Helly numbers of acyclic families

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Abstract

The *Helly number* of a family of sets with empty intersection is the size of its largest inclusion-wise minimal sub-family with empty intersection. Let $\mathcal{F}$ be a finite family of open subsets of an arbitrary locally arc-wise connected topological space Γ. Assume that for every sub-family $\mathcal{G} \subseteq \mathcal{F}$ the intersection of the elements of $\mathcal{G}$ has at most r connected components, each of which is a $\mathbb{Q}$-homology cell. We show that the Helly number of $\mathcal{F}$ is at most $r(d_\Gamma + 1)$, where d_Γ is the smallest integer j such that every open set of Γ has trivial $\mathbb{Q}$-homology in dimension j and higher. (In particular $d_{\mathbb{R}^d} = d$.) This bound is best possible. We prove, in fact, a stronger theorem where small sub-families may have more than r connected components, each possibly with nontrivial homology in low dimension. As an application, we obtain several explicit bounds on Helly numbers in geometric transversal theory for which only ad hoc geometric proofs were previously known; in certain cases, the bound we obtain is better than what was previously known.

1 Introduction

Helly's theorem [32] asserts that if, in a finite family of convex sets in $\mathbb{R}^d$, any $d+1$ sets have non-empty intersection, then the whole family has non-empty intersection. Equivalently, any finite family of convex sets in $\mathbb{R}^d$ with empty intersection must contain a subfamily of at most $d+1$ sets whose intersection is already empty. This invites to define the ***Helly number*** of a family of sets with empty intersection as the size of its largest sub-family $\mathcal{F}$ such that (i) the intersection of all elements of $\mathcal{F}$ is empty, and (ii) for any proper sub-family $\mathcal{G} \subsetneq \mathcal{F}$, the intersection of the elements of $\mathcal{G}$ is non-empty. Helly's theorem then simply states that any finite family of convex sets in $\mathbb{R}^d$ has Helly number at most $d+1$. (When considering the Helly number of a family of sets, we always implicitly assume that the family has empty intersection.)

Helly himself gave a topological extension of that theorem [33] (see also Debrunner [15]), asserting that any finite good cover in $\mathbb{R}^d$ has Helly number at most $d+1$. (For our purposes, a ***good cover*** is a finite family of open sets where the intersection of any sub-family is empty or contractible.) In this paper, we prove topological *Helly-type theorems* for families of non-connected sets, that is, we give upper bounds on Helly numbers for such families.

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1.1 Our results

Let Γ be a locally arc-wise connected topological space. We let d_Γ denote the smallest integer such that every open subset of Γ has trivial $\mathbb{Q}$-homology in dimension d_Γ and higher; in particular, when Γ is a d-dimensional manifold, we have $d_\Gamma = d$ if Γ is non-compact or non-orientable and $d_\Gamma = d+1$ otherwise (see Lemma 23); for example, $d_{\mathbb{R}^d} = d$. We call a family $\mathcal{F}$ of open subsets of Γ ***acyclic*** if for any non-empty sub-family $\mathcal{G} \subseteq \mathcal{F}$, each connected component of the intersection of the elements of $\mathcal{G}$ is a $\mathbb{Q}$-homology cell. (Recall that, in particular, any contractible set is a homology cell.)[1] We prove the following Helly-type theorem:

Theorem 1. *Let $\mathcal{F}$ be a finite acyclic family of open subsets of a locally arc-wise connected topological space Γ. If any sub-family of $\mathcal{F}$ intersects in at most r connected components, then the Helly number of $\mathcal{F}$ is at most $r(d_\Gamma + 1)$.*

We show, in fact, that the conclusion of Theorem 1 holds even if the intersection of small sub-families has more than r connected components and has non-vanishing homology in low dimension. To state the result precisely, we need the following definition that is a weakened version of acyclicity:

Definition 2. A finite family $\mathcal{F}$ of subsets of a locally arc-wise connected topological space is ***acyclic with slack s*** if for every non-empty sub-family $\mathcal{G} \subseteq \mathcal{F}$ and every $i \geq \max(1, s - |\mathcal{G}|)$ we have $\tilde{H}_i(\bigcap_\mathcal{G}, \mathbb{Q}) = 0$.

Note that, in particular, for any $s \leq 2$, *acyclic with slack s* is the same as *acyclic*. With a view toward applications in geometric transversal theory, we actually prove the following strengthening of Theorem 1:

Theorem 3. *Let $\mathcal{F}$ be a finite family of open subsets of a locally arc-wise connected topological space Γ. If (i) $\mathcal{F}$ is acyclic with slack s and (ii) any sub-family of $\mathcal{F}$ of cardinality at least t intersects in at most r connected components, then the Helly number of $\mathcal{F}$ is at most $r(\max(d_\Gamma, s, t) + 1)$.*

In both Theorems 1 and 3 the openness condition can be replaced by a compactness condition (Corollary 22) under an additional mild assumption. As an application of Theorem 3 we obtain bounds on several ***transversal Helly numbers***: given a family $A_1, \ldots, A_n$ of convex sets in $\mathbb{R}^d$ and letting T_i denote the set of non-oriented lines intersecting A_i, we obtain bounds on the Helly number h of $\{T_1, \ldots, T_n\}$ under certain conditions on the geometry of the A_i. Specifically, we obtain that h is

(i) at most $2^{d-1}(2d-1)$ when the A_i are disjoint parallelotopes in $\mathbb{R}^d$,

(ii) at most 10 when the A_i are disjoint translates of a convex set in $\mathbb{R}^2$, and

(iii) at most $4d-2$ (resp. 12, 15, 20, 20) when the A_i are disjoint equal-radius balls in $\mathbb{R}^d$ with $d \geq 6$ (resp. $d = 2, 3, 4, 5$).

Although similar bounds were previously known, we note that each was obtained through an ad hoc, geometric argument. The set of lines intersecting a convex set in $\mathbb{R}^d$ has the homotopy type of $\mathbb{RP}^{d-1}$, and the family T_i is thus only acyclic with some slack; also, the bound $4d-2$ when

[1] To avoid confusion, we note that an *acyclic space* sometimes refers to a homology cell in the literature (see *e.g.*, Farb [18]). Here, the meaning is different: A family is acyclic if and only if the intersection of every non-empty sub-family has trivial $\mathbb{Q}$-homology in dimension larger than zero; but the intersection needs not be connected.

$d \geq 4$ in (iii) is a direct consequence of the relaxation on the condition regarding the number of connected components in the intersections of small families. Theorem 3 is the appropriate type of generalization of Theorem 1 to obtain these results; indeed, the parameters allow for some useful flexibility (cf. Table 1, page 25).

Organization. Our proof of Theorem 1 uses three ingredients. First, we define (in Section 3) the *multinerve* of a family of sets as a simplicial poset that records the intersection pattern of the family more precisely than the usual nerve. Then, we derive (in Section 4) from *Leray's acyclic cover theorem* a purely homological analogue of the Nerve theorem, identifying the homology of the multinerve to that of the union of the family. Finally, we generalize (in Section 5) a theorem of Kalai and Meshulam [38, Theorem 1.3] that relates the homology of a simplicial complex to that of some of its projections; we use this result to control the homology of the nerve in terms of that of the multinerve. Our result then follows from the standard fact that the Helly number of any family can be controlled by the homology (Leray number) of its nerve. Since in this approach low-dimensional homology is not relevant, the assumptions of Theorem 1 can be relaxed, yielding Theorem 3 (Section 6) which we can apply to geometric transversal theory (Section 7).

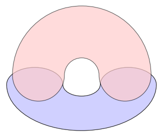
The rest of this introduction compares our results with previous works; Section 2 introduces the basic concepts and techniques that we build on to obtain our results.

1.2 Relation to previous work

Helly numbers and their variants received considerable attention from discrete geometers [14, 16] and are also of interest to computational geometers given their relation to algorithmic questions [2]. The first type of bounds for Helly numbers of families of non-connected sets starts from a "ground" family $\mathcal{H}$, whose Helly number is bounded, and considers families $\mathcal{F}$ such that the intersection of any sub-family $\mathcal{G} \subseteq \mathcal{F}$ is a disjoint union of at most r elements of $\mathcal{H}$. When $\mathcal{H}$ is closed under intersection and *non-additive* (that is, the union of finitely many disjoint elements of $\mathcal{H}$ is never an element of $\mathcal{H}$), the Helly number of $\mathcal{F}$ can be bounded by r times the Helly number of $\mathcal{H}$. This was conjectured (and proven for $r = 2$) by Grünbaum and Motzkin [28] and a proof of the general case was recently published by Eckhoff and Nischke [17], building on ideas of Morris [49]. Direct proofs were also given by Amenta [3] in the case where $\mathcal{H}$ is a finite family of compact convex sets in $\mathbb{R}^d$ and by Kalai and Meshulam [38] in the case where $\mathcal{H}$ is a good cover in $\mathbb{R}^d$ [38].

Matoušek [44] and Alon and Kalai [1] showed, independently, that if $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$ such that the intersection of any sub-family is the union of at most r (possibly intersecting) convex sets, then the Helly number of $\mathcal{F}$ can be bounded from above by some function of r and d. Matoušek also gave a topological analogue [44, Theorem 2] which is perhaps the closest predecessor of Theorem 3: he bounds from above (again, by a function of r and d) the Helly number of families of sets in $\mathbb{R}^d$ assuming that the intersection of any sub-family has at most r connected components, each of which is $(\lceil d/2 \rceil - 1)$-connected, that is, has its ith homotopy group vanishing for $i \leq \lceil d/2 \rceil - 1$.

Our Theorem 1 includes both Amenta's and Kalai-Meshulam's theorems as particular cases but is more general: the figure below shows a family for which Theorem 1 (as well as the topological theorem of Matoušek) applies with $r = 2$, but where the Kalai-Meshulam theorem does not (as the family of connected components is not a good cover). Our result and the Eckhoff-Morris-Nischke



theorem do not seem to imply one another, but to be distinct generalizations of the Kalai-Meshulam theorem. Theorem 3 differs from Matoušek's topological theorem on two accounts. First, his proof uses a Ramsey theorem and only gives a loose bound on the Helly number, whereas our approach gives sharp, explicit, bounds. Second, his theorem is based on the non-embeddability of certain low-dimensional simplicial complexes and therefore allows the connected components to have nontrivial *homotopy* in *high* dimension, whereas Theorem 3 lets them have nontrivial *homology* in *low* dimension.

Very recently, Montejano [48] found a generalization of Helly's topological theorem: if, for each j, $1 \leq j \leq d_\Gamma$, the $(d_\Gamma - j)$th reduced homology group of the intersection of each subfamily of size j vanishes, then the family has non-empty intersection. In particular, he makes no assumption on the intersection of families with more than d_Γ elements but requires that the intersection of each subfamily of size d_Γ must be connected; thus, neither our nor his result implies the other.

The concept of acyclicity with slack appeared previously in the thesis of Hell [31, 30] in a homological condition bounding the *fractional Helly number*. His spectral sequence arguments exploiting this concept are similar to the ones in the proof of our multinerve theorem.

The study of Helly numbers of sets of lines (or more generally, k-flats) intersecting a collection of subsets of $\mathbb{R}^d$ developed into a sub-area of discrete geometry known as geometric transversal theory [61]. The bounds (i)–(iii) implied by Theorem 3 were already known in some form. Specifically, the case (i) of parallelotopes is a theorem of Santaló [52], the case (ii) of disjoint translates of a convex figure was proven by Tverberg [58] with the sharp constant of 5 and the case (iii) of disjoint equal-radius balls was proven with the weaker constant $4d - 1$ (for $d \geq 6$) by Cheong et al. [11]. Each of these theorems was, however, proven through ad hoc arguments and it is interesting that Theorem 3 traces them back to the same principles: controlling the homology and number of the connected components of the intersections of all sub-families.

2 Preliminaries and overview of the techniques

For any finite set X, we denote by $|X|$ its cardinality and by 2^X the family of all subsets of X (including the empty set and X itself). We abbreviate $\bigcap_{t \in A} t$ in $\bigcap_A$ and $\bigcup_{t \in A} t$ in $\bigcup_A$.

Simplicial complex and Nerve. A ***simplicial complex*** X over a (finite) set of ***vertices*** V is a non-empty family of subsets of V closed under taking subsets; in particular, $\emptyset$ belongs to every simplicial complex. An element σ of X is a ***simplex***; its ***dimension*** is the cardinality of σ minus one; a d-simplex is a simplex of dimension d. For a more thorough discussions of simplicial complexes, we refer, e.g., to the book of Matoušek [43, Chapter 1].

The ***nerve*** of a (finite) family $\mathcal{F}$ of sets is the simplicial complex

$$\mathcal{N}(\mathcal{F}) = \left\{ \mathcal{G} \subseteq \mathcal{F} \;\middle|\; \bigcap\nolimits_{\mathcal{G}} \neq \emptyset \right\}$$

with vertex set $\mathcal{F}$. It is a standard fact that the *homology* of a simplicial complex can be defined in several equivalent ways (for example, using simplicial homology or the singular homology of its geometric realization). The ***Nerve theorem*** of Borsuk [5, 7] asserts that if $\mathcal{F}$ is a good cover, then its nerve adequately captures the topology of the union of the members of $\mathcal{F}$; namely, $\mathcal{N}(\mathcal{F})$ has the same homology groups (in fact, the same homotopy type) as $\bigcup_{\mathcal{F}}$.

Bounding Helly numbers using Leray numbers. That the Helly number of a good cover in $\mathbb{R}^d$ is at most $d+1$ can be easily derived from the Nerve theorem. Indeed, let $\mathcal{F}$ be any family of sets with Helly number h; let $\mathcal{G} \subseteq \mathcal{F}$ be an inclusion-wise minimal subfamily with empty intersection with cardinality h. The nerve of $\mathcal{G}$ is $2^{\mathcal{G}} \setminus \{\mathcal{G}\}$, which is the boundary of a $(h-1)$-simplex and therefore has nontrivial homology in dimension $h-2$. On the other hand, assuming that $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, the nerve theorem implies that the good cover $\mathcal{G}$ has the same homology as $\bigcup_{\mathcal{G}}$, which is an open subset of $\mathbb{R}^d$ and therefore has trivial homology in dimension d or larger. This implies that $h-2 < d$, and the bound on the Helly number of $\mathcal{F}$ follows.

