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Long-time Behaviour of Absorbing Boundary Conditions

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A new class of computational far-field boundary conditions for hyperbolic partial differential equations was recently introduced by the authors. These boundary conditions combine properties of absorbing conditions for transient solutions and properties of far-field conditions for steady states. This paper analyses the properties of the wave equation coupled with these new boundary conditions: well-posedness, dissipativity and convergence in time.

1. Introduction

Computational far-field boundary conditions are designed in order to restrict the spatial domain in the numerical computation of partial differential equations. These boundary conditions should generate well-posed problems and be well satisfied by the solution over the original unrestricted domain. They should also be computationally efficient.

In [6] we introduced a new class of computational boundary conditions for hyperbolic differential equations. The purpose of these boundary conditions is to handle the transient phase of a solution as well as a solution close to the steady state. Earlier conditions concentrated on only one of these cases (see [7, 2] and references therein). The basic principles for the design of the new boundary conditions were given in [6], together with numerical computations.

We shall consider here the wave equation in $\mathbb{R}^N$, $N = 2$ or 3:

$$u_{tt} - \Delta u + \alpha^2 u = f(x). \tag{1.1}$$

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The right-hand side depends only on x, α is a non-negative number. The data, f and the initial values, are compactly supported in $\mathbb{R}^N$. It is well known that u converges locally to a steady state satisfying

$$-\Delta v + \alpha^2 v = f. \tag{1.2}$$

We now introduce a domain Ω, with boundary Γ. We suppose that Ω contains the support of the data. The aim of this work is to design a boundary condition on Γ, such that the solution of the equation (1.1) in Ω with this boundary condition will be close to u for both short and long times. To reach the second goal, we write the transparent boundary condition for the steady state v. On the boundary Γ, v satisfies

$$\frac{\partial v}{\partial n} + Kv = 0 \tag{1.3}$$

identically, where K is the Calderon operator for $-\Delta + \alpha^2 I$ in the exterior domain. A good boundary condition for the wave equation on Γ will be

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Ku = 0. \tag{1.4}$$

This is our new boundary condition. It will force u to converge in Ω to the steady state as time tends to infinity. It is local in time and integral along the boundary. We described in [6] a numerical method adapted to this problem.

In section 2, we write the time-dependent and time-independent problems in the adapted frame of Sobolev spaces. In the case where $\alpha = 0$, we shall need weighted Sobolev spaces.

In section 3, we describe briefly the derivation of the absorbing boundary conditions. These boundary conditions are designed to give a good representation of the solution in the transient phase.

In section 4, we give the transparent boundary condition for the steady state by writing the problem as a coupling between Ω and its exterior. Some of the results in sections 2–4 are well known, but included here for completeness.

In section 5, we establish a general result of convergence in time for the homogeneous wave equation (i.e. $f = 0$), with boundary condition (1.4). We prove the exponential decay of the energy of the solution as time tends to infinity. This result is a generalization of the results in control theory (see references in the present paper and in [12]).

In section 6, we use this fundamental result to prove that our new boundary condition forces the convergence to the steady state. Furthermore, we express this boundary condition as a product (in the sense of operators) of an absorbing boundary condition, and the transparent boundary condition for the steady state. These two results show the new boundary condition to be well suited for both short- and long-time computations.

In what follows, Ω will be a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, convex, with boundary Γ $\mathscr{C}^2$ and compact.

2. The free-space problems

We denote by u^* the solution of the dispersive wave equation in $\mathbb{R}^N$:

$$\mathscr{L}u^* = u^*_{tt} - \Delta u^* + \alpha^2 u^* = f(x) \qquad \text{in } [0, T] \times \mathbb{R}^N \tag{2.1a}$$

$$u^*(0) = u^0, \qquad u_t^*(0) = u^1 \qquad \text{in } \mathbb{R}^N, \tag{2.1b}$$

where α is a non-negative constant, and the data u^0, u^1 and f are compactly supported in D. Note that f depends only on the space variable. The well-posedness of this problem is classical.

Theorem 2.1. *For any set of data (u^0, u^1, f) compactly supported in $\mathbb{R}^N$ such that (u^0, u^1) belongs to $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and f belongs to $L^2(\mathbb{R}^N)$, problem* (2.1) *has a unique solution u^* such that $u^* \in \mathscr{C}^0([0, T], H^1(\mathbb{R}^N))$, $u_t^* \in \mathscr{C}^0([0, T], L^2(\mathbb{R}^N))$.*

The proof is a straightforward application of the theory developed in [13], relying on an energy estimate and the Galerkin method.

We now introduce the *steady state*, defined by

$$-\Delta v^* + \alpha^2 v^* = f \qquad \text{in } \mathbb{R}^N. \tag{2.2}$$

We shall write a variational formulation for this problem, in a space V defined in a proper way. If α is non-zero, the adapted space is $H^1(\mathbb{R}^N)$; if $\alpha = 0$, we introduce a weighted space. The definition is different, depending on whether N is 2 or 3 (see [8]).

$$\alpha \neq 0 \qquad V = H^1(\mathbb{R}^N), \tag{2.3a}$$

$$\alpha = 0, N = 2 \quad V = \left\{ u \in D'(\mathbb{R}^2), \frac{1}{[1 + \log(1 + r^2)](1 + r^2)^{1/2}} u \in L^2, \nabla u \in L^2 \right\}, \tag{2.3b}$$

$$\alpha = 0, N = 3 \qquad V = \{u \in D'(\mathbb{R}^3), (1 + r^2)^{-1/2} u \in L^2, \nabla u \in L^2\}. \tag{2.3c}$$

In cases (2.3b,c), V is equipped with a natural norm. Let us notice that for $\alpha = 0$ and $N = 2$, V contains constants. In the following, $\|\cdot\|$ denotes the norm in L^2.

