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ARTIFICIAL BOUNDARY CONDITIONS FOR INCOMPLETELY PARABOLIC PERTURBATIONS OF HYPERBOLIC SYSTEMS*

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Abstract. Artificial boundary conditions are devised for small incompletely parabolic perturbations of hyperbolic systems, which are local, consistent with the hyperbolic equation, well posed, and produce weak boundary layers. The general strategy is applied to the Navier–Stokes system.

Key words. artificial boundary conditions, Navier–Stokes equations

AMS(MOS) subject classifications. 35B25, 35Q10, 35G10

Introduction. A general model for a fluid motion is the following time-dependent compressible Navier–Stokes system:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0,$$

$$\frac{\partial \rho v}{\partial t} + \operatorname{div}(\rho v \cdot v + pI) = \rho g + \operatorname{div} \mu\tau,$$

$$\frac{\partial e}{\partial t} + \operatorname{div}(e+p)v = \rho g \cdot v + \operatorname{div}(K \operatorname{grad} T + \mu v \cdot \tau),$$

where ρ represents the density, p the pressure, T the temperature, and v the velocity of the fluid. τ is the momentum flux density tensor due to friction: $\tau = -\frac{2}{3} I \operatorname{div} v + \operatorname{grad} v + (\operatorname{grad} v)^\tau$. μ and K are the coefficients of viscosity and heat conductivity, respectively. An equation of state relating ρ, p, and T is added to close the system. Those equations are a special case of a class of equations called incompletely parabolic equations.

Although the mathematical analysis of these nonlinear equations is not entirely satisfactory, and due to the increasing complexity of the physical problems involved, the Navier–Stokes model is more and more widely used in today's computational fluid dynamics.

In many problems of interest, the computational domain is infinite, so that an important task is the design and analysis of reliable numerical boundary conditions. Very often the Euler equations have replaced the Navier–Stokes system in computations (i.e., assuming the viscosity and heat conductivity coefficients negligible). In that case stable boundary conditions are provided by prescribing the entering characteristic quantities (see, for instance, [OS]). For better accuracy strategies were described in [EM1], [EM2], and [BT1], [BT2], [BT3], which led to higher-order differential operators on the boundary.

For the Navier–Stokes system, it is well known that more boundary conditions are needed to ensure the well posedness. Considering the Navier–Stokes equation as a perturbation of the Euler system, it has been suggested that extra boundary conditions

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be added to those derived for the Euler system [OS]. The artificial boundary is usually set in a "smooth" region, where the equations can be linearized about a regular state (in general, it is supposed to be constant). The derivation and analysis can then be carried out for the linear equation. In [GS] boundary conditions were built by adding conditions on the normal derivatives to the "hyperbolic" boundary conditions to produce dissipation. In [RS1] and [RS2] "hyperbolic" boundary conditions were tested for a flow over a flat plate to force the convergence to the steady state. More recently in [ABL] Abarbanel, Bayliss, and Lustman worked directly on the Navier–Stokes equation for the flow past an airplane. They decoupled the domain into the boundary layer region and the hyperbolic region, and in the former region used a modal expansion and an approximation of the solution. This approximation is made in the regime of long wavelength.

We develop here a general strategy for the derivation of artificial boundary conditions for incompletely parabolic perturbations of hyperbolic systems. Because of the remark above we shall consider linear systems with constant coefficients. Using the Fourier transform as an essential tool, we shall write artificial boundary conditions for a half-space in such a way that the well posedness and the convergence to the hyperbolic equation are ensured by the well posedness of a reduced hyperbolic problem. The strategy has been introduced in [H] and [HS] for incompressible flows and consists of expanding the modes in terms of the small parameter ν. For the analysis of these boundary conditions we shall rely on the results by Strikwerda in [S] on the well posedness of incompletely parabolic systems, and by Michelson in [M] on the boundary layer expansion and convergence to the "inviscid" equation. This strategy theoretically allows for a convergence up to any accuracy, but the well posedness is not guaranteed (note that in the hyperbolic case, no well-posedness proof is available for general artificial boundary condition; see [EM1]).

Consider an incomplete singular perturbation of a hyperbolic system, i.e.,

$$\frac{\partial w}{\partial t}=\sum_{j=1}^{N} A^{(j)}\frac{\partial w}{\partial x_j}+\nu\sum_{j,k=1}^{N} P^{(jk)}\frac{\partial^2 w}{\partial x_j\,\partial x_k}+F(x,t), \tag{0.1}$$

where the $n\times n$ matrices $P^{(jk)}$ are assumed of the form

$$P^{(jk)}=\begin{pmatrix}\bar P^{(jk)} & 0\\ 0 & 0\end{pmatrix} \tag{0.2}$$

with rank $\bar P^{(jk)}=r$, $\bar P^{(jk)}$ is nonsingular, and $P^{(jk)}=P^{(kj)}$. The matrices $A^{(j)}$ are partitioned in the same way:

$$A^{(j)}=\begin{pmatrix}B^{(j)} & C^{(j)}\\ D^{(j)} & \bar A^{(j)}\end{pmatrix}. \tag{0.3}$$

We require the operator $\partial_t-\sum_{j=1}^N A^{(j)}\partial_j$ to be hyperbolic, the partial operator $\partial_t-\nu\sum_{j=1}^N \bar P^{(jk)}\partial_{jk}$ to be Petrovski parabolic, and the reduced operator $\partial_t-\sum_{j=1}^N \bar A^{(j)}\partial_j$ to be strictly hyperbolic. These assumptions ensure the well posedness of the Cauchy problem. In order to consider an initial boundary value problem in a half-space $x_1>0$ or $x_1<0$, we shall assume that the boundary $\Gamma=\{x_1=0\}$ is noncharacteristic, i.e., that $A^{(1)}$ is nonsingular. Its eigenvalues are denoted by $\lambda_1,\cdots,\lambda_n$ where $\lambda_1,\cdots,\lambda_m$ are negative and $\lambda_{m+1},\cdots,\lambda_n$ are positive. The corresponding eigenvectors are $\Lambda^1,\cdots,\Lambda^n$. For convenience and simplicity, we shall assume that $\bar A^{(1)}$ is a diagonal matrix, with p negative eigenvalues:

$$\bar A^{(1)}=\begin{pmatrix}\bar A^{(1)^-} & 0\\ 0 & \bar A^{(1)^+}\end{pmatrix} \tag{0.4}$$

where

$$\bar{A}^{(1)^-} = \begin{pmatrix} \bar{\lambda}_{1+r} & & \\ & \ddots & \\ & & \bar{\lambda}_{p+r} \end{pmatrix} < 0, \qquad \bar{A}^{(1)^+} = \begin{pmatrix} \bar{\lambda}_{p+r+1} & & \\ & \ddots & \\ & & \bar{\lambda}_{n} \end{pmatrix} > 0.$$

We further assume the existence of a symmetrizer S for the full operator

$$Q = \sum_{j=1}^{N} A^{(j)} \frac{\partial}{\partial x_j} + \nu \sum_{j=1}^{N} P^{(jk)} \frac{\partial^2}{\partial x_j \, \partial x_k}, \tag{0.5}$$

which implies in particular that the symbol of Q,

$$Q(i\xi) = i \sum_{j=1}^{N} A^{(j)} \xi_j - \nu \sum_{j=1}^{N} P^{(jk)} \xi_j \xi_k, \tag{0.6}$$

is diagonalizable through a transformation analytic in ξ. S is a symmetric positive-definite matrix. We shall denote by $\tilde{A}^{(j)}$ and $\tilde{P}^{(jk)}$ the symmetrized matrices $\tilde{A}^{(j)} = SA^{(j)}$, $\tilde{P}^{(jk)} = SP^{(jk)}$. Both the Navier–Stokes and shallow-water systems fulfill all the conditions above.

In § 1 we shall recall the modal analysis for the Cauchy problem. Most results in this section are known (see, for instance, [YS] for Navier–Stokes, [S] for the general case), but we need to set our notation clearly.

In § 2 we derive the local and nonlocal boundary conditions for a half-space. The transparent boundary condition is first written in terms of generalized eigenvalues and eigenfunctions for the system. It is then approximated with respect to the small parameter ν we shall call viscosity for obvious reasons. This yields boundary conditions that are differential of first order in the normal direction, but still integral in time and the tangential derivatives (like the transparent boundary condition for the pure hyperbolic problem). Those boundary conditions are, in turn, approximated by differential operators which are of order zero in time and one in the tangential direction, using the strategy in [EM1].

In § 3, necessary and sufficient conditions for the well posedness of the corresponding initial boundary value problem are set. The same conditions ensure the convergence to the unperturbed hyperbolic problem, with an error estimate. These results are a direct application of the general analysis in [M].

In § 4 the construction above is carried out explicitly for the two-dimensional compressible Navier–Stokes system.

Finally in § 5, we indicate how to produce more accurate boundary conditions. For the sake of clarity, explicit calculations are made in the special case of the two-dimensional linearized shallow water equation. Nevertheless, the construction carries over to any incompletely parabolic system provided the diagonalizability assumption (0.6) is fulfilled.

1. The Cauchy problem.

1.1. Normal modes for the Cauchy problem. The following analysis can be partly found in [S], but we include it here in order to set our notation and to study more particularly the eigenmodes as functions of the parameter ν.

The normal modes are the solutions of (0.1) with $F \equiv 0$, of the type

$$w = e^{st + \xi x_1 + i\eta \cdot y} \Phi, \qquad \operatorname{Re} s \geqq 0,$$

where

$$x=(x_1, \cdots, x_N), \qquad y=(x_2, \cdots, x_N).$$

They satisfy the equation

$$(Q(\xi, i\eta)-sI)\Phi=0. \tag{1.1}$$

Here s and $i\eta$ are the independent variables and ξ is considered as a function of (s, η). The equation in ξ is of order $n+r$. By an abuse of notation, but for simplicity, we shall often refer to ξ as a "generalized eigenvalue."

We shall first need a general lemma on matrices.

LEMMA 1.1. *Let M and S be two matrices of same order n. If M is nonnegative, if S is symmetric positive definite, and moreover if SM is symmetric, then SM is nonnegative.*

Let us recall that a matrix M is nonnegative if for any u, $(Mu, u) \geqq 0$, where $(\cdot, \cdot)$ denotes the usual scalar product. M is positive definite if there exists a constant $\alpha>0$ such that for any u, $(Mu, u) \geqq \alpha\|u\|^2$.

Proof. For convenience, we choose a basis where S is diagonal: $S=\operatorname{diag}(s_1, \cdots, s_n)$, $M=(m_{ij})$. Using the identity $SM=M^TS$, we can write $(Mu, u)=(Nv, v)$, where $u=Sv$, and N is defined by

$$n_{ij}=\frac{1}{2} s_i s_j\left(1+\frac{s_i}{s_j}\right) m_{ij}, \qquad i \leqq j,$$

$$n_{ij}=n_{ji}, \qquad i>j.$$

On the other hand, we can express SM as

$$(SMu, u)=2(Cu, u),$$

where

$$c_{ij}=\frac{1}{s_i+s_j} n_{ij}.$$

We know that a matrix defines a nonnegative bilinear form if and only if all the principal minors are nonnegative. For the matrix $1/(s_i+s_j)$, they are equal for $k \leqq n$ to

$$\frac{\prod_{i<j \leqq k}(s_i-s_j)^2}{2^k \Pi s_i \prod_{i<j \leqq k}(s_i+s_j)^2}.$$

We now use the following classical result: if A and B are two symmetric nonnegative matrices, the matrix whose general term is $a_{ij} b_{ij}$ is also nonnegative. Hence C is nonnegative and the proof is complete.

