A FORMALITY THEOREM FOR POISSON MANIFOLDS

by

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Abstract. — Let M be a differential manifold. Using different methods, Kontsevich and Tamarkin have proved a formality theorem, which states the existence of a Lie homomorphism "up to homotopy" between the Lie algebra of Hochschild cochains on $C^\infty(M)$ and its cohomology $(\Gamma(M,\Lambda TM),\ [-,-]_S)$. Suppose M is a Poisson manifold equipped with a Poisson tensor π; then one can deduce from this theorem the existence of a star product $\star$ on $C^\infty(M)$. In this paper we prove that the formality theorem can be extended to a Lie (and even Gerstenhaber) homomorphism "up to homotopy" between the Lie (resp. Gerstenhaber "up to homtoptopy") algebra of Hochschild cochains on the deformed algebra $(C^\infty(M),*)$ and the Poisson complex $(\Gamma(M,\Lambda TM),\ [\pi,-]_S)$. We will first recall Tamarkin's proof and see how the formality maps can be deduced from Etingof-Kazhdan's theorem using only homotopies formulas. The formality theorem for Poisson manifolds will then follow.

0. Introduction

Let M be a differential manifold. Formality theorems link commutative objects with non-commutative ones. More precisely, one can define two graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The first one $\mathfrak{g}_1 = \Gamma(M,\Lambda TM)$ is the space of multivector fields on M. It is endowed with a graded Lie bracket $[-,-]_S$ called the Schouten bracket (see [**Kos**]).

The space $\mathfrak{g}_1$ can be identified with the cohomology of a cochain complex $\mathfrak{g}_2 = C(A,A) = \bigoplus_{k\geq 0} C^k(A,A)$, the space of regular Hochschild cochains (generated by differential k-linear maps from A^k to A and support preserving), where $A = C^\infty(M)$ is the algebra of smooth differential functions over M. The vector space $\mathfrak{g}_2$ is also endowed with a graded Lie algebra structure given by the Gerstenhaber bracket $[-,-]_G$ [**GV**]. The differential $b = [m,-]_G$ (where $m \in C^2(A,A)$ is the commutative multiplication in A) makes $\mathfrak{g}_2$ into a graded differential Lie algebra and the

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cohomology $H^*(\mathfrak{g}_2, b)$ of $\mathfrak{g}_2$ with respect to b coincides with $\mathfrak{g}_1$. More precisely, one can construct a quasi-isomorphism $\phi^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$, (the Hochschild-Kostant-Rosenberg quasi-isomorphism, see [**HKR**]) between the complexes $(\mathfrak{g}_1, 0)$ and $(\mathfrak{g}_2, b)$; it is defined, for $\alpha \in \mathfrak{g}_1$, $f_1, \cdots, f_n \in A$, by

$$\alpha \mapsto \big((f_1, \ldots, f_n) \mapsto \langle \alpha, df_1 \wedge \cdots \wedge df_n \rangle\big).$$

This map ϕ^1 is not a Lie algebra morphism, but the only obstructions for it to be are given by boundaries for the differential b. In fact, Kontsevich's formality theorem states that ϕ^1 induces a morphism if one relaxes the Lie algebras structures on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ into Lie algebras "up to homotopy" structures. In other words, setting $\phi^0 = m \in C^2(A, A)$, formality theorems can be seen as a construction of a collection of homotopies $\phi^n : \Lambda^n \mathfrak{g}_1 \to \mathfrak{g}_2$ such that ϕ^1 is the Hochschild-Kostant-Rosenberg morphism and the map $\phi = \sum_{n \geq 0} \phi^n : \Lambda^{\cdot} \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfies

$$[\phi, \phi]_G = \phi \circ m_1^{1,1} \tag{0.1}$$

where $m_1^{1,1} : \Lambda^{\cdot} \mathfrak{g}_1 \to \Lambda^{\cdot} \mathfrak{g}_1$ is the canonical extension of the Schouten Lie bracket on $\mathfrak{g}_1$.

The existence of such homotopies was proven by Kontsevich (see [**Ko1**] and [**Ko2**]) and Tamarkin (see [**Ta**]). They use different methods in their proofs. Nevertheless the two approaches are connected (see [**KS**]). In this paper we will use Tamarkin's methods (which are also well explained in [**Hi**]) to obtain a version of the formality theorem when the manifold M is equipped with a Poisson structure. Moreover, as in Tamarkin's proof, we will suppose that $M = \mathbb{R}^n$. In some cases the results can be globalized using techniques of Cattaneo, Felder and Tomassini (see [**CFT**]). More precisely, all our results are valid for an arbitrary manifold up to Section 5 where acyclicity of the de Rham complex only holds for $M = \mathbb{R}^n$ and thus globalization is needed.

One of the goals of this paper is to make the maps ϕ^n given by Tamarkin's proof as explicit as possible. We could try to construct them by induction starting from ϕ^1 (the Hochschild-Kostant-Rosenberg quasi-isomorphism), but we would meet cohomological obstructions to build $(\phi^n)_{n \geq 2}$. Thus a natural idea consists in enlarging the structures in order to reduce the obstructions. More precisely, we know that the graded space $\mathfrak{g}_1 = \Gamma(M, \Lambda TM)$ equipped with the Lie bracket $[-,-]_S$ and the exterior product $\wedge$ has a graded Gerstenhaber algebra structure.

Although the complex $\mathfrak{g}_2$ equipped with the Gerstenhaber bracket and the cup-product is not a Gerstenhaber algebra, Tamarkin [**Ta**] has proved that $\mathfrak{g}_2$ can be endowed with a structure of Gerstenhaber algebras up to homotopy (see Section 1) and established the existence of a quasi-isomorphism of Gerstenhaber algebras up to homotopy between $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

The paper is organized as follows:

- In Section 1, taking our inspiration from the language of operads, we recall the definitions of Lie algebras up to homotopy (L_∞-algebras for short) and reformulate the problem as follows: the (differential) graded Lie algebra structure on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are equivalent to codifferentials d_1 and d_2 on the exterior coalgebras

$\Lambda^\cdot\mathfrak{g}_1$ and $\Lambda^\cdot\mathfrak{g}_2$. A morphism of Lie algebras up to homotopy between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a morphism of differential coalgebras

$$\phi \;:\; (\Lambda^\cdot\mathfrak{g}_1, d_1) \to (\Lambda^\cdot\mathfrak{g}_2, d_2).$$

We will also recall the definition of Gerstenhaber algebras "up to homotopy" or G_∞-algebras (similarly given by a differential d on a peculiar coalgebra $\Lambda^\cdot\underline{\mathfrak{g}^{\otimes\cdot}}$) and morphism between them.

- In Section 2 we recall Tamarkin's construction of the G_∞-structure on $\mathfrak{g}_2$, given by a differential d_2 on $\Lambda^\cdot\underline{\mathfrak{g}_2^{\otimes\cdot}}$.
- In Section 3 we prove (still following Tamarkin's approach) that there exists a G_∞-structure on $\mathfrak{g}_1$, given by a differential d_1' on $\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}$, and that there exists a G_∞-morphism $\psi : (\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}, d_1') \to (\Lambda^\cdot\underline{\mathfrak{g}_2^{\otimes\cdot}}, d_2)$.
- In Section 4 we establish the existence of a G_∞-morphism $\psi' : (\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}, d_1) \to (\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}, d_1')$, where d_1 defines the Gerstenhaber structure on $\mathfrak{g}_1$ described in Section 1. We deduce this fact from the acyclicity of the complex
$$\left(\mathrm{Hom}(\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}), [m_1^{1,1} + m_1^2, -]\right),$$
where $[-,-]$ denotes the graded commutator of morphisms,.
- In Section 5 we prove that the complex $\left(\mathrm{Hom}(\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}), [m_1^{1,1} + m_1^2, -]\right)$ is acyclic for $M = \mathbb{R}^n$ and that homotopy formulas can be written out.
- In Section 6 we show that the G_∞-morphism $\phi = \psi\circ\psi' : (\Lambda^\cdot\underline{\mathfrak{g}_1^{\otimes\cdot}}, d_1) \to (\Lambda^\cdot\underline{\mathfrak{g}_2^{\otimes\cdot}}, d_2)$ induces the desired L_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and also in the same way an associative algebra morphism "up to homotopy" between these two spaces. Moreover when the manifold M is a Poisson manifold equipped with a Poisson tensor field $\pi \in \Gamma(M, \Lambda^2 TM)$ satisfying $[\pi,\pi]_S = 0$, then there exists a star-product on M, that is to say a deformation $m_\star$ of the product m on A whose linear term is π (see [**BFFLS1**] and [**BFFLS2**]). In this case $(\mathfrak{g}_1, [-,-]_S, [\pi,-]_S)$ becomes a graded differential Lie algebra (and even a Gerstenhaber algebra) and $(\mathfrak{g}_2, [-,-]_G, b_\star)$, where $b_\star$ is the Hochschild differential corresponding to the deformed product $m_\star$, is a new graded differential Lie algebra.
- In Section 7 we prove a formality theorem between the two differential graded Lie (and even Gerstenhaber "up to homotopy") algebras $(\mathfrak{g}_{1\star}, [-,-]_S, [\pi,-]_S)$ and $(\mathfrak{g}_{2\star}, [-,-]_G, b_\star)$ following the same steps as in Sections 2, 3 and 4.

Remark: In this paper we emphasize the Lie structures of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, not their associative algebra structures. Hence, our gradings for the spaces $\mathfrak{g}_1$, $\mathfrak{g}_2$, ... are shifted by one from what is usually done in the literature.

1. Definitions and notations

Let $\mathfrak{g}$ be any graded vector space. The exterior coalgebra $\Lambda^\cdot\mathfrak{g}$ is the cofree commutative coalgebra on the vector space $\mathfrak{g}$. In this paper we deal with graded space.

Henceforth, for any graded vector space $\mathfrak{g}$, we choose the following degree on $\Lambda^{\cdot}\mathfrak{g}$: if $X_1, \dots, X_k$ are homogeneous elements of respective degree $|X_1|, \dots |X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$

In particular the component $\mathfrak{g} = \Lambda^1\mathfrak{g} \subset \Lambda^{\cdot}\mathfrak{g}$ is the same as the Lie algebra $\mathfrak{g}$ with degree shifted by one. The coalgebra $\Lambda^{\cdot}\mathfrak{g}$ being cofree, any degree one map $d^k : \Lambda^k\mathfrak{g} \to \mathfrak{g}$ $(k \geq 1)$ extends into a derivation $d^k : \Lambda^{\cdot}\mathfrak{g} \to \Lambda^{\cdot}\mathfrak{g}$ of the coalgebra $\Lambda^{\cdot}\mathfrak{g}$.

Let us recall the definition of Lie algebras "up to homotopy", denoted L_∞-algebras henceforth.

Definition 1.1. — *A vector space $\mathfrak{g}$ is endowed with a L_∞-algebra structure if there are degree one linear maps $m^{1,\dots,1} : \Lambda^k\mathfrak{g} \to \mathfrak{g}$ such that if we extend them to maps $\Lambda^{\cdot}\mathfrak{g} \to \Lambda^{\cdot}\mathfrak{g}$, then $d \circ d = 0$ where d is the derivation*

$$d = m^1 + m^{1,1} + \cdots + m^{1,\dots,1} + \cdots .$$

For more details on L_∞-structures, see [**LS**]. It follows from the definition that a L_∞-algebra structure induces a differential coalgebra structure on $\Lambda^{\cdot}\mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \to \mathfrak{g}$ is a differential.

The Lie algebra structure on $\mathfrak{g}_1 = \Gamma(M, \Lambda TM)$ is given by the Schouten bracket (see [**Kos**]) which extends the Lie bracket of vector fields in the following way:

$$[\alpha, \beta \wedge \gamma]_S = [\alpha, \beta]_S \wedge \gamma + (-1)^{|\alpha|(|\beta|+1)}\beta \wedge [\alpha, \gamma]_S \tag{1.2}$$

for $\alpha, \beta, \gamma \in \mathfrak{g}_1$. For $f \in \Gamma(M, \Lambda^0TM) = C^\infty(M)$ and $\alpha \in \Gamma(M, \Lambda^1TM)$ we set $[\alpha, f]_S = \alpha \cdot f$, the action of the vector field α on f. The grading on $\mathfrak{g}_1$ is defined by $|\alpha| = n \Leftrightarrow \alpha \in \Gamma(M, \Lambda^{n+1}TM)$

We reformulate the graded Lie algebra structure of $\mathfrak{g}_1$ into a L_∞-algebra structure as follows: the Schouten Lie bracket $[-,-]_S$ on $\mathfrak{g}_1$ is equivalent to a (degree one) map $m_1^{1,1} : \Lambda^2\mathfrak{g}_1 \to \mathfrak{g}_1$ that we can extend canonically to $m_1^{1,1} : \Lambda^{\cdot}\mathfrak{g}_1 \to \Lambda^{\cdot}\mathfrak{g}_1$. The Jacobi identity satisfied by $[-,-]_S$ then corresponds to the identity:

$$d_1 \circ d_1 = 0,$$

where $d_1 = m_1^{1,1}$. Hence the map $m_1^{1,1}$ defines a L_∞-algebra structure on $\mathfrak{g}_1$.