Theorem 2.2. *For any $f(x)$ compactly supported belonging to $L^2(\mathbb{R}^N)$, equation* (2.2) *admits a unique solution v^* in V for $\alpha \neq 0$ or $\alpha = 0$ and $N = 3$. If $\alpha = 0$ and $N = 2$, and if in addition $\int_{\mathbb{R}^2} f(x)\mathrm{d}x = 0$, equation* (2.2) *admits a unique solution in $V_{/\mathbb{R}}$.*

Proof. For $\alpha \neq 0$, the result is obvious. If $\alpha = 0$, we use the results in [8]. If $N = 3$, f belongs to the dual space V' of V and (2.2) admits a variational formulation

$$v^* \in V$$

$$(\nabla v^*, \nabla v) = (f, v) \qquad \forall v \in V. \tag{2.4}$$

The space V has been designed such that $\|\nabla u\|$ is a norm on V, equivalent to the norm in V:

$$\|u\|_v^2 = \|(1 + r^2)^{-1/2} u\|^2 + \|\nabla u\|^2$$

and the result follows from the Lax–Milgram theorem.

If $N = 2$, the operator $-\Delta$ is an isomorphism from $V_{/\mathbb{R}}$ onto $(V_{/\mathbb{R}})'$ and f belongs to $(V_{/\mathbb{R}})'$ if and only if $\int_{\mathbb{R}^2} f(x)\mathrm{d}x = 0$. The equation (2.2) then admits the variational formulation

$$v^* \in V_{/\mathbb{R}}$$

$$(\nabla v^*, \nabla v) = (f, v) \qquad \forall v \in V_{/\mathbb{R}}.$$

Since $\|\nabla u\|$ is an equivalent norm on $V_{/\mathbb{R}}$ and (f, v) is linear continuous on $V_{/\mathbb{R}}$, the result follows again from the Lax–Milgram theorem.

Remark 2.1. Since f is compactly supported, v^* can be written using the Green's function

$$v^*(x) = \frac{1}{2\pi}\int \log|x-y| f(y)\mathrm{d}y + p(x), \qquad N = 2 \tag{2.5a}$$

$$v^*(x) = \frac{1}{4\pi}\int \frac{1}{|x-y|} f(y)\mathrm{d}y + p(x), \qquad N = 3 \tag{2.5b}$$

where p is a harmonic polynomial in $\mathbb{R}^N$. For large x, we can make the expansion

$$v^*(x) = \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} f(y)\mathrm{d}y\right)\log|x| + p(x) + O\left(\frac{1}{|x|}\right), \qquad N = 2 \tag{2.6a}$$

$$v^*(x) = \frac{1}{4\pi}\left(\int_{\mathbb{R}^3} f(y)\mathrm{d}y\right)\frac{1}{|x|} + p(x) + O\left(\frac{1}{|x|^2}\right), \qquad N = 3 \tag{2.6b}$$

and v^* belongs to $V_{/\mathbb{R}}$ in $\mathbb{R}^2$ if and only if $\int_{\mathbb{R}^2} f(y)\mathrm{d}y = 0$, and $p(x)$ is a constant. In that case v^* behaves as: $v^*(x) = O(1/|x|) + a$ when $|x| \to +\infty$. For $N = 3$, $v^*(x) = O(1/|x|)$ for any f.

Remark 2.2. In $\mathbb{R}^2$, if f does not satisfy the assumption $\int f(x)\mathrm{d}x = 0$, let $M = \int f(x)\mathrm{d}x$. Let g be any continuous, compactly supported function defined in $\mathbb{R}^N$ such that $\int g(x)\mathrm{d}x = 1$. The problem

$$-\Delta w = g \text{ in } \mathbb{R}^N$$

has a solution which is $\mathscr{C}^0$ in $\mathbb{R}^N$ (thus is not a variational solution). Then $\tilde{v} = v - Mw$ is a solution of

$$-\Delta\tilde{v} = f - Mg$$

and $f - Mg$ satisfies the assumption of Theorem 2.2. Thus v^* can be written as $v^* = \tilde{v} + Mg$. The solution w has a logarithmic behavior at infinity, and $\tilde{v}$ belongs to $V_{/\mathbb{R}}$.

Let us now define the error between the transient state and the steady state as $e = u^* - v^*$. It is solution of the following problem:

$$\mathscr{L}e = e_{tt} - \Delta e + \alpha^2 e = 0 \qquad \text{in } [0, T] \times \mathbb{R}^N \tag{2.7a}$$

$$e(0) = e^0 = u^0 - v,\ e_t(0) = u^1 \qquad \text{in } \mathbb{R}^N. \tag{2.7b}$$

The initial value for e belongs to V (resp. $V_{/\mathbb{R}}$ if $\alpha = 0$ and $N = 2$). We define the energy by

$$E(e, t) = \tfrac{1}{2}\|\nabla e\|^2 + \tfrac{1}{2}\|e_t\|^2 + \tfrac{1}{2}\alpha^2\|e\|^2. \tag{2.8}$$

Corollary 2.3. *Under the hypothesis of Theorems* 2.1 *and* 2.2, *the unique solution e of problem* (2.7) *satisfies the following estimate for any t in $\mathbb{R}^+$*:

$$\|e_t\|^2 + \|e\|_V^2 \leqslant CE(e, 0) \qquad \text{in cases (2.3a) and (2.3c),}$$

$$\|e_t\|^2 + \|e\|_{V_{/\mathbb{R}}}^2 \leqslant CE(e, 0) \qquad \text{in case (2.3b).}$$

Proof. We obtain an energy estimate by multiplying equation (2.7a) by e_t and integrating by parts. It gives

$$E(e, t) = E(e, 0).$$

The quantity $E(e, 0)$ is finite for v belongs to V (resp. $V_{/\mathbb{R}}$ if $\alpha = 0$ and $N = 2$). Since $\|\nabla.\|$ defines an equivalent norm on V (resp. $V_{/\mathbb{R}}$ if $\alpha = 0$ and $N = 2$), the estimate holds.