The ***Leray number*** $L(X)$ of a simplicial complex X with vertex set V is defined as the smallest integer j such that for any $S \subseteq V$ and any $i \geq j$ the reduced homology group $\tilde{H}_i(X[S], \mathbb{Q})$ is trivial. (Recall that $\boldsymbol{X[S]}$ is the sub-complex of X ***induced*** by S, that is, the set of simplices of X whose vertices are in S.) Using this notion, the first part of the above argument can be rephrased as follows:

Lemma 4. *The Helly number of an arbitrary collection of sets exceeds the Leray number of its nerve by at most one.*

The technique of Kalai and Meshulam. Our proof of Theorem 1 extends the key ingredient of the proof by Kalai and Meshulam [38] of the following result:

Theorem 5 (Kalai and Meshulam [38])**.** *Let $\mathcal{H}$ be a good cover in $\mathbb{R}^d$ and $\mathcal{F}$ be a family such that the intersection of every sub-family of $\mathcal{F}$ has at most r connected components, each of which is a member of $\mathcal{H}$; then the Helly number of $\mathcal{F}$ is at most $r(d+1)$.*

Their proof can be summarized as follows. Let $\widetilde{\mathcal{F}}$ denote the family of connected components of elements of $\mathcal{F}$ (strictly speaking, this is a multiset, as an element of $\widetilde{\mathcal{F}}$ may be a connected component of several elements in $\mathcal{F}$; but we can safely ignore this technicality). Now, consider the projection $\widetilde{\mathcal{F}} \to \mathcal{F}$ that maps each element of $\widetilde{\mathcal{F}}$ to the element of $\mathcal{F}$ having it as a connected component. This projection extends to a map $\mathcal{N}(\widetilde{\mathcal{F}}) \to \mathcal{N}(\mathcal{F})$ that is onto, at most r-to-one, and preserves the dimension (that is, maps a k-simplex to a k-simplex). This turns out to imply that $L(\mathcal{N}(\mathcal{F}))$ is at most $rL(\mathcal{N}(\widetilde{\mathcal{F}})) + r - 1$ (Theorem 1.3 of [38], a statement we refer to as the "projection theorem"). Since every element of $\widetilde{\mathcal{F}}$ belongs to $\mathcal{H}$, the multiset $\widetilde{\mathcal{F}}$ is also a good cover in $\mathbb{R}^d$; the Nerve theorem implies that $L(\mathcal{N}(\widetilde{\mathcal{F}}))$ is at most d, and an upper bound of $r(d+1)$ on the Helly number of $\mathcal{F}$ follows.

Čech complexes, Leray's theorem, and multinerves. The assumption that $\mathcal{F}$ is acyclic is strictly weaker than that of Theorem 5. In particular, the family $\widetilde{\mathcal{F}}$ of connected components of members of $\mathcal{F}$ need not be a good cover, and we can no longer invoke the Nerve theorem to bound $L(\mathcal{N}(\widetilde{\mathcal{F}}))$. When a family is not a good cover but merely acyclic, the homology of the union of $\mathcal{F}$ may not be captured by the nerve but is nevertheless related to the homology of the *Čech complex* of the cosheaf given by the connected components of the various intersections, a more complicated algebraic structure. This relation is given by Leray's acyclic cover theorem[2], a central result in (co)sheaf (co)homology, which allows generalizations of the Mayer-Vietoris exact sequence.

[2]see [9, Theorem III.4.13], [21, Section II.5], [39, Proposition 2.8.5] or [8, Theorem 8.9] for the cohomology version and [9, Sections VI.4 and VI.13] for the homology version

We introduce a variant of the nerve where each sub-family of $\mathcal{F}$ defines a number of simplices equal to the number of connected components in its intersection; we call this "nerve with multiplicity" the *multinerve* and encode it as a simplicial poset. For the families that we consider, this multinerve can be interpreted as a Čech complex (of a constant sheaf), and therefore Leray's acyclic cover theorem and its proof apply, yielding a "homology multinerve theorem" (Theorem 8). We then generalize the projection theorem of Kalai and Meshulam to maps from a simplicial poset onto a simplicial complex (Theorem 15).

3 Simplicial posets and multinerves

In this section, we describe how various properties of simplicial complexes can be generalized to simplicial posets; for more thorough discussions of these objects, we refer to the book of Matoušek [43, Chapter 1] for simplicial complexes and to the papers by Björner [4] or Stanley [57] for simplicial posets. We then introduce the *multinerve*, a simplicial poset that generalizes the notion of nerve.

Simplicial posets. A partially ordered set, or ***poset*** for short, is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order on X. We denote by $\boldsymbol{[\alpha,\beta]}$ the ***segment*** defined by α and β in X, that is $[\alpha,\beta] = \{\tau \in X \mid \alpha \preceq \tau \preceq \beta\}$ (similarly, $\boldsymbol{[\alpha,\beta)}$, $\boldsymbol{(\alpha,\beta]}$, and $\boldsymbol{(\alpha,\beta)}$ denote the segments where one or both extreme elements are omitted, and $\boldsymbol{(\alpha,\cdot]}$ denotes the set of simplices $\tau \neq \alpha$ such that $\alpha \preceq \tau$). A ***simplicial poset***[3] is a poset $(X, \preceq)$ that (i) admits a *least element* 0, that is $0 \preceq \sigma$ for any $\sigma \in X$, and such that (ii) for any $\sigma \in X$, there is some integer d such that the lower segment $[0,\sigma]$ is isomorphic to the poset of faces of a d-simplex, that is, $2^{\{0,\ldots,d\}}$ partially ordered by the inclusion; d is the ***dimension*** of σ.

The elements of a simplicial poset X are called its ***simplices***. We call ***vertices*** the simplices of dimension 0 and we say that τ is *contained in* (or a ***face*** of) σ if $\tau \preceq \sigma$. For any fixed simplex σ with set of vertices V_σ, the map associating to any $\tau \in [0,\sigma]$ the set of vertices it contains is a bijection from $[0,\sigma]$ onto 2^{V_σ}. From now on we will omit the partial order and simply say that "X is a simplicial poset" when there is no need from the context to state explicitly what partial order is considered.

It turns out that simplicial posets lie in-between simplicial complexes and the more general notions of Δ-sets and simplicial sets as used in algebraic topology. Specifically:

- *Simplicial complexes are simplicial posets.* The simplices of a simplicial complex, ordered by inclusion, form a simplicial poset (with $\emptyset$ as least element). Henceforth, by abuse of language, we consider that a simplicial complex *is* a simplicial poset; moreover, any definition we state for simplicial posets is also valid for simplicial complexes. However, in contrast to simplicial complexes, a simplicial poset may have several simplices with the same vertex set (for example, two edges connecting the same vertices in a graph with multiple edges).

- *Simplicial posets are Δ-sets and simplicial sets.* As we shall discuss in detail in Section 4.1, the definition of the face operators for simplicial complexes readily extends to simplicial posets. This makes simplicial posets a particular case of Δ-sets (see for instance [60, Example 8.1.8],

[3] Let us emphasize that we are using the terminology from combinatorics and that a simplicial poset is *not* a simplicial object in the category of posets.

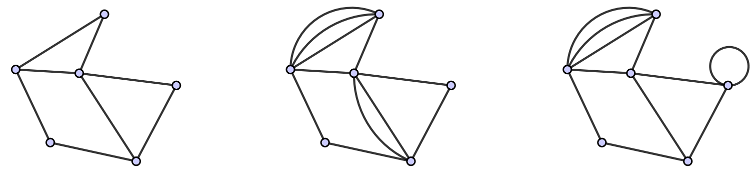


Figure 1: Left: A simplicial complex. Middle: A simplicial poset that is not a simplicial complex. Right: A Δ-set that is neither a simplicial complex nor a simplicial poset.

[19, Section 2.3], or [29, Section 2.1])[4], which are themselves a special case of simplicial sets.[5] However, in contrast to Δ-sets, each d-simplex of a simplicial poset necessarily has $d+1$ distinct vertices.

For instance (see Figure 1), the one-dimensional simplicial complexes are precisely the graphs without loops or multiple edges; the one-dimensional simplicial posets are precisely the graphs without loops (but possibly with multiple edges); and any graph, possibly with loops and multiple edges, is a one-dimensional Δ-set or simplicial set.

Later on, we shall define some concepts for simplicial posets, like their geometric realization or their homology, that are standard for simplicial complexes, Δ-complexes, and simplicial sets. Depending on his or her taste, the reader may view each of these concepts for simplicial posets as an easy extension of the corresponding concept for simplicial complexes, or as a special case of the corresponding concept for Δ-complexes and simplicial sets.

Multinerve. The primary simplicial posets that we will consider are *multinerves*, defined as follows. The ***multinerve*** $\mathcal{M}(\mathcal{F})$ of a finite family $\mathcal{F}$ of subsets of a topological space is the set

$$\mathcal{M}(\mathcal{F}) = \left\{(C, A) \;\middle|\; A \subseteq \mathcal{F} \text{ and } C \text{ is a connected component of } \bigcap\nolimits_A\right\}.$$

By convention, in the case where $A = \emptyset$ is the empty family, we declare the pair $(\bigcap_\emptyset, \emptyset)$ to be equal to $(\bigcup_\mathcal{F}, \emptyset)$ (even though $\bigcup_\mathcal{F}$ may not be connected). Thus $(\bigcap_\emptyset, \emptyset)$ belongs to $\mathcal{M}(\mathcal{F})$ and is the only element in $\mathcal{M}(\mathcal{F})$ for which the second coordinate is the empty set $\emptyset$. We turn $\mathcal{M}(\mathcal{F})$ into a poset by equipping it with the partial order

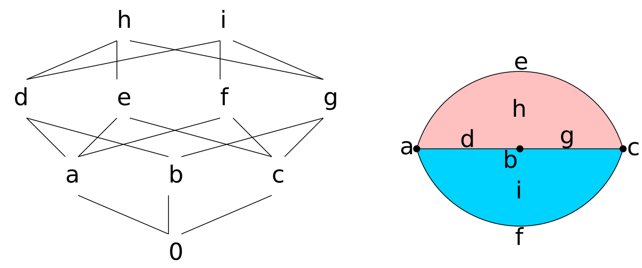
$$(C', A') \preceq (C, A) \iff C' \supseteq C \text{ and } A' \subseteq A.$$

Intuitively, $\mathcal{M}(\mathcal{F})$ is an "expanded" version of $\mathcal{N}(\mathcal{F})$: while $\mathcal{N}(\mathcal{F})$ has one simplex for each non-empty intersecting sub-family, $\mathcal{M}(\mathcal{F})$ has one simplex for each *connected component* of an intersecting sub-family.[6]

[4] The Δ-sets are also called *semi-simplicial sets* in the modern literature, not to be confused with *semi-simplicial complexes* which denoted, in the 1960's, what is nowadays called a simplicial set.

[5] A simplicial poset X equipped with face operators can be turned into a simplicial set $\overline{X}$ by adding all degeneracies of the simplices of X, see for instance [45, Example 1.4] or [19, Example 3.3]. This operation is the left adjoint (induced by left Kan extension) to the forgetful functor from simplicial sets to Δ-sets (obtained by disregarding degeneracy maps); details can be found in [51, Section 1] and [60, Definition 8.1.9]. Thus, any simplicial poset X it is canonically isomorphic to the set of non-degenerate simplices (*i.e.* the core) of its associated simplicial set $\overline{X}$ ([51, Proposition 1.5]).

[6] To get an intuition, it does not harm to assume that, whenever A and A' are different subsets of $\mathcal{F}$, the connected components of $\bigcap_A$ and of $\bigcap_{A'}$ are different. Under this assumption, $\mathcal{M}(\mathcal{F})$ can be identified with the set of all connected components of the intersections of all sub-families of $\mathcal{F}$, equipped with the opposite of the inclusion order.



(a) (b)

Figure 2: (a) A simplicial poset X, represented by its partial order. (b) The geometric realization of X.

More precisely, the image of $\mathcal{M}(\mathcal{F})$ through the projection on the second coordinate $\pi : (C, A) \mapsto A$ is the nerve $\mathcal{N}(\mathcal{F})$; for any $A \in \mathcal{N}(\mathcal{F})$, the cardinality of $\pi^{-1}(A)$ is precisely the number of connected components of $\bigcap_A$. In particular, if the intersection of every subfamily of $\mathcal{F}$ is empty or connected, then $\mathcal{M}(\mathcal{F})$ is (isomorphic to the poset of faces of) $\mathcal{N}(\mathcal{F})$.

Lemma 6. *$\mathcal{M}(\mathcal{F})$ is a simplicial poset. Moreover, the dimension of a simplex (C, A) of $\mathcal{M}(\mathcal{F})$ equals $|A| - 1$.*

Proof. The projection on the second coordinate identifies any lower segment $[(\bigcap_\emptyset, \emptyset), (C, A)]$ with the simplex 2^A. Indeed, let $A' \subseteq A$ and let $C' \subseteq \bigcup_\mathcal{F}$. The lower segment $[(\bigcap_\emptyset, \emptyset), (C, A)]$ contains (C', A') if and only if C' is the connected component of $\bigcap_{A'}$ containing C. Moreover, by definition, $\mathcal{M}(\mathcal{F})$ contains a least element, namely $(\bigcap_\emptyset, \emptyset)$. The statement follows. □

Geometric realization of a simplicial poset. To every simplicial poset X, we associate a topological space $|X|$, its ***geometric realization***, where each d-simplex of X corresponds to a geometric d-simplex (by definition, a geometric (-1)-simplex is empty); see Figure 2 for an example of geometric realization of a simplicial poset, represented by its partial order, and Figure 3 for an example of geometric realization of a multinerve. This notion of geometric realization of a simplicial poset extends that of a simplicial complex, and is also a special case of the geometric realization defined for arbitrary Δ-sets and simplicial sets (see [51, 19], or [45, Chapter III]).[7] However, we can describe a direct construction of the geometric realization of the simplicial poset X as follows. We build up the geometric realization of X by increasing dimension. First, create a single point for every vertex (simplex of dimension 0) of X. Then, assuming all the simplices of dimension up to $d - 1$ have been realized, consider a d-simplex σ of X. The open lower interval $[0, \sigma)$ is isomorphic to the boundary of the d-simplex by definition; we simply glue a geometric d-simplex to the geometric realization of that boundary.