In the same way, the Lie algebra structure on the vector space $\mathfrak{g}_2 = C(A, A)$ is given by the Gerstenhaber bracket $[-,-]_G$ defined, for $D, E \in \mathfrak{g}_2$, by

$$[D, E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1, \dots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E|\cdot i} D(x_1, \dots, x_i, E(x_{i+1}, \dots, x_{i+e}), \dots).$$

The space $\mathfrak{g}_2$ has a grading defined by $|D| = k \Leftrightarrow D \in C^{k+1}(A, A)$ and its differential is $b = [m, -]_G$, where $m \in C^2(A, A)$ is the commutative multiplication on A.

The Gerstenhaber bracket on $C(A, A)$ is equivalent to a map $m_2^{1,1} : \Lambda^2\mathfrak{g}_2 \to \mathfrak{g}_2$ and the differential b is a degree one map $m_2^1 : \mathfrak{g}_2 \to \mathfrak{g}_2$. These maps extends to maps $\Lambda^{\cdot}\mathfrak{g}_2 \to \Lambda^{\cdot}\mathfrak{g}_2$. All identities defining the differential Lie algebra structure on

$\mathfrak{g}_2$ (Jacobi relations for $[-,-]_G$, $b^2=0$, compatibility between b and $[-,-]_G$) can be summarized in the unique relation

$$d_2 \circ d_2 = 0,$$

where $d_2 = m_2^1 + m_2^{1,1}$. Hence the maps b and $[-,-]_G$ defines a L_∞-structure on $\mathfrak{g}_2$. In fact any differential Lie algebra $(\mathfrak{g}, b)$ has a L_∞-structure, with $m_2^1 = b$, $m_2^{1,1}$ is given by its bracket and $m^{1,\dots,1} : \Lambda^{k\geq 3}\mathfrak{g} \to \mathfrak{g} = 0$.

Definition 1.2. — *A L_∞-morphism between two L_∞-algebras $(\mathfrak{g}_1, d_1 = m_1^1 + \dots)$ and $(\mathfrak{g}_2, d_2 = m_2^1 + \dots)$ is a morphism of differential coalgebras*

$$\phi \; : \; (\Lambda^\cdot \mathfrak{g}_1, d_1) \to (\Lambda^\cdot \mathfrak{g}_2, d_2). \tag{1.3}$$

Such a map ϕ is uniquely determined by a collection of maps $\phi^n : \Lambda^n \mathfrak{g}_1 \to \mathfrak{g}_2$ as the differential coalgebras $\Lambda^\cdot \mathfrak{g}_1$ and $\Lambda^\cdot \mathfrak{g}_2$ are cofree. In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are respectively the graded Lie algebra $(\Gamma(M, \Lambda TM),\ [-,-]_S)$ and the differential graded Lie algebra $(C(C^\infty(M), C^\infty(M)),\ [-,-]_G)$ it is easy to check that Definition (0.1) (from the introduction) and Definition (1.3) coincide.

A shuffle (of length n) is a permutation of $\{1,...,n\}$ $(n \geq 1)$ such that there exist $p, q \geq 1$ with $p+q=n$ and the following inequalities hold:

$$\sigma(1) < \dots < \sigma(p), \qquad \sigma(p+1) < \dots < \sigma(p+q).$$

For any permutation σ of $\{1,...,n\}$ and any graded variables $x_1, \dots, x_n$ in $\mathfrak{g}$ (with degree shifted by minus one) we define the sign $\varepsilon(\sigma)$ (the dependence on $x_1, \dots x_n$ is implicit) by the identity

$$x_1 \dots x_n = \varepsilon(\sigma) x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(n)}$$

which holds in the free graded commutative algebra generated by $x_1, \dots x_n$. For any graded vector space $\mathfrak{g}$, each shuffle σ acts on $\mathfrak{g}^{\otimes n}$ by the formula:

$$\sigma \cdot (a_1 \otimes \dots \otimes a_n) = \varepsilon(\sigma) a_{\sigma^{-1}(1)} \otimes \cdots a_{\sigma^{-1}(n)}$$

for $a_0, \cdots, a_n \in \mathfrak{g}$. We denote $\underline{\mathfrak{g}^{\otimes n}}$ the quotient of $\mathfrak{g}^{\otimes n}$ by the image of all the maps $\mathrm{shuf}_{p,q} = \sum \sigma.(-)$, where the sum is over all shuffles of length $n = p+q$ with p, q fixed. The graded vector space $\oplus_{n\geq 0}\underline{\mathfrak{g}^{\otimes n}}$ a quotient coalgebra of the tensor coalgebra $\oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$. It is well known (see [**GK**] for example) that this coalgebra $\oplus_{n\geq 0}\underline{\mathfrak{g}^{\otimes n}}$ is the cofree Lie coalgebra on the vector space $\mathfrak{g}$ (with degree shifted by minus one).

Henceforth, for any space $\mathfrak{g}$, we denote $\Lambda^\cdot \underline{\mathfrak{g}^{\otimes \cdot}}$ the graded space $\bigoplus_{m\geq 1,\ p_1+\dots+p_n=m} \underline{\mathfrak{g}^{\otimes p_1}} \wedge \dots \wedge \underline{\mathfrak{g}^{\otimes p_n}}$. We will use the following grading on $\Lambda^\cdot \underline{\mathfrak{g}^{\otimes \cdot}}$: for $x_1^1, \cdots, x_n^{p_n} \in \mathfrak{g}$, we define

$$|\underline{x_1^1 \otimes \dots \otimes x_1^{p_1}} \wedge \dots \wedge \underline{x_n^1 \otimes \dots \otimes x_n^{p_n}}| = \sum_{i_1}^{p_1} |x_1^{i_1}| + \dots + \sum_{i_n}^{p_n} |x_n^{i_n}| - n.$$

Notice that the induced grading on $\Lambda^\cdot \mathfrak{g} \subset \Lambda^\cdot \underline{\mathfrak{g}^{\otimes \cdot}}$ is the same as the one introduced above. The cobracket on $\oplus \underline{\mathfrak{g}^{\otimes \cdot}}$ and the coproduct on $\Lambda^\cdot \mathfrak{g}$ extend to a cobracket and a coproduct on $\Lambda^\cdot \underline{\mathfrak{g}^{\otimes \cdot}}$. The sum of the cobracket and the coproduct give rise to a Gerstenhaber coalgebra structure on $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$. It is well known that this coalgebra structure is cofree (see [**Gi**],Section 3 for example).

Definition 1.3. — *A structure of Gerstenhaber algebra "up to homotopy" (G_∞-algebra for short) on a graded vector space $\mathfrak{g}$ is given by a collection of degree one maps*

$$m^{p_1,\dots,p_n} \ : \ \underline{\mathfrak{g}^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}^{\otimes p_n}} \to \mathfrak{g}$$

indexed by $p_1, \dots p_n \geq 1$ such that their canonical extension: $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}} \to \Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ satisfies $d \circ d = 0$ where

$$d = \sum_{l \geq 1, \ p_1 + \cdots p_n = l} m^{p_1,\dots,p_n}.$$

More details on G_∞-structures are given in [**Gi**]. Again, as the coalgebra structure of $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ is cofree, the map d makes $\Lambda^{\cdot}\underline{\mathfrak{g}^{\otimes \cdot}}$ a differential coalgebra.

Definition 1.4. — *A morphism of G_∞-algebras between two G_∞-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1) \to (\Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}, d_2)$ of codifferential coalgebras.*

The Lie algebra $\mathfrak{g}_1$ of multivectorfields is in fact a Gerstenhaber algebra that is to say a graded Lie algebra structure with a graded commutative algebra structure (for the same space with grading shifted by -1) and a compatibility between the bracket and the product (expressing that the bracket is a derivation for the product) as in (1.2). On the space $\mathfrak{g}_1 = \Gamma(M, \Lambda TM)$, the commutative structure is given by the exterior product:

$$\forall \alpha, \ \beta \in \Gamma(M, \Lambda TM), \ \alpha \wedge \beta = (-1)^{(|\alpha|+1)(|\beta|+1)} \beta \wedge \alpha. \tag{1.4}$$

We can reformulate the graded Gerstenhaber structure into a G_∞-algebra structure as follows. The graded Lie algebra structure is still given by a map $m_1^{1,1} : \Lambda^2 \mathfrak{g}_1 \to \mathfrak{g}_1$, and the commutative graded algebra structure is given by a map $m_1^2 : \underline{\mathfrak{g}_1^{\otimes 2}} \to \mathfrak{g}_1$ (because $\underline{\mathfrak{g}_1^{\otimes 2}}$ is the quotient of $\mathfrak{g}_1^{\otimes 2}$ by the 2-shuffles, that is to say the elements $a \otimes b + (-1)^{(|a|+1)(|b|+1)} b \otimes a$). The maps $m_1^{1,1}$, m_1^2 above extend into degree one derivations

$$m_1^{1,1}, \ m_1^2 \ : \ \wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}} \to \wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}.$$

All the identities defining the Gerstenhaber-algebra structure on $\mathfrak{g}_1$ can be summarized into the unique relation

$$d_1 \circ d_1 = 0$$

where $d_1 = m_1^{1,1} + m_1^2$. Hence the Gerstenhaber bracket and the exterior product define a G_∞-algebra structure on $\mathfrak{g}_1$. More generally, any Gerstenhaber algebra $(\mathfrak{g}, m, [-,-])$ has a canonical G_∞-structure given by $m^2 = m$, $m^{1,1} = [-,-]$, the other maps being zero.

2. A G_∞-structure on $\mathfrak{g}_2 = C(A,A)$

The Lie algebra $\mathfrak{g}_2$ is also endowed with an associative product. It is the "cup" product $\cup$ defined, for $D, E \in \mathfrak{g}_2$ and $x_1, \dots, x_{|D|+|E|+2} \in A$, by

$$(D \cup E)(x_1, \dots, x_{|D|+|E|+2}) = (-1)^\gamma D(x_1, \dots, x_{|D|+1}) E(x_{|D|+2}, \dots, x_{|D|+|E|+2})$$

where $\gamma = (|E|+1)(|D|+1)$. The projection of this product on the cohomology of $(\mathfrak{g}_2, b)$ is the exterior product $\wedge$, but unfortunately $(\mathfrak{g}_2, [-,-]_G, \cup, b)$ is not a Gerstenhaber algebra. However the relations (1.2), (1.4) are satisfied up to a boundary for b.

Tamarkin stated the existence of a G_∞-structure on $\mathfrak{g}_2$. Our aim in this section is to build this G_∞-structure more explicitly. By Definition 1.3. we have to exhibit a differential d_2 on $\Lambda^{\cdot} \underline{\mathfrak{g}_2^{\otimes \cdot}}$ satisfying, if

$$d_2 = m_2^1 + m_2^{1,1} + m_2^2 + \cdots + m_2^{p_1, \dots, p_n} + \cdots,$$

1. m_2^1 is the map b and $m_2^{1,1}$ is the map $[-,-]_G$.
2. $d_2 \circ d_2 = 0$.

We first reformulate this problem: let $L_2 = \oplus \, \underline{\mathfrak{g}_2^{\otimes n}}$ be the cofree Lie coalgebra on $\mathfrak{g}_2$ (see Section 1 for the notation). Since L_2 is a cofree coalgebra, a Lie bialgebra structure on L_2 is given by degree one maps $l_2^n : \underline{\mathfrak{g}_2^{\otimes n}} \to \mathfrak{g}_2$, corresponding to the differential, and maps $l_2^{p_1,p_2} : \underline{\mathfrak{g}_2^{\otimes p_1}} \wedge \underline{\mathfrak{g}_2^{\otimes p_2}} \to \mathfrak{g}_2$, corresponding to the Lie bracket. These maps extend uniquely into a coalgebra derivation $L_2 \to L_2$ and a coalgebra map $L_2 \wedge L_2 \to L_2$ (still denoted l_2^m and $l_2^{p,q}$). The following lemma is well known.

Lemma 2.1. — *Suppose we have a differential Lie bialgebra structure on the Lie coalgebra L_2, with differential and Lie bracket respectively determined by maps l_2^n and $l_2^{p_1,p_2}$ as above. Then $\mathfrak{g}_2$ has a G_∞-structure given, for all $p, q, n \geq 1$, by*

$$m_2^n = l_2^n, \qquad m_2^{p,q} = l_2^{p,q} \quad \text{and} \quad m_2^{p_1,\dots,p_r} = 0 \text{ for } r \geq 3.$$

Proof: The map $d_2 = \sum_{i \geq 0} l_2^i + \sum_{p_1,p_2 \geq 0} l_2^{p_1,p_2} : \Lambda^{\cdot} L_2 \to \Lambda^{\cdot} L_2$ is the Chevalley-Eilenberg differential on the differential Lie algebra L_2; it satisfies $d_2 \circ d_2 = 0$. □

Thus to obtain the desired G_∞-structure on $\mathfrak{g}_2$, it is enough to define a Lie bialgebra structure on L_2 given by maps l_2^n and $l_2^{p_1,p_2}$ with $l_2^1 = b$ and $l_2^{1,1} = [-,-]_G$.

Let us now give an equivalent formulation of our problem, which is stated in terms of the associated operads in [**Ta**]:

Proposition 2.2. — *Suppose we have a differential bialgebra structure on the cofree tensorial coalgebra $T_2 = \oplus_{n \geq 0} \, \mathfrak{g}_2^{\otimes n}$ with differential and multiplication given respectively by maps $a^n : V^{\otimes n} \to V$ and $a^{p_1,p_2} : V^{\otimes p_1} \otimes V^{\otimes p_2} \to V$. Then we have a differential Lie bialgebra structure on the Lie coalgebra $L_2 = \oplus_{n \geq 0} \, \underline{\mathfrak{g}_2^{\otimes n}}$, with differential and Lie bracket respectively determined by maps l_2^n and $l_2^{p_1,p_2}$ where $l_2^1 = a^1$ and $l_2^{1,1}$ is the anti-symmetrization of $a^{1,1}$.*

A differential bialgebra structure on the cofree tensorial coalgebra $\oplus V^{\otimes n}$ associated to a vector space V is often called a B_∞-structure on V, see [**Ba**].

