INRIA

CENTRE DE ROCQUENCOURT

Institut National
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en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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PARABOLIC WAVE EQUATION APPROXIMATIONS IN HETEROGENEOUS MEDIA

PART I

Alain BAMBERGER
Bjorn ENQUIST
Laurence HALPERN
Patrick JOLY

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IN HETEROGENEOUS MEDIA

Part. I

A. BAMBERGER[*,**], B. ENGQUIST[***], L. HALPERN[**], P. JOLY[*].

INRIA. DOMAINE DE VOLUCEAU - ROCQUENCOURT - B.P. 105 - 78150 - LE CHESNAY CEDEX - FRANCE.
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RESUME

Différentes approximations paraboliques de l'équation des ondes scalaire sont présentées et analysées. Ces équations sont utilisées dans différents domaines, comme la géophysique et l'acoustique sous-marine. Une nouvelle équation parabolique en milieu hétérogène est établie, dont les qualités d'approximation de la réflexion des ondes sur une interface entre deux milieux sont optimales. En effet l'amplitude des ondes réfléchie et réfractée dépend continûment de tous les paramètres, ce qui n'est pas le cas pour les équations classiques. L'existence, l'unicité d'une solution, et des estimations d'énergie sont établies pour cette équation. Les résultats sont donnés aussi bien pour le problème de Cauchy que pour le problème aux limites.

ABSTRACT

Different variants of parabolic approximations of scalar wave equations are derived and their properties are analysed. These equations are of general form which includes the ones used in seismology, underwater acoustics and other applications. A new version of the parabolic approximation is derived for heterogeneous media. It has optimal properties with respect to wave reflection at material interfaces. The amplitudes of the reflected and transmitted waves depend continuously on the interface. Existence, uniqueness and energy estimates are proved.

MOTS CLES

APPROXIMATION PARAXIALE, APPROXIMATION PARABOLIQUE, EQUATIONS DES ONDES.

KEY WORDS

PARAXIAL APPROXIMATION, PARABOLIC APPROXIMATION, WAVE EQUATION.

1 - INTRODUCTION

Parabolic wave equation approximations have been used to describe wave propagation with a preferred direction in a number of applications. The name, parabolic equation, was already used by Leontovitch and Fock [20], and they applied the method to describe electromagnetic waves along the surface of the earth.

Different versions of parabolic approximations to wave equations have later been used as mathematical models for computational algorithms in many other areas. In seismology parabolic wave equation approximations were introduced by Claerbout [6] and have been applied both to scalar and elastic wave propagation [7], [8], [9], [17], [19], [22].

The geophysical applications in seismology have been particularly successful for the inverse problem, in connection to the so-called migration process [8]. Numerical computations based on wave approximations are now a standard part of geophysical data processing.

In underwater acoustics, parabolic approximations have also become an essential tool. The idea was introduced to this field by Tappert and Hardin [26] and has given rise to a lively research activity [5], [12], [23], [24].

In [27], Tappert gives a survey of the applications of parabolic approximation in acoustics as well as in other fields. This article also contains analysis of basic properties of these equations.

Other fields of application for parabolic wave approximations are optics and in particular laser optics, plasma physics, radio waves diffraction problems, and small disturbance calculations of transonic flow [10], [15], [16], [27].

In different fields of application this type of wave equation approximation goes under different names as e.g. "thin beam approximation", "quasi-optical approximation", "15°-approximation", or "one-way wave equation".

Parabolic approximations are also very useful as boundary conditions in computations with the full wave equation [9], [13]. They can be used as "absorbing boundary conditions" when there is no physical boundary, but where an artificial boundary has to be introduced in order to limit the domain of computation. These boundary conditions produce very small artificial reflections at the computational boundary.

The simplest (and most common in applications) form of a paraxial approximation is the parabolic approximation of a scalar wave equation for a homogeneous medium (with paraxial approximations we mean the general class of wave equation approximations which describe wave propagation in a preferred direction [2], [14]). In two space dimensions, the solution $u(x,t)$ of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0 \tag{1.1}$$

with velocity c is approximated by the solution of the parabolic approximation

$$\frac{1}{c}\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t\,\partial x_2} - \frac{c}{2}\frac{\partial^2 u}{\partial x_1^2} = 0 \tag{1.2}$$

This approximation is exact for plane waves traveling in the positive x_2 direction (i.e. "upgoing" waves)

$$u(x,t) = f(x_2 - ct) \quad .$$

The function u above is obviously a solution to both (1.1) and (1.2). Moreover, the approximation (1.2) is good for waves traveling close to the positive x_2 direction.

In a retarded coordinate system $t \to t - \frac{x_2}{c}$, the approximate equation (1.2) becomes :

$$\frac{\partial^2 u}{\partial t\, \partial x_2} - \frac{c}{2}\, \frac{\partial^2 u}{\partial x_1^2} = 0 \tag{1.3}$$

This form is common in applications, i.e. when performing the computations in the frequency domain (see the references to the applications above).

There are two main reasons for using parabolic wave equations instead of the full wave equation.

(i) It is simpler and computationally more efficient in particular in time retarded coordinates (equation (1.3)).

(ii) It only describes the wave propagating in the positive x_2 direction, and not the back reflected wave.

The property (i) is of course important in all applications. The second property (ii) is crucial in many cases. It makes it possible to use the approximation as an evolution equation in x_2, which is not the case for the original wave equation. Property (ii) is also important in the migration process ([8]). Scattered waves are traced backward in time and the reflection from (1.1) would not be physically relevant. Finally, property (ii) is essential when the parabolic approximation is used as boundary condition.

There are two ways of improving the approximation (1.2). It can be generalized in such a way that waves propagating in a direction with a substantial angle to the x_2 axis are also well approximated. This can be done by using higher order differential equations or systems of differential equations. It is studied in [1], [3].

In this paper we shall consider the generalization to variable velocity media. Different types of such parabolic approximations have been derived, based on different principles and objectives. We shall analyze these approximations with respect to their mathematical properties, and in particular the reflection and transmission at an interface. Neither of these approximations has good properties with respect to the amplitudes, and we shall design a new approximation.

In section 2, we recall the parabolic approximation in homogeneous media and we give some mathematical results : existence, uniqueness and energy estimates are proved for the solution of the standard Cauchy problem and also the Cauchy problem with data given on $x_2 = 0$. We also study the phase and group velocities and the fundamental solution.

We write the various existing parabolic approximations in heterogeneous media on a general form in section 3 and we study the reflection and transmission at an interface. We define the transmission conditions and calculate the reflection and transmission coefficients. These coefficients are compared to those for the wave equation and we study the continuity of them with respect to interfaces.

In section 4, we develop a new parabolic approximation which is a good approximation to the wave equation for heterogeneous media with small velocity variations and which has the amplitudes of the reflected and transmitted waves continuous with respect to the location of the interface. As for the homogeneous case energy estimates, existence and uniqueness for the Cauchy problem in time or space are proved. The propagation properties in a variable velocity medium are studied.

Results from numerical experiments are presented in section 5. A numerical approximation of the fundamental solution in homogeneous and heterogeneous media gives quantitative illustrations to the qualitative analysis of the earlier sections. The calculations were performed by F. Collino [11].

Some of the results of this paper were annonced in [14] and some technical details in the proofs are omitted here but are given in the report [2].