4 Homological multinerve theorem

In this section, we prove a generalization of the Nerve theorem stating essentially that the multinerve of an acyclic family, possibly with slack, adequately captures the topology of the union of the family.

[7] In particular, the geometric realization $|X|$ of a simplicial poset X is homeomorphic to the geometric realization of the simplicial set associated to X; the proof of that claim is exactly the same as in Milnor's original paper [47] (see also [19, Example 4.4] and [51, Proposition 2.1]).

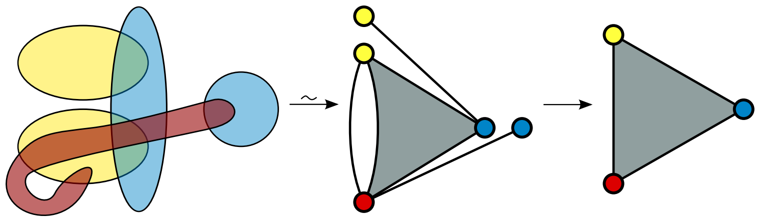


Figure 3: Left: A family $\mathcal{F}$ of subsets of $\mathbb{R}^2$. Middle: The geometric realization of its multinerve $\mathcal{M}(\mathcal{F})$. Right: The geometric realization of its nerve $\mathcal{N}(\mathcal{F})$.

Before we state our result, we briefly recall the definition of the homology groups of a simplicial poset.

4.1 Homology of simplicial posets

The ***homology*** of a simplicial poset can be defined in three different ways: as a direct extension of simplicial homology for simplicial complexes, as a special case of simplicial homology of simplicial sets [45, Section I.2], [22, Section III.2], [60, Definition 8.2], or via the singular homology of its geometric realization; all three definitions are equivalent in that they lead to canonically isomorphic homology groups. We will use both the singular homology viewpoint and the simplicial viewpoint, where the homology is defined via chain complexes. For the reader's convenience, we now quickly recall the definition of the latter. We emphasize that, in this paper, we only consider homology over $\mathbb{Q}$.

Let X be a simplicial poset and assume chosen an ordering on the set of vertices of X. If σ is an n-dimensional simplex, the lower segment $[0, \sigma]$ is isomorphic to the poset of faces of a standard n-simplex $2^{\{0,\dots,n\}}$; here we choose the isomorphism so that it preserves the ordering on the vertices. Thus, we get $n+1$ faces $d_i(\sigma) \in X$ (for $i = 0, \dots, n$), each of dimension $n-1$: namely, $d_i(\sigma)$ is the (unique) face of σ whose vertex set is mapped to $\{0, \dots, n\} \setminus \{i\}$ by the above isomorphism.

For $n \geq 0$, let $C_n(X)$ be the $\mathbb{Q}$-vector space with basis the set of simplices of X of dimension exactly n; furthermore, let $C_{-1}(X) = \{0\}$. Extending the maps d_i by linearity, we get the ***face operators*** $d_i : C_n(X) \to C_{n-1}(X)$. Let $d = \sum_{i=0}^{n} (-1)^i d_i$ be the linear map $C_n(X) \to C_{n-1}(X)$ (which is defined for any $n \geq 0$). The fact that $d \circ d = 0$ is easy and follows from the same argument as for simplicial complexes since it is computed inside the vector space generated by $[0, \sigma]$ which is isomorphic to a standard simplex. The (simplicial) nth homology group $\boldsymbol{H_n(C_\bullet(X), d)}$ is defined as the quotient vector space of the kernel of $d : C_n(X) \to C_{n-1}(X)$ by the image of $d : C_{n+1}(X) \to C_n(X)$.

If, instead of taking $C_{-1}(X) = \{0\}$, we take $C_{-1}(X) = \mathbb{Q}$, and d_0 denotes the linear map that maps each vertex of X to 1, then we obtain the ***reduced*** homology groups [29, Section 2.1].[8] In the sequel, we denote by $\boldsymbol{H_n(O)}$ the ith $\mathbb{Q}$-homology group of O (whether O is a simplicial poset, its associated geometric realization, or a topological space), and by $\boldsymbol{\tilde{H}_n(O)}$ the corresponding reduced homology group.

Remark 7. The equivalence between the simplicial and singular homology viewpoints is standard;

[8] We use the convention that the reduced homology of the empty set is trivial except in dimension -1, where it is $\mathbb{Q}$. In particular, the definition of the Leray number of a simplicial complex, given in Section 2, makes implicitly use of this convention.

see, e.g., [47] or [45, Section 16]. The fact that this direct extension of simplicial homology from simplicial complexes to simplicial posets coincides with the singular homology of the geometric realization of a simplicial poset can be observed as follows. By construction, the chain complex $(C_\bullet(X), d)$ is isomorphic to the *normalized* chain complex of the simplicial set $\overline{X}$ associated to X(see [45, Section 22], [22, Section III.2], and [60, Section 8.3]) which is the quotient vector space of $\mathbb{Q}(\overline{X})$ by the subspace spanned by the degenerate simplices. It is a standard fact that the normalized chain complex has the same homology as the simplicial set (see [45, Theorem 22.1], [60, Theorem 8.3.8]). Thus, the chain complex $(C_\bullet(X), d)$ does compute the homology of $\overline{X}$ and likewise of the geometric realization of X.

4.2 Statement of the multinerve theorem

Our generalization of the Nerve theorem takes the following form:

Theorem 8 (Homological Multinerve Theorem)**.** *Let $\mathcal{F}$ be a family of open sets in a locally arc-wise connected topological space Γ. If $\mathcal{F}$ is* acyclic with slack s *then $\tilde{H}_\ell(\mathcal{M}(\mathcal{F})) \cong \tilde{H}_\ell(\bigcup_\mathcal{F})$ for $\ell = 0$ and any non-negative integer $\ell \geq s$.*

The special case $s = 0$ corresponds to Theorem 1 and is already a generalization of the usual nerve theorem. Actually, since, by definition, acyclic with slack $s = 2$ is the same as acyclic for any $s < 2$, any family that is acyclic with slack $s = 2$ satisfies $\tilde{H}_\ell(\mathcal{M}(\mathcal{F})) \cong \tilde{H}_\ell(\bigcup_\mathcal{F})$ *for all* $\ell \geq 0$. We will need the general case (arbitrary slack) for our applications in geometric transversal theory, where we have to consider families for which intersections of few elements may have non-zero homology in low dimension. The particular case of Theorem 8 where, in addition, the intersection of every subfamily of $\mathcal{F}$ is assumed to be empty or connected (and thus $\mathcal{M}(\mathcal{F}) = \mathcal{N}(\mathcal{F})$), was proved by Hell in his thesis [31, 30] using similar techniques.

The gist of the proof of Theorem 8 is that the chain complex of a multinerve can be interpreted as a Čech complex (Section 4.3) and thus captures the homology of the union by (a special instance of) Leray's acyclic cover theorem. More precisely we use the latter to prove the generalized Mayer-Vietoris argument, which states that the homology of the union can be computed by the data of the singular chain complexes of all the intersections of the family. This is realized by a Čech bicomplex in Section 4.4. The slack conditions ensure, via a standard spectral sequence argument, that the homology of the two Čech (bi)complexes are the same in degree 0 and in degrees s and larger.

The remaining part of the present Section 4 is organized as follows. We prove Theorem 8 in Sections 4.3 and 4.4. In Section 4.5, for completeness, we also give an analogue in *homotopy* of the case $s \leq 2$ (no slack) of Theorem 8. These developments are independent of the subsequent sections, so the reader unfamiliar with algebraic topology and willing to admit Theorem 8 can safely proceed to Section 5.

Remark 9. In the statement of Theorem 8, the assumption that Γ be locally arc-wise connected merely ensures that the connected components and the arc-wise connected components of any open subset of Γ agree. It can be dispensed of by replacing the ordinary homology by the Čech homology (see [9, Section VI.4]). In particular, when the space is not locally arc-wise connected, Lemma 10 below still applies if $H_0(\bigcap_A)$ is replaced by $\check{H}_0(\bigcap_A)$, the $\mathbb{Q}$-vector space generated by the connected components of $\bigcap_A$.

4.3 The chain complex of the multinerve

To compute the homology of a multinerve, we first reformulate its associated chain complex (as given in Section 4.1) in topological terms:

Lemma 10. *The chain complex* $(C_{n\geq 0}(\mathcal{M}(\mathcal{F})), d)$ *is the chain complex satisfying*

$$C_n(\mathcal{M}(\mathcal{F})) = \bigoplus_{\substack{A\subseteq\mathcal{F}\\ |A|=n+1}} H_0(\bigcap_A)$$

whose differential is the linear map $d : C_n(\mathcal{M}(\mathcal{F})) \to C_{n-1}(\mathcal{M}(\mathcal{F}))$ *given by* $d = \sum_{i=0}^{n}(-1)^i d_{A,i}$*, where* $d_{A,i}$ *is the linear map* $d_{A,i} : H_0(\bigcap_A) \to H_0(\bigcap_{A\setminus X_i})$ *induced by the inclusion.*

Proof. By definition, $C_n(\mathcal{M}(\mathcal{F}))$, the n-dimensional part of the chain complex of the multinerve, is the vector space over $\mathbb{Q}$ spanned by the set $\{(C, A) \in \mathcal{M}(\mathcal{F}), |A| = n+1\}$, where C is a connected component of $\bigcap_A$, or equivalently an arc-wise connected component, since Γ is arc-wise locally connected. On the other hand, $H_0(\bigcap_A)$ is canonically isomorphic to the vector space with basis the set of these arc-wise connected components. This implies the first formula. Furthermore, the differential maps (up to sign) a connected component C of $\bigcap_A$ to the connected component C' of $\bigcap_{A'}$ that contains C for any $A' \subset A$ with $|A'| = |A| - 1$. ☐

Given a (locally arc-wise connected) topological space X, the rule that assigns to an open subset $U \subseteq X$ the set $\pi_0(U)$ of its (arc-wise) connected components is a cosheaf on X. Taking $X = \bigcup_{\mathcal{F}}$, and assuming that the elements of $\mathcal{F}$ are open sets in X, the family $\mathcal{F}$ is an open cover of X. It follows from Lemma 10 that the chain complex of $\mathcal{M}(\mathcal{F})$ is isomorphic to the Čech complex $\check{C}(\mathcal{F}, \pi_0)$ of the cosheaf $U \mapsto \pi_0(U)$.

4.4 Proof of the homological multinerve theorem

We write $\left(S_\bullet(X), d^S\right)$ for the singular chain complex of a topological space X that computes its homology. We also write $C_\bullet(\mathcal{M}(\mathcal{F}))$ for the simplicial chain complex computing the simplicial homology of the multinerve $\mathcal{M}(\mathcal{F})$.

For any open subsets $U \subseteq V$ of a (locally arc-wise connected) space X, there is a natural chain complex map $S_\bullet(U) \to S_\bullet(V)$, and thus the rule $U \mapsto S_\bullet(U)$ is a precosheaf on X, but not a cosheaf in general. There is a standard way to replace this precosheaf by a cosheaf. Indeed, following [9, Section VI.12], there is a chain complex of cosheaves $U \mapsto \mathfrak{S}_\bullet(U)$ (where U is an open subset in X) that comes with canonical isomorphisms $H_n(U) \cong H_n(\mathfrak{S}_\bullet(U))$. We write $d^{\mathfrak{S}} : \mathfrak{S}_\bullet(-) \to \mathfrak{S}_{\bullet-1}(-)$ for the differential on $\mathfrak{S}_\bullet(-)$.

We now recall the notion of the *Čech complex* of a (pre)cosheaf associated to a cover, which is just the dual of the more classical notion of Čech complex of a (pre)sheaf; we refer to the classical references [9, Section VI.4], [21, Section II.5.8], [8, Section 11], [39, Remark 2.8.6] for more details on presheaf and precosheaf (co)homology. Let X be a topological space and $\mathcal{U}$ be a cover of X (by open subsets). Also let $\mathfrak{A}$ be a precosheaf of abelian groups on X, that is, the data of an abelian group $\mathfrak{A}(U)$ for every open subset $U \subseteq X$ with corestriction (linear) maps $\rho_{U\subseteq V} : \mathfrak{A}(U) \to \mathfrak{A}(V)$ for any inclusion $U \hookrightarrow V$ of open subsets of X satisfying the coherence rule $\rho_{V\subseteq W} \circ \rho_{U\subseteq V} = \rho_{U\subseteq W}$ for any open sets $U \subseteq V \subseteq W$.

The degree n part of the ***Čech complex*** $\boldsymbol{\check{C}_n(\mathcal{U},\mathfrak{A})}$ of the cover $\mathcal{U}$ with value in $\mathfrak{A}$ is, by definition, $\check{C}_n(\mathcal{U},\mathfrak{A}) := \bigoplus \mathfrak{A}\big(\bigcap_I\big)$ where the sum is over all subsets $I \subseteq \mathcal{U}$ such that $|I| = n+1$ and the intersection $\bigcap_I$ is non-empty. In other words, the sum is over all simplices of dimension n of the nerve of the cover $\mathcal{U}$. The differential d is the sum $d = \sum_{i=0}^{n}(-1)^i d_{I,i}$ where $d_{I,i} : \mathfrak{A}\big(\bigcap_I\big) \to \mathfrak{A}\big(\bigcap_{I\setminus i}\big)$ is defined as in Lemma 10, with $\mathfrak{A}$ instead of H_0.