It is well known (see, for instance, [19]) that the local energy of e converges to zero as time tends to infinity. We shall now seek artificial boundary conditions in a bounded domain, which ensure a convergence (in the energy norm) of the transient state to the steady state. We shall first recall the construction of absorbing boundary conditions (see [7]).

3. The transient state: absorbing boundary conditions

We consider the dispersive wave equation in $\mathbb{R}^N$,

$$\mathscr{L}u = u_{tt} - \Delta u + \alpha^2 u = f(x), \tag{3.1}$$

where f is compactly supported in D.

For the sake of simplicity, we recall here the construction of the absorbing boundary conditions in [7] on a sphere S_R in $\mathbb{R}^2$. The derivation is general and is valid on any boundary Γ of a bounded connected domain in $\mathbb{R}^N$. We assume D to be strictly included in the ball B_R, so that the right-hand side in (3.1) vanishes in a neighbourhood of S_R.

In polar coordinates the wave operator is

$$\mathscr{L}u = u_{tt} - u_{rr} - \frac{1}{r} u_r - \frac{1}{r^2} u_{\theta\theta} + \alpha^2 u.$$

Its symbol is

$$L = -\omega^2 + \xi^2 + \frac{1}{r^2}\eta^2 - \frac{1}{r}\mathrm{i}\xi + \alpha^2, \tag{3.2}$$

where ω, ξ and η are the dual variables of t, r and θ.

The principal symbol is given by

$$L_2 = \xi^2 - \omega^2 + \frac{1}{r^2}\eta^2, \tag{3.3}$$

which can be factorized into

$$L_2 = \left[\xi - \left(\omega^2 - \frac{1}{r^2}\eta^2\right)^{1/2}\right]\left[\xi + \left(\omega^2 - \frac{1}{r^2}\eta^2\right)^{1/2}\right]. \tag{3.4}$$

Following [15], $\mathscr{L}$ can be written as

$$\mathscr{L} = (D_r + \sigma_1)(D_r + \sigma_2) \qquad \text{modulo a } C^\infty \text{ operator},$$

where $D_r = (1/\mathrm{i})\partial_r$. σ_1 and σ_2 are pseudodifferential operators of order 1, given by their symbols $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, with expansions

$$\tilde{\sigma}_i = \sum_{j=-1}^{\infty} \tilde{\sigma}_{i,-j}(r, \theta, t, \omega, \eta).$$

For any j, $\tilde{\sigma}_{i,-j}$ is homogeneous in ω and η of order $-j$. The $\tilde{\sigma}_{i,-j}$ are computed recursively using the composition formula for pseudodifferential operators, which gives

$$\tilde{\sigma}_2 = \left(\omega^2 - \frac{1}{r^2}\eta^2\right)^{1/2} - \frac{i}{2r} + \frac{k_\theta^2}{2r^3}\left(\omega^2 - \frac{1}{r^2}\eta^2\right)^{-1} + \dots . \tag{3.5}$$

$D_r + \sigma_2$ represents the propagation toward the exterior of the sphere S_r. Thus the transparent boundary condition on the sphere S_R is given by

$$(D_r + \sigma_2)u = 0. \tag{3.6}$$

It is not a differential equation in t and θ. For computation, a boundary condition in the form of a differential equation is desirable. We then approximate the symbol of σ_2. For that purpose, we can make two assumptions

(1) $\omega \gg 1$,
(2) $\eta^2/r^2\omega^2 < 1$.

The first hypothesis is a high-frequency assumption. The second one expresses that the waves are propagating in a direction not too far from the normal to the boundary. Truncation of the series in (3.5) after the second term, and approximation of the square root $(1 - \eta^2/r^2\omega^2)^{1/2}$ by 1 gives an approximation

$$\tilde{\sigma}_2^0 = \omega - \frac{i}{2R}.$$

The corresponding approximation of the operator $D_r + \sigma_2$ is

$$D_r + D_t - \frac{i}{2R},$$

and the absorbing boundary condition is

$$u_t + u_r + \frac{u}{2R} = 0. \tag{3.7}$$

If Ω is a domain in $\mathbb{R}^N$ with a smooth compact boundary Γ, the same computation carries over by taking local coordinates and gives the general absorbing boundary condition

$$u_t + u_n + \tfrac{1}{2}H(x)u = 0, \tag{3.8}$$

where $H(x)$ denotes the mean curvature on the surface Γ (see the exact definition in (6.9)) and u_n the normal derivative of u, the normal being oriented to the exterior of Ω.

Higher-order boundary conditions can be obtained as well by more accurate approximations of $\tilde{\sigma}_2$ (see [7]).