Proof: We follow the proof in [**Ta**]. Let V be a finite-dimensional vector space and V^* be the dual space. A differential bialgebra structure on $T = \oplus_{n\geq 0} V^{\otimes n}$ is given by maps $a^n : V^{\otimes n} \to V$ ($n \geq 2$), corresponding to the differential, and maps $a^{p_1,p_2} : V^{\otimes p_1} \otimes V^{\otimes p_2} \to V$ ($p_1, p_2 \geq 0$), corresponding to the product. We can define dual maps of the maps $\sum_{n\geq 0} a^n : T \to T$ and $\sum_{p_1,p_2\geq 0} a^{p_1,p_2} : T \otimes T \to T$, namely $D : \hat{T} \to \hat{T}$ and $\Delta : \hat{T} \to \hat{T}\hat{\otimes}\hat{T}$, where $\hat{T}$ is the completion of the tensor algebra $\oplus_{n\geq 0} V^{*\otimes n}$. The maps D and Δ are given by maps $a^{n*} : V^* \to V^{*\otimes n}$ and $a^{p_1,p_2 *} : V^* \to V^{*\otimes p_1} \otimes V^{*\otimes p_2}$, and define a differential bialgebra structure on the complete free algebra $\hat{T}$. The tensor algebra $\oplus_{n\geq 0} V^{*\otimes n}$ is now graded as follows: $|x| = p$ when $x \in V^{*\otimes p}$.

Similarly, if we consider a differential Lie bialgebra structure on the cofree Lie coalgebra $L = \oplus_{n\geq 0} \underline{V^{\otimes n}}$, the dual maps d and δ of the structure maps $\sum_{n\geq 0} l^n$ and $\sum_{p_1,p_2\geq 0} l^{p_1,p_2}$ induce a differential Lie bialgebra structure on $\hat{L}$, the completion of the free Lie algebra $\oplus_{n\geq 0} Lie(V^*)(n)$ on V^*, where $Lie(V^*)(n)$ is the subspace of element of degree n.

We now replace formally each element x of degree n in $\hat{T}$ (resp. $\hat{L}$) by $h^n x$, where h is a formal parameter. Letting $|h| = -1$, we easily see that a differential associative (resp. Lie) bialgebra structure on the associative (resp. Lie) algebras $(\oplus_{n\geq 0} V^{*\otimes n})[[h]]$ (resp. $(\oplus_{n\geq 0} Lie(V^*)(n))[[h]]$) with the product and coproduct being of degree zero is equivalent to a differential associative (resp. Lie) bialgebra structure on the associative (resp. Lie) algebra $\hat{T}$ (resp. $\hat{L}$). Thus we have a differential free coalgebra $(\hat{T}[[h]], D, \Delta)$.

We can apply now Etingof-Kazhdan's dequantization theorem for graded diffential bialgebras (see [**EK2**] and Appendix of B. Enriquez for a proof in the graded differential "super" case) to our particular case: this proves that there exists a Lie bialgebra $(\hat{L}', [-,-], \delta)$, generated as a Lie algebra by V^* and an injective map $I_{\mathrm{EK}} : \hat{L}'[[h]] \to (\oplus_{n\geq 0} V^{*\otimes n})[[h]]$ such that

1. the restriction $I_{\mathrm{EK}} : V^* \to V^*$ is the identity,
2. $I_{\mathrm{EK}}([a,b]) = I_{\mathrm{EK}}(a) I_{\mathrm{EK}}(b) - I_{\mathrm{EK}}(b) I_{\mathrm{EK}}(a) + O(h)$, for all $a, b \in \hat{L}'[[h]]$,
3. $(\Delta - \Delta^{\mathrm{op}})\, I_{\mathrm{EK}} = h I_{\mathrm{EK}} \delta + O(h^2)$,
4. the maps I_{EK}, δ and $[-,-]$ are given by universal formulas depending only on Δ and the product of $\hat{T}$,
5. if we apply Etingof-Kazhdan's quantization functor (see [**EK1**]) to the Lie bialgebra $(\oplus_{n\geq 0} Lie(V^*)^n[[h]], \delta)$ we get the bialgebra $((\oplus_{n\geq 0} V^{*\otimes n})[[h]], \Delta)$ back.

The last condition implies that $\hat{L}'$ is free as a Lie algebra because $\hat{T}$ is free as an algebra. Moreover, there exist a differential d such that $I_{\mathrm{EK}} \circ d = D \circ I_{\mathrm{EK}}$ so that $(\hat{L}', d, \delta)$ is a differential free Lie coalgebra. Taking now dual maps, we get the result. □

Since the map I_{EK} defined in the precedent proof is the identity on V^*, the first terms a_2^1 and l_2^1 of the differentials will be the same on $T_2 = \oplus \mathfrak{g}_2^{\otimes n}$ and on $L_2 = \oplus_{n\geq 0} \underline{\mathfrak{g}_2^{\otimes n}}$. For the same reason the first term $l_2^{1,1}$ of the Lie bracket on L_2 will be the antisymmetrization of the first term $a_2^{1,1}$ of the cobracket on T_2.

By Proposition 2.2, the problem of defining a Lie bialgebra structure on L_2 given by maps l_2^n and $l_2^{p_1,p_2}$ with $l_2^1 = b$ and $l_2^{1,1} = [-,-]_G$, is equivalent to defining a differential bialgebra structure on T_2 given by maps $a_2^n : \mathfrak{g}_2^{\otimes n} \to \mathfrak{g}_2$ and $a_2^{p_1,p_2} : \mathfrak{g}_2^{\otimes p_1} \otimes \mathfrak{g}_2^{\otimes p_2} \to \mathfrak{g}_2$ where $a_2^1 = b$ and $a_2^{1,1}$ is the product $\{-|-\}$ defined in Section 1. Indeed, the anti-symmetrization of $\{-|-\}$ is by definition $[-,-]_G$. The latter can be achieved using the braces (defined in [**GV**]) acting on the Hochschild cochain complex $\mathfrak{g}_2 = C(A,A)$ for any algebra A. The braces operations are maps $a_2^{1,p} : \mathfrak{g}_2 \otimes \mathfrak{g}_2^{\otimes p} \to \mathfrak{g}_2$ ($p \geq 1$) defined, for all homogeneous $D, E_1, \ldots, E_p \in \mathfrak{g}_2^{\otimes p+1}$ and $x_1, \ldots, x_d \in A$ (with $d = |D| + |E_1| + \cdots + |E_p| + 1$), by

$$a_2^{1,p}(D_1 \otimes (E_1 \otimes \ldots \otimes E_p))(x_1 \otimes \cdots \otimes x_d) =$$

$$\sum (-1)^{\tau} D(x_1, \ldots, x_{i_1}, E_1(x_{i_1+1}, \ldots), \ldots, E_p(x_{i_p+1}, \ldots), \ldots)$$

where $\tau = \sum_{k=1}^{p} i_k(|E_k| + 1)$. The maps $a_2^{1,p} : \mathfrak{g}_2 \otimes \mathfrak{g}_2^{\otimes p} \to \mathfrak{g}_2$ and $a_2^{q \geq 2,p} = 0$ give a unique bialgebra structure on the cofree cotensorial algebra $T_2 = \oplus_{n \geq 0} \mathfrak{g}_2^{\otimes n}$. Similarly taking a_2^1 to be the Hochschild coboundary b and a_2^2 to be the cup-product $\cup$, and $a_2^{q \geq 3} = 0$, gives a unique differential bialgebra structure on the tensor coalgebra T_2. Theorem 3.1 in [**Vo**] asserts that these maps yield a differential bialgebra structure on the cofree coalgebra T_2 (the proof is a straightforward computation, also see [**GV**] and [**Kh**]).

Using this result, we can successively apply Proposition 2.2 and Lemma 2.1 to obtain the desired G_∞-structure on $\mathfrak{g}_2$ given by maps $m_2^{p_1,\ldots,p_k}$ such that $m_2^1 = b$ and $m_2^{1,1} = [-,-]_G$. By construction, the maps $m_2^{p_1,\ldots,p_k}$ are 0 for $k > 2$. Moreover, the map m_2^2 coincides, up to a Hochschild coboundary, with the cup-product $\cup$ because, when passing to cohomology, they both give the same map m_1^2, corresponding to the product $\wedge$ of the Gerstenhaber algebra $(\mathfrak{g}_1, [-,-]_S, \wedge)$.

3. A G_∞-morphism ψ : $(\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1') \to (\Lambda^\cdot \underline{\mathfrak{g}_2^{\otimes \cdot}}, d_2)$

The objective of this section is to prove the following proposition.

Proposition 3.1. — *There exist a differential d_1' on $\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}$ and a morphism of differential coalgebras $\psi : (\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1') \to (\Lambda^\cdot \underline{\mathfrak{g}_2^{\otimes \cdot}}, d_2)$ such that the induced map $\psi^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is the Hochschild-Kostant-Rosenberg map of Section 0.*

Proof: For $i = 1, 2$ and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1 + \cdots + p_k = n} \underline{\mathfrak{g}_i^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_i^{\otimes p_k}}$$

and $V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]}$. Let $d_2^{p_1,\ldots,p_k} : \underline{\mathfrak{g}_2^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_2^{\otimes p_k}} \to \mathfrak{g}_2$ be the components of the differential d_2 defining the G_∞-structure of $\mathfrak{g}_2$ (see Definition 1.3) and denote

$d_2^{[n]}$ and $d_2^{[\leq n]}$ the sums

$$d_2^{[n]} = \sum_{p_1+\cdots+p_k=n} d_2^{p_1,\dots,p_k} \qquad \text{and} \qquad d_2^{[\leq n]} = \sum_{p\leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n\geq 1} d_2^{[n]}$. In the same way, we denote

$$d'^{[n]}_1 = \sum_{p_1+\cdots+p_k=n} d'^{p_1,\dots,p_k}_1 \qquad \text{and} \qquad d'^{[\leq n]}_1 = \sum_{1\leq k\leq n} d'^{[k]}_1.$$

We know from Section 1 that a morphism $\psi : (\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1') \to (\Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}, d_2)$ is uniquely determined by its components $\psi^{p_1,\dots,p_k} : \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_k}} \to \mathfrak{g}_2$. Similarly we set

$$\psi^{[n]} = \sum_{p_1+\cdots+p_k=n} \psi^{p_1,\dots,p_k} \qquad \text{and} \qquad \psi^{[\leq n]} = \sum_{1\leq k\leq n} \psi^{[k]}.$$

Again, we have $d_1' = \sum_{n\geq 1} d'^{[n]}_1$ and $\psi = \sum_{n\geq 1} \psi^{[n]}$.

We have to build both the differential d_1' and the morphism of codifferential ψ. In fact we will build the maps $d'^{[n]}_1$ and $\psi^{[n]}$ by induction. For the first terms, we set

$$d'^{[1]}_1 = 0 \qquad \text{and} \qquad \psi^{[1]} = \phi^1,$$

the Hochschild-Kostant-Rosenberg map (see Section 0).

Suppose we have built maps $(d'^{[i]}_1)_{i\leq n-1}$ and $(\psi^{[i]})_{i\leq n-1}$ satisfying

$$\psi^{[\leq n-1]} \circ d'^{[\leq n-1]}_1 = d_2^{[\leq n-1]} \circ \psi^{[\leq n-1]}$$

on $V_1^{[\leq n-1]}$ and $d'^{[\leq n-1]}_1 \circ d'^{[\leq n-1]}_1 = 0$ on $V_1^{[\leq n]}$. These conditions are enough to insure that d_1' is a differential and ψ a morphism of differential coalgebras. If we reformulate the identity $\psi \circ d'_1 = d_2 \circ \psi$ on $V_1^{[n]}$, we get

$$\psi^{[\leq n]} \circ d'^{[\leq n]}_1 = d_2^{[\leq n]} \circ \psi^{[\leq n]}. \tag{3.5}$$

If we take now into account that $d'^{[1]}_1 = 0$, and that on $V_1^{[n]}$ we have $\psi^{[k]} \circ d'^{[l]}_1 = d_2^{[k]} \circ \psi^{[l]} = 0$ for $k + l > n + 1$, the identity (3.5) becomes

$$\psi^{[1]} d'^{[n]}_1 + B = d_2^{[1]} \psi^{[n]} + A \tag{3.6}$$

where $B = \sum_{k=2}^{n-1} \psi^{[\leq n-k+1]} d'^{[k]}_1$ and $A = d_2^{[1]} \psi^{[\leq n-1]} + \sum_{k=2}^{n} d_2^{[k]} \psi^{[\leq n-k+1]}$ (we now omit the composition sign $\circ$). The term $d_2^{[1]}$ in (3.6) is the Hochschild coboundary b. So thanks to the Hochschild-Kostant-Rosenberg theorem (3.6) is equivalent to the cochains $B - A$ being Hochschild cocycles. Therefore, in order to prove existence of $d'^{[n]}_1$ and $\psi^{[n]}$, it is sufficient to prove that

$$d_2^{[1]}(B - A) = 0 \tag{3.7}$$

and to show that for any choice of those maps, we have

$$d'^{[\leq n]}_1 d'^{[\leq n]}_1 = 0 \text{ on } V_1^{[\leq n+1]}. \tag{3.8}$$

• We will first construct ${d'_1}^{[2]}$: for $n = 2$, we get $A = d_2^{[1]}\psi^{[1]} + d_2^{[2]}\psi^{[1]}$ and $B = 0$ so that

$$\psi^{[1]}{d'_1}^{[2]} = d_2^{[1]}(\psi^{[2]} + \psi^{[1]}) + d_2^{[2]}\psi^{[1]}.$$