Specializing to the case $X = \bigcup_{\mathcal{F}}$, we have a canonical cover of $\bigcup_{\mathcal{F}}$ given by the family $\mathcal{F}$. Thus we can now form the Čech complex $\check{C}(\mathcal{F},\mathfrak{S}_\bullet(-))$ of the cosheaf of complexes $U \mapsto \mathfrak{S}_\bullet(U)$. Explicitly, $\check{C}(\mathcal{F},\mathfrak{S}_\bullet(-))$ is the bicomplex $\check{C}_{p,q}(\mathcal{F},\mathfrak{S}_\bullet(-)) = \bigoplus_{|\mathcal{G}|=p+1}\mathfrak{S}_q(\bigcap_{\mathcal{G}})$ with (vertical) differential $d_v : \bigoplus_{|\mathcal{G}|=p+1}\mathfrak{S}_q(\bigcap_{\mathcal{G}}) \to \bigoplus_{|\mathcal{G}|=p+1}\mathfrak{S}_{q-1}(\bigcap_{\mathcal{G}})$ given by $(-1)^p d^{\mathfrak{S}}$ on each factor and with (horizontal) differential given by the usual Čech differential, that is, $d_h : \bigoplus_{|\mathcal{G}|=p+1}\mathfrak{S}_q(\bigcap_{\mathcal{G}}) \to \bigoplus_{|\mathcal{G}|=p}\mathfrak{S}_q(\bigcap_{\mathcal{G}})$ is the alternate sum $d_v = \sum_{i=0}^{|\mathcal{G}|}(-1)^i d_{\mathcal{G},i}$ with the same notations as in Lemma 10.

It is folklore that the homology of the (total complex associated to the) bicomplex is the (singular) homology $H_\bullet(\bigcup_{\mathcal{F}})$ of the space $\bigcup_{\mathcal{F}}$, (see [8, Proposition 15.18] and [8, Proposition 15.8] for its cohomological analogue). More precisely,

Lemma 11. *(Generalized Mayer-Vietoris principle for singular homology) There are natural isomorphisms*

$$H_n^{tot}(\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-))) \cong H_n\Big(\bigcup_{\mathcal{F}}\Big)$$

where $H_n^{tot}(\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-)))$ is the homology of the (total complex associated to the) Čech bicomplex $\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-)))$.

Lemma 11 is essentially the generalization of the Mayer-Vietoris exact sequence to many open sets and boils down, for the case of two open sets, to the usual Mayer-Vietoris long exact sequence[9]. The proof of Lemma 11, given here for completeness, is a direct adaptation of the one given in [8, Section 15] and follows the proof of Leray's acyclic cover theorem [39, Proposition 2.8.5].

Proof. Since $\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-)))$ is a bicomplex, by a standard argument (for instance see [60, Section 5.6] or [8, Section 13, § 3]), the filtration by the columns of $\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-))$ yields a spectral sequence $F^1_{p,q} \Rightarrow H^{tot}_{p+q}(\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-)))$. Since the horizontal differential is the Čech differential, the first page $F^1_{p,q} = \check{H}_p\big(\mathcal{F},\mathfrak{S}_q(-)\big)$ is isomorphic to the Čech homology of the cosheaves $\mathfrak{S}_q(-)$ associated to the cover (of $X = \bigcup_{\mathcal{F}}$) given by the family $\mathcal{F}$. By Proposition VI.12.1 and Corollary VI.4.5 in [9], these homology groups vanish for $p > 0$, that is $F^1_{p,q} = 0$ if $q > 0$ and $F^1_{p,0} \cong \mathfrak{S}_p(\bigcup_{\mathcal{F}})$. The result now follows from an easy application of Leray's acyclic cover [39, Proposition 2.8.5] which boils down to the following argument. Recall that the differential d^1 on the first page $F^1_{\bullet,\bullet}$ is given by the vertical differential $d_v = \pm d^{\mathfrak{S}}$. Since, by definition, $H_n(\mathfrak{S}_\bullet(\bigcup_{\mathcal{F}}), d^{\mathfrak{S}}) \cong H_n(\bigcup_{\mathcal{F}})$, it follows that $F^2_{p,q} = 0$ if $q > 0$ and $F^2_{p,0} \cong H_p(\bigcup_{\mathcal{F}})$. Now, for degree reasons, all higher differentials $d^r : F^r_{\bullet,\bullet} \to F^r_{\bullet,\bullet}$ are zero. Thus $F^\infty_{p,q} \cong F^2_{p,q}$ and it follows that $H^{tot}_n(\check{C}_{\bullet,\bullet}(\mathcal{F},\mathfrak{S}_\bullet(-))) \cong F^2_{n,0} \cong H_n(\bigcup_{\mathcal{F}})$. □

By Lemma 11, there is a converging spectral sequence[10] (associated to the filtration by the rows of $\check{C}(\mathcal{F},\mathfrak{S}_\bullet(-)))$ $E^1_{p,q} \Rightarrow H_{p+q}(\bigcup_{\mathcal{F}})$ such that $E^1_{p,q} = \bigoplus_{|\mathcal{G}|=p+1} H_q(\bigcap_{\mathcal{G}})$ and the differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is (induced by) the horizontal differential d_h. By Lemma 10, there is an

[9] See [8, Section 8.1] for a proof of this fact with de Rham chains instead of $\mathfrak{S}_\bullet(-)$.

[10] The reader may refer to either one of [60, Chapter 5], [8, Chapter 13], [46] or [56, Section 9.1] for details on spectral sequences.

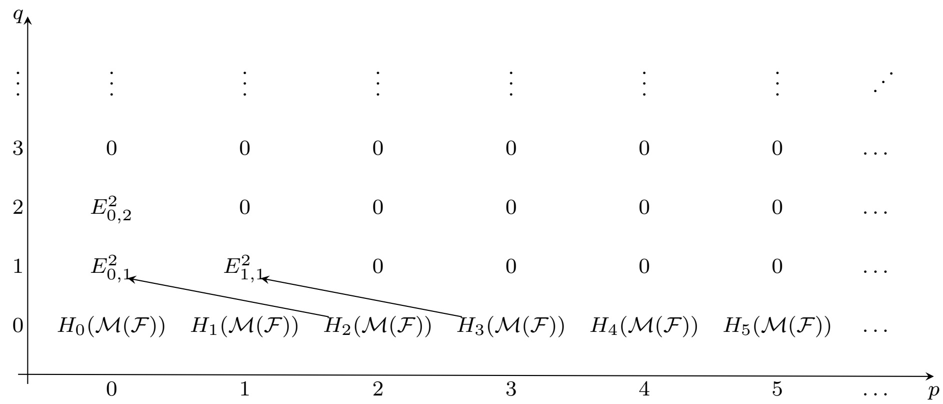


Figure 4: E^2-page of the Čech complex spectral sequence when $\mathcal{F}$ is acyclic with slack $s = 4$. The arrows show the only differential d^2 which can be non-zero.

isomorphism $(E^1_{\bullet,0}, d^1) \cong (C_\bullet(\mathcal{M}(\mathcal{F})), d)$ of chain complexes and thus the bottom line of the page E^2 of the spectral sequence $E^2_{p,0} \cong H_p(\mathcal{M}(\mathcal{F}))$ is the homology of the multinerve of $\mathcal{F}$. The proof of Theorem 8 now follows from a simple analysis of the pages of this spectral sequence.

Proof of Theorem 8. Recall that s is the slack of the family $\mathcal{F}$. By assumption, for any $q \geq \max(1, s-p-1)$ and any sub-family $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| = p+1$, we have $H_q(\bigcap_{\mathcal{G}}) = 0$ and thus $E^1_{p,q} = 0$ for $q \geq \max(1, s-p-1)$. Since, for $r \geq 1$, the differential d^r maps $E^r_{p,q}$ to $E^r_{p-r,q-1+r}$, by induction, we get that the restriction of d^r to $E^r_{p,q}$ is null if both $q \geq 1$ and $p+q \geq s-1$. Further $E^2_{p,0} \cong H_p(\mathcal{M}(\mathcal{F}))$ and, again for degree reasons, it follows that, for $r \geq 2$, $d^r : E^r_{p,0} \to E^r_{p-r,r-1}$ is null if $p \geq s$. See Figure 4 for an example of the E^2-page of the spectral sequence in the case of slack $s = 4$.

Since $E^{r+1}_{\bullet,\bullet}$ is isomorphic to the homology $H_\bullet(E^r_{\bullet,\bullet}, d^r)$, it follows from the above analysis of the differentials d^r that, for $p+q \geq s$ and $q \geq 1$, one has $E^2_{p,q} \cong 0$ and further that $E^2_{p,q} \cong E^3_{p,q} \cong \cdots \cong E^\infty_{p,q}$ for $p+q \geq s$. Now, we use that the spectral sequence converges to $H_\ell(\bigcup_{\mathcal{F}})$. Hence, for any $\ell \geq s$, we find

$$H_\ell\Big(\bigcup\nolimits_{\mathcal{F}}\Big) \cong \bigoplus_{p+q=\ell} E^\infty_{p,q} \cong \bigoplus_{p+q=\ell} E^2_{p,q} \cong E^2_{\ell,0} \cong H_\ell(\mathcal{M}(\mathcal{F})).$$

It remains to identify the degree 0 homology. Note that, for $r \geq 2$, d^r necessarily vanishes on $E^r_{0,0}$ for degree reasons and further, since $-1+r \geq 1$, that $E^r_{\bullet,0} \cap d^r\big(E^r_{p,q}\big) = \{0\}$. Thus, we also have $E^2_{0,0} \cong E^3_{0,0} \cong \cdots \cong E^\infty_{0,0}$ and it follows, as for the case $\ell \geq s$, that $H_0\Big(\bigcup_{\mathcal{F}}\Big) \cong E^2_{0,0} \cong H_0(\mathcal{M}(\mathcal{F}))$. □

4.5 Side note: a homotopic multinerve theorem

It is natural to wonder if Theorem 8 has a counterpart in homotopy. (Like for homology, the homotopy of a simplicial poset can be defined for instance as a special case of the homotopy of simplicial sets, that is, as the homotopy type of its geometric realization.) For completeness, we give the following analogue of the case $s \leq 2$ (no slack):

Theorem 12 (Homotopy Multinerve Theorem)**.** *Let $\mathcal{F}$ be a finite family of sets in a topological space Γ. Assume that each element in the family is a triangulable space such that all finite intersections are sub-triangulations. If the intersection of every subfamily of $\mathcal{F}$ is the disjoint union of finitely many contractible sets, then $\mathcal{M}(\mathcal{F})$ and $\bigcup_{\mathcal{F}}$ are homotopy equivalent.*

The idea of the proof of Theorem 12 is folklore; see, for instance, the proof of [29, Corollary 4G3].

Proof. Let X denote the subset of $\bigcup_{\mathcal{F}} \times |\mathcal{M}(\mathcal{F})|$ defined as

$$X = \bigcup_{(C,A)\in\mathcal{M}(\mathcal{F})} C \times |(C,A)|,$$

where $|(C,A)|$ is the geometric realization of the simplex $(C,A) \in \mathcal{M}(\mathcal{F})$. (This construction is sometimes called the *Mayer-Vietoris blowup complex.*)

Let π_1 denote the projection on the first coordinate, so that $\pi_1(X) = \bigcup_{\mathcal{F}}$. Let $p \in \bigcup_{\mathcal{F}}$. A point $q \in |\mathcal{M}(\mathcal{F})|$ satisfies $(p,q) \in X$ if and only if $q \in |(C,A)|$ for some C containing p; it follows that

$$\pi_1^{-1}(p) \quad = \quad \{p\} \times \bigcup_{\substack{(C,A)\in\mathcal{M}(\mathcal{F})\\ p\in C}} |(C,A)| \quad = \quad \{p\} \times |\{(C,A) \in \mathcal{M}(\mathcal{F}) \text{ s.t. } p \in C\}|;$$

in particular, $\pi_1^{-1}(p)$ is the geometric realization of a simplicial poset isomorphic to a simplex, and every fiber of π_1 is thus contractible. Note that $|\mathcal{M}(\mathcal{F})|$ is the geometric realization of a simplicial set and, by assumption, any element of $\mathcal{F}$ is triangulable, hence the geometric realization of a simplicial set. Since $\bigcup_{\mathcal{F}}$ and X are obtained by gluing together geometric realizations of simplicial sets along geometric realizations of sub-simplicial sets, they are themselves geometric realizations of simplicial sets. Furthermore, the cells of X are products of cells, so the projection π_1 is the geometric realization of a map of simplicial sets. Then X and $\bigcup_{\mathcal{F}}$ are homotopy equivalent by the Vietoris-Begle Theorem (case (3) of Lemma 26).

Similarly, let π_2 denote the projection on the second coordinate, so that $\pi_2(X) = |\mathcal{M}(\mathcal{F})|$. Let $q \in |\mathcal{M}(\mathcal{F})|$ and let (C,A) be the unique simplex of minimum dimension of $\mathcal{M}(\mathcal{F})$ whose geometric realization contains q. Then a point $p \in \bigcup_{\mathcal{F}}$ satisfies $(p,q) \in X$ if and only if $p \in C$, so $\pi_2^{-1}(q) = C \times \{q\}$ is contractible. The cells of $|\mathcal{M}(\mathcal{F})|$ are precisely the sets $|(C,A)|$, hence π_2 is the geometric realization of a map of simplicial sets. Again, X and $|\mathcal{M}(\mathcal{F})|$ are homotopy equivalent by the Vietoris-Begle Theorem (case (3) of Lemma 26). This concludes the proof. □

5 Projection of a simplicial poset

The key ingredient of the proof by Kalai and Meshulam [38] of Theorem 5 is an analysis of the Leray number of the image of a simplicial complex under a simplicial map. More precisely, they show that if projecting a simplicial complex may increase the homology, as measured by the Leray number (see Figure 5 for an example), that accession can be controlled under certain conditions.

In this section, we prove a similar statement for simplicial posets. After introducing some notions of combinatorial topology for simplicial posets (Section 5.1), we state precisely our projection theorem (Section 5.2) and prove it (Sections 5.3–5.6).

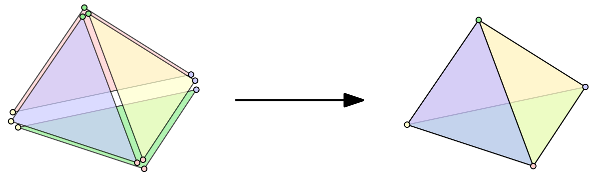


Figure 5: Projecting a simplicial complex can create homology.

5.1 Links, barycentric subdivisions, and simplicial maps

Links. A standard notion in combinatorial topology is that of the *link* of a simplex σ in a simplicial complex X:

$$\mathrm{lk}_X(\sigma) = \{\tau \in X \mid \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in X\}\,.$$

A nice topological feature of the link of σ is that it has the same homotopy type as a neighborhood of σ minus σ itself in the geometric realization of X. This property is instrumental in a technical lemma [37, Proposition 3.1] used in Kalai and Meshulam's proof.

