrak{X}, \mathfrak{Y})$). The morphisms of $\mathfrak{M}_{\mathfrak{X},G_\Sigma}(T)$ are

$$Mor(\mathfrak{M}_{\mathfrak{X},G_\Sigma}(T)) = \Big\{(g, f) \in G_\Sigma \times Mor(\mathrm{Hom}(T \times \Sigma_{g,n,m}, \mathfrak{X}))\Big\}$$

with source and target maps given by $s(g, f) = s(f)$ and $t(g, f) = t(g\cdot f)$ (where $\cdot$ denotes the action of G_Σ). The rule $T \mapsto \mathfrak{M}_{\mathfrak{X},G_\Sigma}(T)$ is easily seen to define a prestack. We let $\left[\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})/G_\Sigma\right]$ be the stackification of $\mathfrak{M}_{\mathfrak{X},G_\Sigma}(-)$. We define in the same way the stack $\left[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\right]$ as the stackification of a fibered groupoid $\mathfrak{L}_{\mathfrak{X},G_\Sigma}(T)$. Since the action of G_Σ on $(\mathrm{L}\mathfrak{X})^n$ is trivial, there is an isomorphism (of fibered groupoids) in between

$$\mathfrak{L}_{\mathfrak{X},G_\Sigma}(T) := \Big\{(g, f) \in G_\Sigma \times Mor(\mathrm{Hom}(T \times \coprod_{i=1}^{n} S^1, \mathfrak{X}))\Big\} \rightrightarrows hom(T \times \coprod_{i=1}^{n} S^1, \mathfrak{X})$$

and $G_\Sigma \times Mor(\mathrm{Hom}(T \times \coprod_{i=1}^n S^1, \mathfrak{X})) \rightrightarrows hom(T \times \coprod_{i=1}^n S^1, \mathfrak{X})$. Hence an isomorphism of stacks $\left[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\right] \cong (\mathrm{L}\mathfrak{X})^n \times \left[\,*\,/G_\Sigma\right]$.

Choosing $g = 1$, the unit of G_Σ induces a prestack morphism $p\colon \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X}) \to \mathfrak{M}_{\mathfrak{X},G_\Sigma}$ which yields the canonical map $p_{G_\Sigma}\colon \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X}) \to \left[\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})/G_\Sigma\right]$ after stackification. This map is shown to be an epimorphism and a G_Σ-torsor as in the usual case of transformation groupoids (see [8, 57, 56]). The case of $(\mathrm{L}\mathfrak{X})^n$ is similar and assertion 3. of the lemma follows immediately.

In order to prove that these quotient stacks are topological stacks, we note that if $X \to \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})$ is a chart for $\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})$ (that is a representable epimorphism from a topological space), then the composition $X \to \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X}) \to \left[\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})/G_\Sigma\right]$ is again a representable epimorphism. The existence of X is given by Proposition 5.1.

By construction, the fibered groupoid $\mathfrak{M}_{\mathfrak{X},G_\Sigma}$ is defined so that the diagram

$$\begin{tikzcd} G_\Sigma \times \mathrm{Hom}(T\times \Sigma_{g,n,m},\mathfrak{X}) \arrow[r] \arrow[d, "pr_2"'] & \mathrm{Hom}(T\times \Sigma_{g,n,m},\mathfrak{X}) \arrow[d, "p"] \\ \mathrm{Hom}(T\times \Sigma_{g,n,m},\mathfrak{X}) \arrow[r, "p"] & \mathfrak{M}_{\mathfrak{X},G_\Sigma}(T) \end{tikzcd}$$

is 2-cartesian. Since the stackification functors commutes with 2-fiber products, it induces the cartesian square asserted in the Lemma. $\square$

Since the map ρ_{in} is equivariant, it passes to the quotient to give rise to a stack morphism $[\rho_{in}/G_\Sigma]\colon [\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma] \longrightarrow [(\mathrm{L}\mathfrak{X})^n/G_\Sigma]$. Furthermore, we have the following lemma.

Lemma 14.6 *Let* $p_{/G_\Sigma} : (\mathrm{L}\mathfrak{X})^n \to [(\mathrm{L}\mathfrak{X})^n/G_\Sigma]$ *be the quotient map (of stacks) given by Lemma 14.5 and similarly for* $\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})$. *The following diagram*

$$\begin{tikzcd} \mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X}) \arrow[r, "\rho_{in}"] \arrow[d] & (\mathrm{L}\mathfrak{X})^n \arrow[d, "p_{/G_\Sigma}"] \\ {[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma]} \arrow[r, "{[\rho_{in}/G_\Sigma]}"] & {[(\mathrm{L}\mathfrak{X})^n/G_\Sigma]}. \end{tikzcd}$$

is a cartesian square.

Proof. Unfolding the definition of $[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma]$ and $[(\mathrm{L}\mathfrak{X})^n/G_\Sigma]$ in the proof of Lemma 14.5, we see that, since ρ_{in} is G_Σ-equivariant, it induces a map of fibered groupoids $\mathfrak{M}_{\mathfrak{X},G_\Sigma}(T) \to \mathfrak{L}_{\mathfrak{X},G_\Sigma}(T)$. After stackification, we get the map $[\rho_{in}/G_\Sigma]\colon [\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma] \longrightarrow [(\mathrm{L}\mathfrak{X})^n/G_\Sigma]$. Furthermore, the diagram pictured in the Lemma follows from the same diagram of fibered groupoids. In order to check that this diagram is 2-cartesian, we note that both vertical arrows are G_Σ-torsors (by Lemma 14.5) and the lemma follows from the usual interpretation of transformation fibered groupoid as groupoids of torsors recalled in Section 1.4 (also see [8, 57]). $\square$

Remark 14.7 Lemma 14.5 and Lemma 14.6 (as well as the constructions underlined there) basically follow because we are considering strict actions of a topological group on a stack (induced by the mapping stack construction). These statements are actually particular cases of more general statements about quotient of topological stacks by (topological) groups which will be studied by the second and third author in a work in progress.

By Lemma 14.6, any bivariant class in

$$H^\bullet\Big([\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma] \overset{[\rho_{in}/G_\Sigma]}{\longrightarrow} [(\mathrm{L}\mathfrak{X})^n/G_\Sigma]\Big)$$

can be pulled-back along $p_{/G_\Sigma}$.

Lemma 14.8 *There is a bivariant class*

$$\sigma^{\mathfrak{X}}_{g,n,m} \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\big[\operatorname{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big] \overset{[\rho_{in}/G_\Sigma]}{\longrightarrow} \big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]\Big)$$

whose pullback $p^*_{/G_\Sigma}\big(\sigma^{\mathfrak{X}}_{g,n,m}\big)$ *is the class* ${r_{g,n,m}}_*\big(\theta^{\mathfrak{X}}_{g,n,m}\big)$, *see* (14.4).

Proof. An element $g \in G_\Sigma$ acts on $\Sigma_{g,n,m}$ and thus on $\iota_{g,n,m}(c_{g,n,m})$. The image $\sigma \cdot \iota_{g,n,m}(c_{g,n,m})$ is a chord diagram diffeomorphic to $c_{g,n,m}$, by a diffeomorphism fixing the boundary circles of $c_{g,n,m}$. Further $\Sigma_{g,n,m}$ also retracts on $\sigma \cdot \iota_{g,n,m}(c_{g,n,m})$ and, indeed, $x \mapsto g \cdot \iota_{g,n,m}(r_{g,n,m}(g^{-1}\cdot x))$ is a retraction of $\Sigma_{g,n,m}$ on $c_{g,n,m}$.

A proof similar to the one of Lemma 14.5 shows that we can form the quotient stack $\big[\operatorname{Map}(c_{g,n,m},\mathfrak{X})/G_\Sigma\big]$ induced by the above action on $c_{g,n,m}$. In particular the action of G_Σ factors through an action of the orientation and circles preserving diffeomorphism $Diff^+_\partial(c_{g,n,m})$ group of $c_{g,n,m}$. Note that the n disjoint circles are pointwisely fixed by an element of $Diff^+_\partial(c_{g,n,m})$. It follows that this group actually acts on the disjoint union of the trees $\coprod_{\mathcal{T}(c_{g,n,m}} t$ (and preserving the circular vertices).

Thus, applying Lemma 14.6 and its proof we get the diagram of pullback squares

$$\begin{array}{ccccc}
\operatorname{Map}(c_{g,n,m},\mathfrak{X}) & \xrightarrow{r_{g,n,m}} & \operatorname{Map}(\Sigma_{g,n,m},\mathfrak{X}) & \xrightarrow{\rho_{in}} & (\mathrm{L}\mathfrak{X})^n \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{/G_\Sigma}} \\
\big[\operatorname{Map}(c_{g,n,m},\mathfrak{X})/G_\Sigma\big] & \overset{[r/G_\Sigma]}{\longrightarrow} & \big[\operatorname{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big] & \xrightarrow{[\rho_{in}/G_\Sigma]} & \big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big].
\end{array} \tag{14.5}$$

For simplicity, we also denote ρ_{in} the composition $\rho_{in}\circ r_{g,n,m}$. This pullback square will allow us to reduce the statement of the lemma to an analogue statement with $c_{g,n,m}$ instead of $\Sigma_{g,n,m}$. Indeed, assume we have a class

$$\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m} \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\big[\operatorname{Map}(c_{g,n,m},\mathfrak{X})/G_\Sigma\big] \overset{[\rho_{in}/G_\Sigma]}{\longrightarrow} \big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]\Big)$$

such that $p^*_{/G_\Sigma}\big(\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}\big) = \theta^{\mathfrak{X}}_{g,n,m}$. Then, since pushfoward and pullback commute in a bivariant theory (see Axiom A.13 in Appendix B.2), we get that

$$p^*_{/G_\Sigma}\Big([r/G_\Sigma]_*\big(\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}\big)\Big) = {r_{g,n,m}}_*\big(\theta^{\mathfrak{X}}_{g,n,m}\big).$$

Thus to finish the proof of the Lemma, it suffices to define the class $\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}$ satisfying $p^*_{/G_\Sigma}\big(\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}\big) = \theta^{\mathfrak{X}}_{g,n,m}$ (where $\theta^{\mathfrak{X}}_{g,n,m}$ is the class defined by the identity (14.2)). To do so, we follow the proof of lemma 14.4 to get an orientation class

$$\theta_{d_{/G_\Sigma}} \in H^{-d\chi(\Sigma_{g,n,m})}\big(\prod_{t\in\mathcal{T}(c_{g,n,m})} \big[\operatorname{Map}(t,\mathfrak{X})/G_\Sigma\big] \to \big[\mathfrak{X}^{\mathcal{V}(c_{g,n,m})}/G_\Sigma\big]\big).$$

Each tree is a deformation retract onto any of its circular vertex, hence we have, for each $t \in \mathcal{T}(c_{g,n,m})$, a factorisation

$$\begin{tikzcd}
\big[\mathfrak{X}^{\mathcal{T}(c_{g,n,m})}/G_\Sigma\big] \arrow[r, "\prod r^*_{t\,/G_\Sigma}"] \arrow[dr, bend right=10, "\Delta^{(\#\mathcal{V}(c_{g,n,m})-\#\mathcal{T}(c_{g,n,m}))}_{/G_\Sigma}"'] & \prod_{t\in\mathcal{T}(c_{g,n,m})}\big[\mathrm{Map}(t,\mathfrak{X})/G_\Sigma\big] \arrow[d, "d_{c_{g,n,m}\,/G_\Sigma}"] \\
 & \big[\mathfrak{X}^{\mathcal{V}(c_{g,n,m})}/G_\Sigma\big]
\end{tikzcd}$$

where the $r^*_{t\,/G_\Sigma}$ are retractions. Note that, similar to the proof of Lemma 14.5.3., there are topological stacks isomorphisms

$$\big[\mathfrak{X}^{\mathcal{V}(c_{g,n,m})}/G_\Sigma\big] \cong \mathfrak{X}^{\mathcal{V}(c_{g,n,m})} \times \big[\,*\,/G_\Sigma\big]$$

and

$$\big[\mathfrak{X}^{\mathcal{T}(c_{g,n,m})}/G_\Sigma\big] \cong \mathfrak{X}^{\mathcal{T}(c_{g,n,m})} \times \big[\,*\,/G_\Sigma\big]$$

since the action of G_Σ on the circular vertices is trivial. Further, under this isomorphism, the map $d_{c_{g,n,m}\,/G_\Sigma}$ is identified with $d_{c_{g,n,m}} \times \mathrm{id}_{/G_\Sigma}$. Thus, by Proposition 7.7, an orientation class θ (as the one given by Lemma 14.4) in $H^{-d\chi(\Sigma_{g,n,m})}(\mathfrak{X}^{\mathcal{T}(c_{g,n,m})} \to \mathfrak{X}^{\mathcal{V}(c_{g,n,m})})$ determines an orientation

$$\theta \otimes 1 \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\mathfrak{X}^{\mathcal{T}(c_{g,n,m})} \times \big[\,*\,/G_\Sigma\big] \overset{d_{c_{g,n,m}}\times \mathrm{id}}{\longrightarrow} \mathfrak{X}^{\mathcal{V}(c_{g,n,m})} \times \big[\,*\,/G_\Sigma\big]\Big).$$

Furthermore, it follows from the proof of Proposition 7.7 that the pullback $p^*_{/G_\Sigma}(\theta \otimes 1)$ is the original orientation class θ. Taking, as in the proof of Lemma 14.4,

$$\theta_{d_{/G_\Sigma}} = \big(\prod r^*_{t\,/G_\Sigma}\big)_*(\theta \otimes 1) \tag{14.6}$$

to be the pushforward of the orientation class $\theta \otimes 1$, we see that

$$p^*_{/G_\Sigma}\big(\theta_{d_{/G_\Sigma}}\big) \;=\; \theta_{d_{c_{g,n,m}}}, \tag{14.7}$$

that is, the pullback $p^*_{/G_\Sigma}\big(\theta_{d_{/G_\Sigma}}\big)$ is the class $\theta_{d_{c_{g,n,m}}}$ of Lemma 14.4 (once again using that pullback and pushforward commute in a bivariant theory).

Now, let us consider the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Map}(c_{g,n,m},\mathfrak{X}) \arrow[rr, "\rho_{in}"] \arrow[dl] \arrow[dd] & & (\mathrm{L}\mathfrak{X})^n \arrow[dl, "p_{/G_\Sigma}"] \arrow[dd, "\mathrm{ev}_{\mathcal{V}}"] \\
\big[\mathrm{Map}(c_{g,n,m},\mathfrak{X})/G_\Sigma\big] \arrow[rr, "{[\rho_{in}/G_\Sigma]}"] \arrow[dd] & & \big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big] \arrow[dd, "\mathrm{ev}_{\mathcal{V}\,/G_\Sigma}"] & \\
 & \prod\limits_{t\in\mathcal{T}(c_{g,n,m})}\mathrm{Map}(t,\mathfrak{X}) \arrow[rr, "d_{c_{g,n,m}}"] \arrow[dl] & & \mathfrak{X}^{\mathcal{V}(c_{g,n,m})} \arrow[dl, "p_{/G_\Sigma}"] \\
\prod\limits_{t\in\mathcal{T}(c_{g,n,m})}\big[\mathrm{Map}(t,\mathfrak{X})/G_\Sigma\big] \arrow[rr, "d_{c_{g,n,m}\,/G_\Sigma}"] & & \big[\mathfrak{X}^{\mathcal{V}(c_{g,n,m})}/G_\Sigma\big] &
\end{tikzcd} \tag{14.8}$$

The bottom vertical and lower horizontal square are pullbacks and so is the top horizontal square. It follows that the front vertical square is a pullback too. We denote by $\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}$ the pullback ${\mathrm{ev}_{\mathcal{V}}}^*_{/G_\Sigma}(\theta_{d_{G_\Sigma}})$ in

$$H^{-d\chi(\Sigma_{g,n,m})}\Big(\big[\mathrm{Map}(c_{g,n,m},\mathfrak{X})/G_\Sigma\big] \overset{[\rho_{in}/G_\Sigma]}{\longrightarrow} \big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]\Big)$$

of the class (14.6)

$$\theta_{d_{G_\Sigma}} \in H^{-d\chi(\Sigma_{g,n,m})}\Big(\prod_{t\in\mathcal{T}(c_{g,n,m})}\big[\mathrm{Map}(t,\mathfrak{X})/G_\Sigma\big] \to \big[\mathfrak{X}^{\mathcal{V}(c_{g,n,m})}/G_\Sigma\big]\Big)$$

defined above. By definition of $\theta^{\mathfrak{X}}_{g,n,m}$ (see identity (14.2)), one has the identities

$$\begin{aligned}
\theta^{\mathfrak{X}}_{g,n,m} &= \mathrm{ev}^*_{\mathcal{V}}(\theta_{d_{c_{g,n,m}}}) \\
&= \mathrm{ev}^*_{\mathcal{V}}\left(p^*_{/G_\Sigma}(\theta_{d_{/G_\Sigma}})\right) \text{ by relation (14.7)} \\
&= p^*_{/G_\Sigma}\big({\mathrm{ev}_{\mathcal{V}}}^*_{/G_\Sigma}(\theta_{d_{/G_\Sigma}})\big) \text{ by diagram (14.8)} \\
&= p^*_{/G_\Sigma}\big(\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}\big)
\end{aligned}$$

which prove that the class $\widetilde{\sigma}^{\mathfrak{X}}_{g,n,m}$ satisfies the expected identity and thus finishes the proof of the lemma. $\Box$

Defining the homology conformal field theory and Proof of Theorem 14.2 The inclusion of the outgoing boundary components $\coprod_{i=1}^m S^1 \hookrightarrow \Sigma_{g,n,m}$ is also G_Σ-equivariant. Thus, similar to the ingoing boundary case, we get a stack morphism

$$\big[\rho_{out}/G_\Sigma\big] \colon \big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big] \longrightarrow \big[(\mathrm{L}\mathfrak{X})^m/G_\Sigma\big].$$

Since G_Σ acts trivially on $\big(\mathrm{L}\mathfrak{X}\big)^m$, the terminal map $G_\Sigma \to \{1\}$ yields a stack morphism

$$[(\mathrm{L}\mathfrak{X})^m/G_\Sigma] \to [(\mathrm{L}\mathfrak{X})^m/\{1\}] \cong \big(\mathrm{L}\mathfrak{X}\big)^m.$$

By Section 9.1, the bivariant class $\sigma^{\mathfrak{X}}_{g,n,m}$ given by Lemma 14.8, yields a Gysin map

$$(\sigma^{\mathfrak{X}}_{g,n,m})^! \colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^n/G_\Sigma\big]\big) \to H\big(\big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big]\big).$$

Composing with the homology pushforward of the two preceeding stacks morphisms, we get, for every (isomorphism class of a) surface $\Sigma_{g,n,m}$, the following linear map:

$$\begin{aligned}
\mu_{\Sigma_{g,n,m}} \colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^n/G_\Sigma\big]\big) &\overset{(\sigma^{\mathfrak{X}}_{g,n,m})^!}{\longrightarrow} H\big(\big[\mathrm{Map}(\Sigma_{g,n,m},\mathfrak{X})/G_\Sigma\big] \\
&\overset{[\rho_{out}/G_\Sigma]_*}{\longrightarrow} H\big(\big[(\mathrm{L}\mathfrak{X})^m/G_\Sigma\big]\big) \to H\big(\big(\mathrm{L}\mathfrak{X}\big)^m\big) \qquad (14.9)
\end{aligned}$$

This map indeed defines the (non-unital, non-counital) homological conformal field theory structure of $H_\bullet(\mathrm{L}\mathfrak{X})$ asserted by Theorem 14.2 as we will now prove (recall from Lemma 14.5 that $\big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big] \cong (\mathrm{L}\mathfrak{X})^n \times \big[\,*\,/G_\Sigma\big]$).

Proof of theorem 14.2. We wish to define the d-dimensional homological conformal field theory structure by assigning to any positive integer n, the graded space $H_\bullet\big((\mathrm{L}\mathfrak{X})^n\big)$. First note that, by Lemmas 14.4 and 14.8, the map (14.9) above

$$\mu_{\Sigma_{g,n,m}} \colon H_\bullet\big(\big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]\big) \to H_{\bullet+d\chi(\Sigma_{g,n,m})}\big((\mathrm{L}\mathfrak{X})^m\big)$$

is of degree $-d\chi(\Sigma_{g,n,m})$. Further, since $G_\Sigma = Diff^+_{n,m}(\Sigma_{g,n,m})$ acts trivially on the incoming boundary of $\Sigma_{g,n,m}$ and thus on $(\mathrm{L}\mathfrak{X})^n$, there is (see Lemma 14.5) a canonical isomorphism of topological stacks

$$\big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big] \cong (\mathrm{L}\mathfrak{X})^n \times \big[\,*\,/G_\Sigma\big]$$

and thus natural isomorphisms

$$H_\bullet\big(\big[(\mathrm{L}\mathfrak{X})^n/G_\Sigma\big]\big) \cong H_\bullet\big((\mathrm{L}\mathfrak{X})^n\big) \otimes H_\bullet(BG_\Sigma) \cong H_\bullet\big((\mathrm{L}\mathfrak{X})^n\big) \otimes H_\bullet(B\Gamma_{n,m}(\Sigma)).$$

It follows that the maps $\mu_{\Sigma_{g,n,m}}$ induce, for any $\Sigma_{g,n,m} \in \mathfrak{M}_{n,m}$, a well defined map from $H_\bullet\big((\mathrm{L}\mathfrak{X})^n\big)$ to $H_\bullet\big((\mathrm{L}\mathfrak{X})^m\big)$, which we still denote $\mu_{\Sigma_{g,n,m}}$. This map has the correct degree shifting with respect to the twisting local coefficient system $\det^{\otimes d}$. Furthermore, since we are only considering closed boundaries, the bundle corresponding to this local system is oriented by [34] and locally trivial over the *category* $\mathfrak{M}_{n,m}$ by [26, 27]. It follows that in order to check that the rule $n \mapsto H_\bullet\big((\mathrm{L}\mathfrak{X})^n\big)$ together with the maps $\mu_{\Sigma_{g,n,m}}$ defines a symmetric monoidal functor from $\mathcal{C}^{nu,nc}_{\mathfrak{M},\det^{\otimes d}}$ to the category of graded vector spaces (*i.e.* a non-unital non-counital d-dimensional homological conformal field theory), it suffices to check the behavior of the maps $\mu_{\Sigma_{g,n,m}}$ with respect to disjoint union and gluing of surfaces. This will be done below similarly to the proof of associativity and coassociativity of the loop product in Theorems 10.1 and 12.5 as well as in Chataur-Menichi [18].