Thus ${d'_1}^{[2]}$ is the image of $d_2^{[2]}$ through the projection on the cohomology of $\mathfrak{g}_2$ and as the Hochschild-Kostant-Rosenberg map $\psi^{[1]}$ is injective from $\mathfrak{g}_1 = H(\mathfrak{g}_2, b = d_2^{[1]})$ to $\mathfrak{g}_2$, we get

$${d'_1}^{[2]} = d_1^{[2]}.$$

• Let us prove (3.7): we have $d_2^{[1]}(-A) = -\sum_{k=2}^{n} d_2^{[1]} d_2^{[k]} \psi^{[\leq n-k+1]}$. Using $d_2 d_2 = 0$, we get

$$\begin{aligned} d_2^{[1]}(-A) &= \sum_{k=2}^{n}\left(\sum_{l=2}^{k} d_2^{[l]} d_2^{[k+1-l]}\right)\psi^{[\leq n-k+1]} \\ &= \sum_{l=2}^{n} d_2^{[l]}\left(\sum_{k=l}^{n} d_2^{[k+1-l]}\psi^{[\leq n-k+1]}\right). \end{aligned}$$

Clearly, we have $\sum_{k=l}^{n} d_2^{[k+1-l]}\psi^{[\leq n-k+1]} = \sum_{k=1}^{n-l+1} d_2^{[k]}\psi^{[\leq n-k+2-l]}$. Using once again $d_1^{[a]}d_1^{[b]}\psi^{[c]} = 0$ on $V_1^{[n]}$ for $a+b+c > n+2$, we add terms $(\psi^{[n-k+2-l+k']})_{0\leq k'\leq k-1}$ to $\psi^{[\leq n-k+2-l]}$ without changing the previous equality. Thus we have

$$d_2^{[1]}(-A) = \sum_{l=2}^{n} d_2^{[l]}\left(\sum_{k=1}^{n-l+1} d_2^{[k]}\right)\psi^{[\leq n+1-l]} = \sum_{l=2}^{n} d_2^{[l]} d_2^{[\leq n+1-l]}\psi^{[\leq n+1-l]}.$$

Since $\left(d_2^{[l]}\right)_{l\geq 2}$ map $V_2^{[\leq k]}$ into $V_2^{[\leq k-1]}$, the previous equality has non-trivial terms only on $V_1^{[\leq n-1]}$. Thus we can apply the induction hypothesis $\psi^{[\leq k]}{d'_1}^{[\leq k]} = d_2^{[\leq k]}\psi^{[\leq k]}$ on $V_1^{[\leq k]}$ for $k \leq n-1$. We get

$$d_2^{[1]}(-A) = \sum_{l=2}^{n} d_2^{[l]}\psi^{[\leq n+1-l]}{d'_1}^{[\leq n+1-l]}.$$

We have now

$$d_2^{[1]}(B-A) = d_2^{[1]}\sum_{k=2}^{n-1}\psi^{[\leq n-k+1]}{d'_1}^{[k]} + \sum_{l=2}^{n} d_2^{[l]}\psi^{[\leq n+1-l]}{d'_1}^{[\leq n+1-l]}.$$

The term corresponding to $l = n$ vanishes since ${d'_1}^{[1]} = 0$. Using a previous argument on $V_1^{[n]}$ for $d_2^{[a]}\psi^{[b]}{d'_1}^{[c]}$, we add maps $\psi^{[p+p']}$ $(p' \geq 0)$ to $\psi^{[\leq p]}$. If we then reindex the sum with respect to the terms ${d'_1}^{[l]}$, we get

$$d_2^{[1]}(B-A) = \sum_{l=2}^{n-1} d_2^{[\leq n+1-l]}\,\psi^{[\leq n-1]}{d'_1}^{[l]}.$$

Therefore we have proved that $d_2^{[1]}(B-A) = \sum_{l=2}^{n-1} d_2^{[\leq n+1-l]}\,\psi^{[\leq n+1-l]}{d'}_1^{[l]}$. Since ${d'}_1^{[l]}\left(V_1^{[\leq k]}\right) \subset V_1^{[\leq k-1]}$, we can again apply the induction hypothesis, thus getting

$$d_2^{[1]}(B-A) = \sum_{l=2}^{n-1} \psi^{[\leq n+1-l]}{d'}_1^{[\leq n+1-l]}\,{d'}_1^{[l]} = 0$$

because ${d'}_1^{[1]} = 0$ and ${d'}_1^{[\leq n-1]}{d'}_1^{[\leq n-1]} = 0$ on $V_1^{[\leq n]}$,again by the induction hypothesis.

• We will finally prove (3.8) that is to say ${d'}_1^{[\leq n]}{d'}_1^{[\leq n]} = 0$ on $V_1^{[\leq n+1]}$. As $\psi^{[1]}$ is a quasi-isomorphism between $(\mathfrak{g}_1, 0)$ and $(\mathfrak{g}_2, b = d_2^{[1]})$, this is equivalent to say that $\psi^{[1]}\,{d'}_1^{[\leq n]}\,{d'}_1^{[\leq n]}$ is a boundary on $V_1^{[\leq n+1]}$. Using a previous degree argument, we get the following identity on $V_1^{[\leq n+1]}$:

$$\psi^{[1]}\,{d'}_1^{[\leq n]}\,{d'}_1^{[\leq n]} = \psi^{[\leq n]}\,{d'}_1^{[\leq n]}\,{d'}_1^{[\leq n]}.$$

By definition of ${d'}_1^{[\leq n]}$ we can write $\psi^{[\leq n]}\,{d'}_1^{[\leq n]} = d_2^{[\leq n]}\,\psi^{[\leq n]}$ as ${d'}_1^{[\leq n]}$ maps $V_1^{[\leq n+1]}$ to $V_1^{[\leq n]}$. Thus it is sufficient to prove that $d_2^{[\leq n]}\psi^{[\leq n]}{d'}_1^{[\leq n]}$ is a boundary when restricted to $V_1^{[\leq n+1]}$.

Now we have

$$d_2^{[\leq n]}\psi^{[\leq n]}{d'}_1^{[\leq n]} = b\psi^{[\leq n]}{d'}_1^{[\leq n]} + \sum_{2\leq k\leq n} d_2^{[\leq k]}\psi^{[\leq n]}{d'}_1^{[\leq n]}.$$

Since $\sum_{2\leq k\leq n} d_2^{[\leq k]}$ maps $V_2^{[\leq k]}$ to $V_2^{[\leq k-1]}$, the linear combination of maps

$$\sum_{2\leq k\leq n} d_2^{[\leq k]}\psi^{[\leq n]}{d'}_1^{[\leq n]}$$

has non-trivial summands only on $V_1^{[\leq n+1]}$. On the latter space we have

$$\sum_{2\leq k\leq n} d_2^{[\leq k]}\,\psi^{[\leq n]}\,{d'}_1^{[\leq n]} = \sum_{2\leq k\leq n} d_2^{[\leq k]}\,d_2^{[\leq n]}\,\psi^{[\leq n]},$$

by definition of ${d'}_1^{[\leq n]}$. Hence, the following identities hold on $V_1^{[\leq n+1]}$:

$$\begin{aligned} d_2^{[\leq n]}\,\psi^{[\leq n]}\,{d'}_1^{[\leq n]} &= b\psi^{[\leq n]}\,{d'}_1^{[\leq n]} - b\,d_2^{[\leq n]}\,\psi^{[\leq n]} + d_2^{[\leq n]}\,d_2^{[\leq n]}\,\psi^{[\leq n]} \\ &= b\,\psi^{[\leq n]}\,{d'}_1^{[\leq n]} - b\,d_2^{[\leq n]}\,\psi^{[\leq n]} \end{aligned}$$

as $d_2 d_2 = 0$. □

Conclusion : The only tool we have used in this section is the existence of a quasi-isomorphism between the complexes $(\mathfrak{g}_1, 0)$ and $(\mathfrak{g}_2, b)$. Since we know explicit homotopy formulas for such a quasi-isomorphism (see [**DL**], [**Ha**]), we obtain explicit formulas for ${d'}_1^{[k]}$ and $\psi^{[k]}$.

4. A G_∞-morphism $\psi' : (\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1) \to (\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1')$

In this section, we will prove the following proposition.

Proposition 4.1. — *If the complex* $\left(\mathrm{Hom}(\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, \Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}), [m_1^{1,1} + m_1^2, -]\right)$ *is acyclic, then there exists a* G_∞*-morphism* $\psi' : (\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1) \to (\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1')$ *such that the induced map* ${\psi'}^{[1]} : \mathfrak{g}_1 \to \mathfrak{g}_1$ *is the identity.*

We will use the same notations for $V_1^{[n]}$, $V_1^{[\leq n]}$, ${d'}_1^{[n]}$ and ${d'}_1^{[\leq n]}$ as in Section 3. We also denote

$$d_1 = \sum_{n \geq 1} d_1^{[n]} \qquad \text{and} \qquad d_1^{[\leq n]} = \sum_{1 \leq k \leq n} d_1^{[k]}$$

and similarly

$$\psi' = \sum_{n \geq 1} \psi^{[n]} \qquad \text{and} \qquad {\psi'}^{[\leq n]} = \sum_{1 \leq k \leq n} {\psi'}^{[n]}.$$

Proof: We will build the maps ${\psi'}^{[n]}$ by induction as in Section 3. For ${\psi'}^{[1]}$ we have to set:

$${\psi'}^{[1]} = \mathrm{Id} \ \text{(the identity map)}.$$

Suppose we have built maps $({\psi'}^{[i]})_{i \leq n-1}$ satisfying

$${\psi'}^{[\leq n-1]} d_1^{[\leq n]} = {d_1'}^{[\leq n]} {\psi'}^{[\leq n-1]}$$

on $V_1^{[\leq n]}$ (${d_1'}^{[\leq n]}$ maps $V_1^{[\leq l]}$ to $V_1^{[\leq l-1]}$). Expliciting the equation $\psi' d_1 = d_1' \psi'$ on $V_1^{[n+1]}$, we get

$${\psi'}^{[\leq n]} \, d_1^{[\leq n+1]} = {d_1'}^{[\leq n+1]} \, {\psi'}^{[\leq n]}. \tag{4.9}$$

If we now take into account that $d_1^{[i]} = 0$ for $i \neq 2$, ${d_1'}^{[1]} = 0$ and that on $V_1^{[n+1]}$ we have ${\psi'}^{[k]} d_1^{[l]} = {d_1'}^{[\leq k]} {\psi'}^{[l]} = 0$ for $k + l > n + 2$, the identity (4.9) becomes

$${\psi'}^{[\leq n]} \, d_1^{[2]} = \sum_{k=2}^{n+1} {d_1'}^{[k]} \, {\psi'}^{[\leq n-k+2]}.$$

We have seen in the previous section that ${d_1'}^{[2]} = {d_1}^{[2]}$. Thus (4.9) is equivalent to

$${d_1}^{[2]} {\psi'}^{[\leq n]} - {\psi'}^{[\leq n]} {d_1}^{[2]} = \left[{d_1}^{[2]}, {\psi'}^{[\leq n]}\right] = -\sum_{k=3}^{n+1} {d_1'}^{[k]} {\psi'}^{[\leq n-k+2]}.$$

Notice that $d_1^{[2]} = m_1^{1,1} + m_1^2$. By the acyclicity of the complex $(\mathrm{End}(\Lambda^\cdot \underline{\mathfrak{g}_1^{\otimes \cdot}}), [d_1^{[2]}, -])$, the construction of ${\psi'}^{[\leq n]}$ will be possible when $\sum_{k=3}^{n+1} {d_1'}^{[k]} {\psi'}^{[\leq n-k+2]}$ is a cocycle in this complex. Thus, to finish the proof, we have to check that

$$\left[d_1^{[2]}, \sum_{k=3}^{n+1} {d_1'}^{[k]} \, {\psi'}^{[\leq n-k+2]}\right] = 0 \text{ on } V_1^{[n+1]}. \tag{4.10}$$

We have

$$D_n = \left[d_1^{[2]}, \sum_{k=3}^{n+1} {d'_1}^{[k]}\, {\psi'}^{[\leq n-k+2]}\right] = \left[d_1^{[2]}, \sum_{k=1}^{n-1} {d'_1}^{[n+2-k]}\, {\psi'}^{[\leq k]}\right].$$

It follows that we can write

$$-D_n = \sum_{k=1}^{n-1} \left[d_1^{[2]}, {d'_1}^{[n+2-k]}\right] {\psi'}^{[\leq k]} - \sum_{k=1}^{n-1} {d'_1}^{[n+2-k]} \left[d_1^{[2]}, {\psi'}^{[\leq k]}\right]. \tag{4.11}$$