Remark 3.1. Any boundary condition of the form $u_t + u_n + a(x)u = 0$ with $a(x) \geqslant 0$ is absorbing: if this boundary condition is prescribed on the boundary Γ of a domain Ω and if $f = 0$, the energy in the domain is a decreasing function of time. This can be easily seen by an energy estimate.

4. The steady-state: transparent boundary condition

We proved in section 2 that the problem

$$-\Delta v^* + \alpha^2 v^* = f \qquad \text{in } \mathbb{R}^N \tag{4.1}$$

has a unique solution in V or $V_{/\mathbb{R}}$, V being defined in (2.3), if f is compactly supported in D and belongs to $L^2(\mathbb{R}^N)$ (with $\int_{\mathbb{R}^2} f(x)\,dx = 0$ if $\alpha = 0$ and $N = 2$).

Let Ω be an open connected domain with smooth compact boundary Γ such that $D \subset \Omega$. Ω' denotes the exterior domain. We define the normal $\mathbf{n}$ as the normal vector exterior to Ω.

Problem (4.1) is equivalent to

$$-\Delta v^* + \alpha^2 v^* = f \qquad \text{in } \Omega, \tag{4.2a}$$

$$-\Delta v^* + \alpha^2 v^* = 0 \qquad \text{in } \Omega', \tag{4.2b}$$

$$v^* \text{ and } \partial v^*/\partial n \qquad \text{continuous on } \Gamma. \tag{4.2c}$$

Consider the exterior problem

$$-\Delta u + \alpha^2 u = 0 \qquad \text{in } \Omega', \tag{4.3a}$$

and

$$u = g \qquad \text{on } \Gamma. \tag{4.3b}$$

We define the space $V(\Omega')$ in the same way as $V(\mathbb{R}^N)$ in section 2 and we have the following result:

Lemma 4.1. (*i*) *For any g in $H^{1/2}(\Gamma)$, problem* (4.3) *admits a unique solution in* $V(\Omega')$, *and the mapping K: $g \to -\partial u/\partial n$ is continuous from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.*

(*ii*) *If $\alpha \neq 0$ or $\alpha = 0$ and $N = 3$, K is strictly coercive, that is there exists a constant β strictly positive such that*

$$\forall g \in H^{1/2}(\Gamma), \qquad \langle Kg, g\rangle_\Gamma \geqslant \beta \| g \|^2_{H^{1/2}(\Gamma)}. \tag{4.4a}$$

(*iii*) *If $\alpha = 0$ and $N = 2$, the mapping K defined by $K\dot{g} = -\partial u/\partial n$ is continuous from $H^{1/2}(\Gamma)_{/\mathbb{R}}$ into $(H^{1/2}(\Gamma)_{/\mathbb{R}})'$ and strictly coercive, that is there exists a constant β strictly positive such that*

$$\forall \dot{g} \in H^{1/2}(\Gamma)_{/\mathbb{R}}, \qquad \langle K\dot{g}, \dot{g}\rangle_\Gamma \geqslant \beta \| \dot{g} \|^2_{H^{1/2}(\Gamma)/\mathbb{R}}, \tag{4.4b}$$

with obvious notation, and $\langle .,. \rangle_\Gamma$ represents the duality between $H^{1/2}(\Gamma)_{/\mathbb{R}}$ and $(H^{1/2}(\Gamma)_{/\mathbb{R}})'$ as well.

(*iv*) *K is symmetric*: $\langle Kg, h\rangle_\Gamma = \langle Kh, g\rangle_\Gamma$, $\forall (h, g) \in H^{1/2}(\Gamma)$.

Proof. (i) The well-posedness of problem (4.3) is classical if $\alpha \neq 0$. It can be found in [8] for $\alpha = 0$. It relies on the fact that the bilinear form

$$a(u, v) = (\nabla u, \nabla v) + \alpha^2 (u, v)$$

defines a scalar product on V, equivalent to the norm in

$$\mathring{V} = \overline{D(\Omega')}^V = \{v \in V,\ v_{/\Gamma} = 0\}.$$

By the Green's formula we have

$$\forall v \in V, \qquad a(u, v) = \langle Kg, v\rangle_\Gamma. \tag{4.5}$$

Since $H^{1/2}(\Gamma)$ is the space of traces on Γ of elements of V, we can write

$$\| Kg \|_{H^{-1/2}(\Gamma)} = \sup_{\varphi \in H^{1/2}(\Gamma)} \frac{|\langle Kg, \varphi\rangle_\Gamma|}{\| \varphi \|_{H^{1/2}(\Gamma)}} = \sup_{\varphi \in H^{1/2}(\Gamma)} \frac{|a(u, v)|}{\| \varphi \|_{H^{1/2}(\Gamma)}}$$

for any extension v of φ in $V(\Omega')$ such that

$$\| v \|_V \leqslant C \| \varphi \|_{H^{1/2}(\Gamma)}.$$

In all cases we have by the Cauchy–Schwarz inequality:

$$|a(u, v)| \leqslant Ca(u, u)^{1/2} \| v \|_V \leqslant Ca(u, u)^{1/2} \| \varphi \|_{H^{1/2}(\Gamma)}$$

and therefore

$$\| Kg \|^2_{H^{-1/2}(\Gamma)} \leqslant Ca(u, u) = C\langle Kg, g\rangle_\Gamma$$

$$\| Kg \|_{H^{-1/2}(\Gamma)} \leqslant C \| g \|_{H^{1/2}(\Gamma)}$$

which proves that K is a continuous operator.