LEMMA 1.2. *For* $\operatorname{Re} s>0$, *there are precisely $(r+p)$ generalized eigenvalues ξ with a negative real part, and $n-p$ with a positive real part.*

Proof. We first prove that there are no purely imaginary generalized eigenvalues. Let us assume that $\xi=i\zeta$, where ζ is a real number. We apply S to (1.1) and multiply by Φ. We now take the real part of the Hermitian product:

$$\nu\left(\left(\sum \tilde{P}^{(jk)} \psi_j \psi_k\right) \Phi, \Phi\right)+\operatorname{Re} s(S\Phi, \Phi)=0, \qquad \psi=(\zeta, \eta_2, \cdots, \eta_n).$$

By assumption, the matrix $\sum P^{(jk)} \psi_j \psi_k$ is nonnegative, and so is $\sum \tilde{P}^{(jk)} \psi_j \psi_k$ by Lemma 1.1. If $\operatorname{Re} s>0$, since $(S\Phi, \Phi)$ is positive, we obtain the contradiction.

Now let N_+ be the number of eigenvalues ξ such that $\operatorname{Re}\xi<0$, and N_- the number of eigenvalues ξ such that $\operatorname{Re}\xi>0$. We have $N_+ + N_- = r+n$, from above. N_+ and N_- are constant functions of η and s. Furthermore, they are constant functions of A and P, as long as, for instance, $P^{(11)}\neq 0$.

Let us choose $\eta=0$ and $B^{(1)}=C^{(1)}=D^{(1)}=0$. Equation (1.1) for ξ reads

$$\det[\nu P^{(11)}\xi^2 + A^{(1)}\xi - sI]=0 \quad \text{or}$$

$$\det[\nu \bar{P}^{(11)}\xi^2 - sI]\cdot \det[\bar{A}^{(1)}\xi - sI]=0.$$

N_+ (respectively, N_-) is then the number of solutions with positive real part (respectively, negative) of the equations

$$\det[\nu\bar{P}^{(11)}\xi^2 - sI]=0, \qquad \det[\bar{A}^{(1)}\xi - sI]=0.$$

The first equation is even in ξ. Hence there are r solutions with positive real part and r with negative real part. Moreover, the second equation reduces to $\xi=s/\bar{\lambda}_i$, which gives p values with a negative real part and $n-r-p$ with a positive real part. So

$$N^+ = r+p, \qquad N^- = r+n-r-p=n-p.$$

We now turn to the behavior of these generalized eigenvalues ξ as the parameter ν tends to zero.

THEOREM 1.1. *If* $\operatorname{Re} s>0$, *as* ν *tends to zero,* r *values of* ξ *tend to infinity as* $1/\nu$, *and* n *values have a finite limit.*

Proof. Let us write $\xi=\alpha+0(\nu)$, where α is a solution to

$$\det\left[A^{(1)}\alpha + \sum_{j\neq 1} iA^{(j)}\eta_j - sI\right]=0. \tag{1.2}$$

By assumption this equation has n solutions, denoted by $\alpha_1(s,\eta),\cdots,\alpha_n(s,\eta)$, and

$$\begin{aligned} &1\leqq j\leqq m, &&\operatorname{Re}\frac{\alpha_j}{s}\leqq 0,\\ &m+1\leqq j\leqq n, &&\operatorname{Re}\frac{\alpha_j}{s}\geqq 0 \end{aligned} \tag{1.3}$$

to any α_j is associated an eigenvector $\Pi^j(s,\eta)$, and $\Pi^1,\cdots,\Pi^n$ span $\mathbf{R}^n$

$$\left(A^{(1)}\alpha_k + \sum_{j\neq 1} iA^{(j)}\eta_j - sI\right)\Pi^k=0. \tag{1.4}$$

If now $\xi=\theta/\nu+O(1)$, θ is a solution of

$$\det(P^{(11)}\theta + A^{(1)})=0, \tag{1.5}$$

which is an equation of degree r in θ, and has r roots $\theta_1\cdots\theta_r$, then $\theta_1,\cdots,\theta_{r+p-m}$ are such that $\operatorname{Re}\theta_j<0$, and $\theta_{r+p-m+1},\cdots,\theta_r$ are such that $\operatorname{Re}\theta_j>0$. The corresponding generalized eigenvectors $\Theta^1\cdots\Theta^r$ are defined by

$$(P^{(11)}\theta_j + A^{(1)})\Theta^j=0. \tag{1.6}$$

In summary, every solution (ξ,Φ) of $(Q(\xi,i\eta)-sI)\Phi=0$ for $\operatorname{Re}s>0$ is such that either

$$\xi(s,\eta,\nu)=\alpha(s,\eta)+O(\nu), \qquad \Phi(s,\eta,\nu)=\Pi(s,\eta)+O(\nu),$$

where

$$\left[A^{(1)}\alpha+\sum_{j\neq 1} i\eta_j A^{(j)}-sI\right]\Pi=0$$

or

$$\xi(s,\eta,\nu)=\frac{1}{\nu}\theta+O(1), \qquad \Phi(s,\eta,\nu)=\Theta+O(\nu),$$

where

$$(P^{(11)}\theta+A^{(1)})\Theta=0.$$

We shall denote by $\xi_1\cdots\xi_m$ the values of ξ of the first form with negative real part, i.e., corresponding to "propagating modes," and $\xi_{m+1}\cdots\xi_{r+p}$ the values of ξ of the second form with negative real parts. We define $\zeta_j(s,\eta)$ and $\psi_j(s,\eta)$ as follows:

$$(1.7)\qquad \begin{aligned} &1\leqq j\leqq m, &&\xi_j(s,\eta,\nu)=\zeta_j(s,\eta)+O(\nu),\\ &m+1\leqq j\leqq r+p, &&\xi_j(s,\eta,\nu)=\frac{1}{\nu}\zeta_j(s,\eta)+O(1)\end{aligned}$$

(ζ_j does not actually depend on s and η if $j\geqq m+1$) and

$$(1.8)\qquad 1\leqq j\leqq r+p, \qquad \Phi^j(s,\eta,\nu)=\Psi^j(s,\eta)+O(\nu),$$

so that

$$(1.9)\qquad \begin{aligned} &1\leqq j\leqq m, &&\zeta_j(s,\eta)=\alpha_j(s,\eta), &&\Psi^j(s,\eta)=\Pi^j(s,\eta),\\ &m+1\leqq j\leqq r+p, &&\zeta_j(s,\eta)=\theta_{j-m}, &&\Psi^j(s,\eta)=\Theta^{j-m}.\end{aligned}$$

1.2. The transmission conditions. Let us first write a weak formulation of (0.1) in a domain Ω with smooth boundary $\partial\Omega$. For any v sufficiently smooth, we multiply (0.1) by S, apply it to v, and integrate on Ω. Using the Green formulas

$$\int_\Omega\left(\tilde{A}^{(j)}\frac{\partial w}{\partial x_j},v\right)+\int_\Omega\left(\tilde{A}^{(j)}w,\frac{\partial v}{\partial x_j}\right)=\int_{\partial\Omega}(\tilde{A}^{(j)}w,v)n_j,$$

$$\begin{aligned}\sum_{j,k=1}^N\int_\Omega\left(\tilde{P}^{(j,k)}\frac{\partial^2 w}{\partial x_j\,\partial x_k},v\right)+\sum_{j,k=1}^N\int_\Omega\left(\tilde{P}^{(j,k)}\frac{\partial w}{\partial x_j},\frac{\partial v}{\partial x_k}\right)\\ =\sum_{j,k=1}^N\int_{\partial\Omega}\left(\tilde{P}^{(j,k)}\frac{\partial w}{\partial x_j},v\right)n_k,\end{aligned}$$

where n is the normal exterior to $\partial\Omega$, we get

$$\begin{aligned}&\int_\Omega\left(S\frac{\partial w}{\partial t},v\right)+\nu\sum_{j,k=1}^N\int_\Omega\left(\tilde{P}^{(j,k)}\frac{\partial w}{\partial x_j},\frac{\partial v}{\partial x_k}\right)\\ &\quad+\frac{1}{2}\sum_{j=1}^N\int_\Omega\left[\left(\tilde{A}^{(j)}\frac{\partial v}{\partial x_j},w\right)-\left(\tilde{A}^{(j)}\frac{\partial w}{\partial x_j},v\right)\right]\\ &=\nu\int_\Gamma\sum_{j,k=1}^N\left(\tilde{P}^{(jk)}\frac{\partial w}{\partial x_j},v\right)n_k+\frac{1}{2}\int_\Gamma\sum_{j=1}^N(A^{(j)}w,v)n_j+\int_\Omega Fv.\end{aligned}$$

We define two bilinear forms a and p, by

$$(1.10)\qquad \begin{aligned} a(v, w) &= \frac{1}{2}\int_{\Omega}\sum_{j=1}^{N}\left[\left(\tilde{A}^{(j)}\frac{\partial v}{\partial x_j}, w\right)-\left(\tilde{A}^{(j)}\frac{\partial w}{\partial x_j}, v\right)\right] dx,\\ p(v, w) &= \int_{\Omega}\sum_{j,k=1}^{N}\left(\tilde{P}^{(j,k)}\frac{\partial w}{\partial x_j}, \frac{\partial v}{\partial x_k}\right),\end{aligned}$$

where a is antisymmetric and p is symmetric, nonnegative (Lemma 1.1).

S being symmetric definite positive, defines a scalar product

$$(1.11)\qquad s(w, v) = \int_{\Omega}(Sw, v)\, dx$$

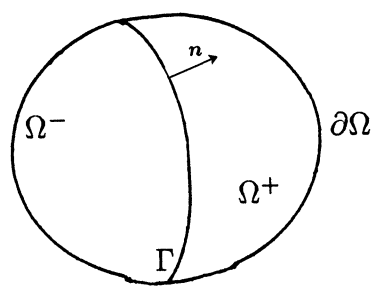
and we can write

$$(1.12)\qquad s\left(\frac{\partial w}{\partial t}, v\right) + \nu p(v, w) + a(v, w) - \int_{\partial\Omega}(S\mathscr{E}w, v)\, d\gamma = \int_{\Omega} Fv,$$

where $\mathscr{E}w$ is the normal stress

$$(1.13)\qquad \mathscr{E}w = \sum_{k=1}^{n}\left(\nu\sum_{j} P^{(jk)}\frac{\partial w}{\partial x_j} + \frac{1}{2}A^{(k)}w\right) n_k.$$

Suppose now that $\Omega = \Omega^- \cup \Omega^+$,



the orientation of n being from Ω^- to Ω^+. We define a^-, a^+, p^-, p^+, s^-, s^+ as we did for a, p, s, but the integral being taken over Ω^- and Ω^+, respectively. Let v be compactly supported in Ω. We thus have

$$(1.14)\qquad s\left(\frac{\partial w}{\partial t}, v\right) + \nu p(v, w) + a(v, w) = \int_{\Omega} Fv,$$

but

$$\begin{aligned} p(v, w) &= p^+(v, w) + p^-(v, w),\\ a(v, w) &= a^+(v, w) + a^-(v, w),\\ s\left(\frac{\partial w}{\partial t}, v\right) &= s^+\left(\frac{\partial w}{\partial t}, v\right) + s^-\left(\frac{\partial w}{\partial t}, v\right),\end{aligned}$$

and we can write

$$(1.15)^{\pm}\quad s^{\pm}\left(\frac{\partial w^{\pm}}{\partial t}, v\right) + \nu p^{\pm}(w^{\pm}, v) + a^{\pm}(w^{\pm}, v) - \int_{\partial\Omega^{\pm}}(S\mathscr{E}w^{\pm}, v)\, d\gamma = \int_{\Omega^{\pm}} F^{\pm}v,$$

where $F^{\pm} = F/\Omega^{\pm}$ and $w^{\pm} = w/\Omega^{\pm}$, so that adding $(1.15)^+$ and $(1.15)^-$, and subtracting from (1.14) we get

$$\int_{\partial\Omega^+} (S\mathscr{E}w, v)\, d\gamma + \int_{\partial\Omega^-} (S\mathscr{E}w, v)\, d\gamma = 0.$$