2 - PARABOLIC APPROXIMATIONS IN HOMOGENEOUS MEDIA

2.1. DERIVATIONS OF THE APPROXIMATION

The analysis in this paper is concerned with the wave equation in two space dimensions, but could be done in $\mathbb{R}^n$ for any n as well.
Consider the scalar wave equation in two space dimensions for a homogeneous medium with velocity $\hat{c} > 0$:

$$(2.1) \qquad \frac{1}{\hat{c}^2} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad .$$

A plane harmonic wave

$$u(x_1, x_2, t) = \exp i(\omega t - k_1 x_1 - k_2 x_2)$$

is a solution to (2.1) if the frequency ω and the wave vector $k = (k_1, k_2)$ satisfy the dispersion relation :

$$(2.2) \qquad \omega^2 = c^2 |k|^2 = c^2 (k_1^2 + k_2^2)$$

From the dispersion relation we can define two frequencies $\omega_+(k)$ and $\omega_-(k)$ corresponding to waves traveling in the positive x_2 direction $(u_+(x,t))$ and the negative x_2 direction $(u_-(x,t))$ respectively.

$$\begin{cases} c \dfrac{k_2}{\omega_+(k)} = + \left(1 - \left(c \dfrac{k_1}{\omega_+(k)}\right)^2\right)^{\frac{1}{2}} \\[2ex] c \dfrac{k_2}{\omega_-(k)} = - \left(1 - \left(c \dfrac{k_1}{\omega_-(k)}\right)^2\right)^{\frac{1}{2}} \end{cases}$$

$$\begin{cases} u(x,t) = u_+(x,t) + u_-(x,t) \\[1ex] u_+(x,t) = \displaystyle\iint_{\mathbb{R}^2} \hat{a}_+(k) \exp i(\omega_+(k)t - k.x)\, dk \\[1ex] u_-(x,t) = \displaystyle\iint_{\mathbb{R}^2} \hat{a}_-(k) \exp i(\omega_-(k)t - k.x)\, dk \end{cases}$$

All solutions with finite energy can be written as a superposition of plane waves related to the frequencies $\omega_+(k)$ and $\omega_-(k)$. The amplitudes $\hat{a}_+(k)$ and $\hat{a}_-(k)$ are given by the Fourier transforms of the initial values at time t_0.

$$\hat{a}_+(k) = \frac{1}{\omega_- - \omega_+}(\omega_- \hat{u}(k,t_0) + i \frac{\partial \hat{u}}{\partial t}(k,t_0))$$

$$\hat{a}_-(k) = \frac{1}{\omega_+ - \omega_-}(\omega_+ \hat{u}(k,t_0) + i \frac{\partial \hat{u}}{\partial t}(k,t_0))$$

The purpose of a paraxial approximation of (2.1) is to provide an equation the solution of which is a good approximation of an essential part of u_+ (or u_-). The ideal equation should have the following dispersion relation :

$$(2.3) \qquad c\,\frac{k_2}{\omega} = +\,(1 - (c\,\frac{k_1}{\omega})^2)^{\frac{1}{2}}$$

This does not correspond to a partial differential equation. If we also want to have a partial differential equation, we must settle for a rational dispersion relation which approximates (2.3). Our design criterion is to require the approximation to be good for propagation directions close to the positive x_2-direction, i.e. for :

$$|c\,\frac{k_1}{\omega}| = |\sin\Theta| \ \text{ small} \quad ; \qquad c\,\frac{k_2}{\omega} \geqslant 0$$

It is natural to use a first order Taylor expansion $(\varepsilon = c\,\frac{k_1}{\omega})$:

$$(1 - \varepsilon^2)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^4) \ ,$$

in order to derive the dispersion relation

$$c\,\frac{k_2}{\omega} = +\,(1 - \frac{1}{2}c^2\frac{k_1^2}{\omega^2}) \ , \text{ or}$$

$$(2.4) \qquad -\omega^2 + ck_2\,\omega + \frac{1}{2}c^2k_1^2 = 0$$

The corresponding partial differential equation is the so-called parabolic approximation (see e.g [27])

$$(2.5) \qquad \frac{\partial^2 u}{\partial t^2} + c\,\frac{\partial^2 u}{\partial x_2 \partial t} - \frac{1}{2}\,c^2\,\frac{\partial^2 u}{\partial x_1^2} = 0 \quad .$$

In figure 2.1 the circular graph of the dispersion relation (2.2) and the graph of the parabolic approximation (2.4) are displayed ($K = \frac{k}{\omega}$).

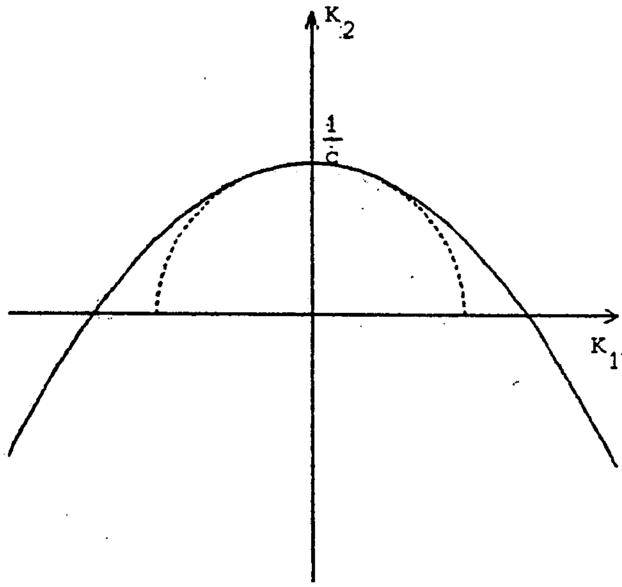


Figure 2.1 : dispersion relations

In geophysics this equation has been called 15° approximation ([8]) since it gave the required accuracy for $\theta \leqslant 15°$. In that field the original technique used when deriving the equation was different. In [8], Claerbout introduced the change of variable $t \to t + \frac{x_2}{c}$.

The function

$$v(x,t) = u(x, t + \frac{x_2}{c})$$

satisfies the equation

$$c^2\,\frac{\partial^2 v}{\partial x_1^2} + c^2\,\frac{\partial^2 v}{\partial x_2^2} - 2c\,\frac{\partial^2 v}{\partial x_2 \partial t} = 0 \quad .$$

The term $c^2 \frac{\partial^2 v}{\partial x_1^2}$ is dropped since it is smaller than the other terms for harmonic waves originally traveling in a direction close to the positive x_2 axis. The resulting equation

$$\frac{\partial^2 v}{\partial t \partial x_2} - \frac{c}{2} \frac{\partial^2 v}{\partial x_1^2} = 0$$

is the equivalent of (2.5) in a moving coordinate frame.

2.2. MATHEMATICAL PROPERTIES OF THE APPROXIMATION

All possible harmonic plane wave solutions of the parabolic approximation are described by the dispersion relation (2.4). Thus, corresponding to a wave vector k there are two frequencies $\omega_+(k)$ and $\omega_-(k)$ with the slowness vectors K^+ and K^- respectively $(K = \frac{k}{\omega})$, K_2^+ is positive, K_2^- is negative.

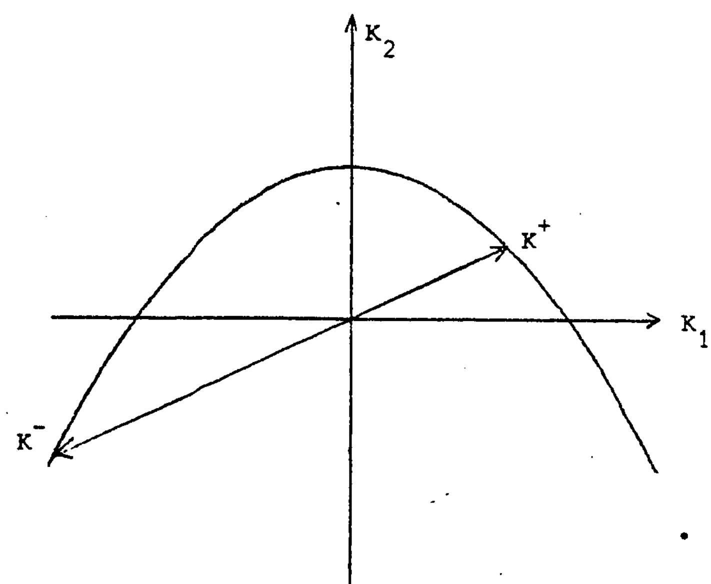


Figure 2.2 : the two propagation modes

The parabolic approximation has been designed such that it can be used as an equation with x_2 considered as an evolution direction. It might therefore seem paradoxical that there remain waves propagating in a direction with a negative x_2 component (corresponding to the vector K^- in figure 2.2).