(ii) We now prove the estimate (4.4a). By definition we have $\langle Kg, g\rangle_\Gamma = a(u, u)$. If $\alpha \neq 0$, or if $\alpha = 0$ and $N = 3$, $a(u, u)$ is a norm on $V(\Omega')$ equivalent to the norm $\| \; \|_V$. Hence,

$$\langle Kg, g\rangle_\Gamma \geqslant \alpha \| u \|^2_V \geqslant \alpha \| g \|^2_{H^{1/2}(\Gamma)}.$$

(iii) If u is the solution of (4.3) with Dirichlet boundary data g, $u + a$ is the solution of (4.3) with Dirichlet boundary data $g + a$, and

$$\frac{\partial u}{\partial n} = \frac{\partial (u + a)}{\partial n}.$$

The operator K then extends to $H^{1/2}(\Gamma)_{/\mathbb{R}}$ by $K\dot{g} = -\partial u/\partial n$. In formula (4.5) we can choose $v = 1$, which gives $\langle Kg, 1\rangle_\Gamma = 0$, and proves that Kg belongs to $(H^{1/2}(\Gamma)_{/\mathbb{R}})'$. For any $(\dot{g}, \dot{h})$ in $H^{1/2}(\Gamma)_{/\mathbb{R}}$, we can define $\langle K\dot{g}, \dot{h}\rangle_\Gamma = \langle Kg, h\rangle_\Gamma$ where g and h are any elements in the class. To prove the coerciveness we write:

$$\langle K\dot{g}, \dot{g}\rangle_\Gamma = \langle Kg, g\rangle_\Gamma = \| \nabla u \|^2 \geqslant C \| \dot{u} \|^2_{V/\mathbb{R}} \geqslant \beta \| \dot{g} \|^2_{H^{1/2}(\Gamma)/\mathbb{R}}.$$

(iv) $\langle Kg, h\rangle_\Gamma = a(u, v)$ for any continuous extension v of g in $V(\Omega')$. A good candidate is clearly v, the solution of the exterior problem in Ω':

$$-\Delta v + \alpha^2 v = 0 \text{ in } \Omega', \qquad v = h \text{ on } \Gamma.$$

In the same way, we have

$$\langle Kh, g\rangle_\Gamma = a(v, u),$$

and since a is symmetric, K is symmetric.

The operator K provides the transparent boundary condition for the steady state. We introduce the problem in Ω

$$\begin{aligned} -\Delta v + \alpha^2 v &= f \qquad \text{in } \Omega, \\ \frac{\partial v}{\partial n} + Kv &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{4.6}$$

Theorem 4.2. *If f belongs to $L^2(\Omega)$ (and in addition $\int_\Omega f(x)\,\mathrm{d}x = 0$ if $\alpha = 0$ and $N = 2$), problem (4.6) has a unique solution v in $H^1(\Omega)$ ($H^1(\Omega)_{/\mathbb{R}}$ if $\alpha = 0$ and $N = 2$) which is the restriction to Ω of v^*.*

Proof. We write (4.6) in the variational form:

$$v \in H^1(\Omega)$$

$$b(v, w) = L(w), \qquad \forall w \in H^1(\Omega)$$

where the bilinear form $b(.,.)$ and the linear form $L(.)$ are given by

$$b(v,w) = (\nabla v, \nabla w) + \alpha^2(v,w) + \langle Kv, w\rangle_\Gamma$$
$$L(w) = (f, w).$$

In all cases, b is bilinear continuous on $H^1(\Omega)$.

If $\alpha = 0$, b is clearly coercive on $H^1(\Omega)$. If $\alpha = 0$ and $N = 3$, we have

$$b(v,v) = \|\nabla v\|^2 + \langle Kv, v\rangle_\Gamma \geqslant C(\|\nabla v\|^2 + \|v\|^2_{H^{1/2}(\Gamma)}).$$

The last term on the right defines on $H^1(\Omega)$ a norm $|||\ \ |||$, equivalent to the norm $\|\ \ \|_1$. The proof is classical, and goes by contradiction. In these two cases we conclude from the Lax–Milgram theorem that (4.7) admits a unique solution.

If $\alpha = 0$ and $N = 2$, we need to introduce $\tilde{H} = H^1(\Omega)_{/\mathbb{R}}$, and define on $\tilde{H}$ the bilinear form b by

$$b(\dot{u}, \dot{v}) = (\nabla u, \nabla v) + \langle Ku, v\rangle_\Gamma \qquad \dot{u}, \dot{v} \in \tilde{H},$$

where u and v are any elements in $\dot{u}$ and $\dot{v}$. It is easy to see that the definition is independent of the elements u and v chosen in the class. The bilinear form b is clearly continuous on $\tilde{H}$, and

$$b(\dot{u}, \dot{u}) = \|\nabla u\|^2 + \langle Ku, u\rangle_\Gamma \geqslant \|\nabla u\|^2 + \beta\|\dot{u}\|^2_{H^{1/2}(\Gamma)/\mathbf{R}} \geqslant C\|\dot{u}\|^2_{H^1(\Omega)/\mathbf{R}}$$

as before.

The bilinear form b fulfils the assumptions for the Lax–Milgram lemma, and L is continuous on $H^1(\Omega)_{/\mathbb{R}}$ since $\int_\Omega f(x)\,\mathrm{d}x = 0$. This completes the study of (4.6). By uniqueness, v is clearly the restriction of v^* to Ω.