Using the induction hypothesis for $\left({\psi'}^{[\leq k]}\right)_{k\leq n-1}$, we get

$$[{d_1}^{[2]}, {\psi'}^{[\leq k]}] = -\sum_{l=3}^{k+1} {d'_1}^{[l]}\, {\psi'}^{[\leq k-l+2]} = -\sum_{l=1}^{k-1} {d'_1}^{[k+2-l]}\, {\psi'}^{[\leq l]}$$

on $V_1^{[\leq k+1]}$. The equation 4.11 then becomes

$$-D_n = \sum_{k=1}^{n-1} \left[d_1^{[2]}, {d'_1}^{[n+2-k]}\right] {\psi'}^{[\leq k]} + \sum_{k=1}^{n-1} {d'_1}^{[n+2-k]} \left(\sum_{l=1}^{k-1} {d'_1}^{[k+2-l]}\, {\psi'}^{[\leq l]}\right).$$

Finally, we have

$$-D_n = \sum_{k=1}^{n-1} \left[d_1^{[2]}, {d'_1}^{[n+2-k]}\right] {\psi'}^{[\leq k]} + \sum_{l=1}^{n-2} \left(\sum_{k=l+1}^{n-1} {d'_1}^{[n+2-k]} {d'_1}^{[k+2-l]}\right) {\psi'}^{[\leq l]}.$$

This implies

$$-D_n = \sum_{k=1}^{n-1} \left(\left[{d'_1}^{[2]}, {d'_1}^{[n+2-k]}\right] + \sum_{p=k+1}^{n-1} {d'_1}^{[n+2-p]} {d'_1}^{[p+2-k]}\right) {\psi'}^{[\leq k]}.$$

But the maps

$$\left[{d'_1}^{[2]}, {d'_1}^{[n+2-k]}\right] + \sum_{p=k+1}^{n-1} {d'_1}^{[n+2-p]}\, {d'_1}^{[p+2-k]} = \sum_{q=2}^{n+2-k} {d'_1}^{[q]}\, {d'_1}^{[n+4-q-k]}$$

are zero because $d'_1 d'_1 = 0$ on $V_1^{[\leq n+2-k]}$. This yields the result. $\square$

5. Acyclicity of the complex $\left(\mathrm{Hom}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}), [m_1^{1,1} + m_1^2, -]\right)$

In this section the manifold M is supposed to be the Euclidian space $\mathbb{R}^m$ for $m \geq 1$. We prove the following proposition:

Proposition 5.1. — *If $M = \mathbb{R}^m$, the cochain complex $\left(\mathrm{End}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}), [m_1^{1,1} + m_1^2, -]\right)$ is acyclic.*

Proof: Since coalgebras maps $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}} \to \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}$ are in one to one correspondence with maps $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}} \to \mathfrak{g}_1$, we are left to check that the cochain complex

$$\Big(\mathrm{Hom}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, \mathfrak{g}_1), [m_1^{1,1} + m_1^2, -]\Big)$$

is acyclic.

First we introduce an "external" bigrading on the cochain complex induced by duality from the following bigrading on $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}$:

$$|x|^{\mathrm{e}} = (p_1 - 1 + \cdots + p_n - 1, n-1)$$

if $x \in \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_n}}$. Let the internal degree of $x \in \mathfrak{g}_1$ be $|x|^{\mathrm{i}} = |x| + 1$, where $|x|$ is the usual degree of an element of $\mathfrak{g}_1$. One recovers the usual degree on $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}$ by

$$|x| = |x|^{\mathrm{e}}_{\mathrm{tot}} + \sum_{i,k} |x_i^k|^{\mathrm{i}}$$

where $|x|^{\mathrm{e}}_{\mathrm{tot}}$ is the sum of the two components of $|x|^{\mathrm{e}}$.

The exterior product m_1^2 makes $\mathfrak{g}_1$ into an associative algebra which is graded commutative for the inner degree. For any vector space V, the space $\mathfrak{g}_1 \otimes V$ is a $\mathfrak{g}_1$-module equipped with a $\mathfrak{g}_1$-action by multiplication on the first factor. Observe that

$$\begin{aligned}\Big(\mathrm{Hom}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, \mathfrak{g}_1), [m_1^{1,1} + m_1^2, -]\Big) &\cong \Big(\mathrm{Hom}_{\mathfrak{g}_1}(\mathfrak{g}_1 \otimes \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, \mathfrak{g}_1), [m_1^{1,1} + m_1^2, -]\Big),\\ &\cong \Big(\mathrm{Hom}_{\mathfrak{g}_1}(\Lambda^{\cdot}_{\mathfrak{g}_1}(\mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes\cdot}}), \mathfrak{g}_1), [m_1^{1,1} + m_1^2, -]\Big)\end{aligned}$$

where $\mathfrak{g}_1$ acts (on the right and on the left) on itself by the multiplication m_1^2.

We now prove acyclicity of this last cochain complex. The codifferential $(\delta)^* = [m_1^{1,1} + m_1^2, -]$ splits in two parts $(\delta_1^2)^* + (\delta_1^{1,1})^* = (\delta)^*$ where $(\delta_1^2)^*$ is the codifferential of bidegree $(1,0)$ induced by m_1^2 and $(\delta_1^{1,1})^*$ is the one of bidegree $(0,1)$ induced by $m_1^{1,1}$. Thus, $\mathrm{Hom}_{\mathfrak{g}_1}(\Lambda^{\cdot}_{\mathfrak{g}_1}\mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes\cdot}}, \mathfrak{g}_1)$ endowed with the bigrading $|-|^{\mathrm{e}}$ is a bicomplex lying in the first quadrant. The bigrading of an element $x \in \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge ... \wedge \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes p_n}}$ is $|x|^{\mathrm{e}} = (p_1 - 1 + ... + p_n - 1, n-1)$.

The codifferential $(\delta_1^2)^*$ is dual to a differential δ_1^2. It is a standard calculation (see [**Lo1**], 1.5 for example) to show that δ_1^2, restricted to each summand $\mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes\cdot}}$, is the usual Harrison boundary, that is to say the image of the Hochschild differential d acting on $\mathfrak{g}_1^{\otimes\cdot+1}$ onto its quotient $\mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes\cdot}}$ by the shuffles.

We now use the fact that $(\mathfrak{g}_1, m_1^2) = (\Gamma(M, \Lambda TM), \wedge)$ is a polynomial algebra. Denote by $\Omega_{\mathfrak{g}_1}$ the module of Kähler differential one-forms of the algebra $\mathfrak{g}_1$. Let $J : \mathfrak{g}_1^{\otimes\cdot+1} \to \Lambda^{\cdot}\Omega_{\mathfrak{g}_1}$ be the map which sends $x_0 \otimes \cdots \otimes x_n$ to $x_0 dx_1 \cdots dx_n$ and $I : \Lambda^{\cdot}\Omega_{\mathfrak{g}_1} \to \mathfrak{g}_1^{\otimes\cdot+1}$ be the anti-symmetrization map given by

$$J(x_0 dx_1 \cdots dx_n) = \sum_{\sigma \in S_n} \frac{(-1)^\sigma}{n!} \varepsilon(\sigma)\, x_0 \otimes x_{\sigma^{-1}(1)} \ldots \otimes x_{\sigma^{-1}(n)}$$

where S_n is the permutation group of $\{1, \cdots, n\}$, $(-1)^\sigma$ is the sign of σ and $\varepsilon(\sigma)$ the Koszul-Quillen sign (see Section 1). It is easy to check that $J \circ I = \mathrm{Id}$.

It is known from [**Ha**] that there exists a homotopy $s : \mathfrak{g}_1^{\otimes \cdot +1} \to \mathfrak{g}_1^{\otimes \cdot +2}$ such that $I \circ J = \mathrm{Id} + d \circ s + s \circ d$. We denote P the natural projection $\mathfrak{g}_1^{\otimes \cdot +1} \to \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$. It is a standard computation (see [**Lo2**]) that the map J factors through $\mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$ to give a map $J' : \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}} \to \Omega_{\mathfrak{g}_1}$. There is also a map $I' = P \circ J : \Omega_{\mathfrak{g}_1} \to \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$. Clearly $J' \circ I' = \mathrm{Id}$. Since the map P commutes with the differential d, the map $s' = P \circ s$ satisfies

$$I' \circ J' = \mathrm{Id} + d \circ s' + s' \circ d.$$

The map s' extends uniquely into a degree 1 homotopy h to $\Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$ so that $\Lambda I' \circ \Lambda J' = \mathrm{Id} + \delta_1^2 \circ h + h \circ \delta_1^2$, where $\Lambda I'$, $\Lambda J'$ are the extensions of the degree zero maps I' and J'. Moreover $\Lambda J' \circ \Lambda I' = \mathrm{Id}$ and $\Lambda^{\cdot}_{\mathfrak{g}_1} \Omega_{\mathfrak{g}_1}$ is a (special) deformation retract of $\Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$(see [**Ha**]). We denote $p : \Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}} \to \Lambda^{\cdot}_{\mathfrak{g}_1} \Omega_{\mathfrak{g}_1}$ the projection. Since $\Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$ is a bicomplex with differential $\delta = \delta^2_{1,1} + \delta_1^{1,1}$, it follows from [**Ka**], Section 3 that there exists a map $u : \Lambda^{\cdot}_{\mathfrak{g}_1} \Omega_{\mathfrak{g}_1} \to \Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}$ and a (degree one) map $H : \Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}} \to \Lambda^{\cdot}_{\mathfrak{g}_1} \mathfrak{g}_1 \otimes \underline{\mathfrak{g}_1^{\otimes \cdot}}[1]$ such that $pu = \mathrm{Id}$ and $up = \mathrm{Id} + \delta H + H\delta$. Hence the cohomology we are looking for is the cohomology of the complex $\left(\mathrm{Hom}_{\mathfrak{g}_1}(\Lambda^{\cdot}_{\mathfrak{g}_1} \Omega_{\mathfrak{g}_1}, \mathfrak{g}_1), \delta_1^{1,1}\right)$ which sits in the complex

$$\left(\mathrm{Hom}(\Lambda^{\cdot} \mathfrak{g}_1, \mathfrak{g}_1), \delta_1^{1,1}\right) \cong \left(\mathrm{Hom}_{\mathfrak{g}_1}(\mathfrak{g}_1 \otimes \Lambda^{\cdot} \mathfrak{g}_1, \mathfrak{g}_1), \delta_1^{1,1}\right).$$

In particular, the differential $\delta_1^{1,1}$ is induced by the usual exterior derivative (see [**HKR**]) on $\mathrm{Hom}_{\mathfrak{g}_1}(\mathfrak{g}_1 \otimes \Lambda^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, \mathfrak{g}_1)$. To finish the proof, we proceed as in [**Ta**] and [**Hi**]. Recall from the introduction that $A = C^\infty(\mathbb{R}^m)$ is the algebra of smooth functions on $\mathbb{R}^m$. Let $\mathrm{Der}(A) = \Omega_A^*$ be the space of smooth derivations on A. Since $\mathfrak{g}_1$ is a A-module, by transitivity of the space of Kähler differentials for smooth manifolds, one has

$$\Omega_{\mathfrak{g}_1} \cong \mathfrak{g}_1 \otimes_A \Omega_A \oplus \Omega_{\mathfrak{g}_1/A}.$$

Since $\mathfrak{g}_1 \cong \Lambda_A^* \mathrm{Der}(A)$, we find that $\Omega_{\mathfrak{g}_1/A} \cong \mathfrak{g}_1 \otimes \mathrm{Der}(A)$ (with grading shifted by minus one on $\mathrm{Der}(A)$). Hence (see [**Ta**].3.5) there is an isomorphism

$$\left(\mathrm{Hom}_{\mathfrak{g}_1}(\Lambda^{\cdot}_{\mathfrak{g}_1} \Omega_{\mathfrak{g}_1}, \mathfrak{g}_1), \delta_1^{1,1}\right) \cong \left(\Lambda^{1+\cdot} \Omega_{\mathfrak{g}_1}, d_{dR}\right)$$

where d_{dR} is de Rham's differential (the degree on the left hand of the isomorphism is the one induced by the inner degree of $\mathfrak{g}_1$). When $\mathfrak{g}_1 = \Gamma(\mathbb{R}^n, \Lambda \mathbb{R}^n)$ this complex is acyclic. □

Remark: At every step of this proof, it is possible to construct explicit homotopy formulas. So the coefficients ${\psi'}^{[n]}$ built in this section can be expressed in an explicit way from the G_∞-structure on $\mathfrak{g}_2$.

Corollary 5.2. — *If* $\mathfrak{g}_1 = \Gamma(\mathbb{R}^m, \Lambda T\mathbb{R}^m)$, *then there exists a* G_∞*-morphism* ψ' : $(\Lambda^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1) \to (\Lambda^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1')$ *such that the induced map* ${\psi'}^{[1]} : \mathfrak{g}_1 \to \mathfrak{g}_1$ *is the identity.*

Proof: It is an immediate consequence of Propositions 4.1 and 5.1. □

6. Consequences when M is a Poisson manifold

From the Sections 3, 4 and 5 we know that the map

$$\phi = \psi \circ \psi' : (\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1) \to (\Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}, d_2)$$

is a G_∞-morphism when $M = \mathbb{R}^m$; in other words, we have the identity

$$\phi \circ d_1 = d_2 \circ \phi \text{ on } \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}. \tag{6.12}$$

Since $\phi : \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}} \to \Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}$ is a coalgebra map, it restricts to the subcoalgebra $\Lambda^{\cdot}\mathfrak{g}_1$ to give a coalgebra map $\Lambda^{\cdot}\mathfrak{g}_1 \to \Lambda^{\cdot}\mathfrak{g}_2$. The restriction of d_1 and d_2 are respectively the codifferential induced by $m_1^{1,1}$ and the codifferential induced by $b + m_2^{1,1}$ (see the end of Section 2). When we restrict these maps to $\Lambda^{\cdot}\mathfrak{g}_1$ and $\Lambda^{\cdot}\mathfrak{g}_2$, the previous equality (6.12) still holds with the difference that, now, d_1 and d_2 are the differential defining the L_∞-structures on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of Section 1. So the restriction of ϕ to these coalgebras yields a morphism of differential coalgebras

$$\phi \ : \ (\Lambda^{\cdot}\mathfrak{g}_1, d_1) \to (\Lambda^{\cdot}\mathfrak{g}_2, d_2).$$

Thus we have constructed the desired L_∞-morphism between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$.