There is however no contradiction since the group velocity vector $V_G(K)$ always has a positive x_2 component.

(2.6) $$V_G(K) = \nabla_k(\omega) = \frac{1}{2 - cK_2}\begin{pmatrix} cK_1 \\ 1 \end{pmatrix}$$

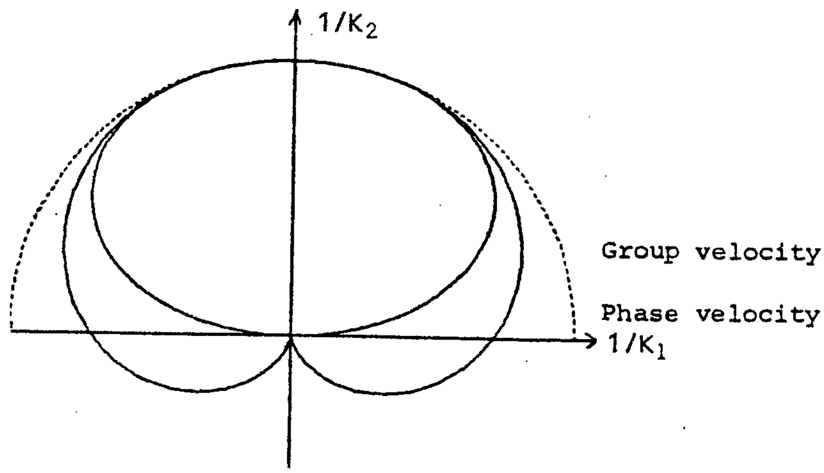


Figure 2.3 : phase velocity, group velocity

The fundamental solution E for (2.5) can be derived from the fundamental solution of the wave equation after a change of variables.

$$\begin{cases} \dfrac{\partial^2 E}{\partial t^2} + c\,\dfrac{\partial^2 E}{\partial x_2 \partial t} - \dfrac{c^2}{2}\,\dfrac{\partial^2 E}{\partial x_1^2} = 0 \\ E(x,0) = 0 \\ \dfrac{\partial E}{\partial t}(x,0) = \delta(x) \end{cases}$$

With $x_1' = \frac{\sqrt{2}}{2}x_1$, $x_2' = x_2 - \frac{c}{2}t$, $t' = t$

$$E'(x',t') = \sqrt{2}\,E(x,t)$$

we get in the primal variables

$$\begin{cases} \dfrac{\partial^2 E'}{\partial t'^2} - \dfrac{c^2}{4}\Delta E' = 0 \\ E'(x',0) = 0 \\ \dfrac{\partial E'}{\partial t'}(x',0) = \delta(x') \end{cases}$$

and thus the fundamental solution E is :

$$E(x,t) = \frac{1}{\pi c} \frac{H(2ctx_2 - x_1^2 - 2x_2^2)}{(2ctx_2 - x_1^2 - 2x_2^2)^{\frac{1}{2}}}$$

where H is the Heaviside function.

The support of E at time t is the interior of the ellipse $\mathscr{E}(t)$ in figure 2.4 whose equation is :

$$(2.7) \qquad 2x_1^2 + 4(x_2 - \frac{c}{2}t)^2 = c^2t^2$$

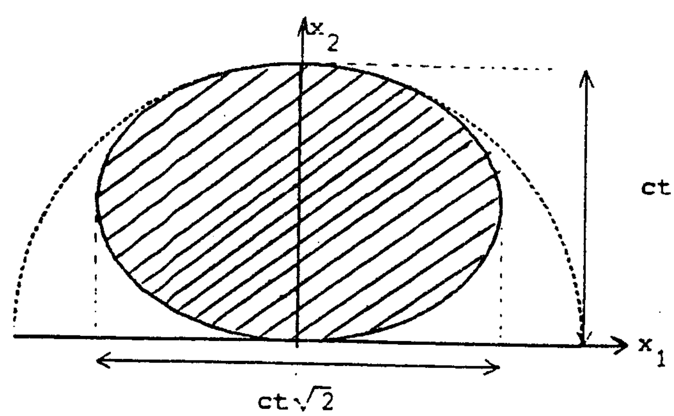


Figure 2.4 : support of the fundamental solutions

Note that the support of the fundamental solution is compact and included in the upper half plane (positive x_2). The singularity along the boundary is of the same kind as that for the regular wave equation.

The parabolic approximation (2.5) has important energy conservation properties. Consider first the Cauchy problem with time as evolution direction.

(2.8a) $$\frac{\partial^2 u}{\partial t^2} + c\,\frac{\partial^2 u}{\partial t\,\partial x_2} - \frac{c^2}{2}\,\frac{\partial^2 u}{\partial x_1^2} = 0 \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$$

(2.8b) $$u(x,0) = u_0(x)$$

$$x \in \mathbb{R}^2$$

(2.8c) $$\frac{\partial u}{\partial t}(x,0) = u_1(x)$$

The following energy identities are valid :

(2.9) $$\frac{1}{2}\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{c^2}{4}\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \frac{1}{2}\int_{\mathbb{R}^2} |u_1|^2 dx + \frac{c^2}{4}\int_{\mathbb{R}^2} \left|\frac{\partial u_0}{\partial x_1}\right|^2 dx$$

(2.10) $$\frac{1}{2}\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x_2}\right|^2 dx + \frac{c^2}{4}\int_{\mathbb{R}^2} \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \frac{1}{2}\int_{\mathbb{R}^2} \left|u_1 + c\,\frac{\partial u_0}{\partial x_2}\right|^2 dx + \frac{c^2}{4}\int_{\mathbb{R}^2} \left|\frac{\partial u_0}{\partial x_1}\right|^2 dx$$

Equation (2.9) is obtained from (2.8a) after multiplying by $\frac{\partial u}{\partial t}$ and integrating over $\mathbb{R}^2$. For the second identity we rewrite (2.8a) on the form

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x_2}\right) - \frac{c^2}{2}\,\frac{\partial^2 u}{\partial x_1^2} = 0 \quad ,$$

multiply by $\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x_2}$ and then integrate over $\mathbb{R}^2$. Consider now the half-space problem for $x_2 > 0$ ($\mathbb{R}^2_+ = \{(x_1,x_2) / x_2 > 0\}$).