Since v is compactly supported in Ω, $\partial\Omega^\pm$ reduces to Γ and $(\mathscr{E}w)^- = (\mathscr{E}w^-)|_\Gamma$ if w is defined in Ω^-, $(\mathscr{E}w)^+ = (\mathscr{E}w^+)|_\Gamma$ if w is defined in Ω^+, and the normal on Γ is exterior to Ω^- (thus interior to Ω^+). The transmission conditions then read

$$(1.16)\qquad \begin{aligned} (\mathscr{E}w)^- &= (\mathscr{E}w)^+ \\ w^- &= w^+ \end{aligned} \quad \text{on } \Gamma,$$

or

$$(1.17)\qquad \sum_{k=1}^{N} \left(\nu \sum_j P^{(jk)} \frac{\partial w^-}{\partial x_j} + \frac{1}{2} A^{(k)} w^- \right) n_k = \sum_{k=1}^{N} \left(\nu \sum_j P^{(jk)} \frac{\partial w^+}{\partial x_j} + \frac{1}{2} A^{(k)} w^+ \right) n_k.$$

In particular, if $\Omega = \mathbf{R}^n$, $\Omega^- = \{x_1 < 0\}$, $\Omega^+ = \{x_1 > 0\}$, so that $\Gamma = \{x_1 = 0\}$, then the transmission conditions are

$$(1.18)\qquad \begin{cases} \nu \displaystyle\sum_{j=1}^{N} \bar{P}^{(j1)} \frac{\partial w^-}{\partial x_j} = \nu \sum_{j=1}^{N} \bar{P}^{(j1)} \frac{\partial w^+}{\partial x_j} & \text{on } \Gamma. \\ w^- = w^+ \end{cases}$$

Condition (1.18) is equivalent to (1.16). The Green formula with the constraints is more useful when we want to prove the well posedness through energy estimates, and it is the reason we include it here. Again, (1.18) can also be written

$$(1.19)\qquad w^- = w^+, \qquad \frac{\partial w^{-I}}{\partial x_1} = \frac{\partial w^{+I}}{\partial x_1},$$

where (w^I, w^{II}) corresponds to the decomposition of the matrices $P^{(jk)}$, i.e.,

$$(1.20)\qquad w^I = (w_1, \cdots, w_r), \qquad w^{II} = (w_{r+1}, \cdots, w_n),$$

but we prefer to use the form (1.18), which seems more fitted to the multidimensional case.

2. Derivation of the artificial boundary conditions. We shall use the transmission conditions we wrote above to derive the transparent boundary condition. Let F and w^0 be compactly supported in Ω^-; consider the Cauchy problem:

$$(2.1)\qquad \begin{cases} \dfrac{\partial w}{\partial t} = \displaystyle\sum_{j=1}^{N} A^{(j)} \frac{\partial w}{\partial x_j} + \nu \sum_{j,k=1}^{N} P^{(jk)} \frac{\partial^2 w}{\partial x_j\, \partial x_k} + F(x, t), \\ w(0) = w^0. \end{cases}$$

It is equivalent to the transmission problem

$$(2.2)\qquad \begin{cases} \dfrac{\partial w^-}{\partial t} - Qw^- = F(x, t) \quad \text{in } \Omega^-, \\ w^-(0) = w^0, \end{cases}$$

$$(2.3)\qquad \begin{cases} \dfrac{\partial w^+}{\partial t} - Qw^+ = 0 \quad \text{in } \Omega^+, \\ w^+(0) = 0 \end{cases}$$

with the transmission conditions

$$\begin{cases} \nu \sum_1^N \bar{P}^{(j1)} \dfrac{\partial w^-}{\partial x_j} = \nu \sum_1^N \bar{P}^{(j1)} \dfrac{\partial w^+}{\partial x_j} \\ w^- = w^+ \end{cases} \quad \text{on } \Gamma. \tag{2.4}$$

2.1. The transparent boundary condition. We introduce the initial boundary value problem in Ω^+:

$$\begin{cases} \dfrac{\partial w}{\partial t} - Qw = 0 \quad \text{in } \Omega^+, & (2.5a) \\ w(t=0) = 0, & (2.5b) \\ \begin{pmatrix} w_1 \\ \vdots \\ w_{r+p} \end{pmatrix} = g \quad \text{on } \Gamma. & (2.5c) \end{cases}$$

THEOREM 2.1. *The boundary value problem* (2.5) *is strongly well posed. The solution is given in Fourier variables* (η, s) *by*

$$\hat{w}(x_1, \eta, s) = \sum_{i=1}^{r+p} \lambda_i e^{\xi_i x_1} \Phi^i, \tag{2.6}$$

where (ξ_i, Φ^i) *are defined in* (1.1) *and the coefficients* λ_i *are determined by the boundary conditions.*

Proof. According to Strikwerda [S], the problem is strongly well posed if and only if the two initial boundary value problems

$$\begin{cases} \dfrac{\partial w^I}{\partial t} = \nu \sum_{j,k=1}^N \bar{P}^{(jk)} \dfrac{\partial^2 w^I}{\partial x_j \, \partial x_k}, \\ w^I = g^I \quad \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial w^{II}}{\partial t} = \sum_{j=1}^N \bar{A}^{(j)} \dfrac{\partial w^{II}}{\partial x_j}, \\ \begin{pmatrix} w_{r+1} \\ \vdots \\ w_{r+p} \end{pmatrix} = \begin{pmatrix} g_{r+1} \\ \vdots \\ g_{r+p} \end{pmatrix} \quad \text{on } \Gamma \end{cases}$$

are strongly well posed. The first problem is a strongly parabolic problem with a Dirichlet boundary condition, and hence is strongly well posed. As for the second one, since $\bar{A}^{(j)}$ is diagonal, the boundary condition reduces to specifying the entering characteristics which is, again, a strongly well posed problem.

Let us now consider Fourier-Laplace transform (2.5a) with respect to t and y. The corresponding variables are (s, η), with $\operatorname{Re} s > 0$. We then get a second-order ordinary differential equation, whose solution is

$$\hat{w} = \sum \lambda_i e^{\xi_i x_1} \Phi^i,$$

where (ξ_i, Φ^i) are given in (1.1), since we supposed that $Q(i\xi)$ was diagonalizable. In order for $\hat{w}$ to be in L^2, the coefficients λ_i must vanish when $\operatorname{Re} \xi_i \geqq 0$. We thus are led to (2.6).

Remark. We have assumed that Q was diagonalizable, so the $(\Phi^i_k)_{1 \leqq i,k \leqq r+p}$ is a nonsingular matrix and thus the boundary condition determines the λ_i's.

THEOREM 2.2. *The transparent boundary condition at* Γ *for the half-space* Ω^- *is*

$$\nu \sum_{j=1}^{N} \bar{P}^{(1j)} \frac{\partial \hat{w}^I}{\partial x_j} = \nu \sum_{j=1}^{r+p} \bar{P}^{(11)} \sum_{i=1}^{r+p} \xi_i M_{ij}^{-1} \Phi^{i^I} \hat{w}_j + \nu \sum_{l \neq 1} i\eta_l \bar{P}^{(l1)} \hat{w}^I, \tag{2.7a}$$

$$\hat{w}_k = \sum_{i=1}^{r+p} \sum_{j=1}^{r+p} M_{ij}^{-1} \hat{w}_j \Phi_k^i, \qquad k = r+p+1, \cdots, n, \tag{2.7b}$$

where (M_{ij}) *is the* $(r+p) \times (r+p)$ *matrix defined by*

$$M_{ij} = \Phi_i^j, \quad \text{and} \quad M^{-1} \text{ is the inverse of } M. \tag{2.8}$$

Proof. w^+ is the solution in Ω^+ of the initial boundary problem (2.5) with $g_k = w_k^-$, $1 \leqq k \leqq r+p$. Theorem 2.1 then enables us to calculate explicitly w_k^+, $r+p+1 \leqq k \leqq n$ and $\partial w^{+^I}/\partial x_j$. The transmission conditions then give the result

$$\nu \sum_{j=1}^{N} \bar{P}^{(1j)} \frac{\partial \hat{w}^I}{\partial x_j} = \nu \bar{P}^{(11)} \frac{\partial \hat{w}^I}{\partial x_1} + \nu \sum_{j \neq 1} \bar{P}^{(1j)} i\eta_j \hat{w}^I.$$

From (2.6) we deduce that

$$\frac{\partial \hat{w}^+}{\partial x_1} = \sum_{i=1}^{r+p} \lambda_i \xi_i \Phi^i \quad \text{on } \Gamma,$$

so that

$$\nu \sum_{j=1}^{N} \bar{P}^{(1j)} \frac{\partial \hat{w}^{+I}}{\partial x_j} = \nu \bar{P}^{(11)} \sum_{j=1}^{r+p} \lambda_j \xi_j \Phi^{j^I} + \nu \sum_{j \neq 1} \bar{P}^{(1j)} i\eta_j \sum_{i=1}^{r+p} \lambda_i \Phi^{i^I},$$

$$\hat{w}_k^+ = \sum_{i=1}^{r+p} \lambda_i \Phi_k^i, \qquad k = r+p+1, \cdots, n.$$

The coefficients λ_j are determined by

$$\sum_{i=1}^{r+p} \lambda_i \Phi_j^i = \hat{w}_j^-, \qquad j = 1, \cdots, r+p.$$

So, if the matrix M is defined by (2.8), we have

$$\lambda_i = \sum_{j=1}^{r+p} M_{ij}^{-1} \hat{w}_j^-$$

and finally $\hat{w}^-$ satisfies (2.7a), (2.7b): (2.7a), (2.7b) is actually the transparent boundary condition.

Remark. If $\bar{A}^{(1)}$ were not diagonal, the same study could be carried over, by choosing an admissible boundary condition (2.5c).

Remark. If $n = r+p$, the transparent boundary condition reduces to (2.7a): the "hyperbolic" part does not require any boundary conditions. (It is the case for instance for Navier–Stokes equation when the flow is supersonic and the boundary is on the outflow.)

2.2. Nonlocal approximate boundary condition. Since we are seeking boundary conditions that are consistent with the hyperbolic problem (i.e., $\nu = 0$), we shall approximate the transparent boundary condition (2.7) with respect to the parameter ν. We thus shall obtain boundary condition relating w and $\partial w^I/\partial x_1$, whose kernel is a nonrational function of s and η, and thus integral in the time variable and the boundary variables. This boundary condition will eventually be approximated by local boundary conditions in § 2.3, using the techniques in [EM1] for hyperbolic problems.

Let us consider the limits as ν tends to zero of the various terms in the right-hand side of (2.7). By (1.8) the vectors Φ^i tend to the corresponding Ψ^i, and hence the matrix M tends to N, where

$$N_{ij}=\Psi^j_i, \qquad 1 \leqq i, j \leqq r+p. \tag{2.9}$$

As for the coefficients $\nu\xi_i$, by (1.9) if $1 \leqq i \leqq m$, $\nu\xi_i$ tends to zero, and if $m+1 \leqq i \leqq n$, $\nu\xi_i$ tends to a finite limit $\zeta_i(s, \eta)$ (which actually does not depend on s, η). Taking the limits in the right-hand side of (2.7) as described, we are led to the boundary condition

$$\nu \sum_{j=1}^{N} \bar{P}^{(1,j)} \frac{\partial \hat{w}^I}{\partial x_j} = \sum_{i=m+1}^{r+p} \bar{P}^{(11)} \sum_{j=1}^{r+p} \zeta_i N_{ij}^{-1} \Psi^{i^I} \hat{w}_j, \tag{2.10a}$$

$$\hat{w}_k = \sum_{i=1}^{r+p} \sum_{j=1}^{r+p} N_{ij}^{-1} \hat{w}_j \Psi^i_k, \qquad r+p+1 \leqq k \leqq n. \tag{2.10b}$$

We shall see in the next section how this boundary condition leads to a well-posed problem in the left half-space Ω^-, whose solution converges as ν tends to zero toward the restriction to Ω^- of the solution of the full hyperbolic problem in $\mathbf{R}^n$. The latter will be done using a boundary layer expansion and the criterion in [M]. Before carrying over the analysis we shall write local boundary conditions. In order to make the mechanisms clear, and since we shall need it later, we shall first recall the derivation of transparent and approximate boundary conditions for the hyperbolic problem.