Remark: Similarly to Definitions 1.1, 1.3, one can define, on a vector space $\mathfrak{g}$, a C_∞-algebra structure given by degree one maps $a^m : \underline{\mathfrak{g}^{\otimes n}} \to \mathfrak{g}$ such that if we extend them to maps $\oplus\underline{\mathfrak{g}}^{\otimes \cdot} \to \oplus\underline{\mathfrak{g}}^{\otimes \cdot}$, then $D = \sum a^m$ satisfies $D \circ D = 0$. In particular, a^2 yields a commutative operation on $\mathfrak{g}$, a^1 a differential and the product a^2 is associative up to homotopies for the differential a^1. Let us then consider the free Lie coalgebras $\oplus_{n\geq 0}\underline{\mathfrak{g}_1}^{\otimes n}$ and $\oplus_{n\geq 0}\underline{\mathfrak{g}_2}^{\otimes n}$. They are also subcoalgebras of respectively $\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}$ and $\Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}$. Hence we can restrict ϕ into a coalgebra map $\phi : \oplus\underline{\mathfrak{g}_1}^{\otimes n} \to \oplus\underline{\mathfrak{g}_2}^{\otimes n}$. Denoting $D_1 = m_1^2$ the codifferential induced by the exterior product $\wedge$, and D_2 the codifferential induced by $\sum_{n\geq 0} m_2^n$, the map ϕ yields a differential coalgebra morphism

$$\phi \ : \ \left(\oplus\underline{\mathfrak{g}_1}^{\otimes n}, D_1\right) \to \left(\oplus\underline{\mathfrak{g}_2}^{\otimes n}, D_2\right),$$

hence, a morphism of C_∞-algebras between $(\mathfrak{g}_1, D_1)$ and $(\mathfrak{g}_2, D_2)$. Through the Etingof-Kazhdan equivalence used in Proposition 2.2, this implies that there is a morphism of A_∞-algebras between $(\mathfrak{g}_1, \wedge)$ and $(\mathfrak{g}_2, \cup)$. More precisely, it means that there is a morphism $\left(\oplus\mathfrak{g}_1^{\otimes n}, \wedge\right) \to \left(\oplus\mathfrak{g}_2^{\otimes n}, b + \cup\right)$ of differential coalgebras between the tensor coalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Details on C_∞ and A_∞-structures can be found in [**GK**] and [**St**].

From now on, we will suppose that the manifold M is a Poisson manifold equipped with a Poisson tensor π (satisfying $[\pi, \pi]_S = 0$). The L_∞-map ϕ allows us to construct a star-product on M (see [**BFFLS1**]). If $\hbar$ is a formal parameter and if we impose ϕ to be $\mathbb{R}[[\hbar]]$-linear, ϕ extends to a L_∞-morphism between $\mathfrak{g}_1[[h]]$ and $\mathfrak{g}_2[[h]]$. Set $\Pi_\hbar = \sum_{n\geq 0} \hbar^n \Lambda^n \pi \in \Lambda^{\cdot}\mathfrak{g}_1$, where $\Lambda^n\pi = \underbrace{\pi \wedge ... \wedge \pi}_{n \text{ times}}$ (here $\wedge$ is not the exterior product of tensor fields but $a \wedge b$ is an element in $\mathfrak{g}_1 \wedge \mathfrak{g}_1$). If we define $m_\star = \phi(\Pi_\hbar)$, we get

$$[m_\star, m_\star]_G = 0. \tag{6.13}$$

This is a consequence of definition (0.1) of a L_∞-morphism given in Section 0 and of the fact that $[\pi,\pi]_S = 0$ implies $m_1^{1,1}(\Pi_\hbar) = 0$. The map $m_\star$ being an element of $\mathfrak{g}_2[[\hbar]]$ of degree one, it defines a bilinear map in $C^2(A,A)[[\hbar]]$, where $C^k(A,A)[[\hbar]]$ denotes the set of k-$\mathbb{R}[[\hbar]]$-linear maps in $C^k(A,A)$. The identity 6.13 implies that $m_\star$ is an associative product on $A[[\hbar]]$. Finally, by definition of ϕ, we have:

$$m_\star = m + \hbar\phi^1(\pi) + \sum_{n\geq 2} \hbar^n \phi^n(\pi,\ldots,\pi),$$

where $\phi^1(\pi) = \{\cdot,\cdot\}$ is the Poisson bracket. This proves that $m_\star$ is a star-product on (M,π).

The spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ can now be endowed with two new structures: the space $(\mathfrak{g}_1, [-,-]_S, [\pi,-]_S)$ becomes a graded differential Lie algebra (and even a Gerstenhaber algebra) whereas $(\mathfrak{g}_2, [-,-]_G, b_\star)$, where $b_\star$ is the Hochschild differential corresponding to the deformed product $m_\star$, is a new graded differential Lie algebra. As in the case when $\pi = 0$, we have the following result *à la* Hochschild-Kostant-Rosenberg.

Theorem 6.1. — *The complexes $(\mathfrak{g}_2[[\hbar]], b_\star)$ and $(\mathfrak{g}_1[[\hbar]], [\hbar\pi, -]_S)$ are quasi-isomorphic.*

Proof: Let us denote $\phi^1_\hbar$ the $\mathbb{R}[[\hbar]]$-linear map $\mathfrak{g}_1[[\hbar]] \to \mathfrak{g}_2[[\hbar]]$ given by

$$\alpha \mapsto \phi^1_\hbar(\alpha) = \sum_{n\geq 0} \hbar^n \phi^{n+1}\left(\Lambda^n\pi \wedge \alpha\right) = \phi_{\mathfrak{g}_1}\left(\sum_{n\geq 0} \hbar^n \Lambda^n \pi \wedge \alpha\right)$$

for $\alpha \in \mathfrak{g}_1$, where $\phi_{\mathfrak{g}_1}$ denotes the projection of ϕ on $\mathfrak{g}_1$. Similarly, we write $\phi_{\mathfrak{g}_1\wedge\mathfrak{g}_1}$ for the projection of ϕ on $\mathfrak{g}_1 \wedge \mathfrak{g}_1$. We get

$$\begin{aligned}
\phi^1_\hbar([\hbar\pi,\alpha]_S) &= \phi_{\mathfrak{g}_1}\left(\sum_{n\geq 0} \hbar^n \Lambda^n\pi \wedge [\hbar\pi,\alpha]_S\right) \\
&= \phi_{\mathfrak{g}_1}\left(\sum_{n\geq 0} \hbar^{n+1} m_1^{1,1}\left(\Lambda^{n+1}\pi \wedge \alpha\right)\right) \\
&= m_2^{1,1}\left(\phi_{\mathfrak{g}_1\wedge\mathfrak{g}_1}\left(\sum_{n\geq 0} \hbar^{n+1}\Lambda^{n+1}\pi \wedge \alpha\right)\right) \\
&= \left[\phi\left(\sum_{n\geq 0}\hbar^n\Lambda^n\pi\right), \phi\left(\sum_{n\geq 0}\hbar^n\Lambda^n\pi\wedge\alpha\right)\right]_G \\
&= [m_\star, \phi^1_\hbar(\alpha)]_G \\
&= b_\star \phi^1_\hbar(\alpha).
\end{aligned}$$

Thus $\phi^1_\hbar$ is a morphism of complexes between $(\mathfrak{g}_1, [\hbar\pi, -]_S)$ and $(\mathfrak{g}_2[[\hbar]], b_\star)$. By definition, we can write $\phi^1_\hbar(\alpha) = \phi^1(\alpha) + \sum_{n\geq 1} \hbar^n \phi^{1,n}_\hbar$ where $\phi^{1,n}_\hbar$ are $\mathbb{R}[[\hbar]]$-linear maps. The proof of the theorem is then a consequence of the following lemma. □

Lemma 6.2. — *Let $\varphi : (B, 0) \to (D, d)$ be a quasi-isomorphism of cochain complexes. Suppose we have two deformed complexes $(B[[\hbar]], b_\hbar)$ and $(D[[\hbar]], d_\hbar)$ with $b_\hbar = \sum_{n\geq 1} \hbar^n b_n$ and $d_\hbar = d + \sum_{n\geq 1} \hbar^n d_n$, where b_i and d_i are $\mathbb{R}[[\hbar]]$-linear maps. Suppose in addition that there exists a morphism of complexes $\varphi_\hbar = \varphi + \sum_{n\geq 1} \varphi_n$ between $(B[[\hbar]], b_\hbar)$ and $(D[[\hbar]], d_\hbar)$, where φ_i are $\mathbb{R}[[\hbar]]$-linear maps. Then $\varphi_\hbar$ is a quasi-isomorphism.*

Proof: Suppose $\delta_\hbar = \sum_{n\geq 0} \hbar^n \delta_n \in D[[\hbar]]$ satisfies $d_\hbar \delta_\hbar = 0$. We will construct $\beta_n \in B$ by induction such that $\beta_\hbar = \sum_{n\geq 0} \beta_n$ satisfies $b_\hbar \beta_\hbar = 0$ and $\varphi_\hbar(\beta_\hbar) = \delta_\hbar$. Since $d_\hbar \delta_\hbar = 0$, we have $d_0 \delta_0 = 0$, so that $\delta_0 = \varphi_0(\beta_0)$, ($\beta_0 \in B$ with $b_0 \beta_0 = 0$). This follows from φ_0 being a quasi-isomorphism between $(B, 0)$ and (D, d_0).
Suppose we have built $\beta_0, \ldots, \beta_n \in B$ such that for all $k \leq n$,

$$\delta_k = \sum_{i=0}^{k} \varphi_i(\beta_{k-i}) \tag{6.14}$$

and

$$\sum_{i=0}^{k} b_i \beta_{k-i} = 0. \tag{6.15}$$

We have shown that Relations (6.14) and (6.15) hold for $k = 0$. We will now construct β_{n+1} such that they hold for $k = n+1$. We can reformulate Relation (6.14) as follows:

$$\varphi_0(\beta_{n+1}) = \delta_{n+1} - \sum_{i=1}^{n+1} \varphi_i(\beta_{n+1-i}).$$

Since φ_0 is a quasi-isomorphism between $(B, 0)$ and (D, d_0), this is equivalent to say:

$$d_0(\delta_{n+1}) - \sum_{i=1}^{n+1} d_0 \varphi_i(\beta_{n+1-i}) = 0. \tag{6.16}$$

Since $d_\hbar \delta_\hbar = 0$ we have $d_0 \delta_{n+1} = -\sum_{i=1}^{n+1} d_i \delta_{n+1-i}$. Therefore

$$\begin{aligned}
(6.16) \iff & \sum_{i=1}^{n+1} d_i \delta_{n+1-i} + \sum_{i=1}^{n+1} d_0 \varphi_i \beta_{n+1-i} = 0 \\
\iff & \sum_{i=1}^{n+1} d_i \sum_{j=0}^{n+1-i} \varphi_j(\beta_{n+1-i-j}) + \sum_{i=1}^{n+1} d_0 \varphi_i \beta_{n+1-i} = 0 \\
\iff & \sum_{i=0}^{n+1} \sum_{j=0}^{n+1-i} d_i \varphi_j(\beta_{n+1-j-i}) = 0
\end{aligned}$$

($\beta_{n+1} = 0$ by convention). Since $\varphi_\hbar$ is a morphism of complexes, we obtain

$$\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i} d_i\varphi_j(\beta_{n+1-j-i}) = \sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i} \varphi_i(b_j\beta_{n+1-j-i}) = \varphi_0\left(\sum_{j=0}^{n+1} b_j\beta_{n+1-j}\right). \tag{6.17}$$

So (6.16) $\Longleftrightarrow \varphi_0\left(\sum_{j=0}^{n+1} b_j\beta_{n+1-j}\right) = 0$. This relation will be satisfied provided we have proved Relation (6.15) for $k = n+1$. As φ_0 is a quasi-isomorphism of complexes between $(B, 0)$ and (D, d_0) we only have to prove that $\varphi_0\left(\sum_{j=1}^{n+1} b_j\beta_{n+1-j}\right)$ is a boundary. Using Relation (6.17), we have

$$\begin{aligned}
\varphi_0\left(\sum_{j=0}^{n+1} b_j\beta_{n+1-j}\right) &= \sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i} d_i\varphi_j(\beta_{n+1-j-i}) \\
&= d_0\sum_{j=0}^{n+1}\varphi_j(\beta_{n+1-j}) + \sum_{i=1}^{n+1}\sum_{j=0}^{n+1-i} d_i\varphi_j(\beta_{n+1-j-i}) \\
&= d_0\sum_{j=0}^{n+1}\varphi_j(\beta_{n+1-j}) + \sum_{i=1}^{n+1} d_i\delta_{n+1-i} \qquad \text{(thanks to (6.14))} \\
&= d_0\sum_{j=0}^{n+1}\varphi_j(\beta_{n+1-j}) - d_0\delta_{n+1}
\end{aligned}$$

because $d_\hbar\,\beta_\hbar = 0$. □

It is clear from the previous proof that we can build explicit homotopy formulas for the map $\phi^1_\hbar$ since in the proof of Theorem 6.1, we had $\phi^1_\hbar = \phi^1 + O(\hbar)$, with ϕ^1 the Hochschild-Kostant-Rosenberg map.