We first deal with the gluing of surfaces. Let $\Sigma_{g,\ell,m} \in \mathfrak{M}_{\ell,n}$ and $\Sigma'_{g',n,m} \in \mathfrak{M}_{n,m}$ be two surfaces. We denote

$$G_\Sigma = Diff^+_{\ell,n}(\Sigma_{g,\ell,n}) \quad \text{and} \quad G'_{\Sigma'} = Diff^+_{n,m}(\Sigma'_{g',n,m})$$

the corresponding diffeomorphisms groups. Since these groups are fixing the boundaries pointwisely, it follows that we have an injective morphisms of topological groups $G_\Sigma \times G'_{\Sigma'} \hookrightarrow H_{\Sigma'\circ\Sigma}$ where $H_{\Sigma\circ\Sigma'} = Diff^+_{\ell,m}(\Sigma'_{g',n,m} \circ \Sigma_{g,\ell,n})$ is the group of oriented diffeomorphisms fixing pointwisely the boundary of the surface $\Sigma'_{g',n,m} \circ \Sigma_{g,\ell,n}$ obtained by gluing $\Sigma_{g,\ell,n}$ and $\Sigma'_{g',n,m}$. For simplicity, henceforth, we denote $\widetilde{\Sigma}_\circ = \Sigma'_{g',n,m} \circ \Sigma_{g,\ell,n}$ this gluing. Since the boundary circles are fixed pointwisely, both $G_\Sigma \times G'_{\Sigma'}$ and $H_{\Sigma'\circ\Sigma}$ acts trivially on $(\mathrm{L}\mathfrak{X})^\ell$

and $\big(\mathrm{L}\mathfrak{X}\big)^m$ so that we have stacks morphisms $\big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma \times G'_{\Sigma'}\big] \to \big(\mathrm{L}\mathfrak{X}\big)^\ell$, $\big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/H_{\Sigma'\circ\Sigma}\big] \to \big(\mathrm{L}\mathfrak{X}\big)^\ell$ defined and that we have these equivariant maps is the same as the ones of Lemma 14.5 and Lemma14.6). The morphism $G_\Sigma \times G'_{\Sigma'} \hookrightarrow H_{\Sigma'\circ\Sigma}$ induces a commutative diagram of stacks

$$\begin{tikzcd}
\big[\,*\,/G_\Sigma\big] \times \big[\,*\,/G'_{\Sigma'}\big] \arrow[r, "\circ"] \arrow[d, hook] & \big[\,*\,/H_{\Sigma\circ\Sigma'}\big] \arrow[d, hook] \\
\mathfrak{M}_{\ell,n} \times \mathfrak{M}_{n,m} \arrow[r, "\circ"] & \mathfrak{M}_{\ell,m}
\end{tikzcd}$$

and a similar diagram after passing to homology with twisted coefficient. We thus have to prove that $\mu_{\Sigma'_{g',n,m}} \circ \mu_{\Sigma_{g,\ell,n}} = \mu_{\widetilde{\Sigma}_\circ}$. The above group morphism $G_\Sigma \times G'_{\Sigma'} \hookrightarrow H_{\Sigma'\circ\Sigma}$ induces an action of $G_\Sigma \times G'_{\Sigma'}$ on $\mathrm{Map}(\widetilde{\Sigma}_\circ, \mathfrak{X})$ and a diagram of cartesian squares (applying, mutatis mutandis, the proof of Lemma 14.6) of topological stacks

$$\begin{tikzcd}
\mathrm{Map}(\widetilde{\Sigma}_\circ, \mathfrak{X}) \arrow[r, "\rho_{in}"] \arrow[d] & \big(\mathrm{L}\mathfrak{X}\big)^\ell \arrow[d] \\
\big[\mathrm{Map}(\widetilde{\Sigma}_\circ, \mathfrak{X})/G_\Sigma \times G'_{\Sigma'}\big] \arrow[r, "{[\rho_{in}/G_\Sigma \times G'_{\Sigma'}]}"] \arrow[d] & \big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma \times G'_{\Sigma'}\big] \arrow[d] \\
\big[\mathrm{Map}(\widetilde{\Sigma}_\circ, \mathfrak{X})/H_{\Sigma'\circ\Sigma}\big] \arrow[r, "{[\rho_{in}/H_{\Sigma'\circ\Sigma}]}"] & \big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/H_{\Sigma'\circ\Sigma}\big]\big]
\end{tikzcd} \tag{14.10}$$

Furthermore, the group $G_\Sigma \times G'_{\Sigma'}$ also acts on $\mathrm{Map}(\Sigma_{g,\ell,n}, \mathfrak{X})$ (through the above action of G_σ and the trivial action of $G'_{\Sigma'}$) and similarly on $\mathrm{Map}(\Sigma'_{g',n,m}, \mathfrak{X})$. Since $\widetilde{\Sigma}_\circ = \Sigma'_{g',n,m} \circ \Sigma_{g,\ell,n}$ is obtained by glueing $\Sigma_{g,\ell,n}$ on $\Sigma'_{g',n,m}$ along the n-disjoint circles of their common boundaries, applying Lemma 5.2 and (the proof of) Lemma 14.6, we get another cartesian square

$$\begin{tikzcd}
\big[\mathrm{Map}(\widetilde{\Sigma}_\circ, \mathfrak{X})/G_\Sigma \times G'_{\Sigma'}\big] \arrow[r, "{[res_{in}\,/G_\Sigma \times G'_{\Sigma'}]}"] \arrow[d, "\widetilde{res}_{out}"'] & \big[\mathrm{Map}(\Sigma_{g,\ell,n}, \mathfrak{X})/G_\Sigma \times G'_{\Sigma'}\big] \arrow[d] \\
\big[\mathrm{Map}(\Sigma'_{g',n,m}, \mathfrak{X})/G_\Sigma \times G'_{\Sigma'}\big] \arrow[r, "{[\rho_{in}\,/G_\Sigma \times G'_{\Sigma'}]}"] & \big[\big(\mathrm{L}\mathfrak{X}\big)^n/G_\Sigma \times G'_{\Sigma'}\big]
\end{tikzcd} \tag{14.11}$$

of stacks (also see [18]), where res_{in} is the restriction map induced by the inclusion $\Sigma_{g,\ell,n} \to \widetilde{\Sigma}_\circ$.

The inclusion of the outgoing boundary of $\Sigma_{g,\ell,n}$ induces a stack morphism

$$\widetilde{\rho}_{mid} \colon \big[\mathrm{Map}(\Sigma_{g,\ell,n}, \mathfrak{X})/G_\Sigma \times G'_{\Sigma'}\big] \to \big[\big(\mathrm{L}\mathfrak{X}\big)^n/G_\Sigma \times G'_{\Sigma'}\big] \to \big[\big(\mathrm{L}\mathfrak{X}\big)^n/G'_{\Sigma'}\big]$$

since the group G_Σ acts trivially on $\big(\mathrm{L}\mathfrak{X}\big)^n$. Similarly, we have topological stacks morphisms

$$\widetilde{\rho}_{out}\colon \big[\operatorname{Map}(\Sigma'_{g',n,m},\mathfrak{X})/G'_{\Sigma'}\big] \to \big[\big(\mathrm{L}\mathfrak{X}\big)^m/G'_{\Sigma'}\big] \to \big(\mathrm{L}\mathfrak{X}\big)^m,$$

$$\widetilde{\widetilde{\rho}}_{out}\colon \big[\operatorname{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/H_{\Sigma'\circ\Sigma}\big] \to \big[\big(\mathrm{L}\mathfrak{X}\big)^m/H_{\Sigma'\circ\Sigma}\big] \to \big(\mathrm{L}\mathfrak{X}\big)^m$$

and a composition of morphisms of topological stacks

$$\begin{aligned}\widetilde{res}_{out}\colon \big[\operatorname{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}\big] &\longrightarrow \big[\operatorname{Map}(\Sigma'_{g',n,m},\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}\big]\\ &\longrightarrow \big[\operatorname{Map}(\Sigma'_{g',n,m},\mathfrak{X})/G'_{\Sigma'}\big].\end{aligned}$$

By Lemma 14.8 applied to the surfaces $\Sigma_{g,\ell,n}$, $\Sigma'_{g',n,m}$ and $\widetilde{\Sigma}_\circ$, there are bivariant classes $\sigma^{\mathfrak{X}}_{g,\ell,n}$, ${\sigma'}^{\mathfrak{X}}_{g',n,m}$ and $\widetilde{\sigma}^{\mathfrak{X}}_\circ$ inducing (as for the definition of $\mu_{\Sigma_{g,n,m}}$), respectively, the Gysin maps

$$(\sigma^{\mathfrak{X}}_{g,\ell,n})^!\colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma\big]\big) \to H\big(\big[\operatorname{Map}(\Sigma_{g,\ell,n},\mathfrak{X})/G_\Sigma\big]\big), \tag{14.12}$$

$$({\sigma'}^{\mathfrak{X}}_{g',n,m})^!\colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^n/G'_{\Sigma'}\big]\big) \to H\big(\big[\operatorname{Map}(\Sigma_{g',n,m},\mathfrak{X})/G'_{\Sigma'}\big]\big), \tag{14.13}$$

$$(\widetilde{\sigma}^{\mathfrak{X}}_\circ)^!\colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/H_{\Sigma'\circ\Sigma}\big]\big) \to H\big(\big[\operatorname{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/H_{\Sigma'\circ\Sigma}\big]\big). \tag{14.14}$$

The first Gysin map above is equivariant with respect to the action of $G'_{\Sigma'}$ since this group acts trivially on $\Sigma_{g,\ell,n}$, thus passes to quotient stack to define a Gysin map

$$(\sigma^{\mathfrak{X}}_{g,\ell,n})^!\colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma\times G'_{\Sigma'}\big]\big) \to H\big(\big[\operatorname{Map}(\Sigma_{g,\ell,n},\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}\big]\big). \tag{14.15}$$

Furthermore, the proof of Lemma 14.8 applied to $\Sigma_{g,\ell,n}$ with respect to the action of the group $G_\Sigma\times G'_{\Sigma'}$ (and not the full $H_{\Sigma'\circ\Sigma}$) also yields a bivariant class $\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}}$ and an associated Gysin map

$$(\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}})^!\colon H\big(\big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma\times G'_{\Sigma'}\big]\big) \to H\big(\big[\operatorname{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}\big]\big). \tag{14.16}$$

The Gysin maps $(\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}})^!$ (morphism (14.16)) and $(\sigma^{\mathfrak{X}}_{g,\ell,n})^!$ (morphism (14.15)) are related as follows. We choose $c_{g,\ell,n}$ to be a chord diagram associated to $\Sigma_{g,\ell,n}$, $c'_{g',n,m}$ to be associated to $\Sigma_{g',n,m}$ and $\widetilde{c}_{g+g',\ell,m} = c'_{g',n,m}\circ c_{g,\ell,n}$ the one associated to $\widetilde{\Sigma}_\circ$ obtained by gluing the two previous ones (see [22] for the composition of chord diagrams). By construction (*i.e.*, use of Lemma 14.8), the bivariant class $\sigma^{\mathfrak{X}}_{g,\ell,n}$ is obtained as a class whose pullback along the stack morphism $\big(\mathrm{L}\mathfrak{X}\big)^\ell \to \big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma\times G'_{\Sigma'}\big]$ is $r_{g,\ell,n,*}\big(\theta^{\mathfrak{X}}_{g,\ell,n}\big)$ where $\theta^{\mathfrak{X}}_{g,\ell,n}$ is given as in formula (14.2) and Lemma 14.4. Similarly, the bivariant class $\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}}$ is obtained as a class whose pullback along the stack morphism $\big(\mathrm{L}\mathfrak{X}\big)^\ell \to \big[\big(\mathrm{L}\mathfrak{X}\big)^\ell/G_\Sigma\times G'_{\Sigma'}\big]$ is $r_{g+g',\ell,m,*}\big(\theta^{\mathfrak{X}}_{g+g',\ell,m}\big)$, where, again the class $\theta^{\mathfrak{X}}_{g+g',\ell,m}$ is given by formula (14.2) and Lemma 14.4.

We now relate the classes $\theta^{\mathfrak{X}}_{g+g',\ell,m}$ and $\theta^{\mathfrak{X}}_{g,\ell,n}$. The chord diagram $\widetilde{c}_{g+g',\ell,m}$ has ℓ-disjoint circles, a set of disjoint trees $\mathcal{T}(c_{g,\ell,n})$ (corresponding to to the trees of $c_{g,\ell,n}$) and an additional set of disjoint trees $\mathcal{T}'(c'_{g',n,m})$ such that the union $\widetilde{\mathcal{T}}(\widetilde{c}_{g+g',\ell,m}) = \mathcal{T}(c_{g,\ell,n})\coprod \mathcal{T}'(c'_{g',n,m})$ is the set of trees associated to $\widetilde{c}_{g+g',\ell,m}$. We define similarly the set of circular vertices $\mathcal{V}(c_{g,\ell,n})$, $\mathcal{V}'(c'_{g',n,m})$ and $\widetilde{\mathcal{V}}(\widetilde{c}_{g+g',\ell,m})$. We thus get a diagram of pullback squares (obtained as for the square (14.1)):

$$\begin{array}{ccccc}
\mathrm{Map}(\widetilde{c}_{g+g',\ell,m},\mathfrak{X}) & \xrightarrow{res^c_{in}} & \mathrm{Map}(c_{g,n,m},\mathfrak{X}) & \longrightarrow & (\mathrm{L}\mathfrak{X})^n \\
\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{V}'}} & & \downarrow{\scriptstyle \mathrm{ev}_{\widetilde{\mathcal{V}}}} \\
\prod\limits_{t\in\widetilde{\mathcal{T}}(\widetilde{c}_{g+g',\ell,m})}\mathrm{Map}(t,\mathfrak{X}) & \longrightarrow & \mathfrak{X}^{\mathcal{V}'(c'_{g',n,m})}\times \prod\limits_{t\in\mathcal{T}(c_{g,n,m})}\mathrm{Map}(t,\mathfrak{X}) & \longrightarrow & \mathfrak{X}^{\widetilde{\mathcal{V}}(\widetilde{c}_{g+g',\ell,m})}
\end{array} \tag{14.17}$$

The left cartesian square above (14.17) and the proof of Lemma 14.4 gives us a bivariant class

$$\theta^{\mathfrak{X}}_{res^c_{in}} = \mathrm{ev}^*_{\mathcal{V}'}(\theta^{\mathfrak{X}}_{d_{res^c_{in}}}) \in H\Big(\mathrm{Map}(\widetilde{c}_{g+g',\ell,m},\mathfrak{X}) \stackrel{res^c_{in}}{\longrightarrow} \mathrm{Map}(c_{g,n,m},\mathfrak{X})\Big)$$

Since the classes $\theta^{\mathfrak{X}}_{g+g',\ell,m}$ and $\theta^{\mathfrak{X}}_{g,\ell,n}$ are also induced by pullbacks along $\mathrm{ev}_{\widetilde{\mathcal{V}}}$ of classes given by Lemma 14.4 (corresponding to the bottom line of (14.17) and the various chord diagrams involved here), it follows from the functoriality of pullbacks (Axiom A.3 in Appendix B.2) that $\theta^{\mathfrak{X}}_{g+g',\ell,m} = \theta^{\mathfrak{X}}_{res^c_{in}}\cdot\theta^{\mathfrak{X}}_{g,\ell,n}$.

Since the following diagram (induced by the various restriction maps)

$$\begin{array}{ccc}
\mathrm{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X}) & \xrightarrow{res_{in}} & \mathrm{Map}(\Sigma_{g,\ell,n},\mathfrak{X}) \\
{\scriptstyle r_{g+g',\ell,m}}\uparrow & & \uparrow{\scriptstyle r_{g,\ell,n}} \\
\mathrm{Map}(\widetilde{c}_{g+g',\ell,m},\mathfrak{X}) & \xrightarrow{res^c_{in}} & \mathrm{Map}(c_{g,\ell,n},\mathfrak{X})
\end{array}$$

is commutative, it follows that

$$r_{g+g',\ell,m,*}\big(\theta^{\mathfrak{X}}_{g+g',\ell,m}\big) = r_{g+g',\ell,m,*}\big(\theta^{\mathfrak{X}}_{res^c_{in}}\big)\cdot r_{g,\ell,n,*}\big(\theta^{\mathfrak{X}}_{g,\ell,n}\big).$$

Unfolding the argument of the proof of Lemma 14.8, we get that the bivariant class $r_{g+g',\ell,m,*}\big(\theta^{\mathfrak{X}}_{res^c_{in}}\big)$ is the pullback of a class

$$\sigma^{\mathfrak{X}}_{res_{in}} \in H\Big(\big[\mathrm{Map}(\Sigma_{g,\ell,n},\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}\big] \stackrel{\left[res_{in}/G_\Sigma\times G'_{\Sigma'}\right]}{\longrightarrow} \big[\mathrm{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}\big]\Big)$$

and further that the Gysin maps induced by the classes $\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}}$, $\sigma^{\mathfrak{X}}_{g,\ell,n}$ and $\sigma^{\mathfrak{X}}_{res_{in}}$ satisfy the identity

$$(\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}})^! \quad = \quad (\sigma^{\mathfrak{X}}_{res_{in}})^!\circ(\sigma^{\mathfrak{X}}_{g,\ell,n})^!. \tag{14.18}$$

Now, using the above defined maps, we can consider the following diagram

$$\begin{tikzcd}
H\big([(\mathrm{L}\mathfrak{X})^\ell/G_\Sigma\times G'_{\Sigma'}]\big) \arrow[r, "(\sigma^{\mathfrak{X}}_{g,\ell,n})^!"] \arrow[dd] \arrow[dr, "(\widetilde{\sigma}^{\mathfrak{X}}_{G_\Sigma\times G'_{\Sigma'}})^!"'] & H\big([\,\mathrm{Map}(\Sigma_{g,\ell,n},\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}]\big) \arrow[r, "\widetilde{\rho}_{mid*}"] \arrow[d, "(\sigma^{\mathfrak{X}}_{res_{in}})^!"] \arrow[dr, phantom, "(2)"] & H\big([(\mathrm{L}\mathfrak{X})^n/G'_{\Sigma'}]\big) \arrow[d, "(\sigma^{\mathfrak{X}}_{g',n,m})^!"] \\
\arrow[r, phantom, "(1)"] & H\big([\,\mathrm{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/G_\Sigma\times G'_{\Sigma'}]\big) \arrow[r, "\widetilde{res}_{out*}"'] \arrow[d] \arrow[dr, phantom, "(3)"] & H\big([\,\mathrm{Map}(\Sigma'_{g',n,m},\mathfrak{X})/G'_{\Sigma'}]\big) \arrow[d, "\widetilde{\rho}_{out*}"] \\
H\big([(\mathrm{L}\mathfrak{X})^\ell/H_{\Sigma'\circ\Sigma}]\big) \arrow[r, "(\widetilde{\sigma}^{\mathfrak{X}}_\circ)^!"'] & H\big([\,\mathrm{Map}(\widetilde{\Sigma}_\circ,\mathfrak{X})/H_{\Sigma'\circ\Sigma}]\big) \arrow[r, "\widetilde{\widetilde{\rho}}_{out*}"'] & H\big((\mathrm{L}\mathfrak{X})^m\big)
\end{tikzcd} \tag{14.19}$$

The left vertical map ad the bottom line represents the map $\mu_{\Sigma'_{g',n,m}\circ\Sigma_{g,\ell,n}}$ while the top line and right vertical composition represents the composition $\mu_{\Sigma'_{g',n,m}}\circ\mu_{\Sigma_{g,\ell,n}}$.

Thus to prove the glueing property, it is enough to show that the diagram (14.19) is commutative. The commutativity of the upper left triangle is precisely identity (14.18) above. The commutativity of the trapezoid labeled (1) follows from the tower of pullback squares (14.10) and naturality of Gysin maps with respect to towers of pullback squares, since all the classes involved are obtained by pullback from a class induced by Lemma 14.8. The commutativity of the square labeled (3) follows from the fact that the restriction to the outgoing boundary of $\widetilde{\Sigma}_\circ$ coincides with the restriction to the outgoing boundary of $\Sigma'_{g',n,m}$. Finally the square labeled (2) is commutative thanks to the naturality of Gysin maps applied to the cartesian square (14.11).

We are left to the case of disjoint union of surfaces, that is, to prove that

$$\mu_{\Sigma_{g,n,m}\coprod\Sigma'_{g',n',m'}} = \pm\,\mu_{\Sigma_{g,n,m}}\otimes\mu_{\Sigma'_{g',n',m'}}$$

where the sign is induced by the Koszul rule and the local coefficient $\det^{\otimes d}$. The sign follows as in [18, 34, 26, 27]. Let $c(g,n,m)$ and $c'(g',n',m')$ be chord diagrams representing respectively $\Sigma_{g,n,m}$ and $\Sigma'_{g',n',m'}$. Then the disjoint union $c(g,n,m)\coprod c'(g',n',m')$ is a chord diagram representing $\Sigma_{g,n,m}\coprod\Sigma'_{g',n',m'}$ and moreover, the diffeomorphism group $Diff^+_{n+n',m+m'}(\Sigma_{g,n,m}\coprod\Sigma'_{g',n',m'})$ is the cartesian product of $G_\Sigma = Diff^+_{n,m}(\Sigma_{g,n,m})$ and $G_{\Sigma'} = Diff^+_{n',m'}(\Sigma'_{g',n',m'})$. Let $\theta_{d_{c_{g,n,m}}}$ and $\theta_{d_{c'_{g',n',m'}}}$ be the respective strong orientation classes given by Lemma 14.4 applied to $\Sigma_{g,n,m}$ and $\Sigma'_{g',n',m'}$. For simplicity we simply denote $\Sigma=\Sigma_{g,n,m}$, $\Sigma'=\Sigma'_{g',n',m'}$, $c=c_{g,n,m}$ and $c'=c'_{g',n',m'}$. Since we are working over a field, by Proposition 7.7, we get that the tensor product

$$\theta_{d_{c_{g,n,m}}}\otimes\theta_{d_{c'_{g',n',m'}}}\in H^{-d\chi(\Sigma\coprod\Sigma')}\Big(\prod_{\mathcal{T}(c\coprod c')}\mathrm{Map}(t,\mathfrak{X})\xrightarrow{d_c\times d_{c'}}\mathfrak{X}^{\mathcal{V}(c\coprod c')}\Big)$$

identifies with the class $\theta_{d_{c_{g,n,m}\coprod c'_{g',n',m'}}}$ given by Lemma 14.4 applied to the surface $\Sigma_{g,n,m}\coprod\Sigma_{g',n',m'}$. This is a consequence of the fact that this class is induced by taking the products of the diagrams inducing $\theta_{d_{c_{g,n,m}}}$ and $\theta_{d_{c'_{g',n',m'}}}$. It follows that the same property holds for the classes obtained by applying Lemma 14.8, namely $\sigma^{\mathfrak{X}}_{g+g',n+n',m+m'} = \sigma^{\mathfrak{X}}_{g,n,m} \otimes \sigma^{\mathfrak{X}}_{g,n,m}$. Further the restriction to the incoming and outgoing boundary yields a commutative diagram

$$\begin{tikzcd}
(\mathrm{L}\mathfrak{X})^{n+n'} \arrow[d, "\cong"'] & \mathrm{Map}(\Sigma_{g,n,m} \coprod \Sigma'_{g',n',m'}, \mathfrak{X}) \arrow[l, "\rho_{in}"'] \arrow[r, "\rho_{out}"] \arrow[d, "\cong"'] & (\mathrm{L}\mathfrak{X})^{m+m'} \arrow[d, "\cong"] \\
(\mathrm{L}\mathfrak{X})^{n} \times (\mathrm{L}\mathfrak{X})^{n'} & \mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X}) \times \mathrm{Map}(\Sigma'_{g',n',m'}, \mathfrak{X}) \arrow[l, "\rho_{in}\times\rho_{in}"] \arrow[r, "\rho_{out}\times\rho_{out}"'] & (\mathrm{L}\mathfrak{X})^{m} \times (\mathrm{L}\mathfrak{X})^{m'}.
\end{tikzcd}$$

It now follows similarly to the case of the glueing of surfaces (and actually more easily) that the operation $\mu_{\Sigma_{g,n,m} \coprod \Sigma'_{g',n',m'}}$ is $\mu_{\Sigma_{g,n,m}} \otimes \mu_{\Sigma'_{g',n',m'}}$.

Now we only need to identify the operations of the **BV**-structure and Frobenius structure with the one given by the homological conformal field theory we just defined. By [33, 34, 18], we know that the **BV**-operator $H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ is induced by the generator of degree 1 in the homology $H_\bullet(\mathfrak{M}_{1,1})$ corresponding to the diffeomorphism of a cylinder given by the Dehn twist along a generator of the degree 1 homology of the cylinder $\Sigma_{0,1,1}$. In other words, this generator is induced by the fundamental class of S^1, and passing to the quotient stack $\mathrm{Map}(\Sigma_{0,1,1}, \mathfrak{X}) \to \big[\mathrm{Map}(\Sigma_{0,1,1}, \mathfrak{X})/S^1\big]$, we see that the action of this generator on $\rho_{out}\big(\mathrm{Map}(\Sigma_{0,1,1}, \mathfrak{X})\big) = \mathrm{L}\mathfrak{X}$ coincides with the operator D of Theorem 13.2.

Now the product and the coproduct are respectively given by pair of pants (with different incoming and outgoing boundaries). The first one correspond to the chord diagram with two circles and one edge connecting them while the second one correspond to the chord diagram with one circle and one diameter. They are given by degree 0 homology classes in $H_\bullet(\mathfrak{M}_{2,1})$ and $H_\bullet(\mathfrak{M}_{1,2})$, thus to identify them, it is enough to consider the Gysin maps obtained via Lemma 14.4 before passing to the quotient by the diffeomorphim groups. Unfolding the proof of Lemma 14.4, we see that these Gysin maps coincides with the ones defining the loop product in Section 10.1 and loop coproduct in Theorem 12.5. □

15 Remarks on brane topology for stacks

Brane topology, as coined by Sullivan and Voronov [25], is an higher dimensional analogue of string topology defined for free mapping sphere spaces instead of free loop space. Many aspects of Brane topology for manifolds have been studied in [25, 17, 45, 41] for instance. Roughly, Brane topology is concerned with the algebraic structure of the homology of $M^{S^n} = \mathrm{Map}(S^n, M)$, where M is an oriented manifold and S^n the standard n-dimensional sphere. In this section we sketch how to apply our general machinery to define Brane topological operations for oriented stacks.