This notion can be extended to simplicial posets: the link of σ in a simplicial poset X would be the set of simplices τ disjoint from σ and such that σ and of τ are all contained in at least one simplex of X. However, it is not hard to prove that the above topological property does not always hold for simplicial posets. For example, consider the link of a vertex of the simplicial poset made of two vertices and two edges connecting them.

Barycentric subdivisions. Instead, we will work on the barycentric subdivision of X. Recall that to any (not necessarily simplicial) poset $(P, \preceq)$ is associated a simplicial complex $\Delta(P)$ called the ***order complex*** of P: the vertices of $\Delta(P)$ are the elements of P, and its d-simplices are the totally ordered subsets of P of size $d+1$ (also called its ***chains***). The ***barycentric subdivision*** $\boldsymbol{\mathrm{sd}(X)}$ of a simplicial poset X with least element 0 is defined to be $\Delta(X \setminus \{0\})$, the order complex of $X \setminus \{0\}$. The vertices of $\mathrm{sd}(X)$ are the non-empty simplices of X and every chain of d faces of distinct dimension contained in one another form a $(d-1)$-simplex of $\mathrm{sd}(X)$. This generalizes the barycentric subdivision for simplicial complexes.

Let us remark that as for simplicial complexes, a geometric realization of $\mathrm{sd}(X)$ can be obtained from a subdivision of the geometric realization of X, as follows (see Figure 6(b)). The barycentric subdivision of a 0-simplicial poset (which is also a simplicial complex) is itself. Let $d \geq 1$; assume that the $(d-1)$-skeleton of X (the simplicial sub-poset of X obtained by keeping only its simplices of dimension at most $d-1$) has already been subdivided. We now explain how to subdivide a d-simplex σ of X. Let v be a new vertex in the interior of the geometric realization of σ. The $(d-1)$-simplices on the boundary of σ have already been subdivided; let B_σ be the set of these subdivided $(d-1)$-simplices. For every $(d-1)$-simplex τ in B_σ, we insert in $\mathrm{sd}(X)$ the d-simplex whose vertices are v and those of τ. Together, these simplices form a subdivision of σ. By induction, every d-simplex of X is subdivided into $(d+1)!$ d-simplices. In particular, the geometric realization of a simplicial poset X is homeomorphic to the geometric realization of the simplicial complex $\mathrm{sd}(X)$.

Given $\sigma \in X$, we denote by $\boldsymbol{D_X(\sigma)}$ the sub-complex of $\mathrm{sd}(X)$ that is the order complex of $[\sigma, \cdot]$; similarly we denote by $\boldsymbol{\dot{D}_X(\sigma)}$ the sub-complex of $\mathrm{sd}(X)$ that is the order complex of $(\sigma, \cdot]$ (see Figure 6(c)); in particular, $\dot{D}_X(0) = \mathrm{sd}(X)$. We will use the fact that $D_X(\sigma)$ (as a sub-complex

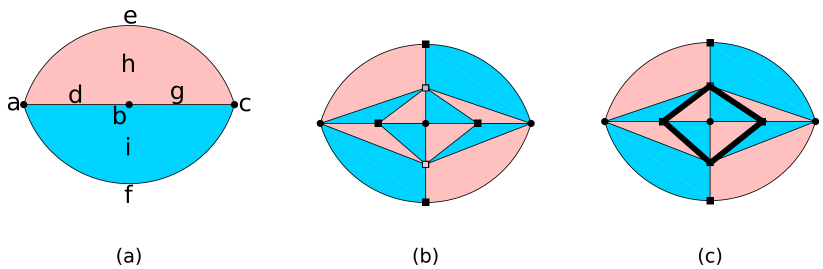


Figure 6: (a) The geometric realization of a simplicial poset X (follow-up of Figure 2). (b) The geometric realization of $\mathrm{sd}(X)$, which also equals $\dot{D}_X(0)$. (c) $\dot{D}_X(b)$ is a 1-dimensional simplicial complex that is a cycle of length four (in black bold lines).

of $\mathrm{sd}(X)$) is a cone (actually, its geometric realization retracts to the geometric realization of the simplex σ) and is therefore contractible. Kalai and Meshulam [38] use that when X is a simplicial complex, $\dot{D}_X(\sigma)$ is isomorphic to the barycentric subdivision of the link of σ in X. This property is, again, false for simplicial posets; in our proof, we find a way to avoid all uses of the notion of link.

Simplicial maps. Let $\varphi : X \to Y$ be a map between two simplicial posets X and Y. We say that φ is ***simplicial*** if, for every simplex σ of X, $\varphi([0, \sigma])$ is *exactly* $[0, \varphi(\sigma)]$. The notion of simplicial maps between simplicial posets extends the notion of simplicial maps for simplicial complexes. In particular, any simplicial map between two posets induces a simplicial map between their barycentric subdivisions, and (therefore) also a continuous map between their geometric realizations. By abuse of language, we speak of a simplicial map from a simplicial poset X to a simplicial complex Y to mean a simplicial map from X to Y *seen as* a simplicial poset.

We say that a simplicial map φ between two simplicial posets is ***dimension-preserving*** if, for any $\sigma \in X$, the dimension of $\varphi(\sigma)$ equals the dimension of σ. This implies that φ maps bijectively $[0, \sigma]$ onto $[0, \varphi(\sigma)]$. All the simplicial maps considered in this paper will be dimension-preserving. Finally, we also say that φ is ***at most r-to-one*** if for any $\sigma \in Y$ the set $\varphi^{-1}(\sigma)$ has cardinality at most r.

5.2 Statement of the projection theorem

If X is a simplicial poset with vertex set V and $S \subseteq V$, the ***induced simplicial sub-poset $X[S]$*** is the poset of elements of X whose vertices are in S, ordered by the order of X. The ***Leray number*** of the simplicial poset X is the smallest integer j such that for any $S \subseteq V$ and any $i \geq j$ the reduced homology group $\tilde{H}_i(X[S], \mathbb{Q})$ is trivial. Like the Nerve theorem bounds the Leray number of the nerve of an open good cover, our Multinerve Theorem bounds the Leray number of the multinerve of an acylic family:

Corollary 13. *If $\mathcal{F}$ is a finite acyclic family of open sets in a locally arc-wise connected topological space Γ, then the Leray number of $\mathcal{M}(\mathcal{F})$ is at most d_Γ.*

Proof. Let $\mathcal{G}$ be a sub-family of $\mathcal{F}$. Since $\mathcal{M}(\mathcal{F})[\mathcal{G}] = \mathcal{M}(\mathcal{G})$ and $\mathcal{G}$ is also acyclic, Theorem 8 yields that $\tilde{H}_\ell(\mathcal{M}(\mathcal{F})[\mathcal{G}]) = \tilde{H}_\ell(\mathcal{M}(\mathcal{G})) \cong \tilde{H}_\ell(\bigcup_\mathcal{G})$ for any $\ell \geq 0$. If in addition we assume $\ell \geq d_\Gamma$, then $\tilde{H}_\ell(\bigcup_\mathcal{G}) = 0$ since $\bigcup_\mathcal{G}$ is an open set in Γ. The statement follows. $\square$

However, Lemma 4 relates the Helly number of $\mathcal{F}$ to the Leray number of its nerve, not of its multinerve. The main result of this section bounds the Leray number of the nerve in terms of a refinement of the Leray number of the multinerve. Specifically, let X be a simplicial poset with vertex set V; we define $J(X)$ to be the smallest integer ℓ such that for every $j \geq \ell$, every $S \subseteq V$, and every simplex σ of $X[S]$, we have $\tilde{H}_j(\dot{D}_{X[S]}(\sigma)) = 0$. If X is a simplicial complex then $L(X) = J(X)$: this follows from [37, Proposition 3.1] and from the isomorphism between $\dot{D}_{X[S]}(\sigma)$ and the barycentric subdivision of the link of σ in $X[S]$. We cannot decide if the same holds for simplicial posets but will, in fact, only need the following easy inequality.

Lemma 14. *If X is a simplicial poset, then $L(X) \leq J(X)$.*

Proof. Let $S \subseteq V$ and let 0 be the least element of X. By definition, $\dot{D}_{X[S]}(0)$ is the barycentric subdivision of $X[S]$. Thus, by definition of $J(X)$, for every $j \geq J(X)$, we have $\tilde{H}_j(X[S]) = 0$. Thus $L(X) \leq J(X)$. □

The purpose of this section is to prove the following projection theorem.

Theorem 15. *Let $r \geq 1$. Let π be a simplicial, dimension-preserving, surjective, at most r-to-one map from a simplicial poset X onto a simplicial complex Y. Then $L(Y) \leq rJ(X) + r - 1$.*

The special case of Theorem 15 when X is a simplicial complex was proven by Kalai and Meshulam [38, Theorem 1.3] in a slightly different terminology. We note that already in this context the bound on $L(Y)$ is tight (see the remark after Theorem 1.3 of [38]). Since Y is a simplicial complex, $L(Y) = J(Y)$ and the conclusion of the theorem can be rewritten as $J(Y) \leq rJ(X)+r-1$; however, we will not use this result.

In the remaining part of this section, we prove Theorem 15. Specifically, we describe how the proof of Kalai and Meshulam [38, Theorem 1.3], once it is reformulated in our terminology, extends, mutantis mutandis, to the case of simplicial posets. The reader not interested in the proof of Theorem 15 can safely proceed to Section 6, where Theorem 15 will be applied to the case where X is the multinerve of our acyclic family and Y is its nerve.

5.3 Structure of the proof

The proof of the projection theorem of Kalai and Meshulam [38, Theorem 1.3] uses properties of the *multiple k-point space* of $\pi : X \to Y$ (defined below) in two independent steps, each using a different spectral sequence[11]. The first step relates the homology of Y to that of the multiple k-point space. The second, more combinatorial step, aims at controlling the topology of the multiple k-point space in terms of the topology of X.

For the proof of their projection theorem, Kalai and Meshulam assume that X is a subset of the join of disjoint 0-complexes $V_1 * \ldots * V_m$, where π maps each vertex of V_i to the ith vertex of Y. Instead, we assume that $\pi : X \to Y$ is dimension-preserving. This assumption is equivalent in the context of simplicial complexes (as can be seen by taking $V_i = \pi^{-1}(i)$ for each vertex i) and remains meaningful for simplicial posets.

[11] The reader may refer to either one of [60, Chapter 5], [8, Chapter 13], [46] or [56, Section 9.1] for details on spectral sequences.

5.4 The image computing spectral sequence

The first spectral sequence considered [38, Theorem 2.1] is due to Goryunov-Mond [25] and uses only topological properties of the geometric realization and the fact that we are considering homology with coefficient in the field $\mathbb{Q}$ of rational numbers. It thus extends verbatim to the setting of simplicial posets.

Specifically, for $k \geq 1$, the ***multiple k-point space*** M_k of X is

$$M_k = \left\{(x_1, \dots, x_k) \in |X|^k \text{ s.t. } \pi(x_1) = \cdots = \pi(x_k)\right\}.$$

Note that there is a natural action of S_k, the symmetric group on k letters, on M_k by permutation, and thus on the homology $H_\bullet(M_k)$ as well. We denote

$$\mathrm{Alt}\, H_n(M_k) = \{v \in H_n(M_k),\ \sigma \cdot v = \mathrm{sgn}(\sigma)v \text{ for all } \sigma \in S_k\}.$$

Recall that $\pi : X \to Y$ satisfies the assumption of Theorem 15. Hence the (geometric realization of the) simplicial map π has finite fibers with the sets $\pi^{-1}(y)$ (for any $y \in Y$) being of cardinality at most r, we have the following result, which is the same as Theorem 2.1 in [38].

Theorem 16 (Goryunov-Mond)**.** *There is a homology spectral sequence $E^r_{p,q}$ converging to $H_\bullet(Y)$ such that*

$$E^1_{p,q} = \begin{cases} \mathrm{Alt}\, H_q(M_{p+1}) & \text{for } 0 \leq p \leq r-1,\ 0 \leq q \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, intuitively, to show that Y has trivial homology in dimension large enough, it suffices to show that it is the case for the multiple point set.

5.5 Homology of multiple point sets

We now argue that $H_q(M_{p+1}) = 0$ for q large enough. Let $X_1, \dots, X_k$ be induced simplicial subposets of X. Define

$$M(X_1, \dots, X_k) = \{(x_1, \dots, x_k) \in |X_1| \times \cdots \times |X_k|, \quad \pi(x_1) = \cdots = \pi(x_k)\}.$$

Note that $M(X_1, \dots, X_k) = M_k$. We are actually mainly interested in the case $X_1 = \cdots = X_k = X$ but it is more convenient to have different indices for bookkeeping issues in the proof. In our setting, the analogue of Proposition 3.1 in [38] is the following.