(2.11a) $$\frac{\partial^2 u}{\partial t^2} + c\,\frac{\partial^2 u}{\partial t\,\partial x_2} - \frac{c^2}{2}\,\frac{\partial^2 u}{\partial x_1^2} = 0 \qquad (x,t) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \ .$$

(2.11b) $$u(x,0) = u_0(x)$$

$$x \in \mathbb{R}^2_+$$

(2.11c) $$\frac{\partial u}{\partial t}(x,0) = u_1(x)$$

(2.11d) $$u(x_1,0,t) = g(x_1,t) \qquad (x_1,t) \in \mathbb{R} \times \mathbb{R}_+ \ .$$

The following energy identities are appropriate when x_2 is considered as an evolution direction

$$(2.12)\quad \begin{cases} \int_0^T \int_{\mathbb{R}} |\frac{\partial u}{\partial t}(x_1,x_2,t)|^2 dx_1 dt + \frac{1}{c}\int_0^{x_2}\int_{\mathbb{R}} |\frac{\partial u}{\partial t}(x_1,\zeta,T)|^2 dx_1 d\zeta + \\ + \frac{c}{2}\int_0^{x_2}\int_{\mathbb{R}} |\frac{\partial u}{\partial x_1}(x_1,\zeta,T)|^2 dx_1 d\zeta = \int_0^T\int_{\mathbb{R}} |\frac{\partial g}{\partial t}(x_1,t)|^2 dx_1 dt + \\ + \frac{1}{c}\int_0^{x_2}\int_{\mathbb{R}} |u_1(x_1,\zeta)|^2 dx_1 d\zeta + \frac{c}{2}\int_0^{x_2}\int_{\mathbb{R}} |\frac{\partial u_0}{\partial x_1}(x_1,\zeta)|^2 dx_1 d\zeta \end{cases}$$

$$(2.13)\quad \begin{cases} \frac{c}{2}\int_0^T\int_{\mathbb{R}} |\frac{\partial u}{\partial x_1}(x_1,x_2,t)|^2 dx_1 dt + \int_0^{x_2}\int_{\mathbb{R}} |(\frac{\partial u}{\partial x_2} + \frac{1}{c}\frac{\partial u}{\partial t})(x_1,\zeta,T)|^2 dx_1 d\zeta + \\ + \frac{1}{2}\int_0^{x_2}\int_{\mathbb{R}} |\frac{\partial u}{\partial x_1}(x_1,\zeta,t)|^2 dx_1 d\zeta = \frac{c}{2}\int_0^T\int_{\mathbb{R}} |\frac{\partial g}{\partial x_1}(x_1,t)|^2 dx_1 dt + \\ + \int_0^{x_2}\int_{\mathbb{R}} |(\frac{\partial u_0}{\partial x_2} + \frac{u_1}{c})(x_1,\zeta)|^2 dx_1 d\zeta + \frac{1}{2}\int_0^{x_2}\int_{\mathbb{R}} |\frac{\partial u_0}{\partial x_1}(x_1,\zeta)|^2 dx_1 d\zeta \end{cases}$$

The derivation is similar to the derivation for (2.9) and (2.10).

3 - PARABOLIC APPROXIMATIONS IN HETEROGENEOUS MEDIA

3.1. DIFFERENT DERIVATIONS OF THE APPROXIMATIONS

In this section, we shall describe three different ways of deriving the parabolic approximations that have appeared in the literature and which lead to three different equations.

Consider the scalar wave equation on the form :

$$(3.1)\qquad \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \ .$$

(A more general variable coefficient formulation will be introduced in section 4.1).

The simplest way of deriving a parabolic approximation for (3.1) is the method of frozen coefficients. The equation is regarded as locally homogeneous and c is replaced by $c(x)$ in formula (2.5).

$$(3.2)\qquad \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c(x)}\frac{\partial^2 u}{\partial t\partial x_2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0$$

In section 2.1 we outlined a technique for the derivation in the homogeneous case following [8]. This technique which is based on a change of variables can be extended to a variable velocity. The transformed function is given by

$$v(x,t) = u(x,t + T(x))$$

$$T(x) = \int_0^{x_2} \frac{d\zeta}{c(x_1,\zeta)}$$

The equation (3.1) expressed in v has the form

$$\left\{\begin{aligned} &2c\,\frac{\partial^2 v}{\partial t\partial x_2} + c^2\,\frac{\partial^2 v}{\partial x_2^2} + c^2\,\frac{\partial}{\partial x_2}\left(\frac{1}{c}\right)\frac{\partial v}{\partial t} + \\ &\qquad - c^2\left(\frac{\partial^2 v}{\partial x_1^2} - 2\,\frac{\partial T}{\partial x_1}\,\frac{\partial^2 v}{\partial t\partial x_1} + \left(\frac{\partial T}{\partial x_1}\right)^2\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t}\,\frac{\partial^2 T}{\partial x_1^2}\right) = 0\end{aligned}\right.$$

As in section 2.1 the $\frac{\partial^2 v}{\partial x_2^2}$ term is dropped and the inverse of the change of variables is performed. The resulting approximate equation is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^{\frac{1}{2}}}\frac{\partial}{\partial x_2}\left(\frac{1}{c^{\frac{1}{2}}}\frac{\partial u}{\partial t}\right) - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0 \tag{3.3}$$

The third technique we shall discuss is based on the expansion of pseudo-differential operators. It follows the derivation of absorbing boundary conditions in [13]. These boundary conditions are paraxial wave equations with x_2 in the direction normal to the boundary.

For smooth velocity c the hyperbolic operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ can be decomposed into a product of two pseudo-differential operators. One of these operators corresponds to waves traveling in the positive x_2 direction and the other in the negative x_2 direction. The first terms in the asymptotic expansion for the one with waves traveling in the positive x_2 direction lead to :

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^{\frac{1}{2}}}\frac{\partial}{\partial x_2}\left(\frac{1}{c^{\frac{1}{2}}}\frac{\partial u}{\partial t}\right) - \frac{1}{2c}\frac{\partial}{\partial x_1}\left(c\,\frac{\partial u}{\partial x_1}\right) = 0 \tag{3.4}$$

The details are given in [2].

The three approximations derived above can be written in the following general form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c\varphi(c)}\frac{\partial}{\partial x_2}\left(\varphi(c)\,\frac{\partial u}{\partial t}\right) - \frac{1}{2\psi(c)}\frac{\partial}{\partial x_1}\left(\psi(c)\,\frac{\partial u}{\partial x_1}\right) = 0 \tag{3.5}$$

where φ and ψ are smooth positive functions. When the velocity c is constant this equation reduces to the parabolic approximation (2.5). In heterogeneous media it generalizes the equations defined above :

$$\left\|\begin{array}{lll} \text{equation (3.2)} : & \varphi(c) = 1 , & \psi(c) = 1 \\ \text{equation (3.3)} : & \varphi(c) = c^{-\frac{1}{2}} , & \psi(c) = 1 \\ \text{equation (3.4)} : & \varphi(c) = c^{-\frac{1}{2}} , & \psi(c) = c \end{array}\right.$$

3.2. TRANSMITTED AND REFLECTED WAVE AT A LINEAR INTERFACE

We shall see that the model (3.5) is not sufficient to ensure a good approximation to the wave equation for heterogeneous media with small velocity variations and the continuity of the solution with respect to interfaces between layers.
Our analysis will be concerned with a particular heterogeneous medium. It consists of two homogeneous half-spaces Ω^- and Ω^+, with a velocity c^- and c^+ respectively, separated by an interface $\Gamma(\alpha)$. The unit normal and tangent vectors to the interface are denoted by ν and τ respectively :

$$(3.6) \quad \begin{cases} \tau = (\cos\alpha, \sin\alpha) \\ \nu = (-\sin\alpha, \cos\alpha) \\ \Gamma(\alpha) = \{x,\ x.\nu = 0\} \\ \Omega^- = \{x,\ x.\nu < 0\} \\ \Omega^+ = \{x,\ x.\nu > 0\} \end{cases}$$

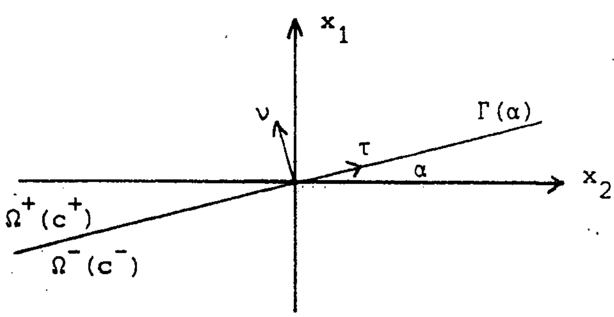


Figure 3.1 : description of the medium

We shall investigate the reflection and transmission of a harmonic plane wave at $\Gamma(\alpha)$ and compare it to the case of the wave equation (2.1). The continuity with respect to α will be studied.
We first derive the transmission conditions for (3.5) at $\Gamma(\alpha)$.