By Proposition 5.1, for any topological stack $\mathfrak{X}$, the mapping stack $\mathrm{Map}(S^n, \mathfrak{X})$ is a topological stack. Here S^n is the n-dimensional sphere. Let

$\mathrm{ev}_0\colon \mathrm{Map}(S^n, \mathfrak{X}) \to \mathfrak{X}$ be the map induced by the evaluation in 0, the based point of S^n. There is a standard pinching map $p_{S^n}\colon S^n \to S^n \vee S^n$ obtained by collapsing an equator to the based point. This map is homotopy coassociative. By Lemma 5.2, if $\mathfrak{X}$ is Hurewicz, there is a cartesian square

$$\begin{tikzcd}
\mathrm{Map}(S^n \vee S^n, \mathfrak{X}) \arrow[r] \arrow[d] & \mathrm{Map}(S^n, \mathfrak{X}) \times \mathrm{Map}(S^n, \mathfrak{X}) \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_0"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

and thus, if $\mathfrak{X}$ is oriented of dimension d, a Gysin map $\Delta^!\colon H_\bullet\big(\mathrm{Map}(S^n, \mathfrak{X}) \times \mathrm{Map}(S^n, \mathfrak{X})\big) \to H_{\bullet - d}\big(\mathrm{Map}(S^n \vee S^n, \mathfrak{X})\big)$. Composing $\Delta^!$ with the map induced by the pinching map p_{S^n}, yields the *brane product*

$$\begin{aligned}\star_{S^n}\colon H\big(\mathrm{Map}(S^n, \mathfrak{X})\big)^{\otimes 2} &\cong H\big(\mathrm{Map}(S^n, \mathfrak{X}) \times \mathrm{Map}(S^n, \mathfrak{X})\big) \\ &\xrightarrow{\Delta^!} H\big(\mathrm{Map}(S^n \vee S^n, \mathfrak{X})\big) \xrightarrow{(p_{S^n})_*} H\big(\mathrm{Map}(S^n, \mathfrak{X})\big). \end{aligned} \quad (15.1)$$

Proposition 15.1 *Let $\mathfrak{X}$ be an oriented stack of dimension d. The brane product makes the shifted homology $\mathbb{H}_\bullet\big(\mathrm{Map}(S^n, \mathfrak{X})\big) = H_{\bullet + d}\big(\mathrm{Map}(S^n, \mathfrak{X})\big)$ a graded commutative algebra.*

Proof. The argument is the same as the ones of the proof of Theorem 10.1 and Proposition 10.9. $\square$

Remark 15.2 The brane product (15.1) can be "twisted" by a class $\alpha \in \bigoplus_{n\geq 0} H^n(\mathrm{Map}(S^n \vee S^n, \mathfrak{X}))$ similarly to the loop product as in Section 10. The analogue for brane product of Theorem 10.3 holds true, the proof being the same. For instance, the twisted brane product is associative if the class α satisfies the 2-cocycle condition (10.3).

Let $\phi^k\colon S^n \to S^n$ be the k^{th}*-iterated power map* defined by the composition

$$\phi^k\colon S^n \xrightarrow{p^{(k)}} \bigvee_{i=1}^{k} S^n \xrightarrow{\bigvee \mathrm{id}} S^n \quad (15.2)$$

where $p^{(k)}\colon S^n \to \bigvee S^n$ is the k^{th}-iterated pinching map and $\bigvee \mathrm{id} : \bigvee S^n \to S^n$ is the identity on each sphere of the bouquet $\bigvee S^n$. The map ϕ^k induced by precomposition maps $\lambda^k\colon \mathrm{Map}(S^n, \mathfrak{X}) \to \mathrm{Map}(S^n, \mathfrak{X})$, $f \mapsto \lambda^k(f) = f \circ \phi^k$. We have

$$\lambda^k \circ \lambda^\ell \quad = \quad \lambda^{k\ell}. \quad (15.3)$$

Theorem 15.3 *Let $\mathfrak{X}$ be an oriented (Hurewicz) stack of dimension d and assume $n \geq 2$. Then the maps $\lambda^k\colon \mathbb{H}_\bullet(\mathrm{Map}(S^n, \mathfrak{X})) \to \mathbb{H}_\bullet(\mathrm{Map}(S^n, \mathfrak{X}))$ are maps of algebras (for the brane product $\star_{S^n}$).*

Moreover, if the ground ring k contains $\mathbb{Q}$, there is a decomposition

$$\mathbb{H}_\bullet(\mathrm{Map}(S^n, \mathfrak{X})) \cong \prod_{i\geq 0} \mathbb{H}_\bullet^{(i)}(\mathfrak{X}),$$

where $\mathbb{H}_\bullet^{(i)}(\mathfrak{X})$ is the eigenspace of λ^k (with eigenvalue k^i), that makes $\Big(\mathbb{H}_\bullet(\mathrm{Map}(S^n, \mathfrak{X})), \star_{S^n}\Big)$ a bigraded commutative algebra (with respect to the shifted total degree and the grading induced by the decomposition).

Proof. By identity (15.3), the maps $\lambda^k \colon \mathrm{Map}(S^n, \mathfrak{X}) \to \mathrm{Map}(S^n, \mathfrak{X})$ equip $\mathbb{H}_\bullet(\mathrm{Map}(S^n, \mathfrak{X}))$ with the *null* multiplication (and not the brane product $\star_{S^n}$) a λ-ring with trivial multiplication as in [48]. The existence of the decomposition then follows from standard properties of λ-ring, see [5, 48]. Thus in order to prove the Lemma, we are left to prove that the maps λ^k are maps of algebras with respect to the brane product. This is an easy consequence of the commutativity of the following diagram

$$\begin{tikzcd}
H\big(\mathrm{Map}(S^n \coprod S^n, \mathfrak{X})\big) \arrow[r, "\Delta^!"] \arrow[d, "(\vee\,\mathrm{id})\coprod(\vee\,\mathrm{id})"'] & H\big(\mathrm{Map}(S^n \vee S^n, \mathfrak{X})\big) \arrow[r, "p^{(2)}"] \arrow[d, "(\vee\,\mathrm{id})\vee(\vee\,\mathrm{id})"] & \mathrm{Map}(S^n, \mathfrak{X}) \arrow[d, "\vee\,\mathrm{id}"] \\
H\big(\mathrm{Map}(\bigvee_{i=1}^k S^n \coprod \bigvee_{i=1}^k S^n, \mathfrak{X})\big) \arrow[r, "\Delta^!_{\bigvee_{i=1}^k S^n}"] \arrow[d, "p^{(k)}\coprod p^{(k)}"'] & H\big(\mathrm{Map}\,\big((\bigvee_{i=1}^k S^n) \vee (\bigvee_{i=1}^k S^n), \mathfrak{X}\big)\big) \arrow[r, "\bigvee p^{(2)}"] \arrow[d, "p^{(k)}\vee p^{(k)}"] & \mathrm{Map}(\bigvee S^n, \mathfrak{X}) \arrow[d, "p^{(k)}"] \\
H\big(\mathrm{Map}(S^n \coprod S^n, \mathfrak{X})\big) \arrow[r, "\Delta^!"] & H\big(\mathrm{Map}(S^n \vee S^n, \mathfrak{X})\big) \arrow[r, "p^{(2)}"] & \mathrm{Map}(S^n, \mathfrak{X})
\end{tikzcd} \tag{15.4}$$

where the Gysin map $\Delta^!_{\bigvee_{i=1}^k S^n}$ is obtained as the pullback by

$$\mathrm{ev}_0 \times \mathrm{ev}_0 \colon \mathrm{Map}\,\big((\bigvee_{i=1}^k S^n) \coprod (\bigvee_{i=1}^k S^n), \mathfrak{X}\big) \to \mathfrak{X} \times \mathfrak{X}$$

(the evaluation at the base points of each component) of the diagonal $\Delta \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. Let

$$\bigvee p^{(2)} \colon \mathrm{Map}\,\big((\bigvee_{i=1}^k S^n) \vee (\bigvee_{i=1}^k S^n), \mathfrak{X}\big) \to \bigvee_{i=1}^k S^n$$

be the map obtained by applying a permutation on the bouquet of spheres (so that the first and $k+1$-sphere are put next to each other, and then the second sphere with the $k+2$-sphere and so on) and then applying $p^{(2)}$ k-times. It follows immediately from this definition that the top right square of diagram (15.4) is commutative. The left squares are seen to be commutative by applying the naturality of Gysin maps as in the proof of Theorem 10.1. The maps

$$p^{(k)} \circ \vee p^{(2)} : \mathrm{Map}\,\big((\bigvee_{i=1}^k S^n) \vee (\bigvee_{i=1}^k S^n), \mathfrak{X}\big) \to \mathrm{Map}(S^n, \mathfrak{X})$$

and $p^{(2)} \circ (p^{(k)} \vee p^{(k)})$ involved in the lower right square of diagram (15.4) are not equal. However, since $n \geq 2$, they are homotopic to each other (the proof being similar to the commutativity of the higher homotopy groups). The result follows. $\square$

Remark 15.4 In view of Section 17.1, Theorem 15.3 holds for oriented manifolds, which did not seem to be known in the literature according to our knowledge.

The brane product described above shall belong to a bigger algebraic structure. Namely, we believe that the following

Claim Let $\mathfrak{X}$ be an oriented Hurewicz stack of dimension d. Then $\mathbb{H}_\bullet\big(\mathrm{Map}(S^n, \mathfrak{X})\big)$ is an algebra over the homology $H_\bullet(\mathcal{C}ac^{(n)})$ of the n-dimensional cacti operad $\mathcal{C}ac^{(n)}$ (see [25, 67] for the definition).

is true for stacks (the corresponding property for manifolds is due to Sullivan-Voronov [25]).

The case $n = 1$ follows from Theorem 14.2 since the operad $H_\bullet(\mathcal{C}ac^{(1)})$ is the **BV**-operad (see [33, 25, 59]). (Indeed, 1-dimensional cacti can be seen as special kind of chord diagram).

We believe the methods introduced in Section 10 and Section 14 could be applied to prove the above claim provided one has a model for the n-dimensional cacti operad in which cacti are obtained by gluing the various lobes using trees.

Remark 15.5 According to a result of Sullivan and Voronov [25, Theorem 5.1.1], there is an isomorphism of operads between $H_\bullet(\mathcal{C}ac^{(n)})$ and the homology operad $H_\bullet(\mathcal{D}_n^{fr})$ of the framed little n-dimensional disks operad $\mathcal{D}_n^{fr}$ (studied in detail in [59]). Hence the claim, if proved, implies such a structure on $\mathbb{H}_\bullet\big(\mathrm{Map}(S^n, \mathfrak{X})\big)$.

Remark 15.6 As in [25], one can prove that the claim follows from a n-dimensional cactus algebra structure of the free sphere stack $\mathrm{Map}(S^n, \mathfrak{X})$ in the category of *correspondences of topological stacks* (and not of topological stacks). This category *Cor* has Hurewicz topological stacks for objects and morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$ given by diagram $\mathfrak{X} \leftarrow \mathfrak{Z} \rightarrow \mathfrak{Y}$. The composition is defined by taking pullbacks. The proof of [25] applies *verbatim* to the framework of stacks. However, applying this idea to pass to homology is rather subtle as one needs to be careful with Gysin maps and do not seem to be straightforward.

16 Orbifold intersection pairing

In this Section, unless otherwise stated, $\mathfrak{X}$ will be an almost complex orbifold. Then $\Lambda\mathfrak{X}$ is again an almost complex orbifold. In particular, $\mathfrak{X}$ and $\Lambda\mathfrak{X}$ are oriented orbifolds. Care has to be taken, because even if $\mathfrak{X}$ is connected and has constant dimension, $\Lambda\mathfrak{X}$ usually has many components of varying dimension (the so-called twisted sectors). Using the (almost) complex structure and our bivariant theory, we will define a refinement of the hidden loop product, which is the (Poincaré) dual of the orbifold cup-product [19].

Warning 16.1 In this section, all (co)homology groups are taken with coefficients in $\mathbb{C}$, the field of complex numbers. In particular, this is true for singular homology $H_\bullet(\mathfrak{X})$, de Rham cohomology (denoted $H^\bullet_{\mathrm{DR}}(\mathfrak{X})$) and compactly supported de Rham cohomology (denoted $H^\bullet_{\mathrm{DR,\,c}}(\mathfrak{X})$).

16.1 Poincaré duality and orbifolds

For (any) oriented orbifolds, there is the **Poincaré duality homomorphism** $\mathcal{P} : H_i(\mathfrak{X}) \to H^{d-i}(\mathfrak{X})$, see [6]. Here $\mathfrak{X}$ is an oriented orbifold which has constant (real) dimension $d = \dim(\mathfrak{X})$. Let us recall briefly the definition of the Poincaré duality homomorphism, see [6] for details. There is the canonical inclusion $H_i(\mathfrak{X}) \hookrightarrow \big(H^i(\mathfrak{X})\big)^*$ which is an isomorphism if $H_i(\mathfrak{X})$ is finite dimensional. Since $\mathfrak{X}$ is of dimension d, there is the Poincaré duality isomorphism $\big(H^i_{\mathrm{DR}}\big)^* \xrightarrow{\sim} H^{d-i}_{\mathrm{DR,\,c}}$, see [6]. Let $\mathrm{inc}\colon H^\bullet_{\mathrm{DR,\,c}}(\mathfrak{X}) \to H^\bullet_{\mathrm{DR}}(\mathfrak{X})$ be the canonical map. The Poincaré duality homomorphism $\mathcal{P}$ is the composition

$$H_i(\mathfrak{X}) \longrightarrow (H^i(\mathfrak{X}))^* \xrightarrow{\sim} (H^i_{\mathrm{DR}}(\mathfrak{X}))^* \xrightarrow{\sim} H^{d-i}_{\mathrm{DR,\,c}}(\mathfrak{X}) \xrightarrow{\mathrm{inc}} H^{d-i}_{\mathrm{DR}}(\mathfrak{X}) \xrightarrow{\sim} H^{d-i}(\mathfrak{X}). \quad (16.1)$$

If the orbifold $\mathfrak{X}$ is proper, then $\mathcal{P}\colon H_\bullet(\mathfrak{X}) \to H^{d-\bullet}(\mathfrak{X})$ is an isomorphism.

Recall that the inertia stack $\Lambda\mathfrak{X}$ usually has many components of varying dimension. The inverse map $I\colon \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ is the isomorphism defined for any object (x, φ) in $\Lambda\mathfrak{X}$, where x is an object of $\mathfrak{X}$ and φ an automorphism of X, by $I(x, \varphi) = (x, \varphi^{-1})$. In the language of groupoids, if $\mathfrak{X}$ is presented by a Lie groupoid $\mathbb{X}$, the map I is presented by the map $(\gamma, \alpha) \mapsto (\gamma^{-1}, \beta)$ for any $(\gamma, \alpha) \in S\mathbb{X} \times_{X_0} X_1$.

The age is a locally constant function $\mathrm{age}\colon \Lambda\mathfrak{X} \to \mathbb{Q}$. If $\mathfrak{X} = [M/G]$ is a global quotient with G a finite group, then

$$\Lambda\mathfrak{X} = \left[\left(\coprod_{g \in G} M^g \right) /G \right]$$

and for $x \in M^g$, the age is equal to $\sum k_j$ if the eigenvalues of g on T_xM are $\exp(2i\pi k_j)$ with $0 \leq k_j < 1$. The age does not depend on which way $\mathfrak{X}$ is considered as a global quotient. So it is well-defined on $\Lambda\mathfrak{X}$ for any arbitrary almost complex orbifold, because any such $\mathfrak{X}$ can be locally written as a global quotient $[M/G]$. Similarly, the dimension is a locally constant function $\dim\colon \Lambda\mathfrak{X} \to \mathbb{Z}$. The age and the dimension are related by the formula (for instance see [19, 29])

$$\dim \;=\; d - 2\,\mathrm{age} - 2\,\mathrm{age} \circ I \quad (16.2)$$

where $I\colon \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ is the inverse map (as above). The **orbifold homology** of $\mathfrak{X}$ is

$$H^{\mathrm{orb}}_{\bullet}(\mathfrak{X}) = H_{\bullet-2\,\mathrm{age}\circ I}(\Lambda\mathfrak{X}) = \bigoplus_{q\in\mathbb{Q}} H_{\bullet-2q}\big([\Lambda\mathfrak{X}]_{\mathrm{age}\circ I=q}\big)$$

where $[\Lambda\mathfrak{X}]_{\mathrm{age}=n}$ is the component of $\Lambda\mathfrak{X}$ for which the age is equal to n. In plain english, we define the orbifold homology of $\mathfrak{X}$ to be the homology of $\Lambda\mathfrak{X}$ with a local degree shifting given, on a component of a certain fixed age, by $-2\mathrm{age}\circ I$. According to formula (16.2), the local degree shifting is also equal to $\dim -d + 2\mathrm{age}$.

The orbifold cohomology is $H^{\bullet}_{\mathrm{orb}}(\mathfrak{X}) = H^{\bullet-2\,\mathrm{age}}(\Lambda\mathfrak{X})$ (see [19, 29]). Note that the shift of degrees are not the same, but rather are Poincaré dual. Indeed, since $\Lambda\mathfrak{X}$ is an oriented orbifold, there is the Poincaré duality homomorphism $\mathcal{P}\colon H_{\bullet}(\Lambda\mathfrak{X}) \to H^{\bullet}(\Lambda\mathfrak{X})$ obtained as the composition (16.1) above on every connected component of $\Lambda\mathfrak{X}$. Since $\Lambda\mathfrak{X}$ has in general several connected components of different dimensions, this is not a graded map with respect to the usual grading. However, we have the following.

Lemma 16.2 *The Poincaré duality homomorphism*

$$H_{\bullet}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{P}} H^{\bullet}(\Lambda\mathfrak{X})$$

maps $H^{\mathrm{orb}}_{i}(\mathfrak{X})$ into $H^{d-i}_{\mathrm{orb}}(\mathfrak{X})$. We call it the **orbifold Poincaré duality homomorphism** $\mathcal{P}^{\mathrm{orb}}\colon H^{\mathrm{orb}}_{i}(\mathfrak{X}) \to H^{d-i}_{\mathrm{orb}}(\mathfrak{X})$.

Proof. It follows from formula (16.2). □

16.2 Orbifold intersection pairing and hidden loop product

Recall that, if $\mathfrak{X}$ is a manifold, then the homology $H_{\bullet}(\mathfrak{X})$ has the intersection pairing and the cohomology $H^{\bullet}(\mathfrak{X})$ has the cup-product. The Poincaré duality homomorphism is an algebra map. However, if $\mathfrak{X}$ is not compact, the intersection ring and cohomology ring may be very different from each other (for instance, if $\mathfrak{X}$ is not compact, $H_{\bullet}(\mathfrak{X})$ has no unit).

Chen-Ruan [19] defined the orbifold cup-product on the cohomology $H^{\bullet}_{\mathrm{orb}}(\mathfrak{X})$ of an almost complex orbifold $\mathfrak{X}$, generalizing the cup-product for manifolds. We will define the analogue of Chen-Ruan orbifold product in homology. Our construction generalizes the intersection pairing for manifolds. Note that we do not assume our orbifolds to be compact.

Our definition of the orbifold intersection pairing is as follows. There are the canonical maps $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ and $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ (see Section 11.1) and a Gysin homomorphism $j^{!}\colon H(\Lambda\mathfrak{X}\times\Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$ (in homology) because j is strongly oriented. Note that this Gysin maps is *not* the same as the one obtained by pulling back (as in Section 11.1) the orientation class of the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ in general.

The main ingredient in the definition of Chen-Ruan orbifold cup-product is the so called *obstruction bundle* whose construction is explained in detail in [19] and [29]. Another very nice reference for this is [44]. The obstruction bundle is a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ denoted $\mathfrak{O}_{\mathfrak{X}}$. We denote $\mathfrak{e}_{\mathfrak{X}} = e(\mathfrak{O}_{\mathfrak{X}})$ the Euler class of $\mathfrak{O}_{\mathfrak{X}}$. The **orbifold intersection pairing** is the composition:

$$\begin{aligned} H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{j^!} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \\ \xrightarrow{\cap \mathfrak{e}_{\mathfrak{X}}} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m_*} H(\mathfrak{X}). \quad (16.3) \end{aligned}$$

Theorem 16.3 *Suppose $\mathfrak{X}$ is an almost complex orbifold of (real) dimension d.*

1. *The orbifold intersection pairing defines a bilinear pairing*

$$H_i^{\mathrm{orb}}(\mathfrak{X}) \otimes H_j^{\mathrm{orb}}(\mathfrak{X}) \xrightarrow{\Cap} H_{i+j-d}^{\mathrm{orb}}(\mathfrak{X}).$$

2. *The orbifold intersection pairing $\Cap$ is associative and graded commutative.*

3. *The orbifold Poincaré duality homomorphism $\mathcal{P}^{\mathrm{orb}} \colon H_\bullet^{\mathrm{orb}}(\mathfrak{X}) \longrightarrow H^{d-\bullet}_{\mathrm{orb}}(\mathfrak{X})$ is a homomorphism of $\mathbb{C}$-algebras, where $H^{d-\bullet}_{\mathrm{orb}}(\mathfrak{X})$ is equipped with the orbifold cup-product [19].*

Recall that graded commutative means that, for any $x \in H_i([\Lambda\mathfrak{X}]_{\mathrm{age}\circ I=k}) \subset H^{\mathrm{orb}}_{i+2k}(\mathfrak{X})$ and $y \in H_j([\Lambda\mathfrak{X}]_{\mathrm{age}\circ I=\ell}) \subset H^{\mathrm{orb}}_{j+2\ell}(\mathfrak{X})$, one has

$$x \Cap y = (-1)^{(i+2k)(j+2\ell)} y \Cap x.$$

Proof.

1. By Riemann-Roch, the obstruction bundle $\mathfrak{O}_{\mathfrak{X}}$ satisfies the following well-known formula (see [29] Lemma 1.12 and [19] Lemma 4.2.2):

$$\mathrm{rank}(\mathfrak{O}_{\mathfrak{X}}) = 2(\mathrm{age} \circ p_1 + \mathrm{age} \circ p_2 - \mathrm{age} \circ m) + \dim_2 - \dim \circ m \qquad (16.4)$$

where $p_1, p_2 : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ are the projections on the first and second factor respectively, $\dim_2 \colon \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \mathbb{Z}$ is the dimension function of the orbifold $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ and rank : $\mathfrak{O}_{\mathfrak{X}} \to \mathbb{Z}$ is the rank function of the vector bundle $\mathfrak{O}_{\mathfrak{X}}$ (as a real vector bundle). Since $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ has codimension equal to $\dim \circ p_1 + \dim \circ p_2 - \dim_2$, the result follows from formula (16.4) and formula (16.2).

