From the Quasi-static to the Dynamic Maxwell's Model in Micromagnetism

Laurence Halpern* Stéphane Labbé†

Abstract

A commonly used model for ferromagnetic materials in the quasistatic regime is the Landau-Lifshitz system coupled with the so-called quasistatic Maxwell's equations. By an appropriate scaling, we justify this approach and we propose a new asymptotic expansion. This suggest a new numerical method.

1 The micromagnetism model

The magnetic material fills a bounded domain Ω in $\mathbb{R}^3$. The evolution of the magnetization field is governed by the Landau-Lifshitz system

$$\frac{\partial \mathbf{M}}{\partial T} = -\gamma\mu_0 \left(\mathbf{M} \times \mathbf{H}_T + \frac{\alpha}{|\mathbf{M}|}\mathbf{M} \times (\mathbf{M} \times \mathbf{H}_T) \right) \text{ in } \Omega, \tag{1}$$

with initial condition $\mathbf{M}^{(0)}$. $\mathbf{M}$ is the magnetization field ; it vanishes outside Ω, and has a prescribed length in Ω

$$|\mathbf{M}(\mathbf{X}, T)| = |\mathbf{M}(\mathbf{X}, 0)| = M_S \text{ a.e in } \Omega. \tag{2}$$

μ_0 is the magnetic permeability, γ the Larmor precession factor, and α a dimensionless dumping factor. They are all positive factors. The total magnetic field $\mathbf{H}_T$ is a linear function of $\mathbf{M}$. It is the sum of three magnetic contributions (we consider the external field to be zero) : the exchange field $\mathbf{H}_{ex} = A\Delta\mathbf{M}$, the anisotropy field $\mathbf{H}_a = -K\,\mathbf{u} \times (\mathbf{M} \times \mathbf{u})$, where $\mathbf{u}$ is the direction of anisotropy, and K and A are physical positive constants. Finally the Maxwell's field $\mathbf{H}$ solves the system of equations whose unknowns are the magnetic field $\mathbf{H}$, the electric field $\mathbf{E}$, and the electrostatic charge ρ, the magnetization field $\mathbf{M}$ being given

$$\begin{aligned} &\varepsilon(\mathbf{X})\frac{\partial \mathbf{E}}{\partial T} + \sigma(\mathbf{X})\mathbf{E} - \textbf{rot } \mathbf{H} = \mathbf{0}, \\ &\mu_0 \frac{\partial}{\partial T}(\mathbf{H} + \mathbf{M}) + \textbf{rot } \mathbf{E} = \mathbf{0}, \end{aligned} \tag{3}$$

with prescribed initial conditions. Furthermore we have for all time the following constraints

$$\textbf{div } (\varepsilon(\mathbf{X})\mathbf{E}) = \rho, \quad \textbf{div } (\mu_0(\mathbf{H} + \mathbf{M})) = 0. \tag{4}$$

*University of Paris 13, Département de Mathématiques, Institut Galilée,93430 Villetaneuse, France, halpern@math.univ-paris13.fr .

†University of Paris 11, Laboratoire de Mathématique, Bat. 425, 91405 Orsay, France, labbe@math.u-psud.fr.

ε_0 is the permittivity in the vacuum, ε_r the relative permittivity of the material, and the value of $\varepsilon(x)$ is $\varepsilon_0\varepsilon_r$ in Ω, ε_0 in the exterior. $\sigma(x)$ is the conductivity of the material; it vanishes outside Ω.

The total field is thus given by

$$\mathbf{H}_T(\mathbf{M}) = -K\,\mathbf{u}\times(\mathbf{M}\times\mathbf{u}) + A\Delta\mathbf{M} + \mathbf{H}(\mathbf{M}). \tag{5}$$

The system we consider here is composed of (1,...,5), with the mandatory constraints (4) and initial conditions.