THEOREM 3.1 : When $\alpha = 0$, *there is one transmission condition, given by :*

$$[\varphi(c)u]_{\Gamma(\alpha)} = 0 \tag{3.7}$$

When $\alpha \neq 0$, *there are two transmission conditions, defined by :*

$$\begin{cases} [u]_{\Gamma(\alpha)} = 0 \\ [\,\theta(c)\,\frac{\partial u}{\partial t} + \frac{1}{2}\,\psi(c)\,\frac{\partial u}{\partial x_1}\sin\alpha\,]_{\Gamma(\alpha)} = 0 \end{cases} \tag{3.8}$$

where $[\;]_{\Gamma(\alpha)}$ *denotes the jump across the interface, and the function* $\theta(c)$ *is defined by :*

$$\begin{cases} \frac{d\theta}{dc} = \psi(c)\;(c\varphi(c))^{-1}\,\frac{d\varphi}{dc} \\ \theta(1) = 1 \end{cases} \tag{3.9}$$

REMARK 3.1 : The number of transmission conditions is discontinuous at $\alpha = 0$. Moreover the conditions themselves are discontinuous since neither of conditions (3.8) converges to the condition (3.7) when α goes to zero. This is of course due to the fact that the equation is of first order in x_2 and second order in x_1.

PROOF : We begin with the case $\alpha = 0$. Then the equation reduces to :

$$\frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} + \frac{1}{c\varphi(c)}\,\frac{\partial^2}{\partial t\partial x_2}(\varphi(c)u) - \frac{1}{2}\,\frac{\partial^2 u}{\partial x_1^2} = 0$$

The highest order of singularity in the equation is achieved by the x_2 derivative. Thus the transmission condition is the same as for the equation :

$$\frac{\partial}{\partial x_2}(\varphi(c)u) = 0$$

which is clearly relation (3.7).

When $\alpha \neq 0$, we define new coordinates along and normal to the interface :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and then rewrite the equation in the new coordinates as :

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{\sin\alpha}{c}\frac{\partial^2 u}{\partial t \partial X_1} - \cos^2\alpha \frac{\partial^2 u}{\partial X_1^2} +$$

$$+ \frac{\sin\alpha\cos\alpha}{2}\frac{\partial^2 u}{\partial X_1 \, \partial X_2} +$$

$$+ \frac{\cos\alpha}{c\varphi(c)}\frac{\partial^2}{\partial t \partial X_2}(\varphi(c)u) + \frac{\sin\alpha\cos\alpha}{2\psi(c)}\frac{\partial^2}{\partial X_1 \, \partial X_2}(\psi(c)u) +$$

$$- \frac{\sin^2\alpha}{2\psi(c)}\frac{\partial}{\partial X_2}\left(\psi(c)\frac{\partial u}{\partial X_2}\right) = 0$$

The highest order singularity is contained in the last term and, if u is not continuous, cannot be compensated by any of the others. Thus u is continuous and satisfies the first condition in (3.8). The highest order singularity is now the sum S of the last three terms :

$$S = \frac{\cos\alpha}{c\varphi(c)}\frac{\partial^2}{\partial t \partial X_2}(\varphi(c)u) + \frac{\sin\alpha\cos\alpha}{2\psi(c)}\frac{\partial^2}{\partial X_1 \, \partial X_2}(\psi(c)u) +$$

$$- \frac{\sin^2\alpha}{2\psi(c)}\frac{\partial}{\partial X_2}\left(\psi(c)\frac{\partial u}{\partial X_2}\right)$$

Using the function θ defined in (3.9), a straightforward calculation gives :

$$S = \frac{1}{\psi(c)}\frac{\partial}{\partial X_2}\left[\cos\alpha\,\theta(c)\frac{\partial u}{\partial t} + \frac{\sin\alpha}{2}\psi(c)\left(\cos\alpha\frac{\partial u}{\partial X_1} - \sin\alpha\frac{\partial u}{\partial X_2}\right)\right]$$

$$+ \left(\frac{1}{c} - \frac{\theta(c)}{\psi(c)}\right)\frac{\partial^2 u}{\partial t \partial X_2}$$

Again, if u is continuous, the highest singularity is in the X_2 derivative, unless the bracket is continuous. We then get (3.8) by noticing that

$$\cos\alpha \frac{\partial u}{\partial X_1} - \sin\alpha \frac{\partial u}{\partial X_2} = \frac{\partial u}{\partial x_1} \ .$$ ■

We are now able to study the reflection and transmission of harmonic plane waves at the interface $\Gamma(\alpha)$.

Let u^I be an incident wave in Ω^-, that is :

(3.10) $$u^I = \exp i(\omega t - k^I.x)$$

where the real vector k^I and the frequency ω are connected by the dispersion relation in Ω^- :

(3.11) $$-\frac{1}{(c^-)^2}\ \omega^2 + \frac{1}{c^-}\ \omega k_2^I + \frac{1}{2}\ (k_1^I)^2 = 0 \ .$$

and are such that the group velocity vector is going toward the interface, i.e.

(3.12) $$V_G\left(\frac{k^I}{\omega}\right).n > 0 \ .$$

The reflected and transmitted waves are defined by their wave vectors k^R and k^T and the reflection and transmission coefficients $R(\alpha)$ and $T(\alpha)$ such that the solution of equation (3.5) is equal to :

(3.13) $$u = \begin{cases} u^I + R(\alpha)\exp i(\omega t - k^R.x) & \text{in } \Omega^- \\ T(\alpha)\exp i(\omega t - k^T.x) & \text{in } \Omega^+ \end{cases}$$

and satisfies the transmission condition(s).

The reflected and transmitted wave vectors k^R and k^T are determined by the following conditions :

- (ω,k^R) satisfies the dispersion relation (3.11) in Ω^-.
- (ω,k^T) satisfies the dispersion relation in Ω^+ :

(3.14) $$-\frac{1}{(c^+)^2}\ \omega^2 + \frac{1}{c^+}\omega k_2 + \frac{1}{2}k_1^2 = 0 \quad .$$

. The projections of k^R and k^T on $\Gamma(\alpha)$ are equal to the projection of k^I :

(3.15) $$k^R.\tau = k^T.\tau = k^I.\tau \quad .$$

. k^R is reflected in Ω^-, k^T is transmitted in Ω^+ :

(3.16) $$V_G(\frac{k^R}{\omega}).\nu < 0 \qquad ; \qquad V_G(\frac{k^T}{\omega}).\nu > 0$$

The cases $\alpha = 0$ and $\alpha \neq 0$ are inherently different, as stated in the lemma.

<u>LEMMA 3.1</u> : *(i) For an horizontal interface there is no reflected wave, and there is one transmitted wave.*

(ii) For an oblique interface there are always one reflected wave and one transmitted wave.

<u>PROOF</u> :

(i) Suppose α is equal to zero. Then the reflected wave vector would be such that :

$$\begin{cases} k_1^R = k_1^I \\ -\dfrac{1}{(c^-)^2}\,\omega^2 + \dfrac{1}{c^-}k_2^R\,\omega + \dfrac{1}{2}(k_1^R)^2 = 0 \end{cases}$$

and k_2^R would be equal to k_2^I. Thus, from (3.16), there is no reflected wave. The transmitted wave vector solves the system :

$$\begin{cases} k_1^T = k_1^I \\ -\dfrac{1}{(c^+)^2}\,\omega^2 + \dfrac{1}{c^+}k_2^T\,\omega + \dfrac{1}{2}(k_1^T)^2 = 0 \end{cases}$$

this gives an unique solution k_2^T, and it can readily be checked that the corresponding wave is transmitted in Ω^+.