2. Since $\mathrm{flip}(\mathfrak{O}_{\mathfrak{X}}) \cong \mathfrak{O}_{\mathfrak{X}}$ (for instance see [29]) and $\mathfrak{e}_{\mathfrak{X}}$ is of even degrees (thus strictly commutes with any class), the commutativity follows as in the proof of 11.1. It remains to prove the associativity. Consider the cartesian

diagrams

$$\begin{tikzcd}[column sep=tiny] & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \arrow[dr, "m_{12}"] & \\ \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "j_{(12)3}"] \arrow[dr, "m_{12}"'] & & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \\ & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "j"'] & \end{tikzcd} \tag{16.5}$$

$$\begin{tikzcd}[column sep=tiny] & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[dr, "m_{23}"] & \\ \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "j_{1(23)}"] \arrow[dr, "m_{23}"'] & & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \\ & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "j"'] & \end{tikzcd} \tag{16.6}$$

The map $j_{(12)3}, j_{1(23)}$ are the canonical embeddings induced by $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ (applied, respectively, to the last two and first two factors). The maps m_{ii+1} $(i = 1, 2)$ are induced by multiplication of the components $i, i+1$. Similarly we denote

$$p_{ij} : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}, \quad i \neq j$$

the map (p_i, p_j) induced by the projections on the component i and j and

$$j_{12} = j \times \mathrm{id} : \Lambda\mathfrak{X}^{\times 3} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times \Lambda\mathfrak{X},$$

$$j_{23} = \mathrm{id} \times j : \Lambda\mathfrak{X}^{\times 3} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$$

the respective embeddings. The excess bundle $\mathfrak{E}_{12}$ associated to diagram (16.5) is defined as follows . There is a canonical map from the normal bundle $N_{j_{(12)3}}$ of

$$\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \overset{j_{(12)3}}{\to} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$$

to the restriction $m_{12}^* N_j$ of the normal bundle of $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \overset{j}{\to} \Lambda\mathfrak{X}$. By definition $\mathfrak{E}_{12} = \mathrm{Coker}(N_{j_{(12)3}} \hookrightarrow m_{12}^* N_j)$. Similarly, there is the excess bundle $\mathfrak{E}_{23} = \mathrm{Coker}(N_{j_{(1(23)}} \hookrightarrow m_{23}^* N_j)$ associated to diagram (16.6). The proof of Theorem 10.8 together with the commutativity of $\mathfrak{e}_{\mathfrak{X}}$ with any class, shows that

$$\begin{aligned} (\alpha \Cap \beta) \Cap \gamma &= m_* \left(j^! \big(m_{12*}\big(j_{12}^!(\alpha \times \beta \times \gamma) \cap p_{12}^* \mathfrak{e}_{\mathfrak{X}}\big)\big) \cap \mathfrak{e}_{\mathfrak{X}}\right) \\ &= m_* \left(m_{12*}\big(j_{(12)3}{}^!\big((j_{12}^!(\alpha \times \beta \times \gamma) \cap p_{12}^* \mathfrak{e}_{\mathfrak{X}}\big) \cap e(\mathfrak{E}_{12})\big) \cap \mathfrak{e}_{\mathfrak{X}}\right) \\ &= m_*^{(2)} \left(j^{(2)}{}^!(\alpha \times \beta \times \gamma) \cap p_{12}^* \mathfrak{e}_{\mathfrak{X}} \cap e(\mathfrak{E}_{12}) \cap m_{12}^* \mathfrak{e}_{\mathfrak{X}}\right). \end{aligned}$$

The second line follows from the excess bundle formula (see Proposition 9.5) applied to diagram (16.5). Similarly,

$$\alpha \Cap (\beta \Cap \gamma) = m_*^{(2)} \left(j^{(2)}{}^!(\alpha \times \beta \times \gamma) \cap p_{23}^* \mathfrak{e}_{\mathfrak{X}} \cap e(\mathfrak{E}_{23}) \cap m_{23}^* \mathfrak{e}_{\mathfrak{X}}\right).$$

Hence we need to prove that the bundles $\mathfrak{O}_{\mathfrak{X}}$ and $\mathfrak{E}_{ij}$ satisfy the following identity

$$p_{12}^*(\mathfrak{O}_{\mathfrak{X}}) + m_{12}^*(\mathfrak{O}_{\mathfrak{X}}) + \mathfrak{E}_{12} \quad = \quad p_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + m_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + \mathfrak{E}_{23} \tag{16.7}$$

in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

The main property of the obstruction bundle $\mathfrak{O}_{\mathfrak{X}}$ is that it precisely satisfies an "affine cocycle condition" see Equation (16.11) below. In fact, there are two cartesian squares (for $i = 1, 2$), analogous to (16.5), (16.6)

$$\begin{tikzcd} & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[dr, "m"] & \\ \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "p_{ii+1}"] \arrow[dr, "m_{ii+1}"'] & & \Lambda\mathfrak{X} \\ & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \arrow[ur, "p_i"'] & \end{tikzcd} \tag{16.8}$$

Since $p_{12} = p_{12} \circ j_{(12)3}$ and $p_1 = p_1 \circ j$, it is easy to check that the "excess" bundles associated to diagram (16.8) for $i = 1, 2$ coincide with $\mathfrak{E}_{12}$ and $\mathfrak{E}_{23}$ respectively. Indeed, we have the following identities

$$\mathfrak{E}_{12} \ = \ p_{12}^* m^* T_{\Lambda\mathfrak{X}} + T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} - p_{12}^* T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} - m_{12}^* T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}}, \tag{16.9}$$
$$\mathfrak{E}_{23} = p_{23}^* m^* T_{\Lambda\mathfrak{X}} + T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} - p_{23}^* T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} - m_{23}^* T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}}. \tag{16.10}$$

in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

It follows from Lemma 4.3.2 and Proposition 4.3.4 in [19] (also see Lemma 1.20 and Proposition 1.25 of [29] for more details) that $\mathfrak{O}_{\mathfrak{X}}$ satisfies the following "associativity" equation

$$p_{12}^*(\mathfrak{O}_{\mathfrak{X}}) + m_{12}^*(\mathfrak{O}_{\mathfrak{X}}) + \mathfrak{E}_{12} \quad = \quad p_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + m_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + \mathfrak{E}_{23} \tag{16.11}$$

in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. This is precisely identity (16.7); the associativity of $\Cap$ follows.

3. Since $\dim : \Lambda\mathfrak{X} \to \mathbb{Z}$ is always even, $\mathcal{P}$ commutes with the cross product. Using general argument on the Poincaré duality homomorphism in [6], Proposition 9.3 and tubular neighborhood (see Section 8), it is straightforward that $\mathcal{P} \circ f^! = f^* \mathcal{P}$ for any strongly oriented map of orbifolds $f : \mathfrak{X} \to \mathfrak{Y}$. Hence the following diagrams are commutative

$$\begin{tikzcd} H^\bullet(\Lambda\mathfrak{X}) \otimes H^\bullet(\Lambda\mathfrak{X}) \arrow[r, "\times"] & H^\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \arrow[r, "j^*"] & H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \\ H_\bullet(\Lambda\mathfrak{X}) \otimes H_\bullet(\Lambda\mathfrak{X}) \arrow[r, "\times"] \arrow[u, "\mathcal{P}"] & H_\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \arrow[r, "j^!"] \arrow[u, "\mathcal{P}"] & H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}), \arrow[u, "\mathcal{P}"] \end{tikzcd}$$

$$\begin{array}{ccccc}
H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) & \xrightarrow{\cup\mathfrak{e}_{\mathfrak{X}}} & H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) & \xrightarrow{m_!} & H^\bullet(\mathfrak{X}) \\
{\scriptstyle\mathcal{P}}\uparrow & & {\scriptstyle\mathcal{P}}\uparrow & & {\scriptstyle\mathcal{P}}\uparrow \\
H_\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) & \xrightarrow{\cap\mathfrak{e}_{\mathfrak{X}}} & H_\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) & \xrightarrow{m_*} & H_\bullet(\mathfrak{X}).
\end{array}$$

Now the result follows from Lemma 16.5 below. □

Remark 16.4 If $\mathfrak{X}$ is compact, the orbifold Poincaré duality map is a linear isomorphism, thus an isomorphism of algebras according to Theorem 16.3.3.

Lemma 16.5 *The Chen-Ruan orbifold cup-product [19] is the composition*

$$H^\bullet(\Lambda\mathfrak{X})\otimes H^\bullet(\Lambda\mathfrak{X}) \xrightarrow{\times} H^\bullet(\Lambda\mathfrak{X}\times\Lambda\mathfrak{X}) \xrightarrow{i^*} H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$$
$$\xrightarrow{\cup\mathfrak{e}_{\mathfrak{X}}} H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) \xrightarrow{m^!} H^\bullet(\Lambda\mathfrak{X}).$$

Proof. The Chen-Ruan pairing in [19] is defined, for compact orbifolds, by the formula

$$\langle \alpha\cup_{\mathrm{orb}}\beta,\gamma\rangle_{\mathrm{orb}} = \int_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} p_1^*(\alpha)\cup p_2^*(\beta)\cup m^*(I^*(\gamma))\cup\mathfrak{e}_{\mathfrak{X}}. \quad (16.12)$$

Until the end of this proof, let us write μ for the pairing given by the formula of Proposition 16.5. We compute $\langle\mu(\alpha,\beta),\gamma\rangle_{\mathrm{orb}}$. Denoting $\int_{\Lambda\mathfrak{X}}$ the orbifold integration map defined in [19], we find

$$\begin{aligned}
\langle\mu(\alpha,\beta),\gamma\rangle_{\mathrm{orb}} &= \int_{\Lambda\mathfrak{X}} \mu(\alpha,\beta)\cup I^*(\gamma)\\
&= \int_{\Lambda\mathfrak{X}} m^!\big(p_1^*(\alpha)\cup p_2^*(\beta)\cup\mathfrak{e}_{\mathfrak{X}}\big)\cup I^*(\gamma)\\
&= \int_{\Lambda\mathfrak{X}} m^!\big(p_1^*(\alpha)\cup p_2^*(\beta)\cup\mathfrak{e}_{\mathfrak{X}}\cup m^*(I^*(\gamma))\big)\\
&= \int_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} p_1^*(\alpha)\cup p_2^*(\beta)\cup m^*(I^*(\gamma))\cup\mathfrak{e}_{\mathfrak{X}}.
\end{aligned}$$

By nondegeneracy of the orbifold pairing, we get $\alpha\cup_{\mathrm{orb}}\beta=\mu(\alpha,\beta)$. □

Similarly to the twisted hidden loop product 11.1, we now introduce orbifold intersection pairing twisted by a cohomology class.

Definition 16.6 Let $\alpha\in H^\bullet(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$ be a (not necessarily homogeneous) cohomology class. We define the **orbifold intersection pairing twisted by** α, denoted $\Cap^\alpha$, to be the composition

$$H(\Lambda\mathfrak{X})\otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X}\times\Lambda\mathfrak{X}) \xrightarrow{j^!}$$
$$H(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) \xrightarrow{\cap(\mathfrak{e}_{\mathfrak{X}}\cup\alpha)} H(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) \xrightarrow{m_*} H(\mathfrak{X}).$$

For a vector bundle $\mathfrak{E}$ over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$, we call $\Cap^{e(\mathfrak{E})}$ the orbifold intersection pairing twisted by $\mathfrak{E}$.

With similar notations as for Theorem 10.3, we prove the following.

Proposition 16.7

1. *If α satisfies the cocycle condition:*
$$p_{12}^*(\alpha) \cup (m \times 1)^*(\alpha) = p_{23}^*(\alpha) \cup (1 \times m)^*(\alpha)$$
in $H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, then $\Cap^\alpha \colon H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X})$ is associative.

2. *If $\mathfrak{E}$ is a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ which satisfies the cocycle condition*
$$p_{12}^*(\mathfrak{E}) + (m \times 1)^*(\mathfrak{E}) = p_{23}^*(\mathfrak{E}) + (1 \times m)^*(\mathfrak{E}) \tag{16.13}$$
in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$, then $\Cap^{e(\mathfrak{E})}$ is associative.

Proof. It follows as Theorem 11.3 and Theorem 16.3.2. $\Box$

We will now explain the relationship between the hidden loop product and the orbifold intersection product. Namely, the first one is obtained by twisting the second one by an explicit vector bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

The inverse map $I \colon \Lambda\mathfrak{X} \overset{\sim}{\to} \Lambda\mathfrak{X}$ induces the "inverse" obstruction bundle $\mathfrak{O}_{\mathfrak{X}}^{-1} = (I \times_{\mathfrak{X}} I)^*(\mathfrak{O}_{\mathfrak{X}})$ which is also a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. We let $\mathfrak{N}_{\mathfrak{X}}$ be the normal bundle of the regular embedding $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \overset{m}{\to} \Lambda\mathfrak{X}$.

Theorem 16.8 *For any almost complex orbifold $\mathfrak{X}$, the hidden loop product coincides with the orbifold intersection pairing twisted by $\mathfrak{O}_{\mathfrak{X}}^{-1} \oplus \mathfrak{N}_{\mathfrak{X}}$, i.e., for any $x \in H_\bullet(\Lambda\mathfrak{X})$,*
$$x \star y = x \Cap^{e(\mathfrak{O}_{\mathfrak{X}}^{-1} \oplus \mathfrak{N}_{\mathfrak{X}})} y.$$

The proof will reduce to the following two lemmas.

The first lemma relates the hidden loop product with the Gysin homomorphism $j^! \colon H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ and yet another canonical bundle on $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. We define the *full excess bundle* $\mathfrak{F}_{\mathfrak{X}}$ as the excess bundle associated to the cartesian diagram

$$\begin{tikzcd} & \Lambda\mathfrak{X} \arrow[dr, "\mathrm{ev}"] & \\ \Lambda\mathfrak{X}_{\mathfrak{X}}\Lambda\mathfrak{X} \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & \mathfrak{X} \\ & \Lambda\mathfrak{X} \arrow[ur, "\mathrm{ev}"'] & \end{tikzcd}$$

i.e., $\mathfrak{F}_{\mathfrak{X}} = \mathrm{Coker}\left(N_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \overset{p_1}{\to} \Lambda\mathfrak{X}} \hookrightarrow N_{\Lambda\mathfrak{X} \overset{\mathrm{ev}}{\to} \mathfrak{X}}\right)$.

Lemma 16.9 *Let $\mathfrak{X}$ be an almost complex orbifold. The hidden loop product $\star : H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X})$ is equal to the composition*

$$H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{j^!} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$$
$$\xrightarrow{\cap e(\mathfrak{F}_{\mathfrak{X}})} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m_*} H(\Lambda\mathfrak{X}).$$

PROOF. Apply the excess formula (Proposition 9.5). □

The next Lemma relates the four bundles introduced above.

Lemma 16.10 *The obstruction bundle satisfies the identity*

$$\mathfrak{O}_{\mathfrak{X}} + \mathfrak{N}_{\mathfrak{X}} + \mathfrak{O}_{\mathfrak{X}}^{-1} = \mathfrak{F}_{\mathfrak{X}}$$

in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

PROOF. Recall that $\mathfrak{O}_{\mathfrak{X}}$ is a solution of equation (16.11):

$$p_{12}^*(\mathfrak{O}_{\mathfrak{X}}) + m_{12}^*(\mathfrak{O}_{\mathfrak{X}}) + \mathfrak{E}_{12} \quad = \quad p_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + m_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + \mathfrak{E}_{23}$$

in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

For any permutation $\tau \in \Sigma_3$ of the set $\{1,2,3\}$, there is a map $\mathfrak{T}_\tau : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ defined as the composition

$$\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{(p_1,p_2,I\circ m)} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{\widetilde{\tau}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{p_{12}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X},$$

where $\widetilde{\tau}$ is the permutation of factors induced by τ. It is well-known (see [19], [29] Lemma 1.10) that $\mathfrak{T}_\tau^*(\mathfrak{O}_{\mathfrak{X}}) \cong \mathfrak{O}_{\mathfrak{X}}$.

Let r be the map

$$(p_1, p_2, I \circ p_2) \colon \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}.$$

Note that

$$p_{12} \circ r = \mathrm{id}, \qquad (m_{12} \circ r) = I \circ \mathfrak{T}_{(13)}, \qquad p_{23} \circ r = (p_2, I \circ p_2).$$

and furthermore $r^* m_{23}^* T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} \cong p_1^*(T_{\Lambda\mathfrak{X}})$. It follows (using Equation (16.9), Equation (16.10) and $\mathfrak{N}_{\mathfrak{X}} = m^* T_{\Lambda\mathfrak{X}} - T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}}$) that the pullback of Equation (16.11) along r yields the identity

$$\begin{aligned} \mathfrak{O}_{\mathfrak{X}} + \mathfrak{O}_{\mathfrak{X}}^{-1} + \mathfrak{N}_{\mathfrak{X}} \quad &= \quad \mathrm{ev}^* T_{\mathfrak{X}} - T_{\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}} - p_1^* T_{\Lambda\mathfrak{X}} - p_2^* T_{\Lambda\mathfrak{X}} \\ &\quad\; + r^* p_{23}^*(\mathfrak{O}_{\mathfrak{X}}) + r^* m_{23}^*(\mathfrak{O}_{\mathfrak{X}}) \end{aligned}$$

in the K-theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. Since the right-hand side of the first line is isomorphic to $\mathfrak{F}_{\mathfrak{X}}$, it suffices to prove that $r^* p_{23}^*(\mathfrak{O}_{\mathfrak{X}})$ and $r^* m_{23}^*(\mathfrak{O}_{\mathfrak{X}})$ have rank 0. This is an easy application of the Riemann-Roch formula (16.4). □

Remark 16.11 One can easily check that the full excess bundle also satisfies the "associativity" equation (16.11). It thus follows from Lemma 16.10 that the twisting bundle $\mathfrak{N}_{\mathfrak{X}} + \mathfrak{O}_{\mathfrak{X}}^{-1}$ satisfies equation (16.13).

PROOF OF THEOREM 16.8. By Lemma 16.9, it suffices to prove that $e(\mathfrak{F}_{\mathfrak{X}}) = \mathfrak{e}_{\mathfrak{X}} \cup e(\mathfrak{O}_{\mathfrak{X}}^{-1} \oplus \mathfrak{N}_{\mathfrak{X}})$ which is trivial by Lemma 16.10. $\Box$

Remark 16.12 Let us sum up the philosophy underlying Theorem 16.8. In Section 11.1, we have defined an associative product on $H_\bullet(\Lambda\mathfrak{X})$ by using the Gysin map $\Delta^!\colon H_\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \to H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ for general oriented stacks. In the case of orbifolds, one can directly use the regular embedding $j\colon \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}\times\Lambda\mathfrak{X}$ (more precisely the regular embeddings associated to the various connected components of $\Lambda\mathfrak{X}$) to define a Gysin map $j^!$. Due to the excess formula (Proposition 9.5), the map $j^!$ does not induce an associative product, so that to get such a product one needs to twist this map by a class satisfying the "affine cocycle" condition (16.7). In the case of almost complex orbifold, the obstruction bundle is a small bundle satisfying this equation. The excess formula ensures that the Gysin map $\Delta^!$ is the Gysin map $j^!$ twisted by the Euler class of a bundle (the full excess bundle by Lemma 16.9). Then Lemma 16.10 proves that the obstruction bundle is a (virtual) subbundle of the full excess bundle and indeed, measures the difference between these two bundles and hence between the two Gysin maps.

Remark 16.13 According to Theorem 16.8, Theorem 16.3.3 and Remark 16.4, if $\mathfrak{X}$ is compact, the orbifold Poincaré duality homomorphism $\mathcal{P}^{\mathrm{orb}}$ induces an isomorphism of algebras between the hidden loop algebra $\big(\mathbb{H}_\bullet(\Lambda\mathfrak{X}), \star\big)$ and the orbifold cohomology equipped with Chen-Ruan orbifold cup-product twisted by the class $e(\mathfrak{O}_X^{-1}X \oplus \mathfrak{N}_{\mathfrak{X}})$. A nice interpretation of this isomorphism has recently been found by González, Lupercio, Segovia and Uribe [37]. They proved that the hidden loop product of compact complex orbifolds is isomorphic to the Chen Ruan product of the cotangent bundle $T^*\mathfrak{X}$ of $\mathfrak{X}$.

16.3 Examples of orbifold interesection pairing

Let us consider now a few examples. Note that the orbifold cup-product has been computed for many compact orbifolds. For instance, one can refer to [19, 29, 20, 36]. Complex toric orbifolds were dealt on in [43, 35]. By Theorem 16.3, the intersection pairing coincides with the orbifold cup-product in these cases.

Example 16.14 Let G be a finite group. Then $[*/G]$ is a complex orbifold. It follows from Theorem 16.8 that the intersection product is the same as the hidden loop product in that case. Note that all the ages are trivial and thus $H_i^{\mathrm{orb}}(\mathfrak{X}) = H_i(\Lambda\mathfrak{X})$ for all i's. Thus the algebra $(H_\bullet^{\mathrm{orb}}, \Cap)$ is concentrated in degree 0 and isomorphic to the center $Z(\mathbb{C}[G])$ of the group algebra $\mathbb{C}[G]$, see Example 17.7.

Now let V be a $\mathbb{C}$-linear representation of G of dimension $n > 0$. Then the quotient stack $[V/G]$ is a complex orbifold. Since all invariant subspaces V^g ($g \in G$) are contractible, the homology $H_\bullet(\Lambda[V/G])$ is still concentrated in degree 0 and is isomorphic to $(\mathbb{C}[G])_G$ as a vector space. However, the ages are no longer trivial (and depends on the specific representation) and add an interesting combinatorial grading to $(\mathbb{C}[G])_G$. It follows that

$$H_\bullet^{\mathrm{orb}}(G) = \bigoplus_{[g]\in C(G)} \mathbb{C}[2\mathrm{age}(g^{-1})]$$

where $C(G)$ is the set of conjugacy classes of G and $\mathbb{C}[q]$ means $\mathbb{C}$ placed in degree q. Since the age is invariant by conjugation, $Z(\mathbb{C}[G])$ (which is isomorphic to $H_\bullet^{\mathrm{orb}}(G)$ as a vector space) inherits a non-trivial grading induced by the age. Since $[V/G]$ is of dimension $n > 0$, the hidden loop product is trivial. Similarly, the pairing $[g] \Cap [h] = 0$ if either V^g or V^h is non-trivial, while if $V^g = V^h = \{0\}$, $[g] \Cap [h]$ is non-zero and behaves as in the case of $[*/G]$. It follows that there is an isomorphism of graded algebras

$$(H_\bullet^{\mathrm{orb}}([V/G]), \Cap) \cong Z_{V,G} \oplus Ann_{V,G}$$

where $Z_{V,G}$ is the (graded by the age) subalgebra of $Z(\mathbb{C}[G])$ generated by the elements with trivial invariant subspace and $Ann_{V,G}$ is the complementary subspace equipped with the zero multiplication. Finally, the orbifold cohomology ring $H^\bullet_{\mathrm{orb}}(\mathfrak{X})$ is isomorphic, as a graded ring, to $Z(\mathbb{C}[G])$ equipped with the age grading. Thus for a generic representation none of the three rings are isomorphic as rings, though they are isomorphic as vector spaces.

Example 16.15 We consider the Kummer surface orbifold $\mathfrak{K} = \big[(S^1)^4/\mathbb{Z}/2\mathbb{Z}\big]$ where $\mathbb{Z}/2\mathbb{Z}$ acts diagonally by $z \mapsto 1/z$ (on each factor). Then $\mathfrak{K}$ is a complex orbifold with 16 orbifolds points $\big\{\big((-1)_1^\epsilon, \dots, (-1)_4^\epsilon\big)\big\}$. We identify the above sets with $(\mathbb{Z}/2\mathbb{Z})^4$ (and we will denote $\epsilon = (\epsilon_1, \dots, \epsilon_4)$ an element in $(\mathbb{Z}/2\mathbb{Z})^4$). The age of the generator τ of $\mathbb{Z}/2\mathbb{Z}$ is one at every orbifold point. Thus, as a graded vector space,

$$H_\bullet^{\mathrm{orb}}(\mathfrak{K}) \cong H_\bullet(\mathfrak{K}) \oplus \big(\mathbb{C}[-2]\big)\big\langle(\mathbb{Z}/2\mathbb{Z})^4\big\rangle$$

where $\mathbb{C}[-2]$ is the vector space $\mathbb{C}$ placed in homological degree 2. Since the coefficient ring is $\mathbb{C}$, the cohomology of $\mathfrak{K}$ is computed by the cohomology of invariant form $\Omega\big((S^1)^4\big)^{\mathbb{Z}/2\mathbb{Z}}$ and its homology by the homology of $\mathbb{Z}/2\mathbb{Z}$-coinvariant chains. Hence

$$H_\bullet(\mathfrak{K}) \cong \mathbb{C}\cdot 1 \oplus \mathbb{C}[-2]\langle a_{i,j},\, 1 \le i < j \le 4\rangle \oplus \mathbb{C}[-4]\alpha_{\mathfrak{K}}$$

where $\alpha_{\mathfrak{K}}$ is the fundamental class of $\mathfrak{K}$ and $a_{i,j}$ is the class given by the cross product of the fundamental classes of the i^{th} and j^{th} circles in $\big(S^1\big)^4$. Thus

$$H_\bullet^{\mathrm{orb}}(\mathfrak{K}) \cong \mathbb{C}\cdot 1 \oplus \mathbb{C}[-2]\langle a_{i,j},\, 1 \le i < j \le 4\rangle \oplus \big(\mathbb{C}[-2]\big)\big\langle(\mathbb{Z}/2\mathbb{Z})^4\big\rangle \oplus \mathbb{C}[-4]\alpha_{\mathfrak{K}}$$

The computation of the intersection pairing is similar to the computations in Section 17.3. We find that $\alpha_{\mathfrak{K}}$ is a unit for the orbifold intersection pairing and that $a_{i,j} \Cap a_{k,l} = 1$ if the 4 indices are distinct and 0 otherwise. The (degree 2) generators $\epsilon \in (\mathbb{Z}/2\mathbb{Z})^4$ comes from the degree 0 homology of the corresponding orbifold point. Hence $\epsilon \Cap \epsilon = 1$ and $\epsilon \Cap \epsilon' = 0$ if $\epsilon \neq \epsilon'$. A degree argument also shows that $\epsilon \Cap a_{i,j} = 0$. This finishes the description of the whole intersection pairing of $\mathfrak{K}$. The hidden loop product of $\mathfrak{K}$ is a little bit different since $\epsilon \star \epsilon = 0$ for the latter product.