Remark 6.3: Lemma 6.2 also holds for two cochain complexes (B, b) and (D, d), where $b_\hbar = b + \sum_{n\geq 1}\hbar^n b_n$ with $b \neq 0$, but we have no explicit homotopy formulas.

The cochains complexes $(B[[\hbar]], b_\hbar)$ and $(D[[\hbar]], d_\hbar)$ are filtered by the powers of $\hbar$. The p-component of the filtration is

$$F^p(B[[\hbar]]) = \hbar^p B[[\hbar]] \subset B[[\hbar]]$$

and similarly for $D[[\hbar]]$. The filtrations are decreasing and the differentials $b_\hbar$ and $d_\hbar$ respects the filtrations as well as the morphism $\varphi_\hbar$. Therefore, there are two spectral sequences with first terms given by

$$EB_1^{p,q} = H^{p+q}(F^qB[[\hbar]]/F^{q+1}B[[\hbar]])$$

and

$$ED_1^{p,q} = H^{p+q}(F^qD[[\hbar]]/F^{q+1}D[[\hbar]])$$

converging respectively to $H^{\cdot}(B[[\hbar]], b_\hbar))$ and $H^{\cdot}(D[[h]], d_\hbar)$. The morphism $\varphi_\hbar$ induces a map of spectral sequences $\varphi_1^\hbar : EB_1^{\cdot,\cdot} \longrightarrow ED_1^{\cdot,\cdot}$.

It is easy to check that $EB_1^{p,q} \cong H^{p+q}(B)\hbar^q$, $ED_1^{p,q} \cong H^{p+q}(D)\hbar^q$ and that the map $\varphi_1^\hbar$ is induced by the quasi-isomorphism φ, hence is an isomorphism for all p, q.

As the spectral sequences are strongly convergent in the sense of [**CE**], Section 15.2, it follows that $\varphi_\hbar$ induces an isomorphism $H^\cdot(B[[\hbar]], b_\hbar)) \cong H^\cdot(D[[h]], d_\hbar)$.

Now, we are in the situation of Section 1: we have two graded differential Lie algebras $(\mathfrak{g}_1, [-,-]_S, [\pi,-]_S)$ and $(\mathfrak{g}_2, [-,-]_G, b_\star)$ such that $H(\mathfrak{g}_2, b_\star) \cong H(\mathfrak{g}_1, [\pi,-]_S)$. The quasi-isomorphism $\phi^1_\hbar$ is not a Lie algebra morphism. The aim of the next section is to construct a L_∞-morphism between $(\mathfrak{g}_1, [\pi,-]_S)$ and $(\mathfrak{g}_2, b_\star)$.

7. A formality theorem for a Poisson manifold

In this section we define $d_{1\star}$ the map $d_{1\star} = m_1^{1,1} + m_1^2 + \hbar[\pi,-]_S : \Lambda^\cdot \underline{\mathfrak{g}_{1\star}^{\otimes^\cdot}} \to \Lambda^\cdot \underline{\mathfrak{g}_{1\star}^{\otimes^\cdot}}$, where $\mathfrak{g}_{1\star} = \mathfrak{g}_1[[\hbar]]$. As for the case $\pi = 0$, we will construct a G_∞-structure on $\mathfrak{g}_{2\star} = \mathfrak{g}_2[[\hbar]]$ given by a differential $d_{2\star} : \Lambda^\cdot \underline{\mathfrak{g}_{2\star}^{\otimes^\cdot}} \to \Lambda^\cdot \underline{\mathfrak{g}_{2\star}^{\otimes^\cdot}}$, where $d_{2\star} = m^1_{2\star} + m_2^{1,1} + \cdots$ with $m^1_{2\star}$ corresponding to the differential $b_\star = [m_\star, -]_G$. We will also prove, following the same steps as in the $\pi = 0$ case, that there exists a G_∞-morphism between the G_∞-algebras $(\mathfrak{g}_{1\star}, d_{1\star})$ and $(\mathfrak{g}_{2\star}, d_{2\star})$.

Theorem 7.1. — *One can build a G_∞-structure on $\mathfrak{g}_2[[\hbar]]$ determined by a differential $d_{2\star} : \Lambda^\cdot \underline{\mathfrak{g}_{2\star}^{\otimes^\cdot}} \to \Lambda^\cdot \underline{\mathfrak{g}_{2\star}^{\otimes^\cdot}}$ with $d_{2\star} = m^1_{2\star} + m^{1,1}_{2\star} + \cdots + m^{p_1,\ldots,p_n}_{2\star} + \cdots$, where*

$$m^{p_1,\ldots,p_n}_{2\star} : \underline{\mathfrak{g}_2[[\hbar]]^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_2[[\hbar]]^{\otimes p_n}} \to \underline{\mathfrak{g}_2[[\hbar]]^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_2[[\hbar]]^{\otimes p_n}},$$

$m^1_{2\star} = b_\star = [m_\star, -]_G$ and $m^{1,1}_{2\star} = m_2^{1,1}$ is the Gerstenhaber bracket.

Proof: We can use the same arguments as in Section 2. Thanks to Lemma 2.1, it is enough to define a differential Lie bialgebra structure on the cofree Lie coalgebra $L_{2\star} = \oplus \underline{\mathfrak{g}_2[[\hbar]]^{\otimes n}}$. Etingof-Kazhdan's dequantization and quantization theorems can be used in the same way to prove it is enough to have a differential bialgebra structure on the cofree tensorial coalgebra $T_{2\star} = \oplus \mathfrak{g}_2[[\hbar]]^{\otimes n}$ since the correspondence in Proposition 2.2 was given by universal formulas.

So we want now to define a bialgebra structure on $T_{2\star}$ given by maps $a^n_{2\star}$ and $a^{p_1,p_2}_{2\star}$ such that $a^1_{2\star} = b_\star$ and $a^{1,1}_{2\star}$ is the product $\{-|-\}$ defined in Section 1. This can be done using braces as in the end of Section 2. This is because the braces inducing a bialgebra structure on $\mathfrak{g}_2$ are independent of the algebra structure on $\mathfrak{g}_2$. Thus a G_∞-structure can be built on $\mathfrak{g}_\star$ with $m^{p_1,\ldots,p_n}_{2\star} = 0$ for $n > 2$. ☐

We can now state the analogue of Proposition 3.1.

Theorem 7.2. — *There exist a G_∞-structure on $\mathfrak{g}_1[[\hbar]]$ corresponding to a differential $d'_{1\star} : \Lambda^\cdot \underline{\mathfrak{g}_{1\star}^{\otimes^\cdot}} \to \Lambda^\cdot \underline{\mathfrak{g}_{1\star}^{\otimes^\cdot}}$ and a morphism of differential coalgebras*

$$\psi_\star : (\Lambda^\cdot \underline{\mathfrak{g}_{1\star}^{\otimes^\cdot}}, d'_{1\star}) \to (\Lambda^\cdot \underline{\mathfrak{g}_{2\star}^{\otimes^\cdot}}, d_{2\star})$$

such that the induced map $\mathfrak{g}_1 \to \mathfrak{g}_2$ is the Hochschild-Kostant-Rosenberg map.

Proof: We will follow the proof of Proposition 3.1 and use the same notations. Let us denote

$$V_{i\star}^{(n)} = \sum_{k=0}^{n} \hbar^k V_i^{[n-k]} \qquad \text{and} \qquad V_{i\star}^{(\leq n)} = \sum_{k=0}^{n} V_{i\star}^{(k)}.$$

There is a decomposition $d_{2\star} = \sum_{k\geq 0} \hbar^k d_{2\star}^{\{k\}}$. We denote $d_{2\star}^{\{k\}p_1,\dots,p_l} : \mathfrak{g}_2^{p_1} \wedge \cdots \wedge \mathfrak{g}_2^{p_l} \to \mathfrak{g}_2$ the components of $d_{2\star} : \Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}} \to \Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}$. Similarly, we denote $d_{2\star}^{\{k\}[n]}$ the map from $\Lambda^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}$ to itself defined by

$$d_{2\star}^{\{k\}[n]} = \sum_{p_1+\cdots+p_l=n} d_{2\star}^{\{k\}p_1,\dots,p_l}.$$

We have the obvious identity $d_{2\star}^{\{k\}} = \sum_{n\geq 1} d_{2\star}^{\{k\}[n]}$. We can now define

$$d_{2\star}^{(m)} = \sum_{k+n=m} d_{2\star}^{\{k\}[n]}$$

and set

$$d_{2\star} = \sum_{m\geq 1} d_{2\star}^{(m)}, \quad d_{2\star}^{(\leq m)} = \sum_{i=1}^{m} d_{2\star}^{(i)}.$$

In the same way we set

$$d'_{1\star} = \sum_{m\geq 1} {d'_{1\star}}^{(m)}, \quad {d'_{1\star}}^{(m)} = \sum_{k+n=m} {d'_{1\star}}^{\{k\}[n]}, \quad {d'_{1\star}}^{(\leq m)} = \sum_{i=1}^{m} {d'_{1\star}}^{(i)},$$

$$\psi_\star = \sum_{m\geq 1} {\psi_\star}^{(m)}, \quad {\psi_\star}^{(m)} = \sum_{k+n=m} {\psi_\star}^{\{k\}[n]} \quad \text{and} \quad {\psi_\star}^{(\leq m)} = \sum_{i=1}^{m} {\psi_\star}^{(i)}.$$

The proof of Proposition 3.1 can now be reproduced, formally replacing the superscripts $[-]$ with $(-)$. We can build maps $d'_{1\star}$ and $\psi_\star$ by induction setting ${d'_{1\star}}^{(1)} = 0$ and $\psi_\star^{(1)} = \phi^1$, the Hochschild-Kostant-Rosenberg map. The proof then relies again only on the fact that ϕ^1 is a quasi-isomorphism of complexes from $(\mathfrak{g}_1, 0)$ to $(\mathfrak{g}_2, b = m_2^1 = m_{2\star}^{(1)})$ (for which we have homotopy formulas). Moreover, at order two we have again

$${d'_{1\star}}^{(2)} = {d_{1\star}}^{(2)} = \hbar[\pi, -]_S + d_1^{1,1} + d_1^2.$$

□

Using the grading $(-)$ along the lines of the proof of Theorem 4.1, we can prove in the same way the following.

Theorem 7.3. — *If the complex* $\left(\mathrm{End}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}), \left[m_1^{1,1} + m_1^2 + \hbar[\pi, -]_S, -\right]\right)$ *is acyclic, then there exists a G_∞-morphism $\psi' : (\Lambda^{\cdot}\underline{\mathfrak{g}_{1\star}^{\otimes\cdot}}, d_{1\star}) \to (\Lambda^{\cdot}\underline{\mathfrak{g}_{1\star}^{\otimes\cdot}}, d'_{1\star})$ such that the induced map $\mathfrak{g}_1 \to \mathfrak{g}_2$ is the identity.*

Theorems 7.2 and 7.3 hold for arbitrary Poisson manifolds.

Corollary 7.4. — *If $\mathfrak{g}_1 = \Gamma(\mathbb{R}^m, \Lambda T\mathbb{R}^m)$, there exists a G_∞-morphism*

$$\psi' : (\Lambda^{\cdot}\underline{\mathfrak{g}_{1\star}^{\otimes\cdot}}, d_{1\star}) \to (\Lambda^{\cdot}\underline{\mathfrak{g}_{1\star}^{\otimes\cdot}}, d'_{1\star}).$$

Proof: Using Theorem 7.3 it is enough to check that the cochain complex

$$\Big(\mathrm{Hom}(\Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}, \Lambda^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}), \Big[m_1^{1,1} + m_1^2 + \hbar[\pi, -]_S, -\Big]\Big)$$

is for $M = \mathbb{R}^m$. This follows from Proposition 5.1 and the following Lemma 7.5. □

Lemma 7.5. — *If a complex (C, d_0) is acyclic, then, for any differential $d_\star = d_0 + \sum_{i\geq 1} \hbar^i d_i$, the $\mathbb{R}[[\hbar]]$-linear complex $(C[[\hbar]], d_0 + \sum_{i\geq 1} \hbar^i d_i)$ is acyclic.*

This follows from Remark 6.3. However, as we wish to be able to construct explicit homotopies we give another proof.

Proof: Suppose we have $x = \sum_{i\geq 0} \hbar^i x_i \in C[[\hbar]]$ satisfying

$$d_\star x = 0. \tag{7.18}$$

We will construct by induction $y = \sum_{i\geq 0} \hbar^i y_i$ satisfying $x = d_\star y$.