2.3. Absorbing boundary conditions for the hyperbolic problem. Here we will follow the lines drawn in [EM1]. We keep the notation and assumptions set in the first section. The hyperbolic system is

$$\frac{\partial w}{\partial t} = \sum_{j=1}^{N} A^{(j)} \frac{\partial w}{\partial x_j} + F. \tag{2.11}$$

By Laplace–Fourier transform in t and y, the solutions of this equation in the full space when $F=0$ are given by

$$\hat{w} = \sum_{i=1}^{N} \lambda_i \, e^{\alpha_i x_1} \Pi^i, \tag{2.12}$$

where (α_i, Π^i) are the eigenvalues and eigenvectors defined in § 1

$$\left(A^{(1)} \alpha_k + \sum_{j \neq 1} i A^{(j)} \eta_j - sI \right) \Pi^k = 0. \tag{2.13}$$

If $(\operatorname{Re} s \operatorname{Re} \alpha_j) \leqq 0$ (respectively, $(\operatorname{Re} s \operatorname{Re} \alpha_j) \geqq 0$), the corresponding mode in (2.12) propagates in the $(x_1 > 0)$-direction (respectively, $x_1 < 0$).

The transparent boundary condition at $x_1 = 0$ for the half-space Ω^- expresses that no wave can propagate from the boundary toward the interior of Ω^-, i.e.,

$$i = m+1, \cdots, n, \qquad \lambda_i = 0. \tag{2.14}$$

Let us define T as the matrix of the eigenvectors:

$$T_{ij}(s, \eta) = \Pi^j_i(s, \eta). \tag{2.15}$$

By (2.15), (2.14) can be rewritten as

$$\forall i = m+1, \cdots, n \quad (T^{-1} \hat{w})_i = 0. \tag{2.16}$$

This is the transparent boundary condition at $x_1 = 0$ for the half-space Ω^-, i.e., the equivalent of (2.7) for $\nu=0$. We shall see in the next section that (2.7) (or (2.10)) actually reduces to (2.16) when ν tends to zero, so that the solution of (0.1) coupled with (2.7) (or (2.10)) tends to the solution of (2.11) with the boundary condition (2.16). This boundary condition is nonlocal in time and space. Following [EM1], we shall make an approximation with respect to the angle of incidence of the wave on the boundary. This is easily achieved by letting $\eta=0$ in (2.13), so that α_k is nothing but s/λ_k where λ_k is an eigenvalue of $A^{(1)}$, and $\Pi^k=\Lambda^k$.

Thus, the first absorbing boundary condition for (2.11) in Ω^- is

$$\forall i=m+1,\cdots,n \quad (\tilde{T}^{-1}w)_i=0, \tag{2.17}$$

where

$$\tilde{T}_{ij}=\Lambda_i^j, \qquad 1\leqq i, \quad j\leqq n,$$

which is simply writing that the entering characteristics of the system are prescribed the value zero on the boundary.

2.4. Local boundary conditions for the full problem. We thus want to make the same kind of approximation in (2.10), and it is now clearer: for $m+1\leqq j\leqq r+p$, neither ζ_j nor Ψ^j depends on (s,η) and therefore both remain unchanged. For $1\leqq j\leqq m$, ζ_j and Ψ^j are approximated by s/λ_j and Λ^j, respectively.

We then define the vector $\tilde{\Psi}^j$, $1\leqq j\leqq r+p$ by

$$\begin{aligned} \tilde{\Psi}^j&=\Lambda^j, && 1\leqq j\leqq m,\\ \tilde{\Psi}^j&=\Psi^j, && m+1\leqq j\leqq r+p \end{aligned} \tag{2.18}$$

and the matrix $\tilde{N}$ by

$$\tilde{N}_{ij}=\tilde{\Psi}_i^j, \qquad 1\leqq i, \quad j\leqq r+p. \tag{2.19}$$

The approximate boundary condition takes the form

$$\begin{cases} \nu \displaystyle\sum_{j=1}^{N} \bar{P}^{(1,j)}\frac{\partial w^I}{\partial x_j}=\sum_{i=m+1}^{r+p}\bar{P}^{(11)}\sum_{j=1}^{r+p}\zeta_i\tilde{N}_{ij}^{-1}\Psi^{i^I}w_j,\\ w_k=\displaystyle\sum_{i=1}^{r+p}\sum_{j=1}^{r+p}\tilde{N}_{ij}^{-1}\tilde{\Psi}_k^i w_j, \qquad r+p+1\leqq k\leqq n. \end{cases} \tag{2.20}$$

This is our local boundary condition. It is of first order in x, and zero order in time. For (2.7) and (2.16), we shall see that it converges to (2.17) when ν tends to zero, so that the solution of (0.1) coupled with (2.20) tends to the solution of (2.11) coupled with the first absorbing boundary condition (2.17).

We shall discuss in § 5 further approximations to these boundary conditions, with respect to either parameters ν or the angle of incidence on the boundary. These boundary conditions will involve higher derivatives in time and the tangential variables.

3. Analysis of the approximate boundary conditions. We shall use here the analysis by Michelson [M] of the well posedness and boundary layer for initial boundary value problems related to parabolic perturbations of hyperbolic equations.

3.1. Well posedness of the boundary value problems. As already pointed out by Strikwerda [S], the well posedness of the initial boundary value problem for (0.1) is equivalent to the well posedness of the purely parabolic problem for $\bar{P}$ and purely hyperbolic problem for $\bar{A}$, provided the boundary conditions satisfy certain decoupling conditions, which are automatically satisfied for boundary conditions of the form (2.7).

Furthermore, if the problem satisfies a uniform Lopatinski condition stated by Michelson in [M], then we can get estimates uniform in ν. Let us define in Ω^- the weighted norms:

$$\|u\|^2_{m_1,m_2,\eta} = \sum_{\substack{|\beta|\leq m_2\\ |\alpha|+|\beta|\leq m_1}} \|(\nu D, \nu\eta)^\alpha (\chi D_{x_1}, D_y, D_t, \eta)^\beta e^{-\eta t} u\|^2_{L^2}, \tag{3.1}$$

where $\chi=\chi(x_1)$ is a fixed smooth nondecreasing function of x_1 such that $\chi(x_1)=x_1$ for $|x_1|<\frac{1}{2}$, and $\chi(x_1)=1$ for $|x_1|>1$. Denote by $|u(x_1,\cdot)|^2_{m_1,m_2,\eta}$ the obvious restriction of the above norm to the hyperplane $x_1=\text{const}$. Let σ be the pseudodifferential operator with symbol $\text{Re}\,(1+\nu s+|\nu\eta|^2)^{1/2}(s=i\omega+\eta)$. If w is partitioned in the natural way mentioned before $w=\binom{w^I}{w^{II}}$, we define v by

$$v=\begin{pmatrix} v^I \\ v^{II}\end{pmatrix}, \quad v^{II}=w^{II}, \quad v^I=\begin{pmatrix}\sigma^{-1}\nu D_{x_1} w^I \\ \sigma w^I\end{pmatrix}. \tag{3.2}$$

We start by writing the decoupled problems.

The parabolic problem is

$$\frac{\partial w^I}{\partial t} = \sum_{j,k=1}^N \bar{P}^{(jk)} \frac{\partial^2 w^I}{\partial x_j\, \partial x_k} + F^I, \tag{3.3a}$$

$$\begin{aligned} &\sum_{j=1}^N \bar{P}^{(1j)} \frac{\partial w^I}{\partial x_j} = 0 \quad \text{on } \Gamma, \\ &w^I(t=0)=0, \end{aligned} \tag{3.3b}$$

and the hyperbolic problem is

$$\frac{\partial w^{II}}{\partial t} = \sum_{j=1}^N \bar{A}^{(j)} \frac{\partial w^{II}}{\partial x^j} + F^{II}, \tag{3.4a}$$

$$\begin{aligned} &\hat{w}_k = \sum_{i=1}^{r+p} \sum_{j=r+1}^{r+p} N^{-1}_{ij} \hat{w}_j \Psi^i_k, \qquad k=r+p+1,\cdots,n \quad \text{on } \Gamma, \\ &w^{II}(t=0)=0, \end{aligned} \tag{3.4b}$$

for the boundary conditions (2.10). For the boundary conditions (2.20) N and Ψ must be replaced by $\tilde{N}$ and $\tilde{\Psi}$, respectively.

Then we have Theorem 3.1.

THEOREM 3.1. *The boundary value problem* (0.1) *coupled with either boundary condition* (2.10) *or* (2.20) *is well posed if and only if the reduced hyperbolic problem* (3.4) *is well posed. Furthermore, if* (3.4) *is well posed in the sense of Kreiss, let integers* $m_1 \geq m_2 \geq 0$ *be such that* $m_1 - m_2 \geq 1$. *Then there exist positive constants* k, ν_0, η_0 *such that for all* $\eta > \eta_0$ *and* $0 \leq \nu < \nu_0$ *the following a priori estimate holds*:

$$\begin{aligned} &\eta\|w\|^2_{m_1,m_2,\eta} + \nu\|D_x w^I\|^2_{m_1,m_2,\eta} + |v(0,\cdot)|^2_{m_1,m_2,\eta} + |\sigma^{1/2} v^I(0,\cdot)|^2_{m_1,m_2,\eta} \\ &\qquad + \sup_{x_1}\,(|w(x_1,\cdot)|^2_{m_1-1,m_2-1,\eta} + |\nu D_x w^I(x_1,\cdot)|^2_{m_1-1,m_2-1,\eta}) \\ &\leq k\eta^{-1}\|F\|^2_{m_1,m_2,\eta}. \end{aligned} \tag{3.5}$$

Proof. The first assertion is a mere consequence of the result by Strikwerda in [S]. As for the second a priori estimate, it follows directly from the general theory on parabolic perturbations for hyperbolic systems by Michelson in [M].

The a priori estimate justifies the boundary layer expansion and proves the convergence of the initial boundary value problems (0.1) coupled with either of the boundary conditions (2.10) or (2.11) as described in § 3.2.

3.2. Boundary layer; Convergence results. A physical phenomenon related to incompletely parabolic approximations of hyperbolic equations with a small parameter ν is the formation of a boundary layer. It is mathematically represented by a formal expansion

$$w(x, t, \nu) = \sum_{i \geqq 0} \nu^i \left(w_i^{(1)}(x, t) + w_i^{(2)}\left(\frac{x_1}{\nu}, y, t\right)\right). \tag{3.6}$$

The functions $w_i^{(1)}$ represent the smooth part of the solution, while the functions $w_i^{(2)}$ represent the boundary layer: they are exponentially decreasing in x_1/ν. Michelson proved in [M] that under the same hypothesis as in Theorem 3.1, the expansion (3.6) was actually valid. We shall apply this result to our particular case.

THEOREM 3.2. *Let $w(x, t, \nu)$ be the solution of* (0.1), (2.10) (*respectively,* (2.20)) *with a sufficiently smooth F. Suppose, as in Theorem* 3.2, *that the reduced hyperbolic problem* (3.4) *is well posed. Then, as ν tends to zero, w converges to the solution u of the hyperbolic problem* (2.11), (2.16) (*respectively,* (2.17)). *More precisely, if $m_1 \geqq m_2 \geqq m_3 \geqq 0$ and $m_1 - m_2 \geqq 1$, we have*

$$\|w(x, t, \nu) - u(x, t)\|_{m_1, m_2, m_3, \eta} \leqq c(\nu + \nu^{(3/2) - m_3}), \tag{3.7}$$

where the norm above is defined by

$$\|u\|^2_{m_1, m_2, m_3, \eta} = \sum_{i=0}^{m_3} \|D^i_{x_1} u\|^2_{m_1 - i, m_2 - i, \eta}. \tag{3.8}$$

Remarks. (1) This result tells even more about the boundary layer: it says that in expansion (3.6) the first term $w_0^{(1)}$ is indeed the solution of the associated hyperbolic problem, and the first term $w_0^{(2)}$ vanishes: the boundary layer is "weak."