Example 16.16 Here is a (rather trivial) non compact example of orbifold product. Let d be a positive integer and consider the weighted projective stack $\mathbb{P}(d,d,\cdots,d) = [V/\mathbb{C}^*]$, where $V := \mathbb{C}^{n+1} - \{(0,0,\cdots,0)\}$ and the action of $\lambda \in \mathbb{C}^*$ is multiplication by $(\lambda^d, \lambda^d, \cdots, \lambda^d)$. Let $\mathfrak{U}$ be an open substack of $\mathbb{P}(d,d,\cdots,d)$. Denoting μ_d the group of d^{th}roots of unity, we have

$$\Lambda\mathfrak{U} = \coprod_{\xi \in \mu_d} \mathfrak{U}.$$

Note that the coarse moduli space of $\mathbb{P}(d,d,\cdots,d)$ is $\mathbb{CP}^n$, and the coarse moduli space of $\mathfrak{U}$ is an open U in $\mathbb{P}^n$. Combining the results of example 16.14 and remark 17.2 we see that the orbifold homology ring of $\mathfrak{U}$ is

$$H_\bullet^{\mathrm{orb}}(\mathfrak{U}) = \mathbb{C}[\mu_d] \otimes H_\bullet(U),$$

where $H_\bullet(U)$ is the usual homology of U endowed with intersection product, and $\mathbb{C}[\mu_d]$ is the group ring of the group μ_d of d^{th} roots of unity (sitting in degree zero).

Example 16.17 We compute the orbifold homology of the weighted projective line $\mathbb{P}(m,n)$. Recall that, for m and n positive integers, $\mathbb{P}(m,n) := [V/\mathbb{C}^*]$, where $V := \mathbb{C}^2 - \{(0,0)\}$ and the action of $\lambda \in \mathbb{C}^*$ on V is by multiplication by (λ^m, λ^n). We do *not* assume n and m to be relatively prime integers.

Note that, except for $\mathbb{CP}^1 = \mathbb{P}(1,1)$, a weighted projective line is never a (orbifold) global quotient as it is simply connected. We can, however, cover $\mathbb{P}(m,n)$ by two global quotient open substacks as follows. Denote the point $[0:1] \in \mathbb{P}(m,n)$ by 0 and the point $[1:0] \in \mathbb{P}(m,n)$ by ∞. Then

$$\mathbb{P}(m,n) - \{\infty\} \cong [\mathbb{C}/\mu_n] \quad \text{and} \quad \mathbb{P}(m,n) - \{0\} \cong [\mathbb{C}/\mu_m],$$

where $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ stands for the group of n^{th} roots of unity. The action of $\xi \in \mu_n$ on $\mathbb{C}$ is given by $x \mapsto \xi^d x$, where $d = \gcd(m,n)$. It follows that the inertia group of the point 0 is μ_n and the inertia group of ∞ is μ_m. The inertia group of every other point is μ_d. Note that μ_d sits inside μ_n as the cyclic subgroup generated by $e^{2i\pi/d} = \left(e^{2i\pi/n}\right)^{n/d}$ and similarly, it also sits inside μ_m as the cyclic subgroup generated by $\left(e^{2i\pi/m}\right)^{m/d}$.

Let us describe the inertia stack $\Lambda\mathbb{P}(m,n)$. Over the point $0 \in \mathbb{P}(m,n)$ consider a disjoint union of n copies of $[*/\mu_n]$ indexed by μ_n. Similarly, over the point ∞ consider a disjoint union of m copies of $[*/\mu_m]$ indexed by μ_m. For every $\psi \in \mu_d$, "join" the copies of $[*/\mu_n]$ and $[*/\mu_m]$ corresponding to $\psi \in \mu_n$ and $\psi \in \mu_m$, respectively, by viewing them as the 0 and the ∞ points of a new copy of $\mathbb{P}(m,n)$ (indexed by ψ). Then, $\Lambda\mathbb{P}(m,n)$ is the union of all these, that is,

$$\Lambda\mathbb{P}(m,n) = \big(\coprod_{\xi\in\mu_n}[*/\mu_n]\big) \cup_{\mu_d} \big(\coprod_{\psi\in\mu_d}\mathbb{P}(m,n)\big) \cup_{\mu_d} \big(\coprod_{\zeta\in\mu_m}[*/\mu_m]\big).$$

The map $\Lambda\mathbb{P}(m,n) \to \mathbb{P}(m,n)$ is the obvious one (it is the identity on each copy of $\mathbb{P}(m,n)$ in $\Lambda\mathbb{P}(m,n)$).

Let us now describe the orbifold homology $H_\bullet^{\mathrm{orb}}(\mathbb{P}(m,n))$. First we determine the ages of the twisted sectors of $\mathbb{P}(m,n)$. For every $\psi \in \mu_d$, the age of $\mathbb{P}(m,n)_\psi$ is zero. For $\xi = e^{2i\pi l/n} \in \mu_n$, the age of $[*/\mu_n]_\xi$ is $\{dl/n\}$. (For $r \in \mathbb{R}$, we define $0 \le \{r\} < 1$ to be the residue of r modulo 1.) Similarly, the age of $[*/\mu_m]_\zeta$, $\zeta = e^{2i\pi k/m} \in \mu_m$, is $\{dk/m\}$. Observe that the ages of two twisted sectors $[*/\mu_n]_\xi$ and $[*/\mu_m]_\zeta$ are distinct unless they lie on some $\mathbb{P}(m,n)_\psi$ (that is, if $\xi = \zeta = \psi$, for some $\psi \in \mu_d$).

The above decomposition of $\Lambda\mathbb{P}(m,n)$ implies that $H_\bullet^{\mathrm{orb}}(\mathbb{P}(m,n))$ is a union of three subrings R_0, R and R_∞ (given by the sectors/copies over the point 0, ∞ and a generic point of $\mathbb{P}(n,m)$) . Only R is a unital subring.

The ring R is the homology of $\coprod_{\psi\in\mu_d}\mathbb{P}(m,n)$ and is isomorphic, as a graded ring, to $\mathbb{C}[\mu_d]\otimes H_\bullet(\mathbb{CP}^1)$, where the elements of the group ring μ_d are sitting in degree 0 (this computation is the one done in Example 16.16). More precisely, if $\beta \in H_0(\mathbb{CP}^1) = \mathbb{C}$ is the homology class of a point and $\alpha \in H_2(\mathbb{CP}^1) = \mathbb{C}$ is the fundamental class, then $\psi\otimes\beta$ corresponds to the homology of a point in $H_0(\mathbb{P}(m,n)_\psi)$ and $\psi\otimes\alpha$ corresponds to the fundamental class of $\mathbb{P}(m,n)_\psi$. In particular, the unit element of R (and also of the whole $H_\bullet^{\mathrm{orb}}(\mathbb{P}(m,n))$ is $1\otimes\alpha$.

The ring R_0 is isomorphic, as a vector space, to $\mathbb{C}[\mu_n]$ (as in Example 16.14). For $\xi_l \in \mu_n$, the basis element $\xi_l \in \mathbb{C}[\mu_n]$ corresponds to the generator of $H_\bullet([*/\mu_n]_{\xi_l}) = H_0(*) = \mathbb{C}$. Its degree is equal to $2 - 2\{-dl/n\}$, where $\xi_l = e^{2i\pi l/n}$. Note that if $\psi = \xi_{\ell n/d} \in \mu_d \subset \mu_n$, then $[*/\mu_n]_\psi \subset \mathbb{P}(n,m)_\psi$ and the class $\xi_{\ell n/d}$ is identified with the degree 0 generator of $H_0(\mathbb{P}(n,m_\psi)$. Every class ξ_i in R_0 is given by (a degree shifting of) an ordinary degree 0 homology class of the point 0 viewed as lying in the ξ_i-fixed point locus 0^{ξ_i} . The map $m\colon \Lambda\mathbb{P}(m,n)\times_{\mathbb{P}(n,m)}\Lambda\mathbb{P}(n,m) \to \mathbb{P}(n,m)$ clearly maps $0^{\xi_i,\xi_j}$ (the intersection locus of the copies of $[*/\mu_n]_{\xi_i}$ and $[*/\mu]_{\xi_j}$ indexed by ξ_i and ξ_j) to $0^{\xi_{i+j}}$ (the copy of $[*/\mu_n]$ indexed by ξ_{i+j}). If ξ_i is in μ_d (that is, it is of the form $\ell n/d$), then $\xi_i \Cap \xi_j = 0$ for degree reasons, since ξ_i is a degree zero homology class. Thus to compute the intersection pairing, it now remains to analyze the obstruction bundle in the other cases. For an integer i, let us denote $0 \le \{i\}_n < n/d$ the residue of i modulo n/d. From (the Riemann-Roch) formula (16.4), we find that, for $\xi_i, \xi_j \notin \mu_d$, the obstruction bundle (over the copies indexed by ξ_i and ξ_j) is of rank 0 if

$$\{i\}_n + \{j\}_n \le n/d$$

and is of rank 2 otherwise. Indeed, if $\{i\}_n + \{j\}_n < n/d$, then $\{d(i+j)/n\} = \{di/n\} + \{dj/n\}$ and $\dim_2 = 0 = \dim\circ m$. If $\{i\}_n + \{j\}_n > n/d$, then $\{d(i+$

$j)/n\} = \{di/n\} + \{dj/n\} - 1$ and $\dim_2 = 0 = \dim \circ m$. If $\{i\}_n + \{j\}_n = n/d$, then $\{d(i+j)/n\} = \{di/n\} + \{dj/n\} - 1$ and $\dim_2 = 0$ but now $\dim \circ m = 2$ (the dimension of the copy $\mathbb{P}(n,m)_{\xi_{i+j}}$).

Thus, it follows that the orbifold intersection pairing $\xi_i \Cap \xi_j$ of basis elements ξ_i and ξ_j is given by

$$\xi_i \Cap \xi_j = \begin{cases} \xi_{i+j} & \text{if } \{i\}_n + \{j\}_n \leq n/d \quad \text{and } \xi_i, \xi_j \notin \mu_d \\ 0 & \text{if } \{i\}_n + \{j\}_n > n/d \quad \text{or } \xi_i \text{ or } \xi_j \in \mu_d \end{cases} \tag{16.14}$$

In particular, R_0 is generated, as a graded ring, by the elements $\xi_1, \xi_{1+n/d}, \dots$. It is easy to describe the action of R on R_0. Indeed, for degree reasons, $(\psi \otimes \beta) \Cap \xi_i = 0$. However, since the fundamental class of $\mathbb{P}(n,m)_{\xi_{\ell n/d}}$ intersects $[*/\mu_n]_{\xi_i}$ for any $\xi_i \in \mu_n$ with a trivial obstruction bundle, we have $(\xi_{\ell n/d} \otimes \alpha) \Cap \xi_i = \xi_{i+\ell n/d}$ (in particular $1 \otimes \beta$ acts as the unit). Summing up, we have

$$\begin{cases} (\psi \otimes \beta) \Cap \xi_i &= 0 \\ (\xi_{\ell n/d} \otimes \alpha) \Cap \xi_i &= \xi_{i+\ell n/d} \end{cases} \tag{16.15}$$

The ring R_∞ is isomorphic, as a vector space, to $\mathbb{C}[\mu_m]$. For $\zeta_k \in \mu_m$, the basis element $\zeta_k \in \mathbb{C}[\mu_m]$ corresponds to the generator of $H_\bullet([*/\mu_m]_{\zeta_k}) = H_0(*) = \mathbb{C}$. Its degree is equal to $2 - 2\{-dk/m\}$, where $\zeta_k = e^{2i\pi k/m}$. We denote $0 \leq \{i\}_m < m/d$ the residue of any integer i modulo m/d. A computation similar to the one of R_0 shows that, the orbifold intersection pairing $\zeta_p \Cap \zeta_q$ of basis elements $\zeta_p, \zeta_q \in \mu_m$ is given by

$$\zeta_p \Cap \zeta_q = \begin{cases} \zeta_{p+q} & \text{if } \{p\}_m + \{q\}_m \leq m/d \quad \text{and } \zeta_p, \zeta_q \notin \mu_d \\ 0 & \text{if } \{p\}_m + \{q\}_m > m/d \quad \text{or } \zeta_p \text{ or } \zeta_q \in \mu_d \end{cases} \tag{16.16}$$

and that the R action on R_∞ is given by

$$\begin{cases} (\psi \otimes \beta) \Cap \zeta_p &= 0 \\ (\zeta_{\ell m/d} \otimes \alpha) \Cap \zeta_p &= \zeta_{p+\ell m/d}. \end{cases} \tag{16.17}$$

Since the points 0 and ∞ do not intersect, it is immediate that $R_0 \Cap R_\infty = \{0\}$.

From the above descriptions (identifying the sectors $[*/\mu_n]_{\xi_{\ell n/d}}$ with their image in $\mathbb{P}(n,m)_{\xi_{\ell n/d}}$), we find that the orbifold homology $H_\bullet^{\text{orb}}(\mathbb{P}(n,m))$ is the graded ring

$$H_\bullet^{\text{orb}} = R_0 \oplus R \oplus R_\infty / (\xi_{\ell n/d} = \xi_{\ell n/d} \otimes \alpha = \zeta_{\ell m/d} \otimes \alpha = \zeta_{\ell m/d})$$

where the product structure is given by formulas (16.14), (16.15), (16.16) and (16.17).

The reader can check that the orbifold product on $H_\bullet^{\text{orb}}(\mathbb{P}(m,n))$ is Poincaré dual to the Chen-Ruan orbifold cup product in Example 5.3 of [19] (note that *loc. cit.* only considers the relatively prime case). It is not hard to check that the hidden loop product is different. Indeed, the grading is different (concentrated in degree 0 and 2) and the only non zero products of basis elements are those involving $\psi \otimes \beta$.

17 Examples

17.1 The case of manifolds

Smooth manifolds form a special class of differentiable stacks with normally non-singular diagonal (Definition 8.15). Denote by the same letter M a manifold and its associated (topological) stack. The diagonal $\Delta\colon M \to M \times M$ is strongly oriented iff the manifold M is oriented.

Proposition 17.1 *Let M be an oriented manifold. The* **BV***-algebra, Frobenius algebra and (non-unital, non-counital) homological conformal field theories structures of $H_\bullet(\mathrm{L}M)$ given by Theorem 13.2, Theorem 12.5 and Theorem 14.2 coincide with the ones of Chas-Sullivan [15], Cohen-Jones [23], Cohen-Godin [22] and Godin [34].*

Proof. By Proposition 5.7, the free loop stack of M is isomorphic to the free loop space $\mathrm{L}M$. It follows from Proposition 9.3 and Proposition 8.32 (in the case $G = \{1\}$), that the Gysin maps of Sections 10, 12, 13 coincide with the Gysin maps in [23] Section 1 (also see [30] Section 3.1). $\square$

Remark 17.2 When M is an oriented manifold, the hidden loop product on $H_\bullet(\Lambda M) \cong H_\bullet(M)$ is simply the usual intersection pairing. It is also immediate that, if M is an almost complex manifold, the orbifold intersection pairing of M also coincides with the usual intersection pairing.

Remark 17.3 Similarly, the brane product for manifolds given by Proposition 15.1 coincides with Sullivan-Voronov one [25] and Chataur one [17].

17.2 Hidden loop (co)product for global quotient by a finite group

An important class of oriented orbifolds are the global quotients $[M/G]$, where G is a finite group, M is an oriented manifold together with an action of G by orientation preserving diffeomorphisms. In this case, the homology of the inertia stack $H(\Lambda[M/G])$ is well known. Assume that our coefficient ring k is a field of characteristic coprime with $|G|$ (or 0). The inertia stack of $[M/G]$ is represented by the transformation groupoid

$$\coprod_{g\in G} M^g \times G \rightrightarrows \coprod_{g\in G} M^g \qquad (17.1)$$

where the action of $h \in G$ moves $y \in M^g$ to $y \cdot h \in M^{h^{-1}gh}$. Furthermore,

$$\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \cong \big[\coprod_{g,h\in G} M^{g,h}/G\big],$$

where $M^{g,h} = M^g \cap M^h$, and the "Pontrjagin" map $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ is induced by the embeddings $i_{g,h}\colon M^{g,h} \hookrightarrow M^{gh}$. Since $|G|$ is coprime with $\mathrm{char}(k)$, the homology groups of the inertia stack $\Lambda[M/G]$ are

$$H_\bullet(\Lambda[M/G]) \cong H_\bullet\left(\coprod_{g\in G} M^g\right)_G \cong \left(\bigoplus_{g\in G} H_\bullet(M^g)\right)_G .$$

The excess bundle $Ex(M, X, X')$ of the diagram of embeddings

$$\begin{tikzcd}[column sep=small] & X \arrow[dr] & \\ Z = X \bigcap X' \arrow[ur] \arrow[dr] & & M \\ & X' \arrow[ur] & \end{tikzcd}$$

is the cokernel of the bundle map $N_{Z\hookrightarrow X} \hookrightarrow (N_{X'\hookrightarrow M})_{/Z}$. Thus $Ex(M, X, X')$ is the virtual bundle $T_M - T_X - T_{X'} + T_Z$ (each component being restricted to Z). For $g, h \in G$, we denote $Ex(g, h) := Ex(M, M^g, M^h)$. The bundles $Ex(g, h)$ induce a bundle Ex on $\Lambda[M/G] \times_{[M/G]} \Lambda[M/G]$ whose Euler class is denoted $e(Ex)$. Since the diagonal $G \to G\times G$ is a group monomorphism, there is a transfer map

$$\mathrm{tr}_{G\times G}^G\colon \Big(\bigoplus_{g,h\in G} H_\bullet(M^g)\otimes H_\bullet(M^h)\Big)_{G\times G} \to \Big(\bigoplus_{g,h\in G} H_\bullet(M^g)\otimes H_\bullet(M^h)\Big)_G$$

explicitly given (see Equation (9.6)) by

$$\mathrm{tr}_{G\times G}^G(x) = \sum_{g\in G} x\cdot(g, 1).$$

The maps $i_g\colon M^{g,h} \hookrightarrow M^g$, $i_h : M^{g,h} \hookrightarrow M^h$ yield Gysin morphims $(i_g \times i_h)^!\colon H_\bullet(M^g \times M^h) \to H_\bullet(M^{g,h})$.

Proposition 17.4 *The hidden loop product $\star : H(\Lambda[M/G]) \otimes H(\Lambda[M/G]) \to H(\Lambda[M/G])$ is the composition*

$$\left(\bigoplus_{g\in G} H(M^g)\right)_G \otimes \left(\bigoplus_{h\in G} H(M^h)\right)_G \to \left(\bigoplus_{g,h\in G} H(M^g\times M^h)\right)_{G\times G} \xrightarrow{\mathrm{tr}_{G\times G}^G} \left(\bigoplus_{g,h\in G} H(M^g\times M^h)\right)_G$$

$$\xrightarrow{\bigoplus (i_g\times i_h)^!} \left(\bigoplus_{g,h\in G} H(M^{g,h})\right)_G \xrightarrow{\cap e(Ex)} \left(\bigoplus_{g,h\in G} H(M^{g,h})\right)_G \xrightarrow{m_*} \left(\bigoplus_{k\in G} H(M^k)\right)_G$$

The proof of Proposition 17.4 relies on Lemma 17.5 below, which is of independent interest. Note that there is a oriented stack morphism

$$\varphi\colon \Lambda[M/G] \times_{[M/G]} \Lambda[M/G] \cong \left[\coprod_{g,h\in G} M^{g,h}/G\right] \to \Lambda[M/G]\times\Lambda[M/G] \quad (17.2)$$

induced by the groupoid map $(x, g) \mapsto (i_g(x), g, i_h(x), g)$.

Lemma 17.5 *The Gysin map $\varphi^!$ is the composition*

$$\left(\bigoplus_{g,h\in G} H(M^g \times M^h)\right)_{G\times G} \overset{\mathrm{tr}^G_{G\times G}}{\longrightarrow} \left(\bigoplus_{g,h\in G} H(M^g \times M^h)\right)_G \overset{\bigoplus(i_g\times i_h)^!}{\longrightarrow} \left(\bigoplus_{g,h\in G} H(M^{g,h})\right)_G.$$

PROOF. The G-equivariant map $M^{g,h} \to M^g \times M^h$, given by $x \mapsto (i_g(x), i_h(x))$, induces an oriented stack morphism

$$\psi\colon \Lambda[M/G] \times_{[M/G]} \Lambda[M/G] \to \Lambda[M/G] \times_{[*/G]} \Lambda[M/G].$$

By Proposition 9.3, $\psi^! = \oplus(i_g \times i_h)^!$. Then, the result follows from the functoriality of Gysin maps and Lemma 9.4. $\Box$

PROOF OF PROPOSITION 17.4. We use the notations of Section 11.1. The cartesian diagram (11.3) (where $\mathfrak{X} = [M/G]$) and the excess formula 9.5 shows that,

$$\Delta^! = \varphi^!(x) \cap e(Ex).$$

Thus the result follows from Lemma 17.5. $\Box$

Similarly we compute the hidden loop coproduct. For any $g \in G$, the unit $1_g \in H_0(M^g)$ induces a map

$$1_g\colon H(M^g) \to H_0(M^g) \otimes H(M^g) \to H(M^g \times M^g).$$

Proposition 17.6 *The hidden loop coproduct is induced (after passing to G-invariant) by the composition*

$$\bigoplus_{g\in G} H(M^g) \overset{\oplus 1_g}{\longrightarrow} \bigoplus_{g\in G} H(M^g \times M^g) \overset{tr^G_{G\times G}}{\longrightarrow} \bigoplus_{g,h\in G} H(M^h \times M^g) \overset{\oplus i^!_{g,h}}{\longrightarrow} \bigoplus_{g,h\in G} H(M^{g,h})$$
$$\overset{\cap e(Ex)}{\longrightarrow} \bigoplus_{g,h\in G} H(M^{g,h}) \overset{(i_g,i_h)_*}{\longrightarrow} \bigoplus_{g,h\in G} H(M^g \times M^h) \cong \bigoplus_{g,h\in G} H(M^g) \otimes H(M^h).$$

PROOF. Let $\mathbb{X}$ be the transformation groupoid $[M \rtimes G \rightrightarrows M]$. Unfolding the definition of the groupoid $\widetilde{\Lambda\mathbb{X}}$ (see Section 12.3), one finds that $\widetilde{\Lambda\mathbb{X}}$ is the transformation groupoid

$$\left(G \times \coprod_{h\in G} M^h\right) \rtimes G^2 \rightrightarrows G \times \coprod_{h\in G} M^h,$$

where the action of $(h_0, h_{1/2}) \in G^2$ on $(g, m) \in g \times M^h$ is $(h_0^{-1} g h_{1/2}, m.h_0)$. The Morita map $p\colon \widetilde{\Lambda\mathbb{X}} \to \Lambda[M/G]$ (Equation (12.12)) has a section κ defined, for $m \in M^h$ and $h_0 \in G$, by $\kappa(m, h_0) = (h, m, h_0, h_0)$. In particular κ induces an isomorphism in homology and commutes with Gysin maps. Thus the Gysin map $\Delta^!$ of Section 12.3 is the composition of κ_* with the Gysin map associated to the sequence of cartesian diagrams

$$\begin{tikzcd} {[\coprod M^{g,h}/G]} \arrow[r] \arrow[d] & {[G \times \coprod M^h/G]} \arrow[r] \arrow[d] & {[G \times \coprod M^h/G \times G]} \arrow[d] \\ {[M/G]} \arrow[r] & {[M \times M/G]} \arrow[r] & {[M/G] \times [M/G]} \end{tikzcd} \quad . \tag{17.3}$$

By Proposition 9.5 and Lemma 9.4, the Gysin maps associated to the left square and the right square are, respectively, $i^!_{g,h}(-) \cap e(Ex(g,h))$ and $\mathrm{Tr}^G_{G\times G}$. Since $\kappa_* = \oplus 1_g$, the result follows. □

Example 17.7 Consider $[*/G]$ where G is a finite group. By Proposition 5.9, the stack morphism $\Phi\colon \Lambda[*/G] \to \mathrm{L}[*/G]$ (see Lemma 12.14) is an isomorphism. Let k be a field of characteristic coprime with $|G|$. Then

$$H_\bullet(\Lambda[*/G]) = \big(\bigoplus_{g\in G} k\big)_G \cong \big(\bigoplus_{g\in G} k\big)^G \cong Z(k[G])$$

where $Z(k[G])$ is the center of the group algebra $k[G]$. By Propositions 17.4, the isomorphism $H_\bullet(\Lambda[*/G]) \cong Z(k[G])$ is an isomorphism of algebras. By Proposition 17.6, the hidden loop coproduct is given by $\delta([g]) = \sum_{hk=g}[h]\otimes[k]$. Thus the Frobenius algebra structure coincides with the one given by Dijkgraaf-Witten [28].

17.3 String topology of $[S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]$

Consider the euclidean sphere

$$S^{2n+1} = \{|z_0|^2 + \cdots + |z_n|^2 = 1, z_i \in \mathbb{C}\}$$

acted upon by $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ identified with the group generated the reflections across the hyperplanes $z_i = 0$ $(0 \le i \le n)$. Let $\mathfrak{X} = [S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]$ be the induced quotient stack which is obviously an oriented orbifold of dimension $2n+1$. We now describe the Frobenius algebras associated to $\Lambda\mathfrak{X}$ and $\mathrm{L}\mathfrak{X}$. Until the end of this section we denote $R = (\mathbb{Z}/2\mathbb{Z})^{n+1}$.