(ii) If α is not zero, then k^R is the other solution of the system :

$$\begin{cases} k_1 \cos\alpha + k_2 \sin\alpha = k^I.\tau \\ -\dfrac{1}{(c^-)^2}\,\omega^2 + \dfrac{1}{c^-}\,k_2\,\omega + \dfrac{1}{2}\,k_1^2 = 0 \end{cases}$$

which can easily be solved, and k^T is solution of :

$$\begin{cases} k_1^T \cos\alpha + k_2^T \sin\alpha = k^T.\tau \\ -\dfrac{1}{(c^+)^2}\,\omega^2 + \dfrac{1}{c^-}\,k_2^T\,\omega + \dfrac{1}{2}\,(k_1^T)^2 = 0 \end{cases}$$

Replacing k_2^T in the second equation yields a quadratic equation :

$$(3.17) \qquad \left(\frac{k_1^T}{\omega}\right)^2 - \frac{2}{c}\,\frac{\cos\alpha}{\sin\alpha}\left(\frac{k_1^T}{\omega}\right) + \frac{2}{c}\left(\frac{k^I.\tau}{\sin\alpha} - \frac{1}{c}\right) = 0$$

Depending on the sign of the discriminant, the solutions are real or imaginary :

If $\dfrac{k^I.\tau}{\omega} \leq \dfrac{1}{2c^+}\;\dfrac{1+\sin^2\alpha}{\sin\alpha}$,

k_1^T is defined as the real root of equation (3.17) satisfying $V_G\left(\frac{k^T}{\omega}\right).\nu > 0$

If $\dfrac{k^I.\tau}{\omega} > \dfrac{1}{2c^+}\;\dfrac{1+\sin^2\alpha}{\sin\alpha}$,

k_1^T is the imaginary root of equation (3.17) such that $\mathrm{Im}\left(\frac{k^T}{\omega}\right).\nu > 0$, and the corresponding wave is evanescent in Ω^+. ∎

The coefficients $R(\alpha)$ and $T(\alpha)$ are now determined by applying the transmission conditions to (3.13).

LEMMA 3.2 : The reflection and transmission coefficients are given by :

If $\alpha = 0$

$$(3.18) \qquad R(0) = 0 \;\; ; \;\; T(0) = \frac{\varphi(c^-)}{\varphi(c^+)}$$

If $\alpha \neq 0$

$$(3.19)\quad \begin{cases} \left[(\theta(c^+) - \theta(c^-))\, \omega \cos \alpha - \frac{1}{2} (\psi(c^+)\, k_1^T - \psi(c^-)\, k_1^R) \sin \alpha \right] T(\alpha) = \\ \qquad\qquad - \frac{1}{2}\, \psi(c^-)\, (k_1^R - k_1^I) \sin \alpha \\ R(\alpha) = T(\alpha) - 1 \end{cases}$$

We notice that for a horizontal interface, R and T do not depend on the incidence angle.

We now compare these results to the ones obtained for the wave equation (2.1). In that case, there are always one reflected and one transmitted wave (when $c^+ \neq c^-$). The corresponding coefficients depend on the incidence angle. For incident waves hitting the interface normally, the wave equation reflection and transmission coefficients are :

$$(3.20)\qquad T_{\square} = \frac{2}{1 + \frac{c^-}{c^+}} \qquad\qquad R_{\square} = \frac{1 - \frac{c^-}{c^+}}{1 + \frac{c^-}{c^+}}$$

If c^+ is close to c^-, it is natural to require $T(0)$ to approximate $T_{\square}$. A second order Taylor expansion of $T(0)$, $R_{\square}$ and $T_{\square}$ yields :

$$(3.21)\qquad \begin{cases} R_{\square} = \frac{\Delta c}{c^-} + O((\Delta c)^2) \\ R(0) = 0 \end{cases} \qquad\qquad \begin{cases} T_{\square} = 1 + \frac{\Delta c}{2c^-} + O((\Delta c)^2) \\ T(0) = 1 - \Delta c\, \frac{\varphi'(c^-)}{\varphi(c^-)} + O(\Delta c)^2) \end{cases}$$

with $\qquad \Delta c = c^+ - c^-$

For any value of the functions φ and ψ, $T(0)$ and $R(0)$ are first order approximations of $T_{\square}$ and $R_{\square}$ respectively. For a better approximation of $T_{\square}$, we easily get the following result :

THEOREM 3.2 : For a normally incident wave on a horizontal interface, the Taylor expansions of $T_{\square}$ *and* $T(0)$ *coincide up to the second order with respect to* $\frac{\Delta c}{c}$ *if and only if*

(3.22) $$\varphi(c) = c^{-\frac{1}{2}} .$$

The second main point we emphasize here is the continuity of $R(\alpha)$ and $T(\alpha)$ as α tends to zero.

LEMMA 3.4 : For any functions φ *and* ψ, *the limits of* $T(\alpha)$ *and* $R(\alpha)$ *when* α *tends to zero are given by :*

(3.23) $$\begin{cases} \lim\limits_{\alpha \to 0} T(\alpha) = T^* \qquad\qquad \lim\limits_{\alpha \to 0} R(\alpha) = R^* \\ \left[(\theta(c^+) - \theta(c^-)) + \dfrac{\psi(c^-)}{c^-}\right] T^* = \dfrac{\psi(c^-)}{c^-} \\ R^* = T^* - 1 \end{cases}$$

PROOF : If $\frac{k_1^I}{\omega}$ is fixed, we have :

$$\lim_{\alpha \to 0} \sin\alpha \frac{k_1^I}{\omega} = \lim_{\alpha \to 0} \sin\alpha \frac{k_1^T}{\omega} = 0$$

and from the dispersion relation in Ω^- :

$$\lim_{\alpha \to 0} \sin\alpha \frac{k_1^R}{\omega} = \frac{2}{c^-}$$

The result is then achieved by passing to the limit in (3.19) (for details see [2]).

THEOREM 3.3 : (i) $T(\alpha)$ *is continuous as* α *tends to zero for any values of* c^+ *and* c^- *if and only if :*

(3.24) $$\frac{d}{dc}\left(\frac{\psi(c)}{c\varphi(c)}\right) = 0 \quad or \quad \frac{d}{dc}\varphi(c) = 0 .$$

(ii) $R(\alpha)$ *is continuous as* α *tends to zero for any values of* c^+ *and* c^- *if and only if :*

$$\frac{d}{dc}\varphi(c) = 0 \quad . \tag{3.25}$$

PROOF : From (3.18) and (3.23), the condition for $T(\alpha)$ to be continuous is :

$$\forall\, c^-, c^+ ; \left[\theta(c^+) - \theta(c^-) + \frac{\psi(c^-)}{c^-}\right] c^- \varphi(c^-) = \varphi(c^+)\,\psi(c^-)$$

Differentiating with respect to c^+ and using the definition of θ given in (3.9) yields :

$$\forall\, c^-, c^+ ; \quad \frac{d\varphi}{dc}(c^+)\left[\frac{\psi(c^-)}{c^-\varphi(c^-)} - \frac{\psi(c^+)}{c^+\varphi(c^+)}\right] = 0$$

which is (3.24).

Conversely, it is clear that (3.24) implies the continuity of $T(\alpha)$. The same argument holds for $R(\alpha)$.