(2) Boundary condition (2.16) is actually the transparent boundary condition for the hyperbolic problem, so that the solution of (0.1), (2.10) converges to the solution of the Cauchy problem for (2.1).

Proof of Theorem 3.2. According to Michelson [M], the following estimate holds:

$$\|w(x, t, \nu) - w_0^{(1)}(x, t) - w_0^{(2)}(x_1/\nu, y, t)\|_{m_1, m_2, m_3, \eta} \leqq c(\nu + \nu^{(3/2) - m_3}). \tag{3.9}$$

We merely need to check that $w_0^{(1)}$ is u and $w_0^{(2)}$ is zero. These terms are obtained by substituting the expansion (3.6) into the equation and the boundary condition, separating the scales x_1 and x_1/ν, and equating to zero the successive coefficients of the resulting series:

From the equation we deduce that $w_0^{(1)}$ and $w_0^{(2)}$ are solutions of the following equations:

$$A^{(1)} w_0^{(2)} + P^{(11)} \frac{\partial w_0^{(2)}}{\partial(x_1/\nu)} = 0, \tag{3.10}$$

$$\frac{\partial w_0^{(1)}}{\partial t} = \sum_{j=1}^{N} A^{(j)} \frac{\partial w_0^{(1)}}{\partial x_j} + F, \tag{3.11}$$

and $w_i^{(1)}$ and $w_i^{(2)}$ are solutions of

$$\frac{\partial w_i^{(1)}}{\partial t} = \sum_{j=1}^{N} A^{(j)} \frac{\partial w_i^{(1)}}{\partial x_j} + \sum_{j,k=1}^{N} P^{(jk)} \frac{\partial^2 w_{i-1}^{(1)}}{\partial x_j \, \partial x_k}, \tag{3.12}$$

$$
(3.13)\qquad \begin{aligned} &A^{(1)} \frac{\partial w_i^{(2)}}{\partial(x_1/\nu)} + P^{(11)} \frac{\partial^2 w_i^{(2)}}{\partial(x_1/\nu)^2} \\ &= \frac{\partial w_{i-1}^{(2)}}{\partial t} - \sum_{j\neq 1} A^{(j)} \frac{\partial w_{i-1}^{(2)}}{\partial x_j} - 2 \sum_{j\neq 1} P^{(1j)} \frac{\partial^2 w_{i-1}^{(2)}}{\partial(x_1/\nu)\partial x_j} - \sum_{j,k\neq 1} P^{(jk)} \frac{\partial^2 w_{i-2}^{(2)}}{\partial x_j\, \partial x_k} \end{aligned}
$$

with the convention that $w_{i-2}^{(2)} \equiv 0$ if $i=1$. We shall assume here the boundary conditions (2.10) are imposed. The calculations are the same for (2.20). For $x_1 = 0$ we have

$$
(3.14)\qquad \frac{\partial \hat{w}_0^{(2)I}}{\partial(x_1/\nu)} = \sum_{i=m+1}^{r+p} \sum_{j=1}^{r+p} \zeta_i N_{ij}^{-1} \Psi^{iI} (\hat{w}_{0,j}^{(1)} + \hat{w}_{0,j}^{(2)}),
$$

$$
(3.15)\qquad \hat{w}_{0,k}^{(1)} + \hat{w}_{0,k}^{(2)} = \sum_{i,j=1}^{r+p} N_{ij}^{-1} \Psi_k^i (\hat{w}_{0,j}^{(1)} + \hat{w}_{0,j}^{(2)}), \qquad r+p+1 \leqq k \leqq n.
$$

For $l \geqq 1$,

$$
(3.16)\qquad \begin{aligned} &\bar{P}^{(11)} \left[\frac{\partial \hat{w}_l^{(2)I}}{\partial(x_1/\nu)} - \sum_{i=m+1}^{r+p} \sum_{j=1}^{r+p} \zeta_i N_{ij}^{-1} \Psi^{iI} (\hat{w}_{lj}^{(1)} + \hat{w}_{lj}^{(2)}) \right] \\ &\quad + \sum_{j\neq 1} \bar{P}^{(1j)} \left(\frac{\partial \hat{w}_{l-1}^{(1)}}{\partial x_j} + \frac{\partial \hat{w}_{l-1}^{(2)}}{\partial x_j} \right) + \bar{P}^{(11)} \frac{\partial \hat{w}_{l-1}^{(1)}}{\partial x_1} = 0, \end{aligned}
$$

$$
(3.17)\qquad \hat{w}_{l,k}^{(1)} + \hat{w}_{l,k}^{(2)} = \sum_{i,j=1}^{r+p} N_{ij}^{-1} \Psi_k^i (\hat{w}_{l,j}^{(1)} + \hat{w}_{l,j}^{(2)}), \qquad r+p+1 \leqq k \leqq n.
$$

Let us start with (3.10). From this form, we deduce that $w_0^{(2)}$ is a linear combination of the "exponential modes" defined in (1.6). Here $w_0^{(2)}$ is supposed to be exponentially decreasing in Ω^-, so that

$$
w_0^{(2)} = \sum_{j=r+p-m+1}^{r} \lambda_j \Theta^j e^{\theta_j x_1}, \qquad x_1 \leqq 0.
$$

We substitute into (3.14), remembering that for $i = m+1, \cdots, r+p$, (ζ_i, Ψ^i) is actually $(\theta_{i-m}, \Theta^{j-m})$. We thus get

$$
\sum_{j=r+p-m+1}^{r} \theta_j \lambda_j \Theta^{j^I} = \sum_{j=1}^{r+p-m} \sum_{i=1}^{r+p} \theta_j N_{j+m,i}^{-1} (\hat{w}_{0,1}^{(1)} + \hat{w}_{0,i}^{(2)}) \Theta^{j^I}.
$$

This amounts to stating that there exist coefficients α_k such that $\sum_{k=1}^r \alpha_k \Theta^{k^I} = 0$. It implies that $\sum_{k=1}^r \alpha_k \Theta^k = 0$, for equation (1.6) can be written

$$
\begin{aligned} &\bar{P}^{(11)} \theta_k \Theta^{k^I} + B^{(1)} \Theta^{k^I} + C^{(1)} \Theta^{k^{II}} = 0, \\ &D^{(1)} \Theta^{k^I} + \bar{A}^{(1)} \Theta^{k^{II}} = 0. \end{aligned}
$$

And if $\sum \alpha_k \Theta^{k^I} = 0$, then $\sum \alpha_k D^{(1)} \Theta^{k^I} = 0$, so that $\sum \alpha_k \bar{A}^{(1)} \Theta^{k^{II}} = 0$, and since $\bar{A}^{(1)}$ is nonsingular, the result follows. From the assumptions (0.5) on the operator Q, the Θ^k's are independent, and hence the α_k's vanish for any k.

Then $\lambda_j = 0$ for $r+p-m+1 \leqq j \leqq r$, and thus $w_0^{(2)}$ vanishes identically in Ω^-. We substitute into (3.14) and (3.15), which indicates that $w_0^{(1)}$ is a solution of the following problem in Ω^-:

$$
\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A^{(j)} \frac{\partial u}{\partial x_j} + F, \qquad x \in \Omega^-
$$

with the boundary conditions

$$
(3.18a)\qquad \sum_{j=1}^{r+p} N_{ij}^{-1} \hat{u}_j = 0, \qquad i = m+1, \cdots, r+p,
$$

(3.18b) $$\hat{u}_k=\sum_{i,j=1}^{r+p} N_{ij}^{-1}\Psi_k^i\hat{u}_j, \qquad k=r+p+1,\cdots,n.$$

We will now prove that (3.18) implies the transparent boundary condition (2.16) $(T^{-1}\hat{u})_i=0$, $m+1\leqq i\leqq n$:

$$(T^{-1}\hat{u})_i=\sum_{j=1}^{n} T_{ij}^{-1}\hat{u}_j=\sum_{j=1}^{r+p} T_{ij}^{-1}\hat{u}_j+\sum_{j=r+p+1}^{n} T_{ij}^{-1}\hat{u}_j.$$

In the second term of the right-hand side we substitute (3.18b):

$$\sum_{k=r+p+1}^{n} T_{ik}^{-1}\hat{u}_k=\sum_{k=r+p+1}^{n} T_{ik}^{-1}\sum_{l=1}^{r+p}\sum_{j=1}^{r+p} N_{lj}^{-1}\Psi_k^l\hat{u}_j,$$

$$\sum_{k=r+p+1}^{n} T_{ik}^{-1}\hat{u}_k=\sum_{k=1}^{n}\sum_{l=1}^{r+p}\sum_{j=1}^{r+p} N_{lj}^{-1}T_{ik}^{-1}\Psi_k^l\hat{u}_j-\sum_{k=1}^{r+p}\sum_{l=1}^{r+p}\sum_{j=1}^{r+p} N_{lj}^{-1}T_{ik}^{-1}\Psi_k^l\hat{u}_j,$$

but

$$\sum_{l=1}^{r+p} N_{lj}^{-1}\Psi_k^l=\delta_{kj} \quad\text{and}\quad \sum_{k=1}^{r+p}\sum_{l=1}^{r+p}\sum_{j=1}^{r+p} N_{lj}^{-1}\Psi_k^l T_{ik}^{-1}\hat{u}_j=\sum_{j=1}^{r+p} T_{ij}^{-1}\hat{u}_j,$$

so that

$$(T^{-1}\hat{u})_i=\sum_{k=1}^{n}\sum_{l=1}^{r+p}\sum_{j=1}^{r+p} N_{lj}^{-1}T_{ik}^{-1}\Psi_k^l\hat{u}_j.$$

On account of (3.18), the latter reduces to

$$(T^{-1}\hat{u})_i=\sum_{l=1}^{m}\sum_{j=1}^{r+p} N_{lj}^{-1}\hat{u}_j\sum_{k=1}^{n} T_{ik}^{-1}\Psi_k^l.$$

For $1\leqq l\leqq m$, Ψ^l corresponds to the hyperbolic part of Q, so that $\sum_{k=1}^n T_{ik}^{-1}\Psi_k^l=\delta_{il}$ and

$$(T^{-1}\hat{u})_i=0 \quad \text{for } m+1\leqq i\leqq n. \qquad \square$$

4. Application to Navier–Stokes equations. We consider here the two-dimensional compressible Navier–Stokes equations:

(4.1a) $$\rho\frac{du_i}{dt}=-\frac{\partial p}{\partial x_i}+\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\operatorname{div} u\right)\right], \qquad i=1,2,$$

(4.1b) $$\rho\frac{dc_vT}{dt}=-p\operatorname{div} u+\mu\sum_{i,j=1}^{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\operatorname{div} u\right)\frac{\partial u_i}{\partial x_j}+\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right),$$