The hidden loop product has a very simple combinatorial description. Let Δ^n be a n-dimensional standard simplex. Denote $v_0, \dots, v_n$ its $n+1$-vertex and $F_0, \dots, F_n$ its n-faces of dimension $n-1$. In other words $F_i = \Delta(v_0, \dots, \widehat{v_i}, \dots, v_n)$ is the convex hull of all vertices but v_i. More generally we denote $F_{i_1\dots i_k} := F_{i_1} \cap \cdots \cap F_{i_k}$ the subface of dimension $n-k$ given by the convex hull of all vertices but $v_{i_1}, \dots, v_{i_k}$. We assign the degree $2n-2k+1$ to a face $F_{i_1\dots i_k}$ of dimension $n-k$.

Proposition 17.8 *Let k be a ring with $1/2 \in k$. Then $H_\bullet(\Lambda\mathfrak{R})$ is the free k-module with basis indexed by elements $r \in R - \{1\}$ of degree 0 and all faces $F_{i_1 \dots i_k}$ in degree $2(n-k)+1$ (in particular $F_\emptyset = \Delta^n$ has degree $2n+1$), i.e.,*

$$H_\bullet(\Lambda\mathfrak{R}) \quad \cong \quad k^{|R|-1} \oplus \Big(\bigoplus_{\substack{k=0\dots n \\ 0 \le i_1 < \dots < i_k \le n}} k.F_{i_1 \dots i_k} \Big).$$

The hidden loop product $\star$ is defined on the basis by the identities

$$F_{i_1 \dots i_k} \star F_{j_1 \dots j_l} = F_{i_1 \dots i_k} \cap F_{j_1 \dots j_l}$$

if the two subfaces have transversal intersection in Δ^n, and is 0 otherwise. The element $\Delta^n = F_\emptyset$ is set to be the unit and all other products involving a generator of $k^{|R|-1}$ are trivial.

In other words, $H_0(\Lambda\mathfrak{R}) = k^{|R|-1}$, and $H_{2i+1}(\Lambda\mathfrak{R})$ is the free module generated by the subfaces of dimension i of the simplex Δ^n. The product is given by transverse intersection in Δ^n.

Proof. Write s_i (i=0...n) for the reflection across the hyperplane $z_i = 0$. Then, for $0 \le k \le n$,

$$(S^{2n+1})^{s_{i_1} \dots s_{i_k}} \quad \cong \quad \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \, / \sum_{j \neq i_1, \dots i_k} |z_j|^2 = 1 \right\} \cong S^{2n-2k+1}.$$

Thus

$$H_\bullet\left((S^{2n+1})^{s_{i_1} \dots s_{i_k}} \right) \cong k\, V'_{s_{i_1} \dots s_{i_k}} \oplus k\, F'_{i_1 \dots i_k}[2(n-k)+1].$$

Since these generators are R-invariant, $|R|$ is invertible in k and $(S^{2n+1})^{s_0 \dots s_n} = \emptyset$, one has

$$H_\bullet(\Lambda\mathfrak{R}) \cong \bigoplus_{g \in R} H_\bullet\big((S^{2n+1})^g\big)_R \cong \bigoplus_{g - \{1\} \in R} H_\bullet\big((S^{2n+1})^g\big)$$

By Proposition 17.4, the hidden loop product is the composition of $\mathrm{tr}^R_{R \times R}$ with

$$H\big((S^{2n+1})^g \times (S^{2n+1})^h\big) \overset{(i_g \times i_h)^!}{\to} H\big((S^{2n+1})^{g,h}\big) \overset{\cap e(Ex(g,h))}{\longrightarrow}$$
$$H\big((S^{2n+1})^{g,h}\big) \overset{m_*}{\to} H((S^{2n+1})^{gh}) \quad (17.4)$$

Clearly $\mathrm{tr}^R_{R \times R}$ is multiplication by the order of R. Furthermore $F_{i_1 \dots i_k}$ and $F_{j_1 \dots j_l}$ are transversal iff the sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_l\}$ are disjoint iff the submanifolds $(S^{2n+1})^{s_{i_1} \dots s_{i_k}}$ and $(S^{2n+1})^{s_{j_1} \dots s_{j_l}}$ are transversal in (S^{2n+1}). In particular, if $F_{i_1 \dots i_k}$ and $F_{j_1 \dots j_l}$ are transversal,

$$(S^{2n+1})^{s_{i_1} \dots s_{i_k}, s_{j_1} \dots s_{j_l}} = (S^{2n+1})^{s_{i_1} \dots s_{i_k} . s_{j_1} \dots s_{j_l}},$$

the excess bundle is of rank 0, $m_* = \mathrm{id}$ and by Poincaré duality,

$$(i_{s_{i_1}\dots s_{i_k}} \times i_{s_{j_1}\dots s_{j_l}})^! \,(F'_{i_1\dots i_k} \times F'_{j_1\dots j_l}) \quad = \quad F'_{i_1\dots i_k j_1\dots j_l}.$$

If $F_{i_1\dots i_k}$ and $F_{j_1\dots j_l}$ are not transversal, one finds

$$(i_{s_{i_1}\dots s_{i_k}} \times i_{s_{j_1}\dots s_{j_l}})^!(F'_{i_1\dots i_k} \times F'_{j_1\dots j_l}) \quad = \quad F'_{i_1\dots i_k} \cap F'_{j_1\dots j_l} = F'_{\{i_1,\dots,i_k\}\cup\{j_1,\dots,j_l\}}$$

and $(S^{2n+1})^{s_{i_1}\dots s_{i_k} s_{j_1}\dots s_{j_l}}$ contains $(S^{2n+1})^{s_{i_1}\dots s_{i_k}, s_{j_1}\dots s_{j_l}}$ as a submanifold of codimension > 0. It follows that

$$m_*\left(F'_{\{i_1,\dots,i_k\}\cup\{j_1,\dots,j_l\}} \cap e(Ex)\right) = 0$$

for degree reason. Similarly, $F'_{i_1,\dots,i_k} \star g = 0$ for any $g \in R$. The result follows by identifying $F_{i_1,\dots,i_k}$ with $2^{-n-1}F'_{i_1,\dots,i_k}$ as basis element. □

Remark 17.9 It is easy to show that the hidden loop coproduct is trivial. Indeed, for degree reason, only the class of $F_\emptyset$ might be non zero. Proposition 17.6 shows the hidden loop coproduct is induced by the composition

$$H(S^{2n+1}) \overset{\sum i_g^!}{\to} \bigoplus H((S^{2n+1})^g) \overset{\cap \oplus e\left((S^{2n+1})^h\right)}{\to} \bigoplus H((S^{2n+1})^g).$$

Since $(S^{2n+1})^h$ is an odd dimensional sphere, its Euler class is 2-torsion, hence trivial by our assumption on k.

Since $R = (\mathbb{Z}/2\mathbb{Z})^{n+1}$ is abelian, its group algebra is a Frobenius algebra (see Example 17.7 above).

Proposition 17.10 *Let k be a field of characteristic different from 2. There is an isomorphism of* **BV***-algebras as well as Frobenius algebras*

$$H_\bullet(\mathrm{L}\mathfrak{R}) \quad \cong \quad H_\bullet(\mathrm{L}S^{2n+1}) \otimes_k k[(\mathbb{Z}/2\mathbb{Z})^{n+1}]. \tag{17.5}$$

The **BV***-operator on the right hand side is $B \otimes \mathrm{id}$ where $B : H_\bullet(\mathrm{L}S^{2n+1}) \to H_{\bullet+1}(\mathrm{L}S^{2n+1})$ is the* **BV***-operator of the loop homology of S^{2n+1}.*

Proof. According to Proposition 5.9, the free loop stack $\mathrm{L}\mathfrak{R}$ is presented by the groupoid

$$[\coprod_{g\in R} \mathcal{P}_g S^{2n+1} \rtimes R \rightrightarrows \coprod_{g\in R} \mathcal{P}_g S^{2n+1}].$$

Hence

$$H_\bullet(\mathrm{L}\mathfrak{X}) \quad = \quad \big(\bigoplus_{g\in R} H_\bullet(\mathcal{P}_g S^{2n+1})\big)_R.$$

Since R is a subgroup of the connected Lie group $SO(2n+2)$, which acts on S^{2n+1}, for all $g \in R$ there is a continuous path $\rho : [0,1] \to SO(2n+2)$ connecting

g to the identity (that is $\rho(0) = g$, $\rho(1) = 1$). In particular, any path $f \in P_g S^{2n+1}$ can be composed with the path $f(0).\rho(t)$ yielding a loop $\Upsilon_g(f) \in \mathrm{L}S^{2n+1}$. It is a general fact that $\Upsilon_g \colon \mathcal{P}_g S^{2n+1} \to \mathrm{L}S^{2n+1}$ is a G-equivariant homotopy equivalence (see [50] for details). We write

$$\Upsilon : \coprod_{g\in R} \mathcal{P}_g S^{2n+1} \to \coprod_{g\in R} \mathrm{L}S^{2n+1}$$

for the map induced by the maps Υ_g for $g \in R$. Since the G-action on $\mathrm{L}M = P_e M$ is trivial, the isomorphism (17.5) follows.

It remains to prove that the linear isomorphism (17.5) is an isomorphism of Frobenius algebras and **BV**-algebras. To do so, we need the evaluation map $\mathrm{ev}_0 : \mathrm{L}\mathfrak{R} \to \mathfrak{R}$ at the groupoid level. One checks that ev_0 is represented by the maps

$$\mathrm{ev}_g \colon \mathcal{P}_g S^{2n+1} \times R \to S^{2n+1} \times R$$

defined by $\mathrm{ev}_g\big((f,h)\big) = (f(1), h)$. Let $(f,g) \in \mathcal{P}_r S^{2n+1} \times \mathcal{P}_h S^{2n+1}$ be such that $f(1) = g(1)$. The composition of the path $f(-)$ and $g(-)\cdot h$ gives an element $m(f,g) \in \mathcal{P}_{rh} S^{2n+1}$. This composition induces the stack morphism $m : \mathrm{L}\mathfrak{R} \times_{\mathfrak{R}} \mathrm{L}\mathfrak{R} \to \mathrm{L}\mathfrak{R}$. Denote by $\widetilde{m}$ the map

$$\coprod_{g,h\in R} \mathrm{L}S^{2n+1} \times_{S^{2n+1}} \mathrm{L}S^{2n+1} \stackrel{\widetilde{m}}{\to} \coprod_{g\in R} \mathrm{L}S^{2n+1}$$

which maps an element $(\gamma,\gamma') \in \mathrm{L}S^{2n+1} \times_{S^{2n+1}} \mathrm{L}S^{2n+1}$ in the component (g,h) to the element $m(\gamma,\gamma')$ in the component gh. Here m is the usual composition of paths. The map $\Upsilon_g \colon \mathcal{P}_g S^{2n+1} \to \mathrm{L}S^{2n+1}$ induces a commutative diagram of R-equivariant maps

$$\begin{array}{ccccc}
\coprod_{g,h} P_g S^{2n+1} \times P_h S^{2n+1} & \leftarrow & \coprod_{g,h} P_g S^{2n+1} \times_{S^{2n+1}} P_h S^{2n+1} & \xrightarrow{m} & \coprod_g P_g S^{2n+1} \\
\downarrow{\scriptstyle \coprod \Upsilon_g \times \Upsilon_h} & & {\scriptstyle \coprod \Upsilon_g \times \Upsilon_h}\downarrow & & \downarrow{\scriptstyle \coprod \Upsilon_g} \\
\coprod_{g,h} \mathrm{L}S^{2n+1} \times \mathrm{L}S^{2n+1} & \longleftarrow & \coprod_{g,h} \mathrm{L}S^{2n+1} \times_{S^{2n+1}} \mathrm{L}S^{2n+1} & \xrightarrow{\widetilde{m}} & \coprod_g \mathrm{L}S^{2n+1}.
\end{array}$$

Since $\mathrm{L}S^{2n+1} \to S^{2n+1}$, $\mathcal{P}_g S^{2n+1} \to S^{2n+1}$ are fibration, the vertical arrows are R-homotopy equivalences. It follows easily that the map

$$\frac{1}{|R|}\Upsilon : H_\bullet\big([\coprod_{g\in R} \mathrm{L}S^{2n+1}/R]\big) \to H_\bullet(\mathrm{L}S^{2n+1}) \otimes k[R]$$

is a morphism of algebras. One proves similarly that $\dfrac{1}{|R|}\Upsilon$ is a coalgebra map.

Now we need to identify the **BV**-operator. Denote again $\mathrm{L}\mathbb{X}$ the transformation groupoid

$$\mathrm{L}\mathbb{X} := \Big[\coprod_{g\in R} \mathcal{P}_g S^{2n+1} \rtimes R \rightrightarrows \coprod_{g\in R} \mathcal{P}_g S^{2n+1}\Big]$$

and recall Remark 5.10.

Since the stack S^1 is canonically identified with the quotient stack $[\mathbb{R}/\mathbb{Z}]$, the homology $H_\bullet(S^1)$ coincides with the homology of the groupoid $\Gamma' := [\mathbb{R} \rtimes \mathbb{Z} \rightrightarrows \mathbb{R}]$. The 0-dimensional simplex $(0,1) \in \mathbb{R} \rtimes \mathbb{Z} = \Gamma'_1$ defines an element in $C_0(\Gamma'_1) \subset C_1(\Gamma')$ which is the generator of $H_1(S^1)$.

The map $\Gamma' \times \mathrm{L}\mathbb{X} \xrightarrow{\theta} \mathrm{L}\mathbb{X}$ defined, for $(x,n) \in \mathbb{R} \times \mathbb{Z}$, $f \in \mathcal{P}_g$ and $h \in R$, by $\theta(x,n,f,h)(t) = f(t+x).g^n h$ is a groupoid morphism representing the S^1-action on $\mathrm{L}\mathfrak{X}$. Since $\Upsilon(\theta((0,1),f) = f$, Υ commutes with the **BV**-operator. □

Remark 17.11 For the sake of completeness, we recall [24] that, $\mathbb{H}_\bullet(\mathrm{L}S^{2n+1}) \cong k[u,v]$ with $|v| = -2n-1$ and $|u| = 2n$ for $n > 0$, and $\mathbb{H}_\bullet(\mathrm{L}S^{2n+1}) \cong k[[u,u^{-1}]][v]$ if $n = 0$. Thus

$$\mathbb{H}_\bullet(\mathrm{L}[S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]) \cong k[(\mathbb{Z}/2\mathbb{Z})^{n+1}][u,v] \text{ if } n > 0, \quad \text{and}$$

$$\mathbb{H}_\bullet(\mathrm{L}[S^1/\mathbb{Z}/2\mathbb{Z}]) \cong k[[u,u^{-1}]][\tau,v]/(\tau^2 = 1) \quad \text{with } |v| = 1,\ |u| = 0 \text{ if } n = 0.$$

Remark 17.12 The stack morphism $\Phi\colon \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ of Section 12.15 is represented at the groupoid level by $\coprod_{g\in R}(S^{2n+1})^g \hookrightarrow \coprod_{g\in R}\mathcal{P}_g S^{2n+1}$ where $x \in (S^{2n+1})^g$ is identified with a constant path. It follows easily that the Frobenius algebra morphism is given by $\Phi(F_\emptyset) = e$, $\Phi(F_{i_1\ldots i_k}) = 0$ and $\Phi(g) = gv$.

17.4 String topology of $\mathrm{L}[*/G]$ when G is a compact Lie group

Any topological group G naturally defines a topological stack corresponding to the groupoid $[G \rightrightarrows \{*\}]$, which is denoted by $[*/G]$. In this section we study the Frobenius structures on the homology of its loop stack and inertia stack assuming that G is a compact and connected Lie group. It turns out that in this case the two Frobenius structures obtained are indeed isomorphic since $\Lambda[*/G]$ and $\mathrm{L}[*/G]$ are homotopy equivalent. In this section, we assume that G is of dimension d and we will work with real coefficients for (co)homology groups.

First we will identify the homology groups $H_\bullet(\Lambda[*/G])$ and $H_\bullet(\mathrm{L}[*/G])$.

Lemma 17.13 *The inertia stack $\Lambda[*/G]$ is represented by the transformation groupoid $[G \rtimes G \rightrightarrows G]$, where G acts on itself by conjugation, while the stack $\Lambda[*/G] \times_{[*/G]} \Lambda[*/G]$ is represented by the groupoid $[(G \times G) \rtimes G \rightrightarrows G \times G]$ with the diagonal conjugacy action.*

The following result is well known [13].

Lemma 17.14 *The map $\Lambda[*/G] \xrightarrow{\Phi} \mathrm{L}[*/G]$ is an homotopy equivalence.*

Proof. Since G is connected, by Proposition 5.7, $\mathrm{L}[*/G]$ can be represented by the loop group $[\mathrm{L}G \rightrightarrows \{*\}]$. On the other hand, $B\mathrm{L}G \cong \mathrm{L}BG$ is homotopy equivalent to $EG \times_G G$ [9], [13]. This equivalence can be seen as follows. Denote by e the unit of G and let P_eG be the based path space of G, that is P_eG is the set, endowed with the compact-open topology, of paths $[0,1] \xrightarrow{f} G$ such that $f(0) = e$. There is an action of the loop group LG on $EG \times P_eG$ given, for $(x, f) \in EG \times P_eG$ and $\gamma \in LG$, by

$$(e, f) \cdot \gamma = \big(e \cdot \gamma(0), \gamma(0)^{-1} * f * \gamma\big)$$

where $*$ stands for the (pointwise) multiplication in G. This action is clearly free, thus $BLG \cong (EG \times P_eG) \times_{LG} \{\mathrm{pt}\}$. The map $(x, f) \mapsto (x, f(1))$ induces a continuous map $\rho : (EG \times P_eG) \times_{LG} \{\mathrm{pt}\} \to EG \times_G G$. Since G is connected, for any $g \in G$, there is a path $f_g \in P_eG$ with $f_g(1) = g$. The map

$$EG \times G \ni (x, g) \mapsto (x, f_g) \in G \times P_eG$$

induces a well-defined map $\psi\colon EG \times_G G \to (EG \times P_eG) \times_{LG} \{\mathrm{pt}\}$. It is easy to see that ψ is independent of the choice of the f_g's and is a left and right inverse of ρ. Hence the homotopy equivalence $BLG \cong EG \times_G G$ follows. Through the isomorphism in between $\mathrm{L}[*/G]$ and $[*/LG]$ (Proposition 5.7), the map Φ of Lemma 12.14 is transfered to the map

$$\psi\colon B[G/G] \to (EG \times P_eG) \times_{LG} \{\mathrm{pt}\} \cong B[\mathrm{pt}/LG].$$

The result follows. $\square$

As an immediate consequence, we have

Corollary 17.15 *The map* $\Phi_*\colon H_\bullet(\Lambda[*/G]) \to H_\bullet(\mathrm{L}[*/G])$ *is an isomorphism of Frobenius algebras.*

Thus it is sufficient to study the Frobenius structure on the homology of the inertia stack $\Lambda[*/G]$.

According to Remark 12.12, there is a dual Frobenius structure induced on $\big(H^\bullet(\Lambda[*/G]), \star, \delta\big)$. We refer to

$$\delta\colon H^\bullet(\Lambda[*/G]) \to H^\bullet(\Lambda[*/G]) \otimes H^\bullet(\Lambda[*/G])$$

and

$$\star : H^\bullet(\Lambda[*/G]) \otimes H^\bullet(\Lambda[*/G]) \to H^\bullet(\Lambda[*/G])$$

as the dual hidden loop coproduct and dual hidden loop product respectively. Since, it is technically easier, we will describe the Frobenius structure of $H^\bullet(\Lambda[*/G])$. The following result is standard [54]. We write EG for a free G-space which is contractible and $BG = EG \times_G \{*\}$ its classifying space so that $H^\bullet([*/G]) = H^\bullet(BG) = H^\bullet_G(*)$.

Proposition 17.16 *1. The cohomology of G, as a topological space, is*

$$H^\bullet(G) = (\Lambda\mathfrak{g}^*)^G \cong \Lambda(y_1, y_2, \cdots, y_l)$$

2. The cohomology of $[*/G]$ is

$$H^\bullet([*/G]) = (S^*(\mathfrak{g}^*))^G \cong S(x_1, x_2, \cdots, x_l)$$

3. The cohomology of $[G/G]$ is

$$H^\bullet([G/G]) = (S^*(\mathfrak{g}^*))^G \otimes (\Lambda\mathfrak{g}^*)^G \cong S(x_1, x_2, \cdots, x_l) \otimes \Lambda(y_1, y_2, \cdots, y_l)$$

4. The cohomology of $[G \times G/G]$ is

$$\begin{aligned} H^\bullet([G \times G/G]) &= (S^*(\mathfrak{g}^*))^G \otimes (\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}^*))^G \\ &\cong S(x_1, x_2, \cdots, x_l) \otimes \Lambda(y_1, y_2, \cdots, y_l, y_1', y_2', \cdots, y_l'), \end{aligned}$$

5. The cohomology of $[G \times G/G \times G]$ is

$$\begin{aligned} H^\bullet([G \times G/G \times G]) &= (S^*(\mathfrak{g}^* \oplus \mathfrak{g}^*))^G \otimes (\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}^*))^G \\ &\cong S(x_1, x_2, \cdots, x_l, x_1', x_2', \cdots, x_l') \\ &\quad \otimes \Lambda(y_1, y_2, \cdots, y_l, y_1', y_2', \cdots, y_l') \end{aligned}$$

Here $l = rank(G)$, $deg(y_i) = deg(y_i') = 2d_i - 1$, $deg(x_i) = deg(x_i') = 2d_i$ and d_i are the exponents of G.

To compute the Frobenius structure on $H^\bullet(\Lambda[*/G])$, we need an explicit construction of certain Gysin maps.

Let M be an oriented manifold with a smooth $(G \times G)$-action. Consider G as a subgroup of $G \times G$ by embedding it diagonally. In this way, M becomes a G-space and we have a morphism of stacks $[M/G] \to [M/G \times G]$, which is indeed a G-principle bundle. According to Section 9.1, there is a cohomology Gysin map $\Delta_!\colon H^\bullet[M/G] \to H^{\bullet - d}[M/G \times G]$, which should be in a certain sense fibration integration.

Recall that when G is a compact connected Lie group, the cohomology of the quotient stack $H^\bullet([M/G])$ with real coefficients can be computed using the Cartan model $(\Omega_G(M), d_G)$, where $\Omega_G(M) := (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ is the space of G-equivariant polynomials $P\colon \mathfrak{g} \to \Omega(M)$, and

$$d_G(P)(\xi) := d(P(\xi)) - \iota_\xi P(\xi), \quad \forall \xi \in \mathfrak{g}.$$

Here d is the de Rham differential and ι_ξ is the contraction by the generating vector field of ξ. Given a Lie group K and a Lie subgroup $G \subset K$, let G act on K from the right by multiplication and K act on itself from the left by multiplication. The submersion $K \to K/G$ is a principal K-equivariant right G-bundle. There is an isomorphism of stacks $[M/G] \xrightarrow{\sim} [K \times_G M/K]$

which induces an isomorphism in cohomology. It is known [54] that, on the Cartan model, this isomorphism can be described by an induction map Ind_K^G : $\Omega_G(M) \to \Omega_K(K \times_G M)$. Here G acts on $K \times M$ by

$$(k, m) \cdot g = (k \cdot g, g^{-1} \cdot m).$$

The induction map is the composition

$$\Omega_G(M) \stackrel{\mathrm{Pul}}{\to} \Omega_{K\times G}(K \times M) \stackrel{\mathrm{Car}}{\to} \Omega_K(K \times_G M),$$

where $\Omega_G(M) \stackrel{\mathrm{Pul}}{\to} \Omega_{K\times G}(K \times M)$ is the natural pullback map, induced by the projections on the second factor $K \times G \to G$, and

$$\Omega_{K\times G}(K \times M) \xrightarrow{\mathrm{Car}} \Omega_K(K \times_G M)$$

is the Cartan map corresponding to a K-invariant connection for the G-bundle $K \to K/G$ [54]. We now recall the description of this map.