Lemma 17. *$\tilde{H}_j(M(X_1, \dots, X_k)) = 0$ for $j \geq \sum_{i=1}^k J(X_i)$.*

Proof. $M(X_1, \sigma_2, \dots, \sigma_k)$ is homeomorphic to

$$\{x_1 \in |X_1|, \quad \forall i = 2, \dots, k,\ \exists x_i \in |\sigma_i|,\ \pi(x_i) = \pi(x_1)\},$$

since the assumption that π is dimension-preserving guarantees that the $x_2, \dots, x_k$ are uniquely determined. Given $\sigma \in X$, we define $\tilde{\sigma}$ as the set of vertices of X in $\pi^{-1}(\pi(\sigma))$. We thus have the following identification:

$$M(X_1, \sigma_2, \dots, \sigma_k) \cong \left| X_1 \left[\bigcap_{i=2}^k \tilde{\sigma}_i \right] \right|,$$

which extends [38, Equation (3.1)]. Let $n = \sum_{j=2}^{k} \dim(X_j)$; define the sets

$$\mathcal{S}'_p = \left\{ (\sigma_2, \ldots, \sigma_k) \in X_2 \times \cdots \times X_k, \sum_{j=1}^{k} \dim(\sigma_j) \geq n - p \right\}$$

and $\mathcal{S}_p = \mathcal{S}'_p - \mathcal{S}'_{p-1}$ for $0 \leq p \leq n$. Furthermore, for $(\sigma_2, \ldots, \sigma_k) \in \mathcal{S}'_p$, define

$$A_{(\sigma_2,\ldots,\sigma_k)} = M(X_1, \sigma_2, \ldots, \sigma_k) \times D_{X_2}(\sigma_2) \times \ldots \times D_{X_k}(\sigma_k).$$

Now, consider the spaces

$$K_p = \bigcup_{(\sigma_2,\ldots,\sigma_k) \in \mathcal{S}'_p} A_{(\sigma_2,\ldots,\sigma_k)} \quad \subseteq M(X_1, \ldots, X_k) \times \mathrm{sd}(X_2) \times \cdots \times \mathrm{sd}(X_k).$$

Since the $D_{X_i}(\sigma_i)$ are contractible, it follows that the projection on the first coordinate $K_n \to M(X_1, \ldots, X_k)$ is a homotopy equivalence and the homology spectral sequence associated to the filtration $\emptyset \subset K_0 \subset \cdots \subset K_n$ converges to $H_\bullet(M(X_1, \ldots, X_k))$ and is analogous to the one given in [38, Proposition 3.2]. The first page of this spectral sequence writes $E^0_{p,q} = H_{p+q}(K_p, K_{p-1})$. The arguments used in [38, Proposition 3.2] for the identification of the second page, that is the $E^1_{p,q}$-terms, are based on properties of the homology of pairs such as excision and Künneth formula. Since the barycentric subdivision of a simplicial poset is itself a simplicial complex, these arguments extend to our setting and we get that

$$E^1_{p,q} \cong \bigoplus_{\substack{(\sigma_2,\ldots,\sigma_k) \\ \in \mathcal{S}_p}} \bigoplus_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k = p+q}} H_{i_1}\left(X_1 \left[\bigcap_{i=2}^{k} \tilde{\sigma}_i \right] \right) \otimes \bigotimes_{j=2}^{k} H_{i_j}\left(D_{X_j}(\sigma_j), \dot{D}_{X_j}(\sigma_j) \right).$$

In the simplicial complex setting, Kalai and Meshulam then use the isomorphism between $\dot{D}_{X_j}(\sigma_j)$ and the barycentric subdivision of the link of σ_j in X_j together with a characterization of Leray numbers in terms of reduced homology of all links in the simplicial complex [37, Proposition 3.1]. The introduction of $J(X)$ in our setting will circumvent the fact that the notion of link does *not* extend well to simplicial posets. Since $D_{X_j}(\sigma_j)$ is contractible, we still have $H_{i_j}(D_{X_j}(\sigma_j), \dot{D}_{X_j}(\sigma_j)) \cong \tilde{H}_{i_j-1}(\dot{D}_{X_j}(\sigma_j))$. This yields the identification

$$E^1_{p,q} \cong \bigoplus_{\substack{(\sigma_2,\ldots,\sigma_k) \\ \in \mathcal{S}_p}} \bigoplus_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k = p+q}} H_{i_1}\left(X_1 \left[\bigcap_{i=2}^{k} \tilde{\sigma}_i \right] \right) \otimes \bigotimes_{j=2}^{k} \tilde{H}_{i_j-1}\left(\dot{D}_{X_j}(\sigma_j) \right). \tag{1}$$

We now have all the ingredients to finish the proof of the lemma. First note that for a simplicial complex $L(Z) = 0$ implies that Z is a simplex; this is still true if Z is a simplicial poset. Let $m = \sum_{j=1}^{k} J(X_j)$. If $m = 0$, then, by Lemma 14, $M(X_1, \ldots, X_k)$ is isomorphic to a simplex and has no reduced homology in all non-negative dimensions. We can thus assume $m > 0$. Since we have a homology spectral sequence $E^1_{p,q}$ converging to $H_\bullet(M(X_1, \ldots, X_k))$, it suffices to prove that $E^1_{p,q} = 0$ when $p + q = i_1 + \cdots + i_k \geq m$. If $i_1 \geq J(X_1)$, we have $i_1 \geq L(X_1)$ by Lemma 14 and therefore $H_{i_1}\left(X_1\left[\bigcap_{i=2}^k \tilde{\sigma}_i\right]\right)=0$. Furthermore, if $i_j-1\geq J(X_j)$, then by definition we have $\tilde{H}_{i_j-1}(\dot{D}_{X_j}(\sigma))=0$. Thus, if $p+q\geq m=\sum_{j=1}^k J(X_j)$, at least one of the tensors in

$$H_{i_1}\left(X_1\left[\bigcap_{i=2}^k \tilde{\sigma}_i\right]\right)\otimes\bigotimes_{j=2}^k \tilde{H}_{i_j-1}\left(\dot{D}_{X_j}(\sigma_j)\right)$$

is null and it follows that $E^1_{p,q}=0$. This concludes the proof. □

5.6 End of the proof of Theorem 15

Lemma 18. *$\tilde{H}_\ell(Y)=0$ if $\ell\geq rJ(X)+r-1$.*

Proof. If $J(X)=0$, we are left to the case where X is a simplex and there is nothing to prove. Thus we may assume $J(X)>0$. By Theorem 16, it suffices to prove that $E^1_{p,q}\cong \mathrm{Alt}\,H_q(M_{p+1})=0$ if $p+q\geq rJ(X)+r-1$ with $p\leq r-1$ and $q\geq 0$. Since $M_{p+1}\cong M(X_1,\ldots,X_{p+1})$ for $X_1=\cdots=X_{p+1}=X$, by Lemma 17, we have that $H_q(M_{p+1})=0$ for $q\geq(p+1)J(X)$. Now the conditions $p+q\geq rJ(X)+r-1$ and $p\leq r-1$ imply $q\geq rJ(X)\geq(p+1)J(X)$ and thus that $H_q(M_{p+1})=0$. There is nothing left to prove. □

We conclude:

Proof of Theorem 15. Let S be a subset of vertices of Y and let $R=\pi^{-1}(S)$. We apply Lemma 18 with $X[R]$ and $Y[S]$, which also satisfy the hypotheses of the theorem. We obtain $\tilde{H}_\ell(Y[S])=0$ if $\ell\geq rJ(X[R])+r-1$. By definition of J, we have $J(X[R])\leq J(X)$; so we have $L(Y)\leq rJ(X)+r-1$, as desired. □

6 Topological Helly-type theorems for acyclic families

We now put everything together to prove our main results, Theorems 1 and 3, and conclude this section by showing that the openness condition can be replaced, in a slightly less general context, by a compactness condition.

6.1 Proof of Theorem 1

Our first step towards a proof of Theorem 1 is to bound from above the J index of the multinerve of an acyclic family. For future reference, we actually allow the family to have some slack.

Lemma 19. *Let Γ be a locally arc-wise connected topological space. If $\mathcal{F}$ is a finite family of open subsets of Γ that is acyclic with slack s, then $J(\mathcal{M}(\mathcal{F}))\leq\max(d_\Gamma,s)$.*

Proof. Let $\mathcal{G}\subseteq\mathcal{F}$ be a sub-family of $\mathcal{F}$, and let σ be a simplex of $\mathcal{M}(\mathcal{F})[\mathcal{G}]=\mathcal{M}(\mathcal{G})$. We need to prove that $\dot{D}_{\mathcal{M}(\mathcal{G})}(\sigma)$ has trivial reduced homology in dimension $\max(d_\Gamma,s)$ and higher.

Given $\sigma=(C,A)\in\mathcal{M}(\mathcal{G})$, we define $\mathcal{G}_\sigma$ as the non-empty traces of the elements of $\mathcal{G}\setminus A$ on C:

$$\mathcal{G}_\sigma=\{U\cap C\mid U\in\mathcal{G}\setminus A, U\cap C\neq\emptyset\}.$$

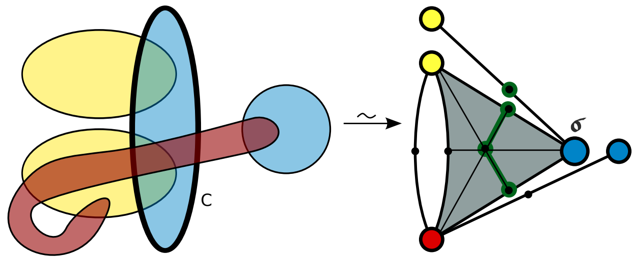


Figure 7: Continuation of Figure 3: On the left, the family $\mathcal{F}$; on the right, the barycentric subdivision $\mathrm{sd}(\mathcal{M}(\mathcal{F}))$ of the multinerve $\mathcal{M}(\mathcal{F})$. In this example, σ is a vertex of $\mathcal{M}(\mathcal{F})$ corresponding to one component C of an object in $\mathcal{F}$. We see that $\dot{D}_{\mathcal{M}(\mathcal{F})}(\sigma)$ (in bold) is a subcomplex of $\mathrm{sd}(\mathcal{M}(\mathcal{F}))$ that is the disjoint union of two homology cells. This is reflected in the fact that G_σ, the trace of the union of the other objects of $\mathcal{F}$ inside C, is also the disjoint union of two homology cells.

(Note that $\mathcal{G}_\sigma$ is a multiset, as a given element may appear more than once.) The map

$$\left\{ \begin{array}{ccc} \mathcal{M}(\mathcal{G}_\sigma) & \rightarrow & [\sigma, \cdot] \\ (C', A') & \mapsto & (C' \cap C, A \cup A') \end{array} \right.$$

is an isomorphism of posets. In particular, $[\sigma, \cdot]$ is a simplicial poset. Both posets have a least element, and removing them yields that $(\sigma, \cdot]$ and $\mathcal{M}(\mathcal{G}_\sigma) \setminus \{(\bigcap_\emptyset, \emptyset)\}$ are isomorphic posets. Taking their order complexes, we get that $\dot{D}_{\mathcal{M}(\mathcal{G})}(\sigma)$ and $\mathrm{sd}(\mathcal{M}(\mathcal{G}_\sigma))$ are isomorphic simplicial complexes; see Figure 7.

Therefore, $\dot{D}_{\mathcal{M}(\mathcal{G})}(\sigma)$ has the same homology as $\mathcal{M}(\mathcal{G}_\sigma)$. Since $\mathcal{F}$ is acyclic (with slack s), the family $\mathcal{G}_\sigma$ is acyclic (with slack s) as well. Theorem 8 now ensures that (in dimension $j \geq s$) the homology of $\mathcal{M}(\mathcal{G}_\sigma)$ is the same as the homology of the union of the elements in $\mathcal{G}_\sigma$. Since $\bigcup_{\mathcal{G}_\sigma}$ is an open subset of Γ, it has homology zero in dimension d_Γ and higher. This concludes the proof. □

Our first Helly-type theorem now follows easily through our projection theorem.

Proof of Theorem 1. Let $\mathcal{F}$ be a finite acyclic family of open subsets of a locally arc-wise connected topological space Γ, and assume that any sub-family of $\mathcal{F}$ intersects in at most r connected components. Let $\mathcal{N}(\mathcal{F})$ and $\mathcal{M}(\mathcal{F})$ denote, respectively, the nerve and the multinerve of $\mathcal{F}$. We consider the projection

$$\pi : \left\{ \begin{array}{ccc} \mathcal{M}(\mathcal{F}) & \rightarrow & \mathcal{N}(\mathcal{F}) \\ (C, A) & \mapsto & A \end{array} \right.$$

(already used in Section 3). The map π is clearly a simplicial, dimension-preserving map. Furthermore, each simplex in the pre-image $\pi^{-1}(\sigma)$ of a simplex $\sigma \in \mathcal{N}(\mathcal{F})$ is of the form (C, σ) where C is a connected component of $\bigcap_\sigma$. The projection π is therefore surjective and at most r-to-one, and we can apply Theorem 15 with $X = \mathcal{M}(\mathcal{F})$ and $Y = \mathcal{N}(\mathcal{F})$. We obtain that $L(\mathcal{N}(\mathcal{F})) \leq rJ(\mathcal{M}(\mathcal{F})) + r - 1$. With Lemma 19, this becomes $L(\mathcal{N}(\mathcal{F})) \leq r(d_\Gamma + 1) - 1$. Since the Helly number of $\mathcal{F}$ is at most $L(\mathcal{N}(\mathcal{F})) + 1$ (Lemma 4), this concludes the proof. □

6.2 Proof of Theorem 3

We also need another (simple) projection theorem for the J index.

Lemma 20. *Let X and Y be two simplicial posets and $k \geq 0$. If there exists a simplicial, dimension-preserving map $f : X \to Y$ whose restriction to the simplices of X of dimension at least k is a bijection onto the simplices of Y of dimension at least k, then $J(Y) \leq \max\big(J(X), k+1\big)$.*

Proof. Since f is simplicial, it induces a map $\tilde{f} : \mathrm{sd}(X) \to \mathrm{sd}(Y)$. We note that $\tilde{f}$ is simplicial and dimension-preserving, since f is simplicial and dimension-preserving.

Any n-simplex of $\mathrm{sd}(Y)$ is a chain of $n+1$ elements of Y of increasing dimensions whose maximal element has therefore dimension at least n. For $n \geq k$, any n-simplex $\tau \in Y$ has a unique pre-image $\sigma \in X$ under f. Thus, for any chain υ in Y with maximal element τ, if υ has a pre-image under f then the maximal element of that pre-image is σ. Since f is simplicial and dimension-preserving, it is a bijection from $[0, \sigma]$ onto $[f(0), \tau]$; it follows that any chain in Y whose maximal element has dimension at least k has one, and only one, pre-image under f. In particular, for any $n \geq k$ we have that $\tilde{f}$ induces a bijection from the n-simplices of $\mathrm{sd}(X)$ onto the n-simplices of $\mathrm{sd}(Y)$.

Now let V be the set of vertices of Y, let S be a subset of V, and let τ be a simplex in $Y[S]$. Let $R = \bigcup_{f^{-1}(S)}$ and let $\{\sigma_1, \ldots, \sigma_p\}$ be the pre-images of τ through f. For every $n \geq k$, the map f induces a bijection between the union of the n-simplices of $X[R]$ containing one of the σ_i, and the set of n-simplices of $Y[S]$ containing τ. (It is actually a disjoint union.) Thus, the same argument as above implies that, for every $n \geq k$, $\tilde{f}$ induces a bijection between the n-simplices of $\bigcup_i \dot{D}_{X[R]}(\sigma_i)$ and those of $\dot{D}_{Y[S]}(\tau)$.