2 Two scalings for the micromagnetism model

We perform the following scaling

$$\mathbf{H} = \bar{h}\hat{\mathbf{h}},\ \mathbf{E} = \bar{e}\hat{\mathbf{e}},\ \rho = \bar{\rho}\hat{R},\ \mathbf{M} = \bar{m}\hat{\mathbf{m}}, \tag{6}$$

and the change of variables

$$\mathbf{X} = \bar{x}\,\mathbf{x},\ T = \bar{t}\,t, \tag{7}$$

where $\bar{x}$ and $\bar{t}$ are the characteristic length and time. By homogeneity, we have the following relations :

$$\bar{m} = \bar{h},\ \mu_0\frac{\bar{h}}{\bar{t}} = \frac{\bar{e}}{\bar{x}},\ \frac{\varepsilon_0\bar{e}}{\bar{x}} = \bar{\rho}. \tag{8}$$

The dimensionless Landau and Lifchitz system is

$$\frac{\partial\hat{\mathbf{m}}}{\partial t} = -\gamma\mu_0\bar{m}\left(\hat{\mathbf{m}}\times\hat{\mathbf{h}}_T + \frac{\alpha}{|\hat{\mathbf{m}}|}\hat{\mathbf{m}}\times(\hat{\mathbf{m}}\times\hat{\mathbf{h}}_T)\right)\ \text{in } \omega, \tag{9}$$

with the constraint $|\hat{\mathbf{m}}(\mathbf{x},t)| = \frac{M_S}{\bar{m}}$ a.e in ω.

By homogeneity in the Landau-Lifshitz system, and linearity in $\mathbf{H}_T$, there appears a new scale $\zeta = \bar{t}\gamma\mu_0$. Applying the new scaling to all variables,

$$\hat{\mathbf{h}} = \zeta\mathbf{h},\ \hat{R} = \zeta R,\ \hat{\mathbf{e}} = \zeta\mathbf{e},\ \hat{\mathbf{m}} = \zeta\mathbf{m}, \tag{10}$$

and choosing $\bar{m} = \bar{t}\gamma\mu_0 M_S$, system (9) becomes

$$\frac{\partial\mathbf{m}}{\partial t} = -\mathbf{m}\times\mathbf{h}_T - \alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{h}_T)\ \text{in } \omega, \tag{11}$$

with the constraint

$$|\hat{\mathbf{m}}(\mathbf{x},t)| = |\hat{\mathbf{m}}(\mathbf{x},0)| = 1\ \text{a.e in } \omega. \tag{12}$$

The total field $\mathbf{h}_T$ is given by the three contributions $\mathbf{h}_a = -K\,\mathbf{u}\times(\mathbf{m}\times\mathbf{u})$, $\mathbf{h}_{ex} = \bar{A}\Delta\mathbf{m} = \frac{A}{\bar{x}^2}\Delta\mathbf{m}$ and $\mathbf{h}$:

$$\mathbf{h}_T(\mathbf{m}) = -K\,\mathbf{u}\times(\mathbf{m}\times\mathbf{u}) + \bar{A}\Delta\mathbf{m} + \mathbf{h}(\mathbf{m}). \tag{13}$$

We set $\eta = \dfrac{\bar{x}}{c\bar{t}}$ where c is the speed of light . In our context, the length of Ω is supposed to be small with respect to the wavelengths. Thus the parameter η is small. With these notations, the Maxwell's system of equations becomes

$$\begin{aligned} &\eta^2\tilde{\varepsilon}\frac{\partial \mathbf{e}}{\partial t}+\eta\tilde{\sigma}\mathbf{e}-\mathbf{rot}\ \mathbf{h}=\mathbf{0},\\ &\frac{\partial}{\partial t}(\mathbf{h}+\mathbf{m})+\mathbf{rot}\ \mathbf{e}=\mathbf{0},\\ &\mathbf{div}\ (\mathbf{h}+\mathbf{m})=0,\quad \mathbf{div}\ (\tilde{\varepsilon}\mathbf{e})=R, \end{aligned} \tag{14}$$

with *ad hoc* initial values. The problem is now ready for asymptotic expansion. Note that it is a kind of singular perturbation for the electric and magnetic fields, in the time variable.

3 Asymptotic expansion for the Maxwell's system

We place ourselves in the linear case, where the magnetization field $\mathbf{m}$ is given, and we consider the system (14). For other asymptotic expansions and scaling concerning Maxwell's equations see [1] and [3]. The well-posedness can be shown using the theory of semi-groups.