Let $\Theta \in \Omega^1(K) \otimes \mathfrak{g}$ be a K-invariant connection on the G-bundle $K \to K/G$. The associated principal G-bundle

$$G \to K \times M \to \frac{K \times M}{G} \cong K \times_G M$$

carries a pullback connection, denoted by the same symbol Θ. We denote $F^\Theta = d\Theta + \frac{1}{2}[\Theta, \Theta]$ its curvature, which is an element in $\Omega^2_K(K \times M) \otimes \mathfrak{g}$. The equivariant momentum map $\mu^\Theta \in (\mathfrak{k}^* \otimes \Omega^0(K))^K \otimes \mathfrak{g}$ is defined by

$$\xi \in \mathfrak{k} \mapsto \mu^\Theta(\xi) = -\iota_\xi \Theta$$

where ι_ξ is the contraction along $\hat{\xi} \in \mathfrak{X}(K)$, the generating vector field of ξ. Then $F^\Theta + \mu^\Theta$ is the equivariant curvature of Θ [10]. Observe that

$$\Omega_{K\times G}(K \times M) \cong (S(\mathfrak{g}^*) \otimes \Omega_K(K \times M))^G$$

is the space of G-equivariant polynomial functions from $\mathfrak{g}$ to $\Omega_K(K\times M)$. Hence if $x \in \mathfrak{g} \otimes \Omega^i_K(K \times M)$ and P is a homogeneous degree q polynomial on $\mathfrak{g}$, then by substitution of variables, we get an element $P(x)$ in $\Omega^{2q+qi}_K(K \times M)$. The Cartan map $\Omega_{K\times G}(K \times M) \to \Omega_K(K \times_G M)$ is the composition

$$\begin{aligned} P \otimes \omega \in (S(\mathfrak{g}^*) \otimes \Omega_K(K \times M))^G \quad &\mapsto \quad P(F^\Theta + \mu^\Theta)\omega \in \Omega_K(K \times M) \\ &\mapsto \quad \mathrm{Hor}\big(P(F^\Theta + \mu^\Theta)\omega\big) \in \Omega_K(K \times_G M), \end{aligned}$$

where $\mathrm{Hor}\colon \Omega(K \times M) \to \Omega(K \times_G M)$ is the horizontal projection with respect to Θ.

If moreover $F^\Theta = 0$ and $K \times M \to K \times_G M$ admits a horizontal section $\sigma\colon K \times_G M \to K \times M$, we have the following lemma.

Lemma 17.17 *Let $P \otimes \omega$ be an element in $(S(\mathfrak{g}^*) \otimes \Omega(M))^G \cong \Omega_G(M)$. Then, $\mathrm{Ind}_K^G(P \otimes \omega) \in \Omega(K \times_G M)$ is the K-equivariant polynomial on $\mathfrak{k}$ with value in $\Omega(K \times_G M)$ defined, for any $\xi \in \mathfrak{k}$, by*

$$\mathrm{Ind}_K^G(P \otimes \omega) \colon \xi \mapsto \sigma^*\big(P(\mu^\Theta(\xi))\mathrm{pr}_2^*(\omega)\big).$$

Proof. First of all,

$$\mathrm{Pul}(P \otimes \omega) \in \big(S\big((\mathfrak{k} \oplus \mathfrak{g})^*\big) \otimes \Omega(K \times M)\big)^{K \times G}$$

is the map $\xi \oplus y \mapsto P(y)\mathrm{pr}_2^*(\omega)$ for any $\xi \in \mathfrak{k}$ and $y \in \mathfrak{g}$. Then, by hypothesis,

$$\mathrm{Hor}\big(P(F^\Theta + \mu^\Theta)\omega\big) = \sigma^* \left(P\big(\mu^\Theta(\xi)\big)\mathrm{pr}_2^*(\omega)\right) \in (S(\mathfrak{k}^*) \otimes \Omega(K \times_G M))^K$$

and the lemma follows. □

Now let K be the cartesian product group $G \times G$. We view G as the diagonal subgroup of K. The K action on itself by left multiplication commutes with the right G-action. We have a principal right G-bundle

$$\begin{array}{rcl} G \longrightarrow & K(= G \times G) & \longrightarrow G \\ & (g, h) & \mapsto gh^{-1}. \end{array}$$

The left Maurer-Cartan form $\Theta_{MC}^L \in \Omega^1(G) \otimes \mathfrak{g}$ on G yields a K-invariant one-form $\Theta = \mathrm{pr}_2^*\left(\Theta_{MC}^L\right) \in \Omega^1(K) \otimes \mathfrak{g}$ by pullback along the projection on the second factor. Then Θ is a $K(= G \times G)$-invariant connection. Moreover it is flat, thus its equivariant curvature reduces to the equivariant momentum $\mu^\Theta \colon \mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g} \to \Omega^0(K) \otimes \mathfrak{g}$.

Lemma 17.18 *For any $(\alpha, \beta) \in \mathfrak{k}(= \mathfrak{g} \oplus \mathfrak{g})$, and $(g, h) \in K(= G \times G)$ one has*

$$\mu^\Theta(\alpha, \beta)|_{(g,h)} = -\mathrm{Ad}_{h^{-1}} \beta.$$

Proof. The generating vector field for the left G-action on G is given, for all $\beta \in \mathfrak{g}$ by

$$\hat{\beta}|_h = \left.\frac{\partial}{\partial t}\right|_{t=0} \exp(t\beta)h = L_h\left(\mathrm{Ad}_{h^{-1}} \beta\right).$$

It follows, for any $(g, h) \in K = G \times G$, that

$$\begin{aligned} \mu^\Theta(\alpha, \beta)|_{(g,h)} &= -\iota_{(\hat{\alpha},\hat{\beta})}(\Theta|_{(g,h)}) \\ &= -\iota_{\hat{\beta}}\, \Theta_{MC}^L\big|_h \\ &= -\Theta_{MC}^L\big|_1 (\mathrm{Ad}_{h^{-1}} \beta) = -\mathrm{Ad}_{h^{-1}} \beta. \end{aligned}$$

□

Let M be a K $(= G \times G)$ space. It is then a G-space. Thus we have an induction map

$$\mathrm{Ind}_{G\times G}^{G} \colon \Omega_G(M) \to \Omega_{G\times G}((G\times G)\times_G M) \cong \Omega_{G\times G}(G\times M).$$

The group $G \times G$ acts on $G \times M$ by

$$(k_1,k_2)\cdot(g,m) = \big(k_1 g k_2^{-1}, (k_1,k_2)\cdot m\big).$$

Lemma 17.19 *1. The map*

$$(G\times G)\times_G M \to G\times M, \quad (k_1,k_2,m) \mapsto \big(k_1 k_2^{-1}, (k_1,k_2)\cdot m\big)$$

is a $(G\times G)$-equivariant diffeomorphism.

2. The map

$$\sigma \colon G\times M \to K\times M, \quad \sigma(g,m) = \big(g, 1, (g^{-1},1)\cdot m\big)$$

is a horizontal section for the principal G-bundle $G \to K\times M \to K\times_G M \cong G\times M$.

As a consequence, we have an isomorphism

$$\Omega_{G\times G}((G\times G)\times_G M) \xrightarrow{\sim} \Omega_{G\times G}(G\times M).$$

Thus there is an induction map

$$\mathrm{Ind}_{G\times G}^{G} \colon \Omega_G(M) \to \Omega_{G\times G}(G\times M).$$

To obtain the Gysin map $H^\bullet([M/G]) \to H^{\bullet-d}([M/G\times G])$, one simply composes the induction map $\mathrm{Ind}_{G\times G}^{G} \colon H^\bullet([M/G]) \to H^\bullet([G\times M/G\times G])$ with the equivariant fiber integration map [4] $H^\bullet([G\times M/G\times G]) \to H^{\bullet-d}([M/G\times G])$ over the first factor G.

Proposition 17.20 *Given a $(G\times G)$-manifold M, the Gysin map*

$$H^\bullet_G(M) \to H^{\bullet-d}_{G\times G}(M)$$

is given, on the Cartan model, by the chain map $\Psi \colon \Omega_G(M) \to \Omega_{G\times G}(M)$, which is defined, for all $P\otimes\omega \in (S(\mathfrak{g}^*)\otimes\Omega(M))^G$, by

$$\Psi(P\otimes\omega) \quad = \quad \left((\xi_1,\xi_2) \mapsto \int_G P(-\xi_2)\varphi^*(\omega)\right), \quad \forall \xi_1,\xi_2 \in \mathfrak{g}, \qquad (17.6)$$

where $\varphi \colon G\times M \to M$ is the map $(g,m) \overset{\varphi}{\mapsto} (g^{-1},1)\cdot m$, and $\int_G$ stands for the fiber integration over the first factor G.

PROOF. The induction map $\mathrm{Ind}_{G\times G}^{G}\colon \Omega_G(M) \to \Omega_{G\times G}(G\times M)$ is a chain level representative of the stack isomorphisms

$$[M/G] \xleftarrow{\sim} [G\times G\times M/G\times G\times G] \xrightarrow{\sim} [G\times G\times_G M/G\times G].$$

induced by Morita equivalences of groupoids. Thus the Gysin map

$$\Delta_!\colon H^\bullet[M/G] \to H^{\bullet-d}[M/G\times G]$$

is the composition of $\mathrm{Ind}_{G\times G}^{G}$ with the Gysin map

$$H^\bullet([G\times M/G\times G]) \to H^{\bullet-d}([M/G\times G])$$

which, by Proposition 9.3, is the equivariant fiber integration.

We now need to express the induction map more explicitly. Recall that, for any $\alpha\in\Omega_G(M)$, $\mathrm{Ind}_{G\times G}^{G}(\alpha)\in\Omega_{G\times G}(G\times M)$. That is, $\mathrm{Ind}_{G\times G}^{G}(\alpha)$ is a polynomial function on $\mathfrak{k}(=\mathfrak{g}\oplus\mathfrak{g})$ valued in $\Omega(G\times M)$.

Write $\varphi\colon G\times M \to M$ for the composition $\varphi = \mathrm{pr}_2\circ\sigma$. Thus $\varphi(g,m) = (g^{-1},1)\cdot m$. According to Lemma 17.17, it suffices to compute $\sigma^*\big(P(\mu^\Theta(\xi_1,\xi_2))\mathrm{pr}_2^*(\omega)\big)$. By Lemma 17.19.3 and Lemma 17.18 we find that

$$\sigma^*(P(\mu^\Theta(\xi_1,\xi_2))) = P(-\xi_2).$$

Now the very definition of φ yields that for any $\alpha = P\otimes\omega\in\Omega_G(M)\cong (S(\mathfrak{g}^*)\otimes\Omega(M))^G$, and $\forall\xi_1,\xi_2\in\mathfrak{g}$,

$$\mathrm{Ind}_{G\times G}^{G}(\alpha)(\xi_1,\xi_2) = P(-\xi_2)\varphi^*(\omega).$$

This concludes the proof. $\square$

Remark 17.21 If we identify an element of $\Omega_G(M)$ with a G-equivariant polynomial $Q\colon\mathfrak{g}\to\Omega(M)$, then Equation (17.6) can be written as follows. $\forall(\xi_1,\xi_2)\in\mathfrak{k}=\mathfrak{g}\oplus\mathfrak{g}$,

$$\Psi(Q)(\xi_1,\xi_2) = \int_G \varphi^*(Q(-\xi_2)).$$

We now go back to our special case.

Denote by $m: G\times G\to G$ and $\Delta\colon G\to G\times G$ the group multiplication and the diagonal map respectively. The diagonal map induces a stack map $\Delta\colon [G\times G/G]\to[G\times G/G\times G]$ and thus a Gysin map

$$\Delta_!\colon H^\bullet([G\times G/G]) \to H^{\bullet-d}([G\times G/G\times G]),$$

which is given by Proposition 17.20. Similarly the group multiplication m induces a stack map $m: [G\times G/G]\to[G/G]$ and thus a Gysin map

$$m_!\colon H^\bullet([G\times G/G]) \to H^{\bullet-d}([G/G]).$$

Since m is G-equivariant, this is the usual G-equivariant Gysin map on manifolds according to Proposition 9.3.

Note that $H^\bullet_G(G)$ is a free module over $H^\bullet([*/G]) \cong S(x_1, \ldots, x_l)$. In fact, $H^\bullet_G(G) = H^\bullet([*/G])[y_1, \ldots, y_l]$ (the y_js are of odd degrees). Thus elements of $H^\bullet_G(G)$ are linear combinations of monomials $y_1^{\epsilon_1}...y_l^{\epsilon_l}$, where each ϵ_j is either 0 or 1. Similarly $H^\bullet_G(G\times G)$ is the free $H^\bullet_G(G)$-module generated by the monomials $y_1^{\epsilon_1}\cdots y_l^{\epsilon_l} {y'_1}^{\epsilon'_1}\ldots {y'_l}^{\epsilon'_l}$.

Lemma 17.22 *The map $m_!$ is a $H^\bullet([*/G])$ linear map defined by*

$$m_!(y_1^{\epsilon_1}...y_l^{\epsilon_l}{y'_1}^{\epsilon'_1}\ldots {y'_l}^{\epsilon'_l}) = y_1^{\epsilon_1+\epsilon'_1-1}...y_l^{\epsilon_l+\epsilon'_l-1}$$

with the convention that $y_j^{-1} = 0$.

Proof. Since $m : G \times G \to G$ is G-equivariant, the Gysin map

$$m_!\colon H^*([G \times G/G]) \to H^*([G/G])$$

is a map of $H^\bullet([*/G])$-module, and by Proposition 9.3, it is the equivariant fiber integration of the principal bundle $G \times G \to G$. It can be represented on the Cartan cochain complex by integration of forms, see [39] for details. In particular $m_!(y_1^{\epsilon_1}...y_l^{\epsilon_l}{y'_1}^{\epsilon'_1}\ldots {y'_l}^{\epsilon'_l})$ is determined by the equation

$$\int_{G\times G} m^*(\alpha) \wedge (y_1^{\epsilon_1}...y_l^{\epsilon_l}{y'_1}^{\epsilon'_1}\ldots {y'_l}^{\epsilon'_l}) \;=\; \int_G \alpha \wedge m_!(y_1^{\epsilon_1}...y_l^{\epsilon_l}{y'_1}^{\epsilon'_1}\ldots {y'_l}^{\epsilon'_l}) \quad (17.7)$$

Since the volume forms on G and on $G \times G$ are respectively given by $y_! \ldots y_l$ and $y_1...y_l y'_1 \ldots y'_l$, Equation (17.7) implies that $m_!\colon H^*(G \times G) \to H^{*-d}(G)$ sends $y_1^{\epsilon_1}...y_l^{\epsilon_l}{y'_1}^{\epsilon'_1}\ldots {y'_l}^{\epsilon'_l}$ to $y_1^{\epsilon_1+\epsilon'_1-1}...y_l^{\epsilon_l+\epsilon'_l-1}$. This finishes the proof. □

The hidden loop product and coproduct on $H_\bullet([G/G])$, by universal coefficient theorem, induces a degree $-d$ coproduct

$$\delta\colon H^\bullet([G/G]) \to H^\bullet([G/G]) \otimes H^\bullet([G/G])$$

and degree $-d$ product

$$\star : H^\bullet([G/G]) \otimes H^\bullet([G/G]) \to H^\bullet([G/G])$$

which makes $H^\bullet([G/G])$ into a Frobenius algebra, called the *dual Frobenius structure* on $H^\bullet([G/G])$.

More explicitly, these two operations are given by the following compositions:

$$\begin{aligned}\delta\colon H^\bullet([G/G]) \overset{m^*}{\to} H^\bullet([G \times G/G]) \;&\overset{\Delta_!}{\to}\; H^{\bullet-d}([G \times G/G \times G])\\ &\to \bigoplus_{i+j=\bullet-d} H^i([G/G]) \otimes H^j([G/G]),\end{aligned}$$

and

$$\begin{aligned}\star : H^\bullet([G/G]) \otimes H^\bullet([G/G]) \cong H^\bullet([G \times G/G \times G]) &\overset{\Delta^*}{\to}\\ H^\bullet([G \times G/G]) &\overset{m_!}{\to} H^{\bullet-d}([G/G]). \quad (17.8)\end{aligned}$$

Theorem 17.23 *Let G be a compact connected Lie group. The dual hidden loop coproduct on $H^\bullet([G/G])$ is trivial. And the dual hidden loop product on $H^\bullet([G/G])$ is given as follows. For any $P(x_1,\ldots,x_l)y_1^{\epsilon_1}...y_l^{\epsilon_l}$ and $Q(x_1,\ldots,x_l)y_1^{\epsilon_1'}...y_l^{\epsilon_l'}$ in $H_\bullet([G/G])$, we have*

$$(P(x_1,\ldots,x_l)y_1^{\epsilon_1}...y_l^{\epsilon_l}) \star \Big(Q(x_1',\ldots,x_l')y_1'^{\epsilon_1'}\ldots y_l'^{\epsilon_l'}\Big)$$

$$= (PQ)(x_1,\ldots,x_l)y_1^{\epsilon_1+\epsilon_1'-1}...y_l^{\epsilon_l+\epsilon_l'-1}$$

with the convention that $y_j^{-1} = 0$.

Proof. On the Cartan model, by Proposition 17.20, the hidden loop coproduct is given by the following composition of chain maps:

$$\Omega_G(G) \xrightarrow{p^*} \Omega_G(G\times G) \xrightarrow{\Psi^*} \Omega_{G\times G}(G\times G) \xrightarrow{\cong} \Omega_G(G)\otimes\Omega_G(G).$$

Here the last map is Künneth formula, and the first map

$$p^*\colon \Omega_G(G) \cong (S(\mathfrak{g}^*)\otimes\Omega(G))^G \to \Omega_G(G\times G) \cong (S(\mathfrak{g}^*)\otimes\Omega(G\times G))^G$$

is $S(\mathfrak{g}^*)$-linear and given by

$$p^*(P\otimes\omega) = P\otimes m^*(\omega), \quad \forall P\otimes\omega\in (S(\mathfrak{g}^*)\otimes\Omega(G))^G.$$

Note that the space

$$\Omega_G(G)\otimes\Omega_G(G) \cong (S(\mathfrak{g}^*)\otimes\Omega(G))^G \otimes (S(\mathfrak{g}^*)\otimes\Omega(G))^G$$

has a $S(\mathfrak{g}^*)^G$-module structure, which is given by multiplication on the second factor: i.e.

$$\forall Q\in S(\mathfrak{g}^*),\ P_1\otimes\omega_1\otimes P_2\otimes\omega_2 \in \Omega_G(G)\otimes\Omega_G(G),$$

one defines $Q\cdot(P_1\otimes\omega_1\otimes P_2\otimes\omega_2)$ by

$$(\xi_1,\xi_2) \ \mapsto\ P_1(\xi_1)\otimes\omega_1\otimes Q(-\xi_2)P_2(\xi_2)\otimes\omega_2 \in \Omega(G)\otimes\Omega(G).$$

By Proposition 17.20, we know that the Gysin map $\Psi^*\colon \Omega_G(G\times G) \to \Omega_{G\times G}(G\times G)$ is indeed a $S(\mathfrak{g}^*)^G$-module map.

There are two kinds of elements $P\otimes\omega$ in $H^\bullet([G/G]) = \big(S(\mathfrak{g}^*)\big)^G\otimes\Lambda(\mathfrak{g})^G$. One consists of those where ω is a top degree form, i.e. a multiple of $y_1\wedge\cdots\wedge y_l$, and the others are those where ω corresponds to a form in $\Omega^{*<d}(G)$. In the latter case, $\Psi(P\otimes\omega)$ vanishes after fiber integration for degree reasons. In the first case, the G-action on G is by conjugation. Since the conjugacy action is trivial in cohomology, $\int_G \varphi^*(\omega) = 0$ and by Proposition 17.20, $\Psi(P\otimes\omega)$vanishes. Hence the dual hidden loop coproduct is trivial.

We now compute the dual hidden loop product. First, by a simple computation, we know that, on the Cartan model, the map $\Delta^* \colon H^*_{G\times G}(G\times G) \to H^*_G(G\times G)$ is given by

$$\Delta^*(P(x_1,\dots,x_l,y_1,\dots,y_l,x'_1,\dots,x_l{}',y'_1,\dots,y'_l)) \\ = P(x_1,\dots,x_l,y_1,\dots,y_l,x_1,\dots,x_l,y'_1,\dots,y'_l).$$

In other words, the map Δ^* is an algebra map that leaves the odd degree generators y_i, y'_j unchanged and send both generators x_i, x'_i $(i=1\dots r)$ to the generator x_i. By Lemma 17.22, one obtains that

$$(m_! \circ \Delta^*)\big(y_1^{\epsilon_1}...y_l^{\epsilon_l}, {y'_1}^{\epsilon'_1}\dots {y'_l}^{\epsilon'_l}\big) \quad = \quad y_1^{\epsilon_1+\epsilon'_1-1}...y_l^{\epsilon_l+\epsilon'_l-1}.$$

The dual hidden loop product now follows from the explicit $S(\mathfrak{g}^*)$-module structure. $\square$

Remark 17.24 It follows that the hidden loop product on $H_\bullet([G/G])$ is trivial while the hidden loop coproduct has a counit given by the fundamental class of G, which is dual of the cohomology class $y_1\dots y_l$.

Remark 17.25 Let G be either a compact Lie group or a discrete group. As we have seen above, the stack $[*/G]$ is strongly oriented and, according to Theorem 12.5, $H_\bullet(\mathrm{L}[*/G])$ is a Frobenius algebra. An alternative approach to string topology for $[*/G]$ has been carried out in Gruher-Salvatore [38]. It would be interesting to find a precise link between the result of this Section 17.4 with those of [38]. Similarly, it would be interesting to find the connection between the construction of our hidden loop product (Theorem 11.1) with that of Abbaspour-Cohen-Gruher [1] for Poincaré duality groups. Another approach to the BV-algebra structure for $\mathrm{L}[*/G]$ was recently studied by Chataur-Menichi [18]. Their results seem to agree with ours.

A Categories fibered in groupoids

The formalism of categories fibered in groupoids provides a convenient framework for working with lax groupoid-valued functors. In this section, we recall some basic facts about categories fibered in groupoids.

Let T be a fixed category. An example to keep in mind is $\mathsf{T} = \mathsf{Top}$, the category of topological spaces. A **category fibered in groupoids** over T is a category $\mathfrak{X}$ together with a functor $\pi\colon \mathfrak{X} \to \mathsf{T}$ satisfying the following properties:

i) For every arrow $f\colon V\to U$ in T, and for every object X in $\mathfrak{X}$ such that $\pi(X)=U$, there is an arrow $F\colon Y\to X$ in $\mathfrak{X}$ such that $\pi(F)=f$.

ii) Given a commutative triangle in T, and a partial lift for it to $\mathfrak{X}$ as in the diagram

$$\begin{tikzcd}
Y \arrow[dr, "F"] & & & & V \arrow[dd, "h"'] \arrow[dr, "f"] & \\
& X & \overset{\pi}{\longrightarrow} & & & U \\
Z \arrow[ur, "G"'] & & & & W \arrow[ur, "g"'] &
\end{tikzcd}$$

there is a unique morphism $H\colon Y \to Z$ such that the triangle commutes and $\pi(H) = h$.

We will often drop the base functor π from the notation and denote a fibered category $\pi\colon \mathfrak{X} \to \mathsf{T}$ by $\mathfrak{X}$.

For a fixed object $T \in \mathsf{T}$, we let $\mathfrak{X}(T)$ denote the category of objects $X \in \mathfrak{X}$ such that $\pi(X) = T$. Morphisms in $\mathfrak{X}(T)$ are morphisms $f\colon X \to Y$ in $\mathfrak{X}$ such that $\pi(f) = \mathrm{id}_T$.

It is easy to see that $\mathfrak{X}(T)$ is a groupoid. This groupoid is sometimes called the *fiber of* $\mathfrak{X}$ *over* T. It is also called the groupoids of T-*points* of $\mathfrak{X}(T)$.

Example A.1

1. Let $\mathsf{T} = \mathsf{Top}$, and let G be a topological group. Let $\mathcal{B}G$ be the category of all principal G-bundles $P \to T$. A morphisms in $\mathcal{B}G$ is a G-equivariant cartesian diagram

$$\begin{tikzcd}
P' \arrow[r] \arrow[d] & P \arrow[d] \\
T' \arrow[r] & T
\end{tikzcd}$$

   The base functor $\mathcal{B}G \to \mathsf{Top}$ is the forgetful functor that sends $P \to T$ to T. Observe that $\mathcal{B}G(T)$ is the groupoid of principal G-bundles over T.

2. Let $\mathsf{T} = \mathsf{Top}$, and let X be a topological space. Let $\mathfrak{X}$ be the category of continuous maps $T \to X$. A morphism in $\mathfrak{X}$ is a commutative triangle

$$\begin{tikzcd}
T' \arrow[rr] \arrow[dr] & & T \arrow[dl] \\
& X &
\end{tikzcd}$$

   The forgetful functor that sends $T \to X$ to T makes $\mathfrak{X}$ a category fibered in groupoids over Top.

   The groupoids $\mathfrak{X}(T)$ is in fact equivalent to a set, namely, the set of continuous maps $T \to X$ (i.e., the set of T-points of X).