5. Long-time behaviour in Ω for the homogeneous wave equation

We consider the homogeneous wave equation

$$u_{tt} - \Delta u + \alpha^2 u = 0 \qquad \text{in } \Omega \times \mathbb{R}_+ \tag{5.1a}$$

$$u(0) = u^0, \qquad u_t(0) = u^1 \tag{5.1b}$$

$$u_t + u_n + Ku = 0 \qquad \text{on } \Gamma \times \mathbb{R}_+ \tag{5.1c}$$

We first establish the well-posedness of problem (5.1). We denote by E the corresponding energy:

$$E(u,t) = \|\nabla u\|^2 + \|u_t\|^2 + \alpha^2\|u\|^2 + \langle Ku, u\rangle_\Gamma. \tag{5.2}$$

Theorem 5.1. *If $E(u,0) < +\infty$, problem (5.1) has a unique solution u such that $u \in \mathscr{C}^0(\mathbb{R}_+, H^1(\Omega))$, $u_t \in \mathscr{C}^0(\mathbb{R}_+, L^2(\Omega))$. Furthermore if (u^0, u^1) belongs to $H^2(\Omega) \times H^1(\Omega)$, the following identity holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u,t) + \|u_t\|^2_{L^2(\Gamma)} = 0. \tag{5.3}$$

The proof is again classical, and relies on the analysis in [13], as in section 2. The energy estimate is obtained by multiplying (5.1a) by u_t and integrating by parts, using (5.1c).

We shall now examine the long-time behavior of u.

Theorem 5.2. *Suppose Γ is $\mathscr{C}^\infty$. If $E(u,0) < +\infty$, there exist two constants M and μ positive such that*

$$E(u,t) \leqslant M e^{-\mu t} E(u,0) \qquad \forall t > 0.$$

This result is not at all straightforward. Such estimates have been proved by Morawetz [15] in the case of exterior domains by using multipliers, and extended in [5, 11] and other work to boundary conditions of the form $u_t + u_n + \lambda u = 0$, where λ is a non-negative constant. It has recently been extended in [3] to more general geometries using microlocal analysis. It is closely related to the exact controllability of the wave equation through the Robin action (see [12]).

The result for our operator K has recently been proved in [4] again using microlocal analysis.

Remark 5.1. Consider the case where $\alpha \neq 0$, with the classical absorbing boundary condition $u_t + u_n = 0$. The energy decreases exponentially in time. Let us define on $H^1(\Omega) \times L^2(\Omega)$ the operator $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ \Delta - \alpha^2 I & 0 \end{pmatrix}$$

$$D(\mathbf{A}) = \{\mathbf{w} \in H^2(\Omega) \times H^1(\Omega), v + u_n = 0\}.$$

Problem (5.1) can be rewritten as

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{A}\mathbf{w}$$

with

$$\mathbf{w}(0) = \mathbf{w}^0 = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}.$$

$\mathbf{A}$ is maximal dissipative, and thus generates a semi-group of contractions: $S(t)$. It is easy to see that $\mathbf{A}$ has a compact resolvant. Therefore the spectrum of $\mathbf{A}$ consists entirely of isolated eigenvalues with finite multiplicity, contained in the half-plane $\operatorname{Re}\lambda \leqslant 0$ (see [20]). Moreover, following [14], it can be proved that there exist four positive constants C_i, $1 \leqslant i \leqslant 4$, such that the eigenvalues of $\mathbf{A}$ lie in a region defined by

$$\{\lambda / |\operatorname{Re}\lambda| \leqslant C_1 |\operatorname{Im}\lambda|^{3/4} + C_2\} \cup \{\lambda / |\operatorname{Im}\lambda| \leqslant C_3 |\operatorname{Re}\lambda|^{1/4} + C_4\}.$$

Eventually, we can prove that there is no eigenvalue lying on the imaginary axis. Let us suppose indeed that $i\lambda$ is an eigenvalue, and u an associated eigenfunction. They satisfy the following equations:

$$\lambda \in \mathbb{R}, \qquad u \in H^2,$$

$$\Delta u + (\lambda^2 - \alpha^2) u = 0 \qquad \text{in } \Omega,$$

$$u = u_n = 0 \qquad \text{on } \Gamma.$$

Thus, by a uniqueness result for the principally normal operators with an elliptic principal part, ([9], p. 224) we conclude that $u \equiv 0$.

The eigenvalues of $\mathbf{A}$ are hence concentrated in the region given in Fig. 1. There is no purely oscillatory mode, unlike in the case of the Neumann or Dirichlet boundary

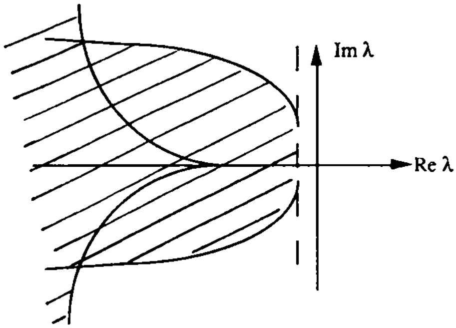

Fig. 1.

condition, and the imaginary part of the eigenvalues is bounded. However, and unfortunately, we cannot conclude that exponential decay results from that study, since the semi-group could possibly generate a continuous spectrum [17]. The exponential decay in time proves that even if S has a continuous spectrum, it is strictly included in the disc of radius 1.

Remark 5.2. The best constant μ in Theorem 5.2 can be given explicitly in the special case where B corresponds to the absorbing boundary condition (3.8), [10].