Furthermore, since they are simplicial and dimension-preserving, both f and $\tilde{f}$ (trivially extended by linearity) commute with the boundary operator. The two previous statements imply that for every $n \geq k+1$, $\tilde{f}$ induces an isomorphism between $H_n(\bigcup_i \dot{D}_{X[R]}(\sigma_i))$ and $H_n(\dot{D}_{Y[S]}(\tau))$. Since the $\dot{D}_{X[R]}(\sigma_i)$ are disjoint subcomplexes of $X[R]$, the homology group $H_n(\bigcup_i \dot{D}_{X[R]}(\sigma_i))$ is $\bigoplus_i (H_n(\dot{D}_{X[R]}(\sigma_i)))$. By definition, all the summands vanish for $n \geq J(X)$. Therefore, $H_n\big(\dot{D}_{Y[S]}(\tau)\big)$ vanishes for any $S \subseteq V$, any $\tau \in Y[S]$, and any $n \geq \max(J(X), k+1)$. □

We can now prove our more general Helly-type theorem.

Proof of Theorem 3. Let Γ be a locally arc-wise connected topological space and let $\mathcal{F}$ be a family of open subsets of Γ that is acyclic with slack s and such that the intersection of any sub-family of $\mathcal{F}$ of size at least t has at most r connected components. Let $\mathcal{N}(\mathcal{F})$ and $\mathcal{M}(\mathcal{F})$ denote, respectively, the nerve and the multinerve of $\mathcal{F}$. We can construct a simplicial poset $\mathcal{M}_{red}(\mathcal{F})$ by identifying together two simplices of $\mathcal{M}(\mathcal{F})$ if and only if they are of the form (C, A) and (C', A') with $A = A'$ and $|A| \leq t-1$. In other words,

$$\mathcal{M}_{red}(\mathcal{F}) = \left\{ A \;\middle|\; A \subseteq \mathcal{F} \text{ has cardinality at most } t-1 \text{ and } \bigcap\nolimits_A \neq \emptyset \right\}$$
$$\cup \left\{ (C, A) \;\middle|\; A \subseteq \mathcal{F} \text{ has cardinality at least } t \text{ and } C \text{ is a connected component of } \bigcap\nolimits_A \right\}.$$

We thus have a surjective map $f : \mathcal{M}(\mathcal{F}) \to \mathcal{M}_{red}(\mathcal{F})$ given by $f(C, A) = (C, A)$ if A has cardinality at least t and $f(C, A) = A$ otherwise. We can make $\mathcal{M}_{red}(\mathcal{F})$ a poset by letting $f(\alpha) \preceq f(\beta)$ whenever $\alpha \preceq \beta$. The poset structure of $\mathcal{M}_{red}(\mathcal{F})$ is similar to the one of the multinerve in Section 3, and the proof of Lemma 6 applies mutatis mutandis to prove that $\mathcal{M}_{red}(\mathcal{F})$ is a simplicial poset. We note that f is simplicial and dimension-preserving. Moreover, for any $n \geq t-1$, f is a bijection from the n-simplices of $\mathcal{M}(\mathcal{F})$ onto the n-simplices of $\mathcal{M}_{red}(\mathcal{F})$. We can thus apply Lemma 20 with $X = \mathcal{M}(\mathcal{F})$, $Y = \mathcal{M}_{red}(\mathcal{F})$, and $k = t-1$, and obtain that $J(\mathcal{M}_{red}(\mathcal{F})) \leq \max(J(\mathcal{M}(\mathcal{F})), t)$. Since $J(\mathcal{M}(\mathcal{F})) \leq \max(d_\Gamma, s)$ by Lemma 19, it follows that $J(\mathcal{M}_{red}(\mathcal{F})) \leq \max(d_\Gamma, s, t)$.

Now, consider the projection $\pi : \mathcal{M}_{red}(\mathcal{F}) \to \mathcal{N}(\mathcal{F})$ that is the identity on simplices of dimension at most $t-2$ and such that for any simplex $(C, A) \in \mathcal{M}_{red}(\mathcal{F})$ of dimension at least $t-1$, $\pi(C, A) = A$. By construction, π is simplicial, dimension-preserving, onto, and at most r-to-one, so we can apply Theorem 15 with $X = \mathcal{M}_{red}(\mathcal{F})$ and $Y = \mathcal{N}(\mathcal{F})$ to obtain that $L(\mathcal{N}(\mathcal{F})) \leq rJ(\mathcal{M}_{red}(\mathcal{F})) + r - 1$. Since $J(\mathcal{M}_{red}(\mathcal{F})) \leq \max(d_\Gamma, s, t)$, we get that $L(\mathcal{N}(\mathcal{F}))$ is at most $r(\max(d_\Gamma, s, t) + 1) - 1$ and the statement now follows from Lemma 4. □

6.3 Extension to compact sets

We finally argue that the openness assumption can be replaced by a compactness assumption under a mild additional condition on the sets.

Lemma 21. *Let $\mathcal{F}$ be a family of subcomplexes of a triangulation T of an arbitrary topological space Γ. Then there exists a family $(O(F))_{F\in\mathcal{F}}$ of open sets in Γ such that, for every $\mathcal{G} \subseteq \mathcal{F}$, the set $\bigcap_{G\in\mathcal{G}} O(G)$ deformation retracts to $\bigcap_{G\in\mathcal{G}} G$.*

Proof. For an arbitrary subcomplex K of T, let $O(K)$ be the union of the *open* simplices of $\mathrm{sd}(T)$ whose closure meets K. (By a slight abuse of notation, we also denote by $O(K)$ the *set* of these simplices.) It is a standard fact [50, Lemma 70.1] that $O(K)$ deformation retracts to K: indeed, every simplex of $O(K)$ has a unique maximal face entirely contained in K; the retraction collapses each such simplex of $O(K)$ towards this maximal face.

Let σ be a simplex in $\mathrm{sd}(T)$. It is thus a chain of simplices in T; let $\min(\sigma)$ be the simplex of T of smallest dimension in this chain. With this notation, $\sigma \in O(K)$ if and only if $\min(\sigma) \in K$ (since K is a subcomplex). In other words,

$$O(K) = \{\sigma \in \mathrm{sd}(T) \mid \min(\sigma) \in K\}.$$

This immediately implies that $O(K)$ is an open set and that, for every sub-family $\mathcal{G}$ of $\mathcal{F}$, we have $\bigcap_{G\in\mathcal{G}} O(G) = O(\bigcap_{\mathcal{G}})$; this latter set retracts to $\bigcap_{\mathcal{G}}$. □

In particular, the condition of being acyclic (with slack s) extends from a family $\mathcal{F}$ to the family $O(\mathcal{F})$. Theorems 1 and 3 therefore extend immediately to subcomplexes of triangulations. We only state the more general version:

Corollary 22. *Let $\mathcal{F}$ be a finite family of subcomplexes of a given triangulation of a locally arc-wise connected topological space Γ. If (i) $\mathcal{F}$ is acyclic with slack s and (ii) any sub-family of $\mathcal{F}$ of cardinality at least t intersects in at most r connected components, then the Helly number of $\mathcal{F}$ is at most $r(\max(d_\Gamma, s, t) + 1)$.*

7 Transversal Helly numbers

Let $\mathcal{H} = \{A_1, \ldots, A_n\}$ be a family of pairwise disjoint convex sets in $\mathbb{R}^d$ and let $T_k(\mathcal{H})$ denote the set of k-dimensional affine subspaces intersecting every member in $\mathcal{H}$. Vincensini [59] conjectured that the Helly number of $\{T_k(A_1), \ldots, T_k(A_n)\}$, the ***$k$-th transversal Helly number*** τ_k of $\{A_1, \ldots, A_n\}$, can be bounded as a function of d and k, generalizing Helly's theorem that corresponds to the case $k = 0$. Vincensini's conjecture is false in such generality but holds in special cases, when the geometry of the A_i is adequately constrained. Understanding which geometric conditions allow for bounded transversal Helly numbers has been one of the focus of geometric transversal theory [14, 16, 34, 61]. In this section we show that Theorem 3 can be used to bound, in a single stroke, three transversal Helly numbers τ_1 previously bounded via ad hoc methods. The parameters used in the applications of Theorem 3 are summarized in Table 1.

For future reference, the following standard lemma bounds the value of d_Γ for some manifolds Γ. The proof can be found in various textbooks, e.g. Greenberg [26, p. 121].

Lemma 23. *Let Γ be a (paracompact) manifold of dimension d. Then $d_\Gamma \leq d+1$. Furthermore, if Γ is non-compact or non-orientable, then $d_\Gamma \leq d$.*

7.1 General remarks

Like most work in geometric transversal theory, we focus on the case $k = 1$, when the subspaces are lines. We therefore give bounds on certain *first transversal Helly numbers*. A line intersecting every member in $\mathcal{H}$ is called a ***line transversal*** to $\mathcal{H}$. We let $T(\mathcal{H}) = T_1(\mathcal{H})$ denote the set of line transversals to $\mathcal{H}$. All lines are *non-oriented*.

The space of lines in $\mathbb{R}^d$ can be considered as a subspace of the space of lines in $\mathbb{RP}^d$, which is the Grassmannian $\mathbb{RG}_{2,d+1}$ of all 2-planes through the origin in $\mathbb{R}^{d+1}$; $\mathbb{RG}_{2,d+1}$ is a manifold of dimension $2d-2$ and can be seen as an algebraic sub-variety of some $\mathbb{RP}^m$ via Grassmann coordinates (also known as Plücker coordinates for $d = 3$). We note that $d_{\mathbb{RG}_{2,d+1}} \leq 2d-1$ by Lemma 23. However, in the applications below, we consider the set Γ of lines in $\mathbb{R}^d$, which is a non-compact submanifold of dimension $2d-2$ of $\mathbb{RG}_{2,d+1}$. It follows that $d_\Gamma \leq 2d-2$, again by Lemma 23.

Let $p : \mathbb{RG}_{2,d+1} \to \mathbb{RP}^{d-1}$ be the map associating each line to its direction. We let $\mathcal{K}(\mathcal{H}) = p(T(\mathcal{H}))$ denote the directions of line transversals to $\mathcal{H}$. As the next lemma shows, the homology of $T(\mathcal{H})$ can be studied through its projection by p.

Lemma 24. *If $\mathcal{H}$ is a finite family of compact convex sets in $\mathbb{R}^d$, then $p|_{T(\mathcal{H})}$ induces an isomorphism in homology. In other words, p induces a bijection between the connected components of $T(\mathcal{H})$ and the connected components of $\mathcal{K}(\mathcal{H})$, and each connected component of $T(\mathcal{H})$ has the same homology as its projection.*

Proof. This essentially follows from the Vietoris-Begle argument: for any direction $\vec{u} \in \mathcal{K}(\mathcal{H})$ the fiber $p^{-1}(\vec{u})$ is contractible, as it is homeomorphic to the intersection of the projections of the members of $\mathcal{H}$ on a hyperplane orthogonal to $\vec{u}$. Furthermore, since $T(\mathcal{H})$ is compact, the restriction $p_{|T(\mathcal{H})}$ is a closed map. Thus Lemma 26(i) (in Appendix A) directly implies the result. □

The number of connected components of $T(\mathcal{H})$ can be bounded under certain conditions on the geometry of the objects in $\mathcal{H}$. A line transversal to a family of disjoint convex sets induces two orderings of the family, one for each orientation of the line; this pair of orderings is called the ***geometric permutation*** of the family induced by the line. A simple continuity argument shows that all lines in a connected component of $T(\mathcal{H})$ induce the same geometric permutation of $\mathcal{H}$. Under certain conditions, this implication becomes an equivalence, and the connected components of $T(\mathcal{H})$ are in one-to-one correspondence with the geometric permutations of $\mathcal{H}$. Various geometric and combinatorial arguments can then be used to bound from above the number of distinct geometric permutations that may exist for one and the same family $\mathcal{H}$.

An ***open thickening*** of a subset H of $\mathbb{R}^d$ is a family $(H^\varepsilon)^{\varepsilon>0}$ such that (i) any H^ε is an open set, (ii) if $\varepsilon < \varepsilon'$, then $H^\varepsilon \subseteq H^{\varepsilon'}$, and (iii) $\bigcap_{\varepsilon>0} H^\varepsilon = H$. For a family $\mathcal{G}$ of subsets of $\mathbb{R}^d$, we

Shape	Previous bound	Our bound	d_Γ	s	t	r
Parallelotopes in $\mathbb{R}^d$ ($d \geq 2$)	$2^{d-1}(2d-1)$ [52]	$2^{d-1}(2d-1)$	$2d-2$	$d+1$	1	2^{d-1}
Disjoint translates of a planar convex figure	5 [58]	10	2	3	4	2
Disjoint unit balls in $\mathbb{R}^d$:						
$d=2$	5 [13]	12	$2d-2$	$d+1$	1	3
$d=3$	11 [11]	15	$2d-2$	$d+1$	1	3
$d=4$	15 [11]	20	$2d-2$	$d+1$	9	2
$d=5$	19 [11]	20	$2d-2$	$d+1$	9	2
$d \geq 6$	$4d-1$ [11]	$4d-2$	$2d-2$	$d+1$	9	2

Table 1: Parameters used to derive bounds on transversal Helly numbers from Theorem 3.

let $\mathcal{G}^\varepsilon = \{H^\varepsilon \mid H \in \mathcal{G}\}$. In the three applications below, we consider transversals to *compact* sets. Since any compact set admits an open thickening, the following lemma will allow us to consider the same problem with *open* sets.