Remark A.2 There are two ways of thinking of a fibered category $\mathfrak{X} \to \mathsf{T}$. One is to think of it as a device for cataloguing the objects parameterized by a *moduli problem* over T. In this case, an object $X \in \mathfrak{X}(T)$ is viewed as a "family parameterized by T."

The second point of view is to think of $\mathfrak{X}$ as some kind of a *space*. In this case, an object in $\mathfrak{X}(T)$ is simply thought of as a T-valued point of $\mathfrak{X}$, that is, a map from T to $\mathfrak{X}$.

The Yoneda type Lemma A.4 clarifies this dual point of view.

Remark A.3

1. Conditions (**i**) and (**ii**) imply that, for every morphism $f\colon T' \to T$ in T, every object $X \in \mathfrak{X}(T)$ has a "pull-back" $f^*(X)$ in $\mathfrak{X}(T')$. The pull-back is unique up to a unique isomorphism. We sometimes denote $f^*(X)$ by $X|_{T'}$.
2. The pull-back functors f^* (whose definition involves making some choices) give rise to a lax groupoid-valued functor $T \mapsto \mathfrak{X}(T)$. Conversely, given a lax groupoid-valued functor on T, it is possible to construct a category fibered in groupoids over T via the so-called Grothendieck construction.

A.1 The 2-category of fibered categories

Categories fibered in groupoids over T form a 2-category. Let us explain how this works.

A *morphism* $f\colon \mathfrak{X} \to \mathfrak{Y}$ of fibered categories is a functor $f\colon \mathfrak{X} \to \mathfrak{Y}$ between the underlying categories such that $\pi_{\mathfrak{Y}} \circ f = \pi_{\mathfrak{X}}$. Given two such morphisms $f, g\colon \mathfrak{X} \to \mathfrak{Y}$, a *2-morphism* $\varphi\colon f \Rightarrow g$ between them is a natural transformation of functors φ from f to g such that the composition $\pi_{\mathfrak{Y}} \circ \varphi$ is the identity transformation from $\pi_{\mathfrak{X}}$ to itself.

With morphisms and 2-morphisms as above, categories fibered in groupoids over T form a 2-category $\mathfrak{Fib}_{\mathsf{T}}$. The 2-morphisms in $\mathfrak{Fib}_{\mathsf{T}}$ are automatically invertible.

The construction in Example A.1.2 can be performed in any category T and it gives rise to a functor $\mathsf{T} \to \mathfrak{Fib}_{\mathsf{T}}$. From now on, we will use the same notation for an object T in T and for its corresponding category fibered in groupoids.

We have the following Yoneda-type lemma.

Lemma A.4 (Yoneda lemma) *Let $\mathfrak{X}$ be a category fibered in groupoids over T, and let T be an object in T. Then, the natural functor*

$$\mathrm{Hom}_{\mathfrak{Fib}_{\mathsf{T}}}(T, \mathfrak{X}) \to \mathfrak{X}(T)$$

is an equivalence of groupoids.

This lemma implies that the functor $\mathsf{T} \to \mathfrak{Fib}_{\mathsf{T}}$ is fully faithful. That is, we can think of the category T as a full subcategory of $\mathfrak{Fib}_{\mathsf{T}}$. For this reason, in the sequel we quite often do not distinguish between an object T and the fibered category associated to it.

A.2 Descent condition

To simplify the exposition, and to avoid the discussion of Grothendieck topologies, we will assume from now on that $\mathsf{T} = \mathsf{Top}$.

We say that a category $\mathfrak{X}$ fibered in groupoids over T is a **stack**, if the following two conditions are satisfied:

i) *Gluing morphisms.* Given two objects X and Y in $\mathfrak{X}$ over a fixed topological spaces T, morphisms between them form a sheaf. That is, the presheaf of sets on T defined by

$$U \mapsto \mathrm{Hom}_{\mathfrak{X}(U)}(X|_U, Y|_U)$$

is a sheaf.

ii) *Gluing objects.* Let T be a topological space, and let $\{U_i\}$ be an open covering of T. Assume we are given objects $X_i \in \mathfrak{X}(U_i)$, together with isomorphisms $\varphi_{ij}\colon X_j|_{U_i \cap U_j} \to X_i|_{U_i \cap U_j}$ in $\mathfrak{X}(U_i \cap U_j)$ which satisfy the cocycle condition

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$

on $U_i \cap U_j \cap U_k$ for every triple of indices i, j and k. Then, there is an object X over T, together with isomorphisms $\varphi_i\colon X|_{U_i} \to X_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$.

The data given in (**ii**) is usually called a *gluing data* or a *descent data*. It follows from (**i**) that the object X in (**ii**) is unique up to a unique isomorphism.

Stacks over T form a full sub 2-category of $\mathfrak{Fib}$.

Example A.5

1. The fibered category $\mathcal{B}G$ of Example A.1.1 is a stack. This is because one can glue principal G-bundles over a fixed space T using a gluing data (and the same thing is true for morphisms of principal G-bundles as well).

2. The fibered category $\mathfrak{X}$ of Example A.1.2 is a stack. This is because, given a collection of continuous maps $f_i\colon U_i \to X$ which are equal over the intersections $U_i \cap U_j$, we can uniquely glue them to a continuous map $f\colon T \to X$.

Note that the cocycle condition over triple intersections does not appear in Example A.5.2. The reason for this is that the fiber groupoids $\mathfrak{X}(U)$ are equivalent to sets. That is, if there is a morphisms between two objects in $\mathfrak{X}(U)$ it has to be unique.

In view of Example A.5.2 (and Lemma A.4), the descent condition for a stack $\mathfrak{X}$ can be interpreted as follows. Let T be a topological space and $\{U_i\}$ an open covering of T. Assume we are given morphisms $f_i\colon U_i \to \mathfrak{X}$, together with 2-isomorphisms $\varphi_{ij}\colon f_j|_{U_i \cap U_j} \Rightarrow f_i|_{U_i \cap U_j}$, satisfying the cocycle condition $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$. (This should be thought of as saying that φ_{ij} are "identifying f_i and f_j along $U_i \cap U_j$.") Then, we can glue f_i to a global map $f\colon T \to \mathfrak{X}$ whose restriction $f|_{U_i}$ to U_i is identified to f_i via a 2-isomorphism $\varphi_i\colon f|_{U_i} \Rightarrow f_i$.

A.3 Quotient stacks

To any topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$ one can associate a stack $[X_0/X_1]$ called the *quotient stack* of the groupoid. A quick definition for this quotient stack is as follows. By definition, $[X_0/X_1]$ is the stack associated to the (fibered category associated to the) presheaf of groupoids

$$T \;\mapsto\; [X_1(T) \rightrightarrows X_0(T)].$$

Since we have not discussed the stack associated to a category fibered in groupoids, we give an alternative description of $[X_0/X_1]$ in terms of principal bundles.

We only describe the case when $\mathbb{X}$ is the action groupoid $[X \times G \rightrightarrows X]$ of the action of a topological group G on a topological space X and refer the reader for the general case to ([57], Section 12). In the case of a group action, the quotient stack is denoted by $[X/G]$.

For a topological space T, the groupoid $[X/G](T)$ of T-points of $[X/G]$ is the groupoid of pairs (P, φ), where P is a principal G-bundle over T, and $\varphi\colon P \to X$ is a G-equivariant map. The morphisms in $[X/G](T)$ are G-equivariant morphisms $f\colon P' \to P$ such that $\varphi' = \varphi \circ f$.

It is easy to verify that $[X/G]$ is a stack. When X is a point, the quotient stack $[*/G]$ coincides with $\mathcal{B}G$ of Example A.1.1 and is called the *classifying stack* of G. Remark that, by Lemma A.4, the groupoid $\mathrm{Hom}(T, \mathcal{B}G)$ of morphisms from T to $\mathcal{B}G$ is equivalent to the groupoid of principal G-bundles over T.

B Generalized Fulton-MacPherson bivariant theories

In this section we recall the axioms of a Fulton-MacPherson bivariant theory. Our theory is slightly more general than the original approach of Fulton-MacPherson in the following ways:

- Since we need to work with stacks, the underlying category of our theory is indeed a 2-category. All fiber products and commutative diagrams should be interpreted in the 2-categorical sense. The associated bivariant groups, however, will be equal for 2-isomorphic moprhisms.

- Fulton-MacPherson have a notion of a ‘confined morphism’ (along which you can push forward bivariant classes) while we believe it is more natural to have ‘confined triangles’.

- Product of bivariant classes are only partially defined.

In the context of this paper, these differences, however, are not crucial and give us the right amount of generality to define the desired Gysin maps.

B.1 The underlying (2-)category

The underlying category of a generalized bivariant theory is a category C (rather, 2-category) with fiber products and a final object. The category C is equipped with the following structure:

- A class of commutative triangles called **confined triangles**

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[dl, "v"] \\ & S & \end{tikzcd}$$

  We usually write this triangle as $X \xrightarrow{f} Y \xrightarrow{v} S$. We sometime refer to the above triangle as *a morphism* $f\colon X \to Y$ *confined relative to* S.

- A class of squares called **independent squares**

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

  Note: we will distinguish the above square from its transpose, so the transpose of an independent square may not be independent.

- A class of morphisms called **adequate**.

We require the following axioms to be satisfied:

A1. A triangle $X \xrightarrow{f} X \xrightarrow{v} Z$ in which $f = \mathrm{id}_X$ is the identity map is confined.

A2. If the inside triangles in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[dl, "w"] \\ & S & \end{tikzcd}$$

are confined, then so is the outside triangle.

B1. Any commutative square in which the top and the bottom morphisms are the identity maps is independent.

B2. Any square obtained from juxtaposition (vertical, or horizontal) of independent squares is independent.

C. If in the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[r, "v'"] \arrow[d, "g"] & S' \arrow[d] \\ X \arrow[r, "f"'] & Y \arrow[r, "v"'] & S \end{tikzcd}$$

the left square (or its transpose) is independent and f is confined relative to S, then f' is confined relative to S'.

D. All isomorphisms are adequate.

Lemma B.1 *Given*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow W$$

if f is confined relative to W then f is confined relative to Z.

PROOF. Use Axioms **B1** and **C**. $\square$

B.2 Axioms for a bivariant theory

A bivariant theory T on such a category C assigns to every morphism $f\colon X \to Y$ in C a graded abelian group $T(X \xrightarrow{f} Y)$, or $T(f)$ for short. We denote the i^{th} graded component, $i \in \mathbb{Z}$, of T by T^i. We sometimes denote an element $\alpha \in T(X \xrightarrow{f} Y)$ by

$$X \underset{\alpha}{\xrightarrow{f}} Y\ .$$

The functor T support three types of operations:

1. *Product.* For every $f\colon X \to Y$ and adequate $g\colon Y \to Z$, there is a product

$$T^i(X \xrightarrow{f} Y) \otimes T^j(Y \xrightarrow{g} Z) \xrightarrow{\cdot} T^{i+j}(X \xrightarrow{g\circ f} Z).$$

2. *Pushforward.* Given a confined triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[dl, "v"] \\ & S & \end{tikzcd}$$

there is a pushforward homomorphism

$$f_*\colon T^i(X \xrightarrow{u} S) \longrightarrow T^i(Y \xrightarrow{v} S).$$

3. *Pullback.* For every independent square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

there is a pullback homomorphism

$$g^*\colon T^i(X \xrightarrow{f} Y) \longrightarrow T^i(X' \xrightarrow{f'} Y').$$

(Observe the abuse of notation.)

These operations should satisfy the following compatibility axioms:

A1. *Product is associative.* Given a diagram

$$\begin{tikzcd} X \arrow[r, "f", "\alpha"'] & Y \arrow[r, "g", "\beta"'] & Z \arrow[r, "h", "\gamma"'] & W \end{tikzcd}$$

where g, h and $h \circ g$ are adequate, we have

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

in $T(h \circ g \circ f)$.

A2. *Pushforward is functorial.* If the triangles in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[dl, "w"] \\ & S & \end{tikzcd}$$

are confined, then

$$(g \circ f)_* = g_* \circ f_*\colon T^i(X \xrightarrow{u} S) \longrightarrow T^i(Z \xrightarrow{w} S).$$

A3. *Pullback is functorial.* If the squares in

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"'] & X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\ Y'' \arrow[r, "h"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

are independent, then

$$(g \circ h)^* = h^* \circ g^*\colon T^i(X \xrightarrow{f} Y) \longrightarrow T^i(X'' \xrightarrow{f''} Y'').$$

A12. *Product and pushforward commute.* Given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\alpha"'] & Y \arrow[r, "g"] & Z \arrow[r, "h", "\beta"'] & W \end{tikzcd}$$

with f confined relative to W and h adequate, we have

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot \beta$$

in $T(h \circ g)$.

A13. *Product and pullback commute.* Given

$$\begin{tikzcd} X' \arrow[r, "h''"] \arrow[d, "f'"'] & X \arrow[d, "f"', "\alpha"] \\ Y' \arrow[r, "h'"] \arrow[d, "g'"'] & Y \arrow[d, "g"', "\beta"] \\ Z' \arrow[r, "h"] & Z \end{tikzcd}$$

with independent squares, g and g' adequate, we have

$$h^*(\alpha \cdot \beta) = h'^*(\alpha) \cdot h^*(\beta)$$

in $T(g' \circ f')$.

A23. *Pushforward and pullback commute.* Given

$$\begin{tikzcd} X' \arrow[r, "h''"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \arrow[dd, bend left, "\alpha"] \\ Y' \arrow[r, "h'"] \arrow[d, "g'"'] & Y \arrow[d, "g"'] \\ Z' \arrow[r, "h"] & Z \end{tikzcd}$$

with independent squares and f confined relative to Z, we have

$$f'_*(h^*\alpha) = h^* f_*(\alpha)$$

in $T(g')$.

A123. *Projection formula.* Given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \arrow[d, phantom, shift left=3, "\alpha"] & \\ Y' \arrow[r, "g"] \arrow[rr, bend right, "\beta"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

with independent square, g adequate and confined relative to Z and $h \circ g$ adequate, we have

$$\alpha \cdot g_*(\beta) = g'_*(g^*\alpha \cdot \beta)$$

in $T(h \circ f)$.

We say a bivariant theory T has **unital** if for every $X \in \mathsf{C}$ there is an element $1_X \in T^0(X \xrightarrow{id} X)$ with the following properties:

- For every $f\colon W \to X$ and every $\alpha \in T(W \xrightarrow{f} X)$, we have $\alpha \cdot 1_X = \alpha$.
- For every $g\colon X \to Y$ and every $\beta \in T(X \xrightarrow{g} Y)$, we have $1_X \cdot \beta = \beta$.
- For every $g\colon X' \to X$, we have $g^*(1_X) = 1_{X'}$.

A bivariant theory T is called **skew-commutative** (respectively, **commutative**), if for any square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \arrow[d, phantom, shift left=3, "\alpha"] \\ Y' \arrow[r, "g", "\beta"'] & Y \end{tikzcd}$$

that is independent or its transpose is independent, g and f are adequate, we have

$$g^*(\alpha) \cdot \beta = (-1)^{\deg(\alpha)\deg(\beta)} f^*(\beta) \cdot \alpha$$

(respectively, $g^*(\alpha) \cdot \beta = f^*(\beta) \cdot \alpha$).

Note that we don't assume the class of adequate morphisms to be closed; that is, if f, g are adequate, $g \circ f$ might not be adequate. However, in practice, it is convenient to specify a (large) closed subclass of adequate maps, called the **strongly adequate** morphisms. In particular, the product of bivariant classes are always defined and associative on the subclass of strongly adequate morphisms.

Using the definitions of Section 7 and results of Sections 4, 6, 6.1, it is straightforward to prove

Theorem B.2 *The bivariant theory of Section 7 is a generalized Fulton-MacPherson bivariant theory.*

Note that, in view of Lemma 6.4 and Example 7.6.1, we can choose the class of strongly adequate morphisms to be the class of strongly proper maps.

References

[1] H. Abbaspour, R. Cohen, K. Gruher, *String topology of Poincaré duality groups*, Geometry & Topology Monographs **13** (2007) 1-10.

[2] H. Abbaspour, M. Zeinalian, *String bracket and flat connections*, Algebr. Geom. Topol. **7** (2007), 197–231

[3] L. Abrams, *Two dimensional topological quantum field theories and Frobenius algebras*, J. Knot theory ramifications **5** (1996), 569–587.

[4] M. Atiyah, R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1983) no. 1, 1–29.

[5] F. Atiyah; D. Tall, *Group representations, λ-rings and the J-homomorphism*, Topology **8** 1969 253–297.

[6] K. Behrend, *Cohomology of stacks*, Intersection theory and moduli, ICTP Lect. Notes, XIX, Trieste (2004), 249–294.

[7] K. Behrend, G. Ginot, B. Noohi, P. Xu, *String topology for loop stacks*, C. R. Acad. Sci. Paris, Ser. I **344** (2007), p 247–252.

[8] K. Behrend, P. Xu, *Differentiable stacks and gerbes*, J. Symplectic Geom. **9** (2011), no. 3, 285–341.

[9] D. Benson, *Representations and Cohomology Volume 2*, Cambridge studies in Advanced Math. 31, (1998), Cambridge University Press.

[10] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften, 298, Springer-Verlag, 1992.

[11] G. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Math. 46 (1972), Academic Press, New York.

[12] G. Bredon, *Topology and Geometry*, Graduate texts in Mathematics 139, (1993), Springer-Verlag.

[13] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics, 107. Birkhäuser Boston, 1993.

[14] A. Cattaneo, J. Fröhlich, B. Pedrini, *Topological field theory interpretation of string topology*, Comm. Math. Phys. **240** (2003), no. 3, 397–421.

[15] M. Chas, D. Sullivan, *String topology*, preprint `GT/9911159`.

[16] M. Chas, D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, The legacy of Niels Henrik Abel, 771–784, Springer, Berlin, 2004.

[17] D. Chataur, *A bordism approach to string topology*, Int. Math. Res. Not. 2005, no. 46, 28292875.

[18] D. Chataur, L. Menichi, *String topology of classifying spaces*, preprint `AT/0801.0174`.

[19] L. Chen, Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004) no. 1, 1–31.

[20] T. Coates, A. Corti,Y.-P. Lee, H.-H. Tseng *The quantum orbifold cohomology of weighted projective spaces*, Acta Math. **202** (2009), no. 2, 139–193.

[21] R. Cohen, *The Floer homotopy type of the cotangent bundle*, `AT/0702852`.

[22] R. Cohen, V. Godin, *A polarized view of string topology*, Topology, Geometry and Quantum Field Theory, London Math. Soc. Lecture Notes **308** (2004), 127–154.

[23] R. Cohen, J. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), no. 4, 773–798.

[24] R. Cohen, J. Jones, J. Yan, *The loop homology algebra of spheres and projective spaces*, Categorical decomposition techniques in algebraic topology, 77–92, Progr. Math., **215**, Birkhäuser, 2004.

[25] R. Cohen, A. Voronov *Notes on string topology*, String topology and cyclic homology, Adv. Courses Math. CRM Barcelona, (2006), 1–95.

[26] K. Costello, *Topological conformal field theories and gauge theories*, Geom. Topol. **11** (2007), 1539–1579.

[27] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214.

[28] R. Dijkgraaf, E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), no. 2, 393–429.

[29] B. Fantechi, L. Göttsche, *Orbifold cohomology for global quotients*, Duke Math. J. **117** (2003), no. 2, 197–227.

[30] Y. Félix, J.-C. Thomas, M. Vigué, *Rational String Topology*, J. Eur. Math. Soc. **9**, (2007), 123–156.

[31] D. S. Freed, M. J. Hopkins, C. Teleman *Twisted equivariant K-theory with complex coefficients*, J. Topology (2008) **1** (1): 16-44.

[32] W. Fulton, R. MacPherson, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc. **31** (1981), no. 243.

[33] E. Getzler, *Batalin-Vilkovisky alegbras and two dimensional topological field theory*, Comm. Math. Phys. **159** (1994), no 2, 265-285.

[34] V. Godin, *Higher string topology operations*, preprint arXiv:0711.4859.

[35] R. Goldin, M. Harada, *Orbifold cohomology of hypertoric varieties*, Internat. J. Math. **19** (2008), no. 8, 927–956

[36] R. Goldin, T. S. Holm, A. Knutson, *Orbifold cohomology of torus quotients*, Duke Math. J. **139** (2007), no. 1, 89–139.

[37] A. González, E. Lupercio, C. Segovia, B. Uribe, *Chen-Ruan Cohomology of cotangent orbifolds and Chas-Sullivan String Topology*, Math. Res. Lett. **14** (2007), no3., 491–501.

[38] K. Gruher, P. Salvatore, *Generalized string topology operations*, to appear Proc. London Math. Soc.; preprint `AT/0602210`.

[39] V. Guillemin, S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present. Springer-Verlag, Berlin, 1999.

[40] W.-Y. Hsiang, *Cohomology theory of topological transformation groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete 85, Springer, Berlin, 1975.

[41] P. Hu, I. Kriz, A. Voronov, *On Kontsevich's Hochschild cohomology conjecture*, Compos. Math. **142** (2006), no. 1, 143–168.

[42] K. Igusa, *higher Franz-Reidemester Torsion*, AMS/IP Studies in Advanced Mathematics, 31, Amer. Math. Soc., Providence, RI, 2002.

[43] Y. Jiang, *The orbifold cohomology ring of simplicial toric stack bundles*, Illinois J. Math. **52** (2008), no. 2, 493–514.

[44] T. Jarvis, R. Kaufmann and T. Kimura, *Stringy K-theory and the Chern character*, Invent. Math. **168** (2007), no. 1, 23–81.

[45] S. Kallel, P. Salvatore, *Rational maps and string topology*, Geometry and Topology **10** (2006) 1579–1606.

[46] M. Kashiwara, P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften 292, Springer-Verlag (1990)

[47] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **136** (1999), 483-496.

[48] J.-L. Loday, *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. Math. **96** (1989), No. 1, 205–230.

[49] E. Lupercio, B. Uribe, *Loop groupoids, gerbes, and twisted sectors on orbifolds*, Orbifolds in mathematics and physics (Madison, WI, 2001), 163–184, Contemp. Math., 310, Amer. Math. Soc. (2002).

[50] E. Lupercio, B. Uribe, M. Xicoténcatl, *The loop orbifold of the symmetric product*, J. Pure Appl. Algebra **211**, no 2, 293–306.

[51] E. Lupercio, B. Uribe, M. Xicoténcatl, *Orbifold String Topology*, Geom. Topol. **12** (2008), no. 4, 2203–2247..

[52] Y. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications, 47,Amer. Math. Soc., Providence, (1999).

[53] M. Markl, S. Shnider and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, 96, Amer. Math. Soc., Providence, RI, 2002

[54] E. Meinrenken, *Equivariant cohomology and the Cartan model*, Encyclopedia of Mathematical Physics, Elsevier (2006).

[55] I. Moerdijk, J. Mrčun, *Lie groupoids, sheaves and cohomology, Poisson geometry, deformation quantisation and group representations*, London Math. Soc. Lecture Note Ser. **323** (2005), 145–272.

[56] B. Noohi, *Mapping stacks of topological stacks*, J. Reine Angew. Math. **646** (2010), 117-133.

[57] B. Noohi, *Foundations of topological stacks, I*, preprint, `AG/0503247`.

[58] B. Noohi *Homotopy types of topological stacks*, arXiv:0808.3799v2 [math.AT].

[59] P. Salvatore, N. Wahl, *Framed discs operads and Batalin Vilkovisky algebras*, Q. J. Math. **54** (2003), no. 2, 213–231.

[60] G. Segal, *Two dimensional conformal field theories and modular functors*, in IXth Int. Congress on Math. Physics (1988) 22–37.

[61] G. Segal, *The definition of conformal field theory*, in Topology, geometry and quantum field theory, 421–577, London Math. Soc. Lecture Note Ser., **308** Cambridge Univ. Press, Cambridge.

[62] A. Strøm, *Note on cofibrations*, Math. Scand. **19** (1966) 11–14.

[63] A. Strøm, *Note on cofibrations II*, Math. Scand. **22** (1968) 130–142.

[64] D. Sullivan, *String Topology: Background and Present State*, `AT/0710.4141`

[65] H. Tamanoi, *A homotopy theoretic proof of the BV identity in loop homology*, `AT/0706.1277`

[66] U. Tillmann, *On the homotopy of stable mapping class group*, Invent. Math., **130**, 257–275.

[67] A. Voronov, *Notes on universal algebra*, Graphs and patterns in mathematics and theoretical physics, 81–103, Proc. Sympos. Pure Math. **73**, Amer. Math. Soc., Providence, (2005).

[68] G.W. Whitehead, *Elements of homotopy theory*, Graduate Text in Mathematics 81, Springer-Verlag, New York, 1978.

[69] Ch. Zhu, *n-groupoids and stacky groupoids*, Int. Math. Res. Not. 2009, no. 21, 40874141.