(4.1c) $$\frac{d\rho}{dt}=-\rho\operatorname{div} u.$$

Here ρ is the density, u_i is the velocity component, p is the pressure, T is the temperature, μ and k are the coefficients of viscosity and heat conductivity, respectively, and c_v is the specific heat at constant volume. The pressure p is related to ρ and T by $p=\rho RT$, where R is the gas constant. We shall introduce γ as the ratio of specific heats, i.e., $\gamma=c_p/c_v$ (recall that $R=c_p-c_v$), and the Prandtl number of the gas $P_r=\mu c_p/k$. P_r is supposed to be constant here. As usual $d/dt=\partial/\partial t+u_1(\partial/\partial x_1)+u_2(\partial/\partial x_2)$. We shall assume that the artificial boundary is sufficiently far from any turbulent regime, so that we can consider (u,ρ,T,p) as a small perturbation of a smooth regime (u,ρ,T,p). Since in our analysis the lower-order derivatives are not of much importance, and the results of Michelson carry over to variable coefficients as well by freezing the coefficients, we shall concentrate here on the case where the reference regime is constant as a function of time and space. Let us call $(\tilde{u}, \tilde{\rho}, \tilde{T}, \tilde{p})$ the perturbation. It is a solution of the following problem:

$$\rho\frac{d\tilde{u}_1}{dt}+RT\frac{\partial\tilde{\rho}}{\partial x_1}+\rho R\frac{\partial\tilde{T}}{\partial x_1}=\mu\left[\frac{4}{3}\frac{\partial^2\tilde{u}_1}{\partial x_1^2}+\frac{\partial^2\tilde{u}_1}{\partial x_2^2}+\frac{1}{3}\frac{\partial^2\tilde{u}_2}{\partial x_1\,\partial x_2}\right], \tag{4.2a}$$

$$\rho\frac{d\tilde{u}_2}{dt}+RT\frac{\partial\tilde{\rho}}{\partial x_2}+\rho R\frac{\partial\tilde{T}}{\partial x_2}=\mu\left[\frac{\partial^2\tilde{u}_2}{\partial x_1^2}+\frac{4}{3}\frac{\partial^2\tilde{u}_2}{\partial x_2^2}+\frac{1}{3}\frac{\partial^2\tilde{u}_1}{\partial x_1\,\partial x_2}\right], \tag{4.2b}$$

$$\rho\frac{d\tilde{T}}{dt}+(\gamma-1)\rho T\operatorname{div}\tilde{u}=\gamma P_r^{-1}\mu\Delta\tilde{T}, \tag{4.2c}$$

$$\frac{d\tilde{\rho}}{dt}+\rho\operatorname{div}\tilde{u}=0. \tag{4.2d}$$

We shall normalize these equations by redefining $\tilde{\rho}$ as $\tilde{\rho}/\rho$ and introducing the undisturbed kinematic viscosity $\nu=\mu/\rho$. So that the equations can be written in the form (0.1)

$$\frac{\partial U}{\partial t}=A^{(1)}\frac{\partial U}{\partial x_1}+A^{(2)}\frac{\partial U}{\partial x_2}+\nu\sum_{j,k=1}^{2}P^{(jk)}\frac{\partial^2 u}{\partial x_j\,\partial x_k}+F(x,t), \tag{4.3}$$

where $U=(\tilde{u}_1, \tilde{u}_2, \tilde{T}, \tilde{\rho})$

$$A^{(1)}=\begin{pmatrix} -u_1 & 0 & -R & -RT\\ 0 & -u_1 & 0 & 0\\ -(\gamma-1)T & 0 & -u_1 & 0\\ -1 & 0 & 0 & -u_1\end{pmatrix},$$

$$A^{(2)}=\begin{pmatrix} -u_2 & 0 & 0 & 0\\ 0 & -u_2 & -R & -RT\\ 0 & -(\gamma-1)T & -u_2 & 0\\ 0 & -1 & 0 & -u_2\end{pmatrix},$$

$$P^{(11)}=\begin{pmatrix} \frac{4}{3} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \gamma P_r^{-1} & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$P^{(22)}=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \frac{4}{3} & 0 & 0\\ 0 & 0 & \gamma P_r^{-1} & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$P^{(12)}=\begin{pmatrix} 0 & \frac{1}{6} & 0 & 0\\ \frac{1}{6} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

They satisfy all the assumptions we made in the Introduction. We shall write here the precise formulation for the local boundary conditions (2.20). We shall start by studying the eigenvalues of $A^{(1)}$, and recalling the corresponding boundary conditions for the Euler equations (cf., for instance, [EM1]).

4.1. The Euler system. It is well known that the eigenvalues of $A^{(1)}$ are

$$\lambda_1 = -u_1 - c, \quad \lambda_2 = \lambda_3 = -u_1, \quad \lambda_4 = -u_1 + c. \tag{4.4}$$

The corresponding eigenvectors are

$$\begin{gathered}\Lambda^1 = \begin{pmatrix} c \\ 0 \\ (\gamma-1)T \\ 1 \end{pmatrix}, \qquad \Lambda^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \\ \Lambda^3 = \begin{pmatrix} 0 \\ 0 \\ T \\ -1 \end{pmatrix}, \qquad \Lambda^4 = \begin{pmatrix} c \\ 0 \\ -(\gamma-1)T \\ -1 \end{pmatrix}.\end{gathered} \tag{4.5}$$

So that $A^{(1)}$ is diagonalizable and the matrix $\tilde{T}$ in (2.17) is

$$\tilde{T} = \begin{pmatrix} c & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ (\gamma-1)T & 0 & T & -(\gamma-1)T \\ 1 & 0 & -1 & -1 \end{pmatrix} \tag{4.6}$$

and

$$A^{(1)} = \tilde{T} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} \tilde{T}^{-1}.$$

The number of boundary conditions required by the system at $x_1 = 0$ in Ω^- depends on whether the flow is super or subsonic and the boundary is inflow or outflow, as summarized below.

The **subsonic** case: $|u_1| < c$.

- **inflow:** $u_1 < 0$.

  $\lambda_1 < 0, \lambda_2, \lambda_3, \lambda_4 > 0, \quad m = 1$: **3 boundary conditions.**

  $(\tilde{T}^{-1}u)_i = 0, \qquad i = 2, 3, 4.$

- **outflow:** $u_1 > 0$.

  $\lambda_1, \lambda_2, \lambda_3 < 0, \quad \lambda_4 > 0, \quad m = 3$: **1 boundary condition.**

  $(\tilde{T}^{-1}u)_4 = 0.$

The **supersonic case:** $|u_1| > c$.

- **inflow:** $u_1 < 0$.

  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0, \quad m = 0$: **4 boundary conditions.**

  $u_i = 0, \qquad i = 1, 2, 3, 4.$

- **outflow:** $u_1 > 0$.

  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0, \quad m = 4$: **0 boundary conditions.**

The matrix $\tilde{T}^{-1}$ is given by

$$\tilde{T}^{-1}=\begin{pmatrix} \frac{1}{2c} & 0 & \frac{1}{2\gamma T} & \frac{1}{2\gamma} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma T} & -\frac{\gamma-1}{\gamma} \\ \frac{1}{2c} & 0 & -\frac{1}{2\gamma T} & -\frac{1}{2\gamma} \end{pmatrix} \tag{4.7}$$

so that the boundary conditions are as follows.

- *Subsonic inflow*:

$$\begin{aligned} &\frac{\tilde{u}_1}{c}-\frac{1}{\gamma}\frac{\tilde{T}}{T}-\frac{1}{\gamma}\frac{\tilde{\rho}}{\rho}=0, \\ &\tilde{u}_2=0, \\ &\frac{1}{\gamma}\frac{\tilde{T}}{T}-\frac{\gamma-1}{\gamma}\frac{\tilde{\rho}}{\rho}=0. \end{aligned} \tag{4.8a}$$

- *Subsonic outflow*:

$$\frac{\tilde{u}_1}{c}-\frac{1}{\gamma}\frac{\tilde{T}}{T}-\frac{1}{\gamma}\frac{\tilde{\rho}}{\rho}=0. \tag{4.8b}$$

- *Supersonic inflow*:

$$\tilde{u}_1=\tilde{u}_2=\tilde{T}=\tilde{\rho}=0. \tag{4.8c}$$

- *Supersonic outflow*:

no boundary condition.

We reintroduced here the density ρ for the sake of consistency. These boundary conditions are stable in the sense of Kreiss (see [EM1]).

4.2. The Navier–Stokes system. Here the number of boundary conditions is $n-p$, where p is the number of negative eigenvalues of $\bar{A}^{(1)}$. $\bar{A}^{(1)}$ reduces to $-u_1$, so that we must distinguish only between the inflow and the outflow cases:

- **Inflow** boundary: $u_1<0$.

$p=0$: **4 boundary conditions.**

- **Outflow** boundary: $u_1>0$

$p=1$: **3 boundary conditions.**

We must determine the other part of the family (Ψ^j), i.e., the θ_j and Θ_j solutions of

$$[P^{(1,1)}\theta+A^{(1)}]\Theta=0.$$

The cubic equation for θ has an immediate root we shall call θ_3:

$$\theta_3=u_1. \tag{4.9}$$

The two other roots are solutions of the quadratic equation

$$-u_1(\tfrac{4}{3}\theta-u_1)(\alpha\theta-u_1)-RT(\alpha\theta-\gamma u_1)=0, \tag{4.10}$$

where, for simplicity, we set

$$\alpha=\gamma P_r^{-1}.$$

We have

$$\theta_1\theta_2=\frac{3}{4\alpha}(u_1^2-c^2),\qquad \theta_1+\theta_2=\frac{3}{4}\frac{[\alpha(u_1^2-c^2/\gamma)+\frac{4}{3}u_1^2]}{\alpha u_1}$$

with $c^2=\gamma RT$.

Recalling that $\gamma>1$, we can determine the signs of the roots. We order θ_1 and θ_2 so that $\theta_1<\theta_2$ (θ_1 cannot be equal to θ_2).

Subsonic case: $\theta_1<0<\theta_2$.
- Inflow case: $\theta_3<0$;
- Outflow case: $\theta_3>0$.

Supersonic case.
- Inflow case: $\theta_1<\theta_2<0$, $\theta_3<0$;
- Outflow case: $0<\theta_1<\theta_2$, $\theta_3>0$.

The corresponding generalized eigenvalues are

$$\Theta^i=\begin{pmatrix} u_1\\ 0\\ (\gamma-1)Tu_1/(\alpha\theta_i-u_1)\\ -1\end{pmatrix},\quad i=1,2,\qquad \Theta^3=\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{4.11}$$

We now have all the elements to (2.20).
- *Subsonic inflow*: $m=1$, $p=0$, $r+p=3$.

$$\zeta_1=\lambda_1,\quad \tilde{\Psi}^1=\Lambda^1,\quad \zeta_2=\theta_1,\quad \tilde{\Psi}^2=\Theta^1,\quad \zeta_3=\theta_3,\quad \tilde{\Psi}^3=\Theta^3$$

and

$$\tilde{N}=\begin{pmatrix} c & u_1 & 0\\ 0 & 0 & 1\\ (\gamma-1)T & U_1 & 0\end{pmatrix},\qquad \tilde{N}^{-1}=\frac{1}{\det\tilde{N}}\begin{pmatrix} -U_1 & 0 & u_1\\ (\gamma-1)T & 0 & -c\\ 0 & \det\tilde{N} & 0\end{pmatrix},$$

where $U_i=(\gamma-1)Tu_1/(\alpha\theta_i-u_1)$, $\det\tilde{N}=(\gamma-1)Tu_1-U_1c$

$$\begin{aligned}
&\nu\frac{\partial\tilde{u}_1}{\partial x_1}+\frac{1}{8}\nu\frac{\partial\tilde{u}_2}{\partial x_2}=\frac{\theta_1}{1-c/(\gamma P_r^{-1}\theta_1-u_1)}\left(\tilde{u}_1-\frac{c}{\gamma-1}\frac{\tilde{T}}{T}\right),\\
&\nu\frac{\partial\tilde{u}_2}{\partial x_1}+\frac{1}{6}\nu\frac{\partial\tilde{u}_1}{\partial x_2}=u_1\tilde{u}_2,\\
&\nu\frac{\partial\tilde{T}}{\partial x_1}=\frac{(\gamma-1)T\theta_1}{\gamma P_r^{-1}\theta_1-u_1-c}\left(\tilde{u}_1-\frac{c}{\gamma-1}\frac{\tilde{T}}{T}\right),\\
&\frac{\tilde{\rho}}{\rho}=\frac{-1}{u_1(1-c/(\gamma P_r^{-1}\theta_1-u_1))}\left[\frac{\gamma P_r^{-1}\theta_1}{\gamma P_r^{-1}\theta_1-u_1}\tilde{u}_1-\frac{u_1+c}{\gamma-1}\frac{\tilde{T}}{T}\right].
\end{aligned} \tag{4.12a}$$

This system reduces to (4.8a) when $\nu=0$.