6. The new boundary condition: analysis

It was recalled in section 2 that the local energy of $u^* - v^*$ converges to zero when time tends to infinity. It is thus desirable to produce a boundary condition on Γ, such that the solution of the wave equation with this boundary condition converges to the solution v of problem (2.5). Consider the inhomogeneous wave equation with the boundary condition (1.4)

$$u_{tt} - \Delta u + \alpha^2 u = f(x) \qquad \text{in } \Omega \times [0, T], \tag{6.1a}$$

$$u(0) = u^0, \qquad u_t(0) = u^1, \tag{6.1b}$$

$$u_t + u_n + Ku = 0 \qquad \text{on } \Gamma \times [0, T], \tag{6.1c}$$

Theorem 6.1. *If f belongs to $L^2(\Omega)$ and if $E(u, 0) < +\infty$, problem (6.1) has a unique solution u such that $u \in \mathscr{C}^0([0, T], H^1(\Omega))$, $u_t \in \mathscr{C}^0([0, T], L^2(\Omega))$.*

The proof is exactly the same as in the previous sections. Consider now the 'steady state':

$$-\Delta \tilde{u} + \alpha^2 \tilde{u} = f \qquad \text{in } \Omega, \tag{6.2a}$$

$$\tilde{u}_n + K\tilde{u} = 0 \qquad \text{on } \Gamma. \tag{6.2b}$$

A straightforward consequence of Theorem 5.2 is the following:

Corollary 6.2. *Under the hypothesis of Theorem 6.1, the energy of $u - \tilde{u}$ converges exponentially to zero as t tends to infinity.*

This result shows that the simplest way to force the convergence of u to the steady state v^* defined in section 2 is to impose on Γ the boundary condition:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial n}+Ku=0, \tag{6.3}$$

This is our new boundary condition.

Remark 6.1. Suppose we replace K by a multiplication operator, $Bu=a(x)u$ with $a(x)\geqslant 0$. According to [3], u converges (in the energy semi-norm), as t tends to infinity, to the solution of

$$\begin{aligned} -\Delta\tilde{u}+\alpha^2\tilde{u}&=f && \text{in } \Omega,\\ \tilde{u}_n+a(x)\tilde{u}&=0 && \text{on } \Gamma, \end{aligned}$$

which is in general different from v^*. However, for the special choice of boundary condition (3.8), that is $a(x)=\frac{1}{2}H(x)$, $\tilde{u}$ can be 'close' to v^* if Ω is sufficiently large (see [2]).

The approximate problem consisting of the wave equation in Ω coupled with the long-time boundary condition (6.3) can be formulated as a transmission problem in $\mathbb{R}^N$:

$$u_{tt}-\Delta u+\alpha^2 u=f(x) \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{6.4a}$$

$$-\Delta w+\alpha^2 w=0 \qquad \text{in } \Omega'\times\mathbb{R}_+, \tag{6.4b}$$

with the initial values

$$u(0)=u^0, \qquad u_t(0)=u^1 \qquad \text{in } \Omega \tag{6.4c}$$

and the transmission conditions

$$u=w, \qquad u_t+u_n-w_n=0 \qquad \text{on } \Gamma. \tag{6.4d}$$

We have just proved that u converges to v (in the energy semi-norm) and we have reached a part of our goal. It remains to see whether (6.3) is still absorbing (in the sense defined in remark 3.1). The following theorem gives a representation of the boundary condition (6.3) as a product of an absorbing boundary condition for the transient state, and the transparent boundary condition for the steady state. This indicates its good properties for both short- and long-time computation (for one-dimensional numerical tests see [6]).

Theorem 6.3. *If u is the solution of the wave equation* (6.1a) *in Ω, the following identity holds on the boundary*:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial n}+\frac{\partial u}{\partial t}+Ku\right)=\left(\frac{\partial}{\partial n}+\frac{\partial}{\partial t}+H\right)\left(\frac{\partial u}{\partial n}+Ku\right) \tag{6.5}$$

where H stands for the mean curvature on Γ.

Proof. Let us first recall that by the definition of K we have

$$\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n}+Ku\right)=\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n}-\frac{\partial w}{\partial n}\right)$$

where w is the solution of the exterior problem

$$
\begin{aligned}
-\Delta w + \alpha^2 w = 0 \qquad & \text{in } \Omega', \\
w = u \qquad & \text{on } \Gamma.
\end{aligned}
$$

We now need some results in differential geometry. The classical definitions on Γ, and the construction of the extension in Ω, as well as the divergence lemma, can be found in [1].

From now on we suppose that Γ is compact. This ensures the existence of a finite atlas on Γ, $\mathscr{F} = \cup (\omega, \psi)$. ω is an open subset of Γ, and ψ is defined on an open subset $\hat{\omega}$ of $\mathbb{R}^{N-1}$: $\psi : \hat{\omega} \to \omega$ is a system of local coordinates. The tangent space $T(x)$ to Γ at point x is generated by the vectors

$$
\mathbf{e}_\alpha(x) = \frac{\partial \psi}{\partial y^\alpha}(y) \qquad \alpha = 1, \ldots, N-1;\ x = \psi(y). \tag{6.6}
$$

Furthermore we suppose that the orientation of $T(x)$ and Γ are consistent, that is $(\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}, \mathbf{n})$ is positive.