REMARK 3.2 : By Theorem 3.2, the parabolic equation is a good approximation of the wave equation if $\varphi(c) = c^{-\frac{1}{2}}$. The continuity of the transmission coefficient then requires $\psi(c)$ to be $c^{\frac{1}{2}}$. But in that case the reflection coefficient is not continuous. Let us notice that for equation (3.2) the coefficients are continuous, but it is not a second order approximation to the wave equation, in the sense stated in Theorem 3.2. In an other hand, equations (3.3) and (3.4) yield a good approximation to the wave equation, while the coefficients are not continuous.

Our conclusion is thus that equation (3.5) is not general enough to satisfy all the criteria discussed above and a more general form is needed.

4 - A NEW PARABOLIC APPROXIMATION IN HETEROGENEOUS MEDIA

As we mentionned in the introduction, the parabolic approximation has been designed so that its dispersion relation approximates that of the wave equation, i.e. we approximate the velocity of waves. One also needs to approximate the amplitudes of waves. In particular, good properties of the amplitudes of reflected and transmitted waves across interfaces are important. Therefore we intend to develop an equation which generalizes equation (2.5) and satisfies the three criteria below :

Consider the medium consisting of $\Omega^-(c^-)$ and $\Omega^+(c^+)$ defined in (3.6). The equation is required to be such that :

(i) When $\alpha = 0$, for a normal incidence, the transmission coefficient is equal to the one of the wave equation, up to the second order with respect to $\frac{\Delta c}{c}$.

(ii) The reflexion and transmission coefficients $R(\alpha)$ and $T(\alpha)$ respectively are continuous with respect to α.

The third criterion applies to any velocity distribution :

(iii) The Cauchy problem is well-posed.

We saw in the previous section that the model (3.4) is unable to satisfy these conditions.

4.1. DERIVATION OF THE EQUATION

We are seeking an equation in the following general family of parabolic approximations :

$$(4.1) \qquad \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c\zeta(c)}\frac{\partial}{\partial x_2}\left(\zeta(c)\frac{\partial u}{\partial t}\right) - \frac{1}{2\chi(c)\xi(c)}\frac{\partial}{\partial x_1}\left(\chi(c)\frac{\partial}{\partial x_1}(\xi(c)u)\right) = 0$$

where ζ, ξ and χ are smooth positive functions of c, and are to be determined such that criteria (i), (ii) and (iii) be satisfied. (Note that this family contains the family (3.5) of the previous section).

The main result is the following :

THEOREM 4.1 : The equation (4.1) *satisfies the conditions* (i) *and* (ii) *if and only if* ζ *and* ξ *are modulo multiplicative constants given by :*

$$(4.2) \qquad \zeta(c) = \xi(c) = c^{-\frac{1}{2}} \quad , \quad \forall\, c \in \mathbb{R}_+^*$$

Moreover if χ *is chosen as :*

$$(4.3) \qquad \chi(c) = c \quad , \quad \forall\, c \in \mathbb{R}_+^*$$

then the condition (iii) *is satisfied and the following energy is constant with respect to time* t *:*

$$(4.4) \qquad E(t) = \frac{1}{2}\iint \frac{1}{c^2}\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{4}\iint c\left|\frac{\partial}{\partial x_1}(c^{-\frac{1}{2}}u)\right|^2 dx$$

Therefore the new parabolic approximation in heterogeneous media, with properties (i), (ii) and (iii), is :

$$(4.5) \qquad \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + c^{-\frac{1}{2}}\frac{\partial}{\partial x_2}\left(c^{-\frac{1}{2}}\frac{\partial u}{\partial t}\right) - \frac{1}{2}c^{-\frac{1}{2}}\frac{\partial}{\partial x_1}\left(c\frac{\partial}{\partial x_1}(c^{-\frac{1}{2}}u)\right) = 0$$

which can be rewritten using an auxiliary unknown :

$$(4.6) \qquad \begin{cases} v = c^{-\frac{1}{2}}u \\ \dfrac{1}{c}\dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t\,\partial x_2} - \dfrac{1}{2}\dfrac{\partial}{\partial x_1}\left(c\dfrac{\partial v}{\partial x_1}\right) = 0 \end{cases}$$

PROOF OF THEOREM 4.1 : We shall treat the three criteria separately.

Part (i) : For $\alpha = 0$, the equation reduces to (3.5). So we have the equivalent result to Theorem 3.2. Equation (4.1) fulfills criterion (i) if and only if the function ξ is given by :

$$\xi(c) = c^{-\frac{1}{2}} \quad ; \quad \forall c \in \mathbb{R}_+^* .$$

Part (ii) : The new function defined by :

$$v = \xi u$$

is solution of the equation :

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} + \frac{\xi}{c\zeta}\frac{\partial}{\partial x_2}\left(\frac{\zeta}{\xi}\frac{\partial v}{\partial t}\right) - \frac{1}{2\chi}\frac{\partial}{\partial x_1}\left(\chi\frac{\partial v}{\partial x_1}\right) = 0$$

This equation belongs to family (3.5) with :

$$\varphi = \frac{\zeta}{\xi} \quad ; \quad \psi = \chi$$

Then seeking a solution u on the form

$$\begin{cases} u = \exp i(\omega t - k^I.x) + R(\alpha)\exp i(\omega t - k^R.x) & \text{in } \Omega^- \\ u = T(\alpha)\exp i(\omega t - k^T.x) & \text{in } \Omega^+ \end{cases}$$

is equivalent to seeking v on the form

$$\begin{cases} v = \xi(c^-)\left[\exp i(\omega t - k^I.x) + R(\alpha)\exp i(\omega t - k^R.x)\right] & \text{in } \Omega^- \\ v = \xi(c^+)\, T(\alpha)\exp i(\omega t - k^T.x) & \text{in } \Omega^+ \end{cases}$$

or, equivalently, if $\tilde{v} = \xi(c^-)^{-1} v$:

$$\begin{cases} \tilde{v} = \exp i(\omega t - k^I.x) + \overline{R}(\alpha)\exp i(\omega t - k^R.x) & \text{in } \Omega^- \\ \tilde{v} = \overline{T}(\alpha)\exp i(\omega t - k^T.x) & \text{in } \Omega^+ \end{cases}$$

where

$$\begin{cases} \overline{R}(\alpha) = R(\alpha) \\ \overline{T}(\alpha) = \dfrac{\xi(c^+)}{\xi(c^-)}\, T(\alpha) \end{cases}$$

We then can apply Lemmas 3.2 and 3.4 to obtain

$$\begin{cases} \overline{T}(0) = \dfrac{\varphi(c^-)}{\varphi(c^+)} & \overline{R}(0) = 0 \\ \overline{T}^* = \dfrac{\dfrac{\psi(c^-)}{c^-}}{\theta(c^+) - \theta(c^-) + \dfrac{\psi(c^-)}{c^-}} & \overline{R}^* = \overline{T}^* - 1 \end{cases}$$

where we have set :

$$\begin{cases} \overline{T}^* = \lim\limits_{\alpha \to 0} \overline{T}(\alpha) \\ \overline{R}^* = \lim\limits_{\alpha \to 0} \overline{R}(\alpha) \end{cases}$$

The condition (ii) can be written :

$$\begin{cases} \lim\limits_{\alpha \to 0} T(\alpha) = T(0) \\ \lim\limits_{\alpha \to 0} R(\alpha) = 0 \end{cases}$$

or, equivalently :

$$\overline{T}(0) = \overline{T}^* \qquad ; \qquad \overline{R}(0) = \overline{R}^*$$

Applying Theorem 3.3 to $\overline{T}$ and $\overline{R}$ proves that the condition (ii) is fulfilled if and only if the functions ζ and ξ are related by :

$$\frac{d}{dc}\left(\frac{\zeta}{\xi}\right) = 0 \quad .$$

The functions ζ and ξ are now defined (modulo a multiplicative constant) which gives the first part of the theorem.