• *Subsonic outflow*: $m=3,\ p=1,\ r+p=4$.

$$\zeta_i=\lambda_i,\quad \tilde{\Psi}^i=\Lambda^i,\quad i=1,2,3,\quad \zeta_4=\theta_1,\quad \tilde{\Psi}^4=\Theta^1,$$

$$\tilde{N}=\begin{pmatrix} c & 0 & 0 & u_1\\ 0 & 1 & 0 & 0\\ (\gamma-1)T & 0 & T & U_1\\ 1 & 0 & -1 & -1\end{pmatrix},$$

$$\tilde{N}^{-1}=\frac{1}{TZ}\begin{pmatrix} T-U_1 & 0 & u_1 & u_1T\\ 0 & TZ & 0 & 0\\ T-U_1-\gamma T & 0 & u_1+c & T(u_1+c)-ZT\\ +\gamma T & 0 & -c & -cT\end{pmatrix},$$

where $Z=\gamma u_1-c(U_1/T-1)$. Condition (2.20) reduces to

$$\nu\sum_{j=1}^{2}\bar{P}^{(1j)}\frac{\partial w^I}{\partial x_j}=\bar{P}^{(11)}\theta_1\left(\sum_{j=1}^{4}\tilde{N}_{4j}^{-1}w_j\right)\Theta^{1^I},$$

or

$$\begin{aligned}&\nu\frac{\partial\tilde{u}_1}{\partial x_1}+\frac{1}{8}\nu\frac{\partial\tilde{u}_2}{\partial x_2}=\frac{c\theta_1u_1}{Z}\left(\frac{\gamma}{c}\tilde{u}_1-\frac{\tilde{T}}{T}-\frac{\tilde{\rho}}{\rho}\right),\\ &\nu\frac{\partial\tilde{u}_2}{\partial x_1}+\frac{1}{6}\nu\frac{\partial\tilde{u}_1}{\partial x_2}=0,\\ &\nu\frac{\partial\tilde{T}}{\partial x_1}=\frac{c\theta_1U_1}{Z}\left(\frac{\gamma}{c}\tilde{u}_1-\frac{\tilde{T}}{T}-\frac{\tilde{\rho}}{\rho}\right).\end{aligned}\tag{4.12b}$$

Again, these equations reduce to (4.8b) when $\nu=0$.

• *Supersonic inflow*: $m=0,\ p=0,\ r+p=3$.

$$\zeta_i=\theta_i,\quad \tilde{\Psi}^i=\Theta^i,\quad i=1,\cdots,3,$$

and

$$\tilde{N}=\begin{pmatrix} u_1 & u_1 & 0\\ 0 & 0 & 1\\ U_1 & U_2 & 0\end{pmatrix}\tilde{N}^{-1}=\frac{1}{u_1(U_1-U_2)}\begin{pmatrix} -U_2 & 0 & u_1\\ +U_1 & 0 & -u_1\\ 0 & u_1(U_1-U_2) & 0\end{pmatrix}.$$

The boundary conditions are

$$\begin{aligned}&\nu\frac{\partial\tilde{u}_1}{\partial x_1}+\frac{1}{8}\nu\frac{\partial\tilde{u}_2}{\partial x_2}=\frac{3}{4u_1}\left(u_1^2-\frac{c^2}{\gamma}\right)\tilde{u}_1+\frac{3c^2}{4\gamma}\frac{\tilde{T}}{T},\\ &\nu\frac{\partial\tilde{u}_2}{\partial x_1}+\frac{1}{6}\nu\frac{\partial\tilde{u}_1}{\partial x_2}=u_1\tilde{u}_2,\\ &\nu\frac{\partial\tilde{T}}{\partial x_1}=\frac{(\gamma-1)}{\alpha}Tu_1\left(\frac{\tilde{u}_1}{u_1}+\frac{1}{\gamma-1}\frac{\tilde{T}}{T}\right),\\ &\frac{\tilde{\rho}}{\rho}=-\frac{\tilde{u}_1}{u_1}.\end{aligned}\tag{4.12c}$$

If $\nu=0$, the system reduces to $\tilde{u}_1=\tilde{u}_2=\tilde{T}=\tilde{\rho}=0$.

- *Supersonic outflow*: $m=4$, $p=1$, $r+p=4$.

$$\zeta_i=\lambda_i, \qquad i=1,\cdots,4$$

so that the boundary condition is

$$\begin{aligned}&\frac{\partial\tilde{u}_1}{\partial x_1}+\frac{1}{8}\frac{\partial\tilde{u}_2}{\partial x_2}=0,\\ &\frac{\partial\tilde{u}_2}{\partial x_1}+\frac{1}{6}\frac{\partial\tilde{u}_1}{\partial x_2}=0,\\ &\frac{\partial\tilde{T}}{\partial x_1}=0.\end{aligned} \tag{4.12d}$$

The results in § 3 apply to these equations as follows.

THEOREM 4.1. *The initial boundary value problem for* (4.2) *and the zero-order boundary conditions* (4.12) *is well posed in* Ω^-. *As the viscosity* ν *tends to zero, the solution converges to the solution of the Euler equation with the corresponding boundary conditions* (4.8), *the* L^2 *norm of the error decreases linearly in* ν.

Proof. We merely need to check that the reduced hyperbolic problem is well posed, which is extremely simple here since $\bar{A}^{(1)}=-u_1$. The boundary condition then reduces to $\tilde{\rho}=0$ (when there is a boundary condition for $\tilde{\rho}$) and the following problem:

$$\begin{aligned}&\frac{\partial\tilde{\rho}}{\partial t}=-u_1\frac{\partial\tilde{\rho}}{\partial x_1}-u_2\frac{\partial\tilde{\rho}}{\partial x_2}, \qquad x_1\leqq 0,\\ &\tilde{\rho}=0, \qquad x_1=0\end{aligned}$$

for $u_1<0$ is obviously well posed.

Remark. In [GS] the authors introduced for the Navier–Stokes compressible equation artificial boundary conditions by requiring them to be dissipative. Furthermore, these boundary conditions produce a weak boundary layer. Therefore Theorem 4.1 also holds in that case. We have not been able to decide whether our boundary conditions are dissipative or not. However for more general systems or higher dimensions, it seems difficult to extend their techniques which consist of studying the boundary form $(\mathscr{E}w, w)$ and matching coefficients of the boundary condition to get the right sign. It does not allow for higher-order boundary conditions either.

5. Higher-order boundary conditions. We discussed earlier the goals of our work: provide boundary conditions which would be (1) local and (2) consistent with the Euler equation. A first step was made in § 3 by an approximation of order zero of the right-hand side in (2.10). We now want to increase the accuracy of our boundary conditions. This means, from our point of view, expand first the transparent boundary condition (2.10) up to higher order in ν. By doing this, we shall keep terms like $\alpha_i(s,\eta)$, for $1\leqq i\leqq m$, where α_i is the traveling mode defined in (1.3). These will correspond to pseudodifferential operators of order 1 on the boundary, which are, of course, far from being local. We thus shall in turn approximate these modes with the techniques described in [EM]. The first realistic approximation is similar to (2.17): we shall set $\eta=0$, and approximate the quantities in (2.7) to first order in ν.

We shall restrict ourselves here to the particular case of the viscous linearized shallow-water system, though the procedure carries over without modification to more general systems, provided they possess a symmetrizer. This property ensures that the eigenvalues ξ_i for the system have an expansion $\xi_i(s,\eta,\nu)=\zeta_i(s,\eta)+\nu\chi_i(s,\eta)+O(\nu^2)$,

$1 \leqq i \leqq m$, and $\xi_i(s, \eta, \nu) = (\zeta_i + \nu\chi_i + O(\nu^2))/\nu$, $m+1 \leqq i \leqq r+p$, with a corresponding expansion for the eigenvectors Ψ_j.

Let us consider the shallow-water equations, linearized about the steady-state $(U, 0)$:

$$\frac{\partial w}{\partial t} = A^{(1)} \frac{\partial w}{\partial x_1} + A^{(2)} \frac{\partial w}{\partial x_2} + \nu\left(P^{(11)} \frac{\partial^2 w}{\partial x_1^2} + P^{(22)} \frac{\partial^2 w}{\partial x_2^2}\right), \tag{5.1}$$

where $w = (u_1, u_2, \varphi)$.

$$A^{(1)} = \begin{pmatrix} -U & 0 & -1 \\ 0 & -U & 0 \\ -c^2 & 0 & -U \end{pmatrix}, \tag{5.2}$$

$$A^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -c^2 & 0 \end{pmatrix}, \tag{5.3}$$

$$P^{(11)} = P^{(22)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.4}$$

with $c > 0$.

In our notation of § 0, $r = 2$, $\bar{P}^{(ij)} = \delta_{ij} I_2$, and $\bar{A}^{(1)} = -U$. The eigenvalues of $A^{(1)}$ are

$$\lambda_1 = -U - c, \quad \lambda_2 = -U, \quad \lambda_3 = -U + c \tag{5.5a}$$

and the corresponding eigenvectors are

$$\Lambda_1 = \begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 1 \\ 0 \\ -c \end{pmatrix}. \tag{5.5b}$$

The solutions of $(P^{(1,1)}\theta + A^{(1)})\Theta = 0$ are

$$\theta_1 = \frac{U^2 - c^2}{U}, \qquad \theta_2 = U, \tag{5.6a}$$

and the corresponding generalized eigenvectors are

$$\Theta_1 = \begin{pmatrix} U \\ 0 \\ -c^2 \end{pmatrix}, \qquad \Theta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{5.6b}$$

The signs of the λ's and θ's depend on whether the flow is sub or supersonic, and ingoing or outgoing, as in the case of the full Navier–Stokes equations.

We shall approximate the generalized eigenvalues and eigenvectors $\xi_i(s, 0, \nu)$ and $\Phi^i(s, 0, \nu)$ up to first order in ν.

For $1 \leqq i \leqq m$:

$$\begin{aligned} &\xi_i(s, \nu) = \zeta_i(s) + \nu\chi_i s^2 + O(\nu^2), \\ &\zeta_i(s) = \frac{s}{\lambda_i}, \end{aligned} \tag{5.7a}$$

$$\begin{aligned} &\Phi^i(s, \nu) = \Psi^i + \nu \Xi^i s + O(\nu^2), \\ &\Psi^i = \Lambda^i; \end{aligned} \tag{5.7b}$$

For $m+1\leqq i\leqq r+p$:

$$\xi^i(s,\nu)=\frac{1}{\nu}\zeta_i+\chi_{i-m}s+O(\nu),\qquad \zeta_i=\theta_{i-m}, \tag{5.8a}$$

$$\Phi^i(s,\nu)=\Psi^i+\nu\Xi^{i-m}s+O(\nu^2),\qquad \Psi^i=\Theta^{i-m} \tag{5.8b}$$

(in the formulas above, the variable η, being zero, has been omitted).