Since Γ is compact, it is well known that one can define a neighbourhood of Γ by

$$
\begin{aligned}
&\Phi : \Gamma \times]-\varepsilon, \varepsilon[\to U_\varepsilon \\
&(x, v) \to \Phi(x, v) = x + v\mathbf{n}(x). \\
&U_\varepsilon = \{x \in \mathbb{R}^N, \operatorname{Inf}|x - \tilde{x}| < \varepsilon\} \qquad \tilde{x} \in \Gamma.
\end{aligned}
$$

The normal vector $\mathbf{n}$, defined on Γ can be extended to U_ε as a smooth function, compactly supported, with a norm identically one, close to Γ. This extension of $\mathbf{n}$ will enable us to define

$$
\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} - \frac{\partial}{\partial n}\right) \qquad \text{on } \Gamma.
$$

We introduce on Γ the metric tensor $g_{\alpha\beta}$ associated with the system of normal coordinates $(\mathbf{e}_\alpha)_{1 \leq \alpha \leq N-1}$ defined by

$$
g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta \tag{6.7}
$$

and the tensor of curvature defined by

$$
\frac{\partial \mathbf{n}}{\partial y^\alpha} = C_\alpha^\beta \mathbf{e}_\alpha. \tag{6.8}
$$

From C is derived the mean curvature H from

$$
H(x) = \sum_{\alpha=1}^{N-1} C_\alpha^\alpha. \tag{6.9}
$$

If Γ is the sphere of radius R, $H(x)$ is equal to $(N-1)/R$.

We can now define the tangential gradient and divergence: if h is a smooth function defined on Γ, they are given by

$$
\overrightarrow{\operatorname{grad}}_\Gamma h(x) = \sum_{\alpha,\beta} g^{\alpha\beta} \frac{\partial h(y)}{\partial y^\beta} \mathbf{e}_\alpha; \qquad x = \psi(y), \tag{6.10}
$$

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$, and if $\mathbf{u}$ is a vector tangent to Γ then

$$\operatorname{div}_\Gamma \mathbf{u}(x) = \frac{1}{\sqrt{g}} \sum_\alpha \frac{\partial}{\partial y^\alpha} (\sqrt{g}\, u^\alpha)(y); \qquad x = \psi(y), \tag{6.11}$$

where g is the determinant of $(g_{\alpha\beta})$.

We shall denote by γ_0 the trace on Γ, Π the tangential trace on Γ, γ_1 the normal derivative:

$$\begin{aligned} &\gamma_0 \varphi = \varphi_{/\Gamma}, \\ &\Pi \mathbf{u} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \\ &\gamma_1 \varphi = (\overrightarrow{\operatorname{grad}}\, \varphi)\cdot \mathbf{n}. \end{aligned} \tag{6.12}$$

Using the local coordinates in U_ε, it is easy to see that, if φ is a smooth function defined in Ω, we have

$$\Pi\, \overrightarrow{\operatorname{grad}}\, \varphi = \overrightarrow{\operatorname{grad}}_\Gamma\, \gamma_0 \varphi. \tag{6.13}$$

We are now able to link the divergence and the tangential divergence:

Lemma 6.4. *If* $\mathbf{u}$ *is a vector in* $C_0^\infty(U_\varepsilon)$, *one has*

$$\gamma_0 \operatorname{div} \mathbf{u} = \operatorname{div}_\Gamma \Pi \mathbf{u} + H \mathbf{u}\cdot\mathbf{n} + \gamma_1(\mathbf{u}\cdot\mathbf{n}). \tag{6.14}$$

The proof is straightforward and relies on the use of the local coordinates to write the divergence.

We now use the lemma to define:

$$\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}\right).$$

Lemma 6.5. *If* u *is a solution of* (6.4a) *belonging to* $C_0^\infty(\bar{\Omega})$, *and if* w *is the solution of* (6.4b) *belonging to* $C_0^\infty(\bar{\Omega}^c)$, *then*

$$\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}\right) + H\left(\frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}\right) = \gamma_0 u_{tt}. \tag{6.15}$$

Proof of Lemma 6.5. We write successively (6.14) with $\mathbf{u}$ equal to $\overrightarrow{\operatorname{grad}}\, u$ and $\overrightarrow{\operatorname{grad}}\, w$, using (6.13):

$$\gamma_0 \Delta u = \operatorname{div}_\Gamma(\overrightarrow{\operatorname{grad}}_\Gamma\, \gamma_0 u) + H\frac{\partial u}{\partial n} + \gamma_1\left(\frac{\partial u}{\partial n}\right),$$

$$\gamma_0 \Delta w = \operatorname{div}_\Gamma(\overrightarrow{\operatorname{grad}}_\Gamma\, \gamma_0 w) + H\frac{\partial w}{\partial n} + \gamma_1\left(\frac{\partial w}{\partial n}\right),$$

since $u = w$ on Γ, their tangential gradients are equal. Moreover, $\Delta u - \Delta w = u_{tt} + u - w = u_{tt}$ on Γ, which completes the proof of the lemma.

This result allows us to establish (6.5), using the following identity:

$$\left(\frac{\partial}{\partial n} + \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial n} + Ku\right) \equiv \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial n} + Ku + \frac{\partial u}{\partial t}\right) - H\left(\frac{\partial u}{\partial t} + Ku\right).$$

Remark 6.2. The absorbing boundary condition in (6.5) differs from the one used in (3.8) by lower terms only. It is the same on flat boundaries.

Remark 6.3. The new boundary condition is global along the boundary, and the operator K is, in general, not known explicitly. There are several ways to overcome this difficulty in numerical computations: Fourier series in special cases, use of Lagrange multipliers, etc. See [6] and references therein for details.

Remark 6.4. All the results developed in the previous sections carry over without important modifications if, instead of $\mathbb{R}^N$, we consider an exterior domain $\tilde{\Omega}$, with a Dirichlet boundary condition $u^*{}_{/\tilde{\Gamma}} = g$.

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