Lemma 25. *Let $\mathcal{H}$ be a finite family of compact convex sets in $\mathbb{R}^d$ and $\mathcal{H}^\varepsilon$ be an open thickening of $\mathcal{H}$. There exists $\varepsilon > 0$ such that for every $\mathcal{G} \subseteq \mathcal{H}$, the family $\mathcal{G}$ has a common transversal if and only if the family $\mathcal{G}^\varepsilon$ has a common transversal.*

Proof. Let $\mathcal{G} \subseteq \mathcal{H}$. To prove the lemma, it suffices to prove that, if $\mathcal{G}$ has no transversal, then, for $\varepsilon > 0$ small enough, $\mathcal{G}^\varepsilon$ has no transversal. We prove the contrapositive statement: assume that $\mathcal{G}^\varepsilon$ has a transversal for every $\varepsilon > 0$; we will prove that $\mathcal{G}$ has a transversal. There exists a sequence (ε_n) decreasing towards zero, and, for every n, a line (ℓ_n) transversal to $\mathcal{G}^{\varepsilon_n}$: it intersects H^{ε_n} ($H \in \mathcal{G}$) at point $a_{H,n}$. Up to taking a subsequence, we can assume that (ℓ_n) converges towards a line ℓ, and that each sequence $(a_{H,n})$ converges towards some point a_H (by compactness of $\mathbb{RG}_{2,d+1}$, and since the objects are bounded). Of course, each a_H belongs to ℓ, and also to the closure of each H^{ε_n}, hence to H, since H is closed. So $\mathcal{G}$ has a line transversal. □

7.2 Three theorems in geometric transversal theory

We can now deduce three transversal Helly numbers from our main result. The main interest in these derivations is not that the bounds are better; in fact, one matches the previously known bound, one is weaker (10 instead of 5), and the last one is better ($4d-2$ instead of $4d-1$, when $d \geq 6$). They do show, however, that the combinatorial and homological conditions of Theorem 3 may be useful in identifying situations where the transversal Helly numbers are bounded; in fact the question whether our second and third examples afford bounded transversal Helly numbers were raised in the late 1950's and only answered in 1986 and 2006. Refer to Table 1 for a summary of the parameters used in the applications of Theorem 3.

Parallelotopes in arbitrary dimension. Let $\mathcal{H}$ be a finite family of parallelotopes in $\mathbb{R}^d$ with edges parallel to the coordinate axis. Santaló [52] showed that the transversal Helly number τ_1 of $\mathcal{H}$ is at most $2^{d-1}(2d-1)$. Here is how Santaló's theorem can be seen to follow from Theorem 3. We can restrict ourselves to *open* parallelotopes by Lemma 25.

Let D be the set of directions in $\mathbb{RP}^{d-1}$ that are *not* orthogonal to the direction of any coordinate axis. D has exactly 2^{d-1} connected components. Recall that $p^{-1}(D)$ is the set of lines whose direction is in D. When studying the existence of transversals to $\mathcal{H}$, it does not harm to restrict to lines in $p^{-1}(D)$, since the set of transversals to $\mathcal{H}$ is open and since the complement of $p^{-1}(D)$ has empty interior.

For each connected component of D, the set of transversals to $\mathcal{H}$ with direction in this component can be seen to be homeomorphic to the interior of a polytope in a $(2d-2)$-dimensional affine subspace of $\mathbb{R}^{2d}$ by adequate use of ***Cremona coordinates*** [23]. In particular, for any $\mathcal{G} \subseteq \mathcal{H}$, the set $T(\mathcal{G}) \cap p^{-1}(D)$ consists of at most 2^{d-1} contractible components. Moreover, if $\Gamma = p^{-1}(D)$, then $d_\Gamma \leq 2d-2$ by Lemma 23. Theorem 1 now implies an upper bound of $2^{d-1}(2d-1)$ in the Helly number of transversals of parallelotopes.

If we consider the partition of line space into 2^{d-1} regions $R_1, \ldots, R_{2^{d-1}}$ induced by the above partition of $\mathbb{RP}^{d-1}$, the Cremona coordinates recast the set of line transversals in each R_i into a convex set, and Santaló's theorem follows directly from applying Helly's theorem inside each R_i [23]. While this is simpler, we know of no other example where a transversal Helly number is obtained by partitioning the space of lines and identifying a convexity structure in each region. In fact, the definition of convexity structures on the Grassmannian in itself raises several issues [24].

Disjoint translates in the plane. Tverberg [58] showed that for any compact convex subset $D \subset \mathbb{R}^2$ with non-empty interior, the transversal Helly number τ_1 of any finite family $\mathcal{H}$ of disjoint translates of D is at most 5. This settled a conjecture of Grünbaum [27] previously proven in the cases where D is a disk [13] and a square [27], or with the weaker bound of 128 [40]. Tverberg's proof uses in an essential way properties of geometric permutations of collections of disjoint translates of a convex figure [41]. Here, we show how an upper bound of 10 can be easily derived from Theorem 3 and the sole property that the number of geometric permutations of n disjoint translates of a compact convex set with non-empty interior in $\mathbb{R}^2$ is at most 3 in general and at most 2 if $n \geq 4$ [41].

First, remark that instead of translates of a compact convex set, we can consider translates of an *open* convex set (using Lemma 25, by letting H^ε be the set of points at distance strictly less than ε from H). Now, observe that for any $A_i \in \mathcal{H}$ the set $T_i = T(\{A_i\})$ has the homotopy type of $\mathbb{RP}^1$. Moreover, for any sub-family $\mathcal{G} \subseteq \mathcal{H}$ of size at least two, the set of directions in $\mathcal{K}(\mathcal{G})$ corresponding to a given geometric permutation of $\mathcal{G}$ is a connected proper subset of $\mathbb{RP}^1$, and Lemma 24 implies that $T(\mathcal{G})$ is acyclic with slack $s = 3$. Moreover, the number of components in $T(\mathcal{G})$ is at most the maximum number of geometric permutations of $\mathcal{G}$, that is at most 3 in general and at most 2 when $|\mathcal{G}| \geq 4$ [41]. We can therefore apply Theorem 3 with $d_\Gamma = 2$, $s = 3$, $t = 1$ and $r = 3$, getting an upper bound of 12, or with $d_\Gamma = 2$, $s = 3$, $t = 4$ and $r = 2$, obtaining the better bound of 10.

In dimension 3 or more there exist families of disjoint translates of a polyhedron with arbitrarily many connected components of line transversals; in other words, r cannot be bounded. In that setting, indeed, Tverberg's theorem is known not to generalize [36].

Disjoint unit balls in arbitrary dimension. Cheong et al. [11] showed that the transversal Helly number τ_1 of any finite collection $\mathcal{H}$ of disjoint equal-radius closed balls in $\mathbb{R}^d$ is at most $4d-1$. That this number is bounded was first conjectured by Danzer [13] and previously proven for $d = 2$ [13] and $d = 3$ [35] or under various stronger assumptions (see [11] and the discussion therein).

The proof of Cheong et al. [11] combines a characterization of families of geometric permutations of $n \geq 9$ disjoint unit balls with a local application of Helly's topological theorem. Here we show how Theorem 3 and some ingredients of their proofs yield a slightly improved bound.

First, note that by Lemma 25, we can consider open balls with the same radius (say one). Observe that for any $A_i \in \mathcal{H}$ the set $T_i = T(\{A_i\})$ has the homotopy type of $\mathbb{RP}^{d-1}$, and is therefore homologically trivial in dimension d and higher. Then, for any sub-family $\mathcal{G} \subseteq \mathcal{H}$ of size at least two, the set of directions in $\mathcal{K}(\mathcal{G})$ corresponding to a given geometric permutation of $\mathcal{G}$ is convex[12] [6] and therefore contractible. In other words, $\mathcal{K}(\mathcal{G})$ is a disjoint union of contractible sets; so is $T(\mathcal{G})$ by Lemma 24. It follows that for any $\mathcal{G} \subseteq \mathcal{H}$, $T(\mathcal{G})$ is acyclic with slack $d+1$. Moreover, for any d the number of geometric permutations of a family of n disjoint equal-radius balls in $\mathbb{R}^d$ is at most 3 in general and at most 2 when $n \geq 9$ [12]. We can thus apply Theorem 3 with $d_\Gamma = 2d-2$, $s = d+1$, $t = 9$, and $r = 2$, obtaining the upper bound of $2\max(2d-1, 10)$. For $d \geq 6$, this yields the upper bound of $4d-2$, but for $d \in \{2,3,4,5\}$ this bound is only 20. In the case $d = 2$ (resp. $d = 3$) it can be improved to 12 (resp. 15) by using $d_\Gamma = 2d-2$, $s = d+1$, $t = 1$, and $r = 3$.

It is conjectured that any family of 4 or more disjoint equal-radius balls in $\mathbb{R}^d$ has at most two geometric permutations. If this is true, then our bounds would improve to $4d-2$ for any $d \geq 3$. Since the transversal Helly number τ_1 of disjoint equal-radius balls is at least $2d-1$ [10], this number is known up to a factor of 2. Families of n disjoint balls with arbitrary radii in $\mathbb{R}^d$ have up to $\Theta(n^{d-1})$ geometric permutations [55] and their transversal Helly number is unbounded; if the radii are required to be in some fixed interval $[1, \rho]$, this bound reduces to $O(\rho^{\log \rho})$ [62] and Theorem 3 similarly implies that the first transversal Helly number is $O(d\rho^{\log \rho})$, where the constant in the $O()$ is independent of ρ, n and d.

A Homology of spaces with contractible fibers

In some situations, topological (or homological) properties of a topological space X can be understood by considering a projection $p : X \to Y$ with *contractible* fibers. An example from the geometric transversal literature is when X is the set of line transversals to some family of convex sets and p maps a line to its direction. While simple settings allow for elementary proofs (see e.g. the proof of [11, Lemma 14]), standard arguments in algebraic topology lead to more general statements such as Lemma 24 or Theorem 12. In this appendix, we collect some of these arguments, essentially variants of the classical (and generalized) Vietoris-Begle mapping theorem, in the hope that they can be useful in other contexts.

Lemma 26. *(Vietoris-Begle argument) Let $\pi : X \to Y$ be a continuous surjective map from a topological space X onto a topological space Y. We assume that the fiber $\pi^{-1}(y)$ is contractible for every $y \in Y$. Assume* either one *of the following assumptions is satisfied*

1. *X, Y are paracompact Hausdorff and, further, π is closed;*

2. *X and Y are manifolds and π is a submersion;*

3. *X and Y are (the geometric realization of) simplicial sets and $\pi : X \to Y$ is (the geometric realization of) a map of simplicial sets;*

[12] Convexity in $\mathbb{RP}^{d-1}$ is relative to the metric induced through the identification $\mathbb{RP}^{d-1} = \mathbb{S}^{d-1}/\mathbb{Z}_2$.

4. $\pi : X \to Y$ *is a fibration;*

5. $X = \bigcup_{n\geq 0} X_n$ *is a union of closed subsets (with* X_n *in the relative interior of* X_{n+1}*) such that* $\pi_{|X_n} : X_n \to Y$ *is proper with contractible fibers.*

6. X *and* Y *are locally finite CW-complexes and further* π *is proper.*

Then, the natural map $\pi_* : H_n(X) \to H_n(Y)$ *is an isomorphism for all* n*.*

*(*Homotopy enhancement of Vietoris-Begle argument*): in addition, the map* π *is an homotopy equivalence when either assumption 3. or 6. is satisfied or when assumption 4. is satisfied and further,* X *and* Y *are CW-complexes.*

Proof. Let us recall that we work over a characteristic zero field and thus it is equivalent to prove the result in cohomology by the universal coefficient theorem [56, Section 5.5] [26, Theorem 23.28]. The case of assumption 1. reduces to the Vietoris-Begle mapping theorem (see [56, Theorem 15, Section 6.9]). The case of assumption 2. is the main result of [54]. The case of assumption 5. is a corollary of [39, Proposition 2.7.8] applied to a constant sheaf.

In the case of assumption 6, first note that X and Y are locally compact, locally contractible, and have metrizable connected components since they are locally finite CW-complexes [42, Proposition II.3.6, Proposition II.3.8 and Theorem II.6.6]. Further, since π is onto, it induces a surjection of the set of connected components of X to the ones of Y, and this surjection is indeed a bijection since π has contractible (hence connected) fibers. Now the homotopy version (hence the homology version as well) of Vietoris-Begle argument follows by applying the main result of [53] to each connected component of X.

The case of assumption 3. (as well as its homotopic version) is proved in [20] in the case where X, Y are the geometric realizations of simplicial complexes and π is the realization of a simplicial map. The general case of simplicial sets reduces to the previous one since, if X and Y are geometric realizations of simplicial sets, then they are homeomorphic to the geometric realizations of simplicial complexes K and L, and further the geometric realization of any map of simplicial sets is homotopic to the geometric realization of a simplicial map from K to L, see [42, Theorem III.6.1 and Corollary III.6.2].

In the case of assumption 4., the map $\pi : X \to Y$ is a fibration. Further, since $\pi : X \to Y$ has contractible fibers, it follows from the long exact sequence of homotopy groups of a fibration (for instance, see [56, Section 7.2], [29, Theorem 4.40] or [8, Section 17]) that the induced maps $\pi_* : \pi_k(X, x_0) \to \pi_k(Y, y_0)$ are isomorphisms for any k and any choice of a base point $x_0 \in X$ (recall that we assume π to be surjective). Thus $\pi : X \to Y$ is a weak homotopy equivalence and thus induces an isomorphism in (co)homology [56, Theorem 25, Section 7.6]. Since, by Whitehead's Theorem (see [56, Section 7.6]), weak homotopy equivalences between CW-complexes are homotopy equivalences, this concludes the proof. □

Although some spaces satisfy several of the assumptions 1. to 5. simultaneously, these assumptions are not equivalent in general; any of them is enough to ensure the result. Let us give some examples in which Lemma 26 applies.

- If X is (Hausdorff) compact and Y is Hausdorff, then Assumption 1. is automatically satisfied.
- If X and Y are simplicial complexes or Δ-sets, and π is a simplicial map, then they verify Assumption 3.

- Recall that a large class of examples of fibrations are given by fiber bundles [56]. We recall that $\pi : X \to Y$ is a fiber bundle if there exists a topological space F (the fiber) such that any point in Y has a neighborhood U such that $\pi^{-1}(U)$ is homeomorphic to a product $U \times \pi^{-1}(y)$ in such a way that the map $\pi_{|\pi^{-1}(U)}$ identifies with the first projection $U \times \pi^{-1}(y) \to U$. That is, the map $\pi : X \to Y$ is locally trivial with fiber homeorphic to F. In particular, covering spaces, vector bundles, principal group bundles are fibrations.

- If X (Hausdorff) can be covered by an union $\bigcup X_n$ of compact spaces such that the fibers of $p_{|X_n}$ are contractible, then 5. is satisfied and the result of the lemma holds.

- If X is a finite CW-complex and π is cellular, then 6. is satisfied.

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