Relation (7.18) at order 0 gives $d_0 x_0 = 0$; so by hypothesis there exists $y_0 \in C$ such that $x_0 = d_0 y_0$. Suppose we have built y_i for $i \leq n-1$ such that $x_k = \sum_{i=0}^k d_i y_{k-i}$ for all $k \leq n-1$. We want to build y_n such that $x_n = \sum_{i=0}^n d_i y_{n-i}$. From the acyclicity of the complex (C, d_0) this is equivalent to

$$d_0 \left(x_n - \sum_{i=1}^n d_i y_{n-i} \right) = 0. \tag{7.19}$$

We have

$$(7.19) \iff \sum_{i=1}^n d_i x_{n-i} + \sum_{i=1}^n d_0 d_i y_{n-i} = 0.$$

By the induction hypothesis, we obtain

$$\begin{aligned}
(7.19) \iff & \sum_{i=1}^n d_i x_{n-i} - \sum_{i=1}^n \sum_{j=1}^i d_j d_{i-j} y_{n-i} = 0 \\
\iff & \sum_{i=1}^n d_i x_{n-i} - \sum_{j=1}^n d_j \sum_{i=j}^n d_{i-j} y_{n-i} = 0 \\
\iff & \sum_{i=1}^n d_i x_{n-i} - \sum_{j=1}^n d_j \sum_{i=0}^{n-j} d_i y_{n-i-j} = 0 \\
\iff & \sum_{i=1}^n d_i x_{n-i} - \sum_{j=1}^n d_j x_{n-j} = 0.
\end{aligned}$$

This proves the result. □

As in Section 6, it is easy to see that $\phi_\star = \psi'_\star \circ \psi_\star$ is a G_∞-morphism between $\mathfrak{g}_{1\star}$ and $\mathfrak{g}_{2\star}$. Moreover $\phi_\star$ restricts to a L_∞-morphism

$$\widetilde{\phi_\star} : (\mathfrak{g}_1[[\hbar]], \hbar[\pi, -]_S, [-,-]_S) \to (\mathfrak{g}_2[[\hbar]], b_\star, [-,-]_G)$$

and also to a A_∞-morphism

$$\check{\phi_\star} : (\mathfrak{g}_1[[\hbar]], \hbar[\pi, -]_S, \wedge) \to (\mathfrak{g}_2[[\hbar]], b_\star, \sum_{i\geq 2} m^i_{2\star}).$$

If we now restrict the map $\phi_\star$ to $\phi_\star^{[1]} : \mathfrak{g}_1[[\hbar]] \to \mathfrak{g}_2[[\hbar]]$, we have $\phi_\star^{[1]}([\hbar\pi, \alpha]_S) = b_\star \phi_\star^{[1]}(\alpha)$ for any $\alpha \in \mathfrak{g}_1$. So we have constructed another morphism of complexes $(\mathfrak{g}_1[[\hbar]], \hbar[\pi, -]_S) \to (\mathfrak{g}_2[[\hbar]], b_\star)$. According to Lemma 6.2, the map $\phi_\star^{[1]}$ is a quasi-isomorphism which is *a priori* different from the map $\phi^1_\hbar$. We leave the two following questions unanswered:

Question 1: Are the two maps $\phi^1_\hbar$ and $\phi_\star^{[1]}$ the same?

Question 2: To prove the existence of the G_∞-morphism $\phi_\star$, we have used the grading $(-)$ which imposes the initial condition $\psi_\star^{[1]} = \phi^1$, the Hochschild-Kostant-Rosenberg morphism. Is it possible to build a map $\phi_\star$ such that $\phi_\star^{[1]} = \phi^1_\hbar$?

Remark: B. Keller helped us to give a partial answer to this second question, using the following proposition (see K. Lefèvre [**Le**] for a proof in the A_∞ case).

Proposition 7.6. — *Let A and B be two L_∞ (respectively A_∞, or G_∞)-algebras, with structures determined by differentials*

$$d_A \; : \; \Lambda^\cdot A \to \Lambda^\cdot A \text{ (respectively } \underline{A^{\otimes \cdot}} \to \underline{A^{\otimes \cdot}} \text{ or } \Lambda^\cdot \underline{A^{\otimes \cdot}} \to \Lambda^\cdot \underline{A^{\otimes \cdot}}),$$

and d_B defined in the same way. Denote $d_A = \sum_{n\geq 0} d^n_A(\dot{,} \dots, \dot{})$, where d^n_A is a homogeneous component of d_A (in the G_∞ case, we write $d_A = \sum_{l\geq 0, n_1+\cdots+n_p = l} d_A^{n_1,\dots,n_p}$ with $d_A^{n_1,\dots,n_p} : \underline{A^{\otimes n_1}}\Lambda \dots \Lambda \underline{A^{\otimes n_p}} \to B$ and we order the maps $d_i^{n_1,\dots,n_p}$ such that $(n_1,\dots,n_p) \geq (m_1,\dots,m_q) \Leftrightarrow (n_1+\cdots+n_p > m_1+\cdots+m_q)$ or $(n_1+\cdots+n_p = m_1+\cdots+m_q$ and $(n_1,\dots,n_p) \geq (m_1,\dots,m_q)$ for the lexicographic order). Suppose there exists a "twisting" element $a \in A$ such that

$$\sum_{n\geq 0} d^n_A(a,\dots,a) = 0,$$

and a L_∞ (respectively A_∞, or G_∞)-morphism $\varphi = \sum_{n\geq 0} \varphi^n$ (using the same convention as above) between A and B. Then

- **(a) :** = *there exists a "twisted" L_∞ (respectively A_∞, or G_∞)-algebra structure on A with differential $d_{A_a} = \sum_{n\geq 0} d^n_{A_a}$ given by*

$$d^n_{A_a}(\cdot,\dots,\cdot) = \sum_{i\geq 0} d^{n+i}_A(\dots, a, \dots, a, \dots),$$

where the element a is inserted i times;

(b) : *the element $b \in B$ defined by*

$$b = \sum_{n \geq 0} \varphi^n(a, \dots, a)$$

satisfies

$$\sum_{n \geq 0} d_B^n(b, \dots, b) = 0.$$

(c) : *there exists a "twisted" L_∞ (respectively A_∞, or G_∞)-algebra structure on B with differential $d_{B_b} = \sum_{n \geq 0} d_{B_b}^n$ given by*

$$d_{B_b}^n(\cdot, \dots, \cdot) = \sum_{i \geq 0} d_B^{n+i}(\dots, b, \dots, b, \dots)$$

where the element b is inserted i times;

(d) : *there exists a L_∞ (respectively A_∞, or G_∞)-morphism between the two "twisted" L_∞ (respectively A_∞, or G_∞)-algebra structures on A and B given by $\varphi_{ab} = \sum_{n \geq 0} \varphi_{ab}^n$, where*

$$\varphi_{ab}^n(\cdot, \dots, \cdot) = \sum_{i \geq 0} \varphi^{n+i}(\dots, a, \dots, a, \dots)$$

where the element a is inserted i times.

In our case, where $A = \mathfrak{g}_1$ and $B = \mathfrak{g}_2$ and φ is Tamarkin's L_∞ (respectively A_∞, or G_∞)-morphism, and $a = \pi$ (the Poisson tensor field), we can apply the previous proposition, but only in the L_∞ case (because otherwise $\sum d_1^k(\pi, \dots, \pi) \neq 0$), and get a deformed L_∞-morphism between the graded Lie algebras $(\mathfrak{g}_1[[\hbar]], \hbar[\pi, -]_S, [-, -]_S)$ and $(\mathfrak{g}_2[[\hbar]], b_\star, [-, -]_G)$.

8. Appendix (B. Enriquez and P. Etingof): Etingof-Kazhdan's dequantization theorem for graded differential super-bialgebras

In this appendix, we prove the following theorem:

Theorem 8.1. — *We have an equivalence of categories*

$$DQ_\Phi \ : \ \mathrm{DGQUE} \to \mathrm{DGLBA}_h$$

from the category of differential graded quantized universal enveloping super-algebras to that of differential graded Lie super-bialgebras such that if $U \in \mathrm{Ob}(\mathrm{DGQUE})$ and $\mathfrak{a} = DQ(U)$, then $U/hU = U(\mathfrak{a}/h\mathfrak{a})$, where U is the universal algebra functor, taking a differential graded Lie super-algebra to a differential graded super-Hopf algebra.

Recall the Etingof-Kazhdan quantization theorems. We denote by LBA the prop of Lie bialgebras and $\mathrm{UE_{cp}}$ the prop of co-Poisson universal enveloping algebras. $\mathrm{UE_{cp}}$ is a completion of the prop of co-Poisson cocommutative bialgebras. We let h be a formal variable and we set $\mathrm{LBA}_h = \mathrm{LBA}[[h]]$ and $\mathrm{UE}_{\mathrm{cp},h} = \mathrm{UE_{cp}}[[h]]$.

We denote by QUE the prop of quantized universal enveloping algebras. QUE is topologically free over $k[[h]]$, it is a completion of the prop Bialg_h of quasi-cocommutative bialgebras. We have a natural isomorphism $\mathrm{QUE} \underset{k[[h]]}{\otimes} k = \mathrm{UE}_{\mathrm{cp}}$.

Theorem 8.2. — *([**EK1**], [**EK2**]): To any associator Φ, one can attach a prop isomorphism $Q_\Phi : \mathrm{QUE} \to \mathrm{UE}_{\mathrm{cp},h}$, whose reduction* mod h *is*

$$\mathrm{QUE} \underset{k[[h]]}{\otimes} k \xrightarrow{\sim} \mathrm{UE}_{\mathrm{cp}} \xrightarrow{\sim} \mathrm{UE}_{\mathrm{cp},h} \underset{k[[h]]}{\otimes} k.$$

Theorem 8.3. — *(Propic Milnor-Moore theorem) : We have a symmetrization isomorphism*

$$\mathrm{UE}_{\mathrm{cp},h} \xrightarrow{\sim} \widehat{S^{\cdot}}(\mathrm{LBA}_h),$$

where $\widehat{S^{\cdot}}$ is the completed symmetric algebra Schur functor.

This theorem (see [**EE**]) is based on Euler idempotents (see [**Lo3**]).

If P is a prop, we can attach to each symmetric tensor category $\mathcal{S}$, the category of $\mathcal{S}$-representations of P, $\mathrm{Rep}_{\mathcal{S}}(P)$. Objects of $\mathrm{Rep}_{\mathcal{S}}(P)$ are pairs (V, ρ) of $V \in \mathrm{Ob}(\mathcal{S})$ a prop morphism $P \to \mathrm{Prop}(V)$, where $\mathrm{Prop}(V)$ is the prop attached to V (e.g., $\mathrm{Prop}(V)(n, m) = \mathrm{Hom}_{\mathcal{S}}(V^{\otimes n}, V^{\otimes m})$).

Corollary 8.4. — *If $\mathcal{S}$ is any symmetric tensor category, Q_Φ gives rise to an equivalence of categories*

$$DQ_{\Phi,\mathcal{S}} \ : \ \mathrm{Rep}_{\mathcal{S}}(\mathrm{QUE}) \to \mathrm{Rep}_{\mathcal{S}}(\mathrm{LBA}_h),$$

whose reduction mod h *is the "$\mathcal{S}$-primitive part" functor*

$$\mathrm{Rep}_{\mathcal{S}}(\mathrm{UE}_{\mathrm{cp}}) \to \mathrm{Rep}_{\mathcal{S}}(\mathrm{LBA}),$$

adjoint to the "$\mathcal{S}$-universal enveloping algebra" functor $\mathrm{Rep}_{\mathcal{S}}(\mathrm{LBA}) \to \mathrm{Rep}_{\mathcal{S}}(\mathrm{UE}_{\mathrm{cp}})$.

We will work with the symmetric category $\mathcal{S} = \mathrm{Complexes(Vect)}$ of complexes $(V^{\cdot}, d^{\cdot})$ of topologically free $k[[h]]$-modules, where actions of the symmetric groups are the same as those for super-vector spaces ($\underset{i\in\mathbb{Z}}{\oplus} V^i$ is decomposed as $\big(\underset{i\in 2\mathbb{Z}}{\oplus} V^i\big) \oplus \big(\underset{i\in 2\mathbb{Z}+1}{\oplus} V^i\big)$).

Let us describe precisely $\mathrm{Rep}_{\mathcal{S}}(\mathrm{QUE})$ and $\mathrm{Rep}_{\mathcal{S}}(\mathrm{LBA}_h)$ when $\mathcal{S} = \mathrm{Complexes(Vect)}$.

- $\mathrm{Rep}_{\mathcal{S}}(\mathrm{LBA}_h)$ is the category DGLBA_h of complexes $(V^{\cdot}, d^{\cdot})$, together with differential graded maps $\mu : \Lambda^2(V^{\cdot}) \to V^{\cdot}$ and $\delta : V^{\cdot} \to \Lambda^2(V^{\cdot})$, satisfying the Lie bialgebra axioms in the $\mathbb{Z}/2\mathbb{Z}$-graded sense;
- $\mathrm{Rep}_{\mathcal{S}}(\mathrm{QUE})$ is the category DGQUE, obtained as follows. Let DGLA be the category of differential graded Lie super-algebras. The category of their universal enveloping algebras is denoted DGUE; DGUE is a full subsategory of $\mathrm{DGBialg}_{\mathrm{coco}}$, which is the category of differential graded super-cocommutative super-bialgebras. (DGUE is the subclass of algebras A characterized by the Milnor-Moore condition $\underset{n\geq 0}{\cup} \mathrm{Ker}(\mathrm{Id} - \eta \circ \varepsilon)^{\otimes n} \circ \Delta^{(n)} = A$.) DGQUE is the category of formal deformations of DGUE in the category DGBialg of differential graded super-bialgebras.

Corollary 8.4 says that $DQ_{\Phi,\mathcal{S}}$ induces an isomorphism of categories between DGQUE and DGLBA$_h$, which is Theorem 8.1.

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