We expand now R and $\mathbf{m}$ as functions of η,

$$R=\sum_{i=0}^{\infty}\eta^i\ R_i,\quad \mathbf{m}=\sum_{i=0}^{\infty}\eta^i\ \mathbf{m}_i, \tag{15}$$

and we search for $\mathbf{e}$ and $\mathbf{h}$ such that

$$\mathbf{e}=\sum_{i=0}^{\infty}\eta^i\ \mathbf{e}_i,\quad \mathbf{h}=\sum_{i=0}^{\infty}\eta^i\ \mathbf{h}_i. \tag{16}$$

Inserting these expansions into the system (14), we obtain first the so-called quasi-static Maxwell's system

$$\begin{cases} \mathbf{div}\ (\mathbf{h}_0+\mathbf{m}_0)=0,\ \mathbf{rot}\ \mathbf{h}_0=\mathbf{0},\\ \mathbf{rot}\ \mathbf{e}_0=-\dfrac{\partial}{\partial t}(\mathbf{m}_0+\mathbf{h}_0),\ \mathbf{div}\ (\tilde{\varepsilon}\mathbf{e}_0)=R_0, \end{cases} \tag{17}$$

and a sequence of systems for $k\geq 1$

$$\begin{cases} \mathbf{div}\ (\mathbf{h}_k+\mathbf{m}_k)=0,\ \mathbf{rot}\ \mathbf{h}_k=\tilde{\varepsilon}\dfrac{\partial \mathbf{e}_{k-2}}{\partial t}+\tilde{\sigma}\mathbf{e}_{k-1},\\ \mathbf{rot}\ \mathbf{e}_k=-\dfrac{\partial}{\partial t}(\mathbf{h}_k+\mathbf{m}_k),\ \mathbf{div}\ (\tilde{\varepsilon}\mathbf{e}_k)=R_k. \end{cases} \tag{18}$$

(with the convention $\mathbf{e}_{-2}=\mathbf{e}_{-1}=\mathbf{0}$). Using the Helmholtz decomposition in weighted Sobolev spaces, we proved

THEOREM 3.1. *Suppose $\mathbf{m}_k$ belongs to $\mathcal{C}^{p-k+1}(\mathbb{R}^+;\mathbb{L}^2(\omega))$ and R_k belongs to $\mathcal{C}^{p-k}(\mathbb{R}^+,\mathbb{L}^2(\omega))$ for $0\leq k\leq p$. Then problems (17)and (18) have a unique solution $(\mathbf{h}_k,\mathbf{e}_k)$ in $\mathcal{C}^{p-k+1}(\mathbb{R}^+,\mathbb{L}^2(\mathbb{R}^3))\times\mathcal{C}^{p-k}(\mathbb{R}^+,\mathbb{L}^2(\mathbb{R}^3))$ for $0\leq k\leq p$.*

We verify now that the asymptotic expansions really approximate the fields. Let $\check{\mathbf{h}}_p$ and $\check{\mathbf{e}}_p$ be the partial sums, $\tilde{\mathbf{h}}_p$ and $\tilde{\mathbf{e}}_p$ denote the errors, *i.e.*

$$\check{\mathbf{e}}_p=\sum_{i=0}^{p}\eta^i\ \mathbf{e}_i,\tilde{\mathbf{e}}_p=\mathbf{e}-\check{\mathbf{e}}_p;\quad \check{\mathbf{h}}_p=\sum_{i=0}^{p}\eta^i\ \mathbf{h}_i,\tilde{\mathbf{h}}_p=\mathbf{h}-\check{\mathbf{h}}_p. \tag{19}$$

The errors satisfy, for any $p\geq 0$, a system of the type

$$
(20) \quad \begin{cases} \eta^2 \tilde{\varepsilon} \dfrac{\partial \tilde{\mathbf{e}}_p}{\partial t} + \eta \tilde{\sigma} \tilde{\mathbf{e}}_p - \quad \mathbf{rot}\ \tilde{\mathbf{h}}_p = \ 0(\eta^{p+1}), \\ \dfrac{\partial \tilde{\mathbf{h}}_p}{\partial t} + \mathbf{rot}\ \tilde{\mathbf{e}}_p = \ 0(\eta^{p+1}), \\ \tilde{\mathbf{h}}_p(\mathbf{x}, 0) = \tilde{\mathbf{e}}_p(\mathbf{x}, 0) = \mathbf{0}. \end{cases}
$$