The χ_i's and Ξ_i's are obtained by substitution of the expressions above in formula (11) for $\eta=0$:

$$(\xi A^{(1)}+\nu\xi^2P^{(11)}-sI)\Phi=0.$$

To λ_1, λ_2, λ_3 are associated three values of ξ and vectors Φ:

$$\begin{aligned}\tilde{\xi}_1&=\frac{s}{\lambda_1}-\frac{\nu s^2}{2\lambda_1^3}+O(\nu^2),\\ \tilde{\xi}_2&=\frac{s}{\lambda_2}-\frac{\nu s^2}{\lambda_2^3}+O(\nu^2),\\ \tilde{\xi}_3&=\frac{s}{\lambda_3}-\frac{\nu s^2}{2\lambda_2^3}+O(\nu^2),\end{aligned} \tag{5.9a}$$

$$\begin{aligned}\tilde{\Phi}^1&=\Lambda^1+\nu s\begin{pmatrix}-1/2\lambda_1c\\0\\0\end{pmatrix}+O(\nu^2)=\begin{pmatrix}1-\nu s/2\lambda_1c\\0\\c\end{pmatrix}+O(\nu^2),\\ \tilde{\Phi}_2&=\Lambda^2+O(\nu^2)=\begin{pmatrix}0\\1\\0\end{pmatrix}+O(\nu^2),\\ \tilde{\Phi}_3&=\Lambda^3+\nu s\begin{pmatrix}-1/2\lambda_3c\\0\\0\end{pmatrix}+O(\nu^2)=\begin{pmatrix}1-\nu s/2\lambda_3c\\0\\-c\end{pmatrix}+O(\nu^2),\end{aligned} \tag{5.9b}$$

and to θ_1, θ_2 are associated two values of ξ and Φ:

$$\begin{aligned}\tilde{\tilde{\xi}}_1&=\frac{U^2-c^2}{\nu U}+s\frac{U^2+c^2}{U(U^2-c^2)}+O(\nu),\\ \tilde{\tilde{\Phi}}^1&=\Theta^1+\nu s\begin{pmatrix}0\\0\\c^2/(U^2-c^2)\end{pmatrix}+O(\nu^2)\\ &=\begin{pmatrix}U\\0\\-c^2(1-\nu s/(U^2-c^2))\end{pmatrix}+O(\nu^2),\end{aligned} \tag{5.10a}$$

$$\begin{aligned}\tilde{\tilde{\xi}}_2&=\frac{U}{\nu}+\frac{s}{U}+O(\nu),\\ \tilde{\tilde{\Phi}}^2&=\Theta^2+O(\nu^2)=\begin{pmatrix}0\\1\\0\end{pmatrix}+O(\nu^2).\end{aligned} \tag{5.10b}$$

We cannot go any further without dividing the analysis into four cases: subsonic or supersonic, inflow or outflow.

—*Subsonic inflow case*: $-c < U < 0$, $p = 0$.

$$\lambda_1 < 0, \quad \lambda_2, \lambda_3 > 0: \quad m = 1,$$
$$\theta_1 > 0, \qquad \theta_2 < 0.$$

p is equal to zero. We have three boundary conditions

$$\xi_1 = \tilde{\xi}_1, \qquad \Phi^1 = \tilde{\Phi}^1,$$
$$\xi_2 = \tilde{\tilde{\xi}}^2, \qquad \Phi^2 = \tilde{\tilde{\Phi}}^2.$$

By a zero-order approximation of the ξ_i's and Φ^i's, we get the first set of approximated boundary conditions:

$$(5.11)_0 \qquad \begin{cases} \nu \dfrac{\partial u_1}{\partial x_1} = 0, \\[2mm] \nu \dfrac{\partial u_2}{\partial x_1} - U u_2 = 0, \\[2mm] \varphi - c u_1 = 0. \end{cases}$$

By a first-order approximation in ν, we obtain a new set of boundary conditions, which contains differentiation in time:

$$(5.11)_1 \qquad \begin{cases} \nu \dfrac{\partial u_1}{\partial x_1} = -\dfrac{\nu}{U + c} \dfrac{\partial u_1}{\partial t}, \\[2mm] \nu \dfrac{\partial u_2}{\partial x_1} = U u_2 + \dfrac{\nu}{U} \dfrac{\partial u_2}{\partial t}, \\[2mm] \varphi = c u_1 - \dfrac{\nu}{2(U + c)} \dfrac{\partial u_1}{\partial t}. \end{cases}$$

—*Subsonic outflow case*: $0 < U < c$, $p = 1$.

$$\lambda_1, \lambda_2 < 0, \quad \lambda_3 > 0: \quad m = 2,$$
$$\theta_1 < 0, \qquad \theta_2 > 0.$$

The generalized eigenvalues and eigenvectors with negative real parts (when $\operatorname{Re} s > 0$) are

$$\xi_1 = \tilde{\xi}_1, \quad \xi_2 = \tilde{\xi}_2, \quad \xi_3 = \tilde{\tilde{\xi}}_1,$$
$$\Phi^1 = \tilde{\Phi}^1, \quad \Phi^2 = \tilde{\Phi}^2, \quad \Phi^3 = \tilde{\tilde{\Phi}}^1.$$

The analogue to $(5.11)_0$ becomes

$$(5.12)_0 \qquad \nu \frac{\partial u_1}{\partial x_1} = \frac{c - U}{c}(-c u_1 + \varphi), \qquad \nu \frac{\partial u_2}{\partial x_1} = 0,$$

and the first-order boundary condition is

$$(5.12)_1 \qquad \begin{cases} \nu \dfrac{\partial u_1}{\partial x_1} = \dfrac{c - U}{c}(-c u_1 + \varphi) + \dfrac{\nu}{2c^2(c - U)} \dfrac{\partial}{\partial t}(-c^2 u_1 + U\varphi), \\[2mm] \nu \dfrac{\partial u_2}{\partial x_1} = -\dfrac{\nu}{U} \dfrac{\partial u_2}{\partial t}. \end{cases}$$

In the supersonic case, the calculations are much easier.

—*Supersonic inflow case*: $U < -c$, $p = 0$.

$$\lambda_1, \lambda_2, \lambda_3 > 0: \quad m = 0,$$
$$\theta_1, \theta_2 < 0.$$

We have three boundary conditions:

$$(5.13)_0 \qquad \begin{cases} \nu \dfrac{\partial u_1}{\partial x_1} = \dfrac{U^2 - c^2}{U} u_1; \\ \nu \dfrac{\partial u_2}{\partial x_1} = U u_2, \\ \varphi = -\dfrac{c^2}{U} u_1, \end{cases}$$

and

$$(5.13)_1 \qquad \begin{cases} \nu \dfrac{\partial u_1}{\partial x_1} = \dfrac{U^2 - c^2}{U} u_1 + \nu \dfrac{U^2 + c^2}{U(U^2 - c^2)} \dfrac{\partial u_1}{\partial t}, \\ \nu \dfrac{\partial u_2}{\partial x_1} = U u_2 + \dfrac{\nu}{U} \dfrac{\partial u_2}{\partial t}, \\ \varphi = -\dfrac{c^2}{U} u_1 + \dfrac{\nu c^2}{U(U^2 - c^2)} \dfrac{\partial u_1}{\partial t}. \end{cases}$$

—*Supersonic outflow case*: $U > c$, $p = 1$.

$$\lambda_1, \lambda_2, \lambda_3 < 0: \quad m = 3,$$
$$\theta_1, \theta_2 > 0.$$

Here we have two boundary conditions:

$$(5.14)_0 \qquad \nu \frac{\partial u_1}{\partial x_1} = 0, \qquad \nu \frac{\partial u_2}{\partial x_1} = 0,$$

$$(5.14)_1 \qquad \begin{cases} \nu \dfrac{\partial u_1}{\partial x_1} = \dfrac{\nu}{U^2 - c^2} \dfrac{\partial}{\partial t} (-U u_1 + \varphi), \\ \nu \dfrac{\partial u_2}{\partial x_1} = -\dfrac{\nu}{U} \dfrac{\partial u_2}{\partial t}. \end{cases}$$

We shall not repeat here the calculations for the inviscid case, i.e., $\nu = 0$. They are identical to those for the full Euler equations. The local boundary condition (2.17) is in this case:

Subsonic inflow case:

$$(5.15) \qquad u_2 = 0, \qquad \varphi - c u_1 = 0;$$

Subsonic outflow case:

$$(5.16) \qquad \varphi - c u_1 = 0;$$

Supersonic inflow case:

$$(5.17) \qquad u_1 = u_2 = \varphi = 0;$$

Supersonic outflow case:

(5.18) no boundary conditions.

The analogue of Theorem 4.1 holds.

THEOREM 5.1. *The initial boundary value problem for* (5.1) *and any of the boundary conditions* $(5.11+i)_0$, $i=0,\cdots,3$ *is well posed in* Ω^-. *As the viscosity* ν *tends to zero, the solution converges to the solution of the inviscid equation with the corresponding boundary condition* $(5.14+i)$, *and the* L^2*-norm of the error decreases as* $O(\nu)$.

The proof is exactly the same as for Theorem 4.1.

Remark. In this case, the well posedness in the classical sense can be expressed by energy estimates, using the variational formula (1.14). Let us denote by $E(t)$ the quantity defined by

$$E(t)=\frac{1}{2}\iint_{\Omega^-}[c^2(u_1^2+u_2^2)+\varphi^2]\,dx \tag{5.19}$$

and the analogue on Γ:

$$E_\Gamma(t)=\frac{1}{2}\int_\Gamma[c^2(u_1^2+u_2^2)+\varphi^2]\,dx_2. \tag{5.20}$$

The energy equality then reads

$$\begin{aligned}&\frac{dE}{dt}+\nu\iint_{\Omega^-}(\nabla u_1^2+\nabla u_2^2)\,dx\\&=-UE_\Gamma(t)+c^2\int_\Gamma\left(\nu u\cdot\frac{\partial u}{\partial x_1}-u_1\varphi\right)dx_2+\iint_{\Omega^-}Fu\,dx.\end{aligned} \tag{5.21}$$

It can be easily checked in each case that the quantity integrated on Γ is negative.

Unfortunately, the decoupling conditions prescribed in [M] to obtain the well posedness and the error estimates do not apply to our higher-order boundary conditions $(5.11+i)_1$. We have not been able to establish a priori estimates in this case either. However, the formal expansion (3.6) is still available. It is an easy matter to check that for the higher-order boundary conditions $(5.11+i)_1$, the next term in the expansion vanishes. For instance, in the subsonic inflow case, it is due to the fact that $(\partial/\partial x_1+1/(U+c)\,\partial/\partial t)(w_0^{(1)})=0$. *So the boundary layer is weaker than in the former case, and the solution of the corresponding initial boundary value problem converges formally to the solution of the inviscid equation with boundary condition* $(5.14+i)$, *the error being* $O(\nu^2)$.

Remark. Consider the boundary conditions derived from $(5.11+i)_1$ in the four cases by neglecting certain terms:

$$\begin{cases}\nu\dfrac{\partial u_1}{\partial x_1}=-\dfrac{\nu}{U+c}\dfrac{\partial u_1}{\partial t},\\[2ex]\nu\dfrac{\partial u_2}{\partial x_1}=Uu_2+\dfrac{\nu}{U}\dfrac{\partial u_2}{\partial t},\qquad\varphi=cu_1,\end{cases} \tag{5.22}$$

$$\nu\frac{\partial u_1}{\partial x_1}=\frac{c-U}{c}(-cu_1+\varphi),\qquad\nu\frac{\partial u_2}{\partial x_1}=-\frac{\nu}{U}\frac{\partial u_2}{\partial t}, \tag{5.23}$$

$$\begin{cases}\nu\dfrac{\partial u_1}{\partial x_1}=\dfrac{U^2-c^2}{U}u_1+\nu\dfrac{U^2+c^2}{U(U^2-c^2)}\dfrac{\partial u_1}{\partial t},\\[2ex]\nu\dfrac{\partial u_2}{\partial x_1}=u_2+\dfrac{\nu}{U}\dfrac{\partial u_2}{\partial t},\qquad\varphi=-\dfrac{c^2}{U}u_1,\end{cases} \tag{5.24}$$

$$\nu\frac{\partial u_1}{\partial x_1}=0,\qquad\nu\frac{\partial u_2}{\partial x_1}=-\frac{\nu}{U}\frac{\partial u_2}{\partial t}. \tag{5.25}$$

These boundary conditions are well posed in the classical sense: we have neglected the terms which could prevent the energy from decreasing in time. Furthermore, they still give an approximation to the inviscid problem with boundary conditions $(5.14+i)$ in $O(\nu^2)$: the relevant equations are unchanged. However, the last statement remains formal, since the decoupling conditions still do not hold.

So far we have considered approximations to the inviscid equations with the "zero-order" boundary conditions (2.17), which are those used in practice. It is tempting to try to approximate the Euler problem better. This adds new important difficulties: as pointed out in [EM] the choice of the "good" boundary condition in the hyperbolic case is not canonical, and furthermore, it is not clear whether or not it is well posed in the sense of Kreiss. An analysis of such boundary conditions, together with numerical experiments, will be presented in a forthcoming paper.

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