We obtain hyperbolic estimates by multiplying the first equation by $\tilde{\mathbf{e}}$, the second by $\tilde{\mathbf{h}}$, and using Green's formula. The Gronwall lemma leads to the conclusion

THEOREM 3.2. *For any $p \geq 1$, the following error estimates hold : for any positive time τ, there exists a constant C such that*

$$
(21) \quad \begin{array}{l} ||\tilde{\mathbf{h}}_p||_{\mathbb{L}^\infty(0,\tau;\mathbb{L}^2(\mathbb{R}^3))} \leq C\eta^p, \\ ||\tilde{\mathbf{e}}_p||_{\mathbb{L}^\infty(0,\tau;\mathbb{L}^2(\mathbb{R}^3))} \leq C\eta^{p-1},\ ||\tilde{\mathbf{e}}_p||_{\mathbb{L}^2(0,\tau;\mathbb{L}^2(\omega))} \leq C\eta^{p-\frac{1}{2}}. \end{array}
$$

For $p = 0$, the error estimates are weaker : for any positive time τ, there exists a constant C such that

$$
(22) \quad \begin{array}{l} ||\tilde{\mathbf{h}}_0||_{\mathbb{L}^\infty(0,\tau;\mathbb{L}^2(\mathbb{R}^3))} \leq C\sqrt{\eta},\ |\tilde{\mathbf{h}}_0|_{\mathbb{L}^\infty(0,\tau;\mathbb{H}^1(\mathbb{R}^3))} \leq C\eta, \\ ||\mathbf{rot}\ (\tilde{\varepsilon}\tilde{\mathbf{e}}_0)||_{\mathbb{L}^2(0,\tau;\mathbb{L}^2(\omega))} \leq C\eta. \end{array}
$$

4 Asymptotic developpement for the Micromagnetism system coupled with the Maxwell's model

We come back now to the Landau-Lifshitz system (11). Theorems of existence and comments on uniqueness can be found in [5].

The magnetization field $\mathbf{m}$ is now an unknown, with initial value independent of η. Inserting expansions (15) and (16) into (11) and (14), we obtain the first term

$$
(23) \quad \begin{array}{l} \dfrac{\partial \mathbf{m}_0}{\partial t} = -\mathbf{m}_0 \times \mathbf{h}_{T,0} - \alpha \mathbf{m}_0 \times (\mathbf{m}_0 \times \mathbf{h}_{T,0}) \text{ in } \omega, \\ |\mathbf{m}_0| = 1, \mathbf{m}_0(\mathbf{x}, 0) = \mathbf{m}^{(0)}(\mathbf{x}), \text{a.e. in } \omega, \end{array}
$$

and

$$
(24) \quad \mathbf{h}_{T,0} = -K\ \mathbf{u} \times (\mathbf{m}_0 \times \mathbf{u}) + \bar{A}\Delta \mathbf{m}_0 + \mathbf{h}_0
$$

where $\mathbf{h}_0$ is given by the quasi-static Maxwell's system in (17).

The problem (23) is proved to be well-posed in [4]. We first give an energy estimate on the solution to (11) and (14).

THEOREM 4.1. *Let $(\mathbf{m}, \mathbf{e}, \mathbf{h})$ solve the equations (11) and (14). The following energy estimate holds*

$$
(25) \quad \begin{array}{l} \dfrac{1}{2}\dfrac{d}{dt}[\eta^2 \displaystyle\int_{\mathbb{R}^3} \tilde{\varepsilon}|\mathbf{e}|^2\,dx + \int_{\mathbb{R}^3} |\mathbf{h}|^2\,dx + \int_{\mathbb{R}^3} \tilde{A}|\mathbf{grad}\ \mathbf{m}|^2\,dx + \int_{\mathbb{R}^3} K|\mathbf{u}\cdot\mathbf{m}|^2\,dx] \\ \qquad + \eta \displaystyle\int_\omega \tilde{\sigma}|\mathbf{e}|^2\,dx + \int_\omega |\mathbf{m} \times \mathbf{h}_T|^2\,dx = 0. \end{array}
$$

With these estimates, we can prove convergence

THEOREM 4.2. *The solution $(\mathbf{m}, \mathbf{h})$ to (11) (14) converges weak-* to the solution $(\mathbf{m}_0, \mathbf{h}_0)$ of the quasistatic model (23) in $\mathbb{L}^\infty(0,\tau;\mathbb{H}^1(\mathbb{R}^3))$ as η tends to 0(modulo the extraction of a subsequence).*

The proof mimics the proof by Carbou in [2] for the convergence of the complete system towards the quasistatic system as the permittivity ε_0 tends to 0. But we still do not approximate the electric field. Therefore we introduce the other terms in the expansion.

They are given for any $n \geq 0$ by

$$(26)\qquad \begin{aligned} &\frac{\partial \mathbf{m}_n}{\partial t} = -\sum_{k+l=n} \mathbf{m}_k \times \mathbf{h}_{T,l} - \alpha \sum_{k+l=n} \sum_{i+j=k} \mathbf{m}_l \times (\mathbf{m}_i \times \mathbf{h}_{T,j}) \text{ in } \omega, \\ &\sum_{k+l=n} \mathbf{m}_k \cdot \mathbf{m}_l = 0, \mathbf{m}_n(\mathbf{x}, 0) = 0, \text{a.e. in } \omega \end{aligned}$$

and

$$(27)\qquad \mathbf{h}_{T,j} = -K\,\mathbf{u} \times (\mathbf{m}_j \times \mathbf{u}) + \bar{A}\Delta \mathbf{m}_j + \mathbf{h}_j$$

where $\mathbf{h}_j$ is given by (18).

It is a linear equation which can be shown to be well-posed. There is no proof of convergence today.

5 A dynamical method of simulation using finite volume

The idea is to compute the partial sums $(\check{\mathbf{m}}_n, \check{\mathbf{h}}_n, \check{\mathbf{e}}_n)$. Using the fact that for all i in $\mathbb{N}$, $(\mathbf{e}_i, \mathbf{h}_i, \mathbf{m}_i)$ depend only on $(\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)$ for $j \leq i$, we compute the $(\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)$ successively. At each level, we use the same finite volume method in space, but a different scheme in time to compute $(\mathbf{e}_j^n, \mathbf{h}_j^n, \mathbf{m}_j^n)$, approximation of $(\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)$ at time t_n.

For each time step t_n, $(\mathbf{e}_0^n, \mathbf{h}_0^n, \mathbf{m}_0^n)$ is first computed, by an explicit second order Taylor scheme in time for the system (26). It is proved in [4] that there exists a unique time step Δt_n such that the scheme is stable and has optimal convergence. $\mathbf{e}_0^n$ and $\mathbf{h}_0^n$ are obtained by solving a Laplace equation in ω.

Then $(\mathbf{e}_k^n, \mathbf{h}_k^n, \mathbf{m}_k^n)$ for $k > 0$ are computed successively by the following algorithm :

1. Prediction of $\mathbf{h}_k^n$ using $\mathbf{e}_{k-1}^n$, $\mathbf{e}_{k-2}^n$ and $\mathbf{m}_k^{n-1}$ with a first order implicit scheme in (18).
2. Computation of $\mathbf{m}_k^n$ using $\mathbf{h}_k^n$ with a first order implicit scheme in (26).
3. Correction of $\mathbf{h}_k^n$ using $\mathbf{m}_k^n$ in (18) .
4. Computation of $\mathbf{e}_k^n$ by solving (18) .

All the computations in space amount to solving a Poisson equation, for which we have fast solvers (see [4]). The presented algorithm provides an accurate method to compute the solutions to Landau-Lifshitz coupled with Maxwell's equations in ferromagnets.

References

[1] H. Ammari,A. Buffa,J-C. Nédélec, *A justification of eddy currents models for the Maxwell equations*, to appear in SIAM Journal of Appl. Math.

[2] G. Carbou, P. Fabrie, *Time average in micromagnetism*, J. Differ. Equations 147, 2 (1998), pp. 383-409

[3] P. Degond, P-A. Raviart, *An analysis of the Darwin model of approximation to Maxwell's equations*, Forum Math. 4, 1 (1992), pp. 13-44.

[4] S. Labbé,*Simulation numérique du comportement hyperfréquence des matériaux magnétiques*, thèse de l'Université Paris-Nord (1998).

[5] A. Visintin, *On Landau-Lifshitz equation for ferromagnetism*, Japan Journal of Appl. Math., 2 (1985), pp. 62-84.