Part (iii) : In the same way as in (2.9) (i.e. by multiplying by $\frac{\partial u}{\partial t}$ the equation and integrating by part), we obtain the following a priori estimate :

$$\frac{1}{2}\frac{d}{dt}\left[\iint \frac{\chi\xi^2}{c^2}\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\iint \chi\left|\frac{\partial}{\partial x_1}(\xi u)\right|^2 dx\right] +$$

$$+ \iint \frac{\partial}{\partial x_2}\left(\zeta\frac{\partial u}{\partial t}\right)\frac{\chi\xi^2}{c\zeta}\frac{\partial u}{\partial t}\, dx = 0 \quad .$$

that leads to a sufficient condition for an energy to be constant. If the functions ζ, χ, ξ are related by :

$$\chi\xi^2 = c\zeta^2 \qquad , \qquad \forall\, c \in \mathbb{R}_*^+ ,$$

for any solution u to equation (4.1) the following energy is constant :

$$E(t) = \frac{1}{2} \iint \frac{\chi\xi^2}{c^2} \left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{4} \iint \xi \left|\frac{\partial}{\partial x_1} (\zeta u)\right|^2 dx$$

The second part of the Theorem is terminated by noting that, if ξ is equal to ζ, the previous result gives $\chi = c$.

4.2. A GENERAL WAVE EQUATION

For simplicity, we have been working on the reduced wave equation, but the analysis applies to the general wave equation as well :

$$(4.7) \qquad \rho \frac{\partial^2 u}{\partial t^2} - \operatorname{div} (\mu \operatorname{grad} u) = 0$$

where (ρ,μ) are positive functions of x.

We define new coefficients c and σ by :

$$(4.8) \qquad \begin{cases} c = \left(\dfrac{\mu}{\rho}\right)^{\frac{1}{2}} \\ \sigma = (\rho\mu)^{\frac{1}{2}} \end{cases}$$

With these notations, the corresponding parabolic approximation is given by :

$$(4.9a) \qquad \begin{cases} v = \sigma^{\frac{1}{2}} u \\ \dfrac{1}{c} \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t \partial x_2} - \dfrac{1}{2} \dfrac{\partial}{\partial x_1} \left(c \dfrac{\partial v}{\partial x_1}\right) = 0 \end{cases}$$

This is the equation we shall consider all throughout the remainder of this paper and we shall study its mathematical properties.

4.3. WELL-POSEDNESS

To begin with, we consider the Cauchy problem related to equation (4.9a)

$$(4.9a)\qquad \begin{cases} v = \sigma^{\frac{1}{2}} u \\ \dfrac{1}{c}\ \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t \partial x_2} - \dfrac{1}{2}\ \dfrac{\partial}{\partial x_1}\left(c\ \dfrac{\partial v}{\partial x_1}\right) = 0 \end{cases} \qquad \text{in} \quad \mathbb{R}^2 \times [0,T]$$

with the initial data :

$$(4.9b)\qquad \begin{cases} u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^2 \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in} \quad \mathbb{R}^2 \end{cases}$$

For simplicity, we shall also note :

$$(4.10)\qquad v_0 = \sigma^{\frac{1}{2}} u_0 \qquad ; \qquad v_1 = \sigma^{\frac{1}{2}} u_1$$

The functions $\sigma(x)$ and $c(x)$ are bounded from below and above :

$$(4.11)\qquad \begin{cases} 0 < \sigma_* \leqslant \sigma(x) \leqslant \sigma^* < +\infty & \text{a.e. in} \quad \mathbb{R}^2 \\ 0 < c_* \leqslant c(x) \leqslant c^* < +\infty & \text{a.e. in} \quad \mathbb{R}^2 \end{cases}$$

In order to give an existence result, we introduce the Hilbert spaces :

$$(4.12)\qquad H = L^2(\mathbb{R}^2)$$

equipped with the usual scalar product and norm, denoted by $(\ ,\ )$ and $\|\ \|$ respectively, and

$$(4.13)\qquad V = \left\{ v \in H \quad ; \quad \frac{\partial v}{\partial x_1} \in H \right\}$$

equipped with the scalar product

$$(v,w)_V = (v,w) + \left(\frac{\partial v}{\partial x_1}, \frac{\partial w}{\partial x_1}\right)$$

and the norm

$$\|v\|_V = (v,v)_V^{\frac{1}{2}}$$

THEOREM 4.2 : *Under the regularity assumptions* :

$$\begin{cases} (v_0,v_1) \in V \times V \\ -\frac{1}{2} \frac{\partial}{\partial x_1}\left(c \frac{\partial v_0}{\partial x_1}\right) + \frac{\partial v_1}{\partial x_2} \in H \end{cases},$$

the Cauchy problem (4.9a,b) *has a unique (strong) solution* u *such that* :

$$\begin{cases} v \in \mathscr{C}^2(0,T;H) \cap \mathscr{C}^1(0,T;V) \\ -\frac{1}{2} \frac{\partial}{\partial x_1}\left(c \frac{\partial v}{\partial x_1}\right) + \frac{\partial^2 v}{\partial t \partial x_2} \in \mathscr{C}^0(0,T;H) \end{cases}$$

Moreover the following energy is constant as a function of time :

$$(4.14) \qquad E(t) = \frac{1}{2} \iint \frac{1}{c} \left|\frac{\partial v}{\partial t}\right|^2 dx + \frac{1}{4} \iint c \left|\frac{\partial v}{\partial x_1}\right|^2 dx \; .$$

OUTLINE OF PROOF : For further details the reader is referred to [2]. The result is an application of the semi-group theory after rewriting the equation as a system of equations which are of first order in time :

$$(4.15) \qquad \begin{cases} \frac{\partial v}{\partial t} - w = 0 \\ \frac{\partial w}{\partial t} + c \frac{\partial w}{\partial x_2} - \frac{c}{2} \frac{\partial}{\partial x_1}\left(c \frac{\partial v}{\partial x_1}\right) = 0 \end{cases}$$

In the classical way, we introduce the Hilbert space $\mathscr{H} = V \times H$, equipped with the following norm :

$$\|(v,w)\|^2_{\mathscr{H}} = \iint \frac{|v|^2}{c} dx + \iint c \left|\frac{\partial v}{\partial x_1}\right|^2 dx + 2 \iint \frac{|w|^2}{c} dx$$

We define on $\mathscr{H}$ an unbounded operator A by :

$$\begin{cases} D(A) = \left\{(v,w) \in \mathscr{H} \; ; \; \frac{\partial w}{\partial x_1} \in H \; , \; -\frac{1}{2} \frac{\partial}{\partial x_1}\left(c \frac{\partial v}{\partial x_1}\right) + \frac{\partial w}{\partial x_2} \in H\right\} \\ A : D(A) \subset \mathscr{H} \to \mathscr{H} \\ \qquad (v,w) \to A(v,w) = -w, -\frac{c}{2} \frac{\partial}{\partial x_1}\left(c \frac{\partial v}{\partial x_1}\right) + c \frac{\partial w}{\partial x_2} \end{cases}$$

Within this framework, the Cauchy problem becomes :

$$(4.17)\qquad \begin{cases} \dfrac{d}{dt}(v(t),w(t)) + A(v(t),w(t)) = 0 \\ (v(0),w(0)) = (v_0,v_1) \end{cases}$$

By virtue of the Hille-Yoshida theorem, our result is a straightforward consequence of the following lemma.

LEMMA 4.1 : For $\lambda \geqslant \dfrac{1}{2\sqrt{2}}$, $A + \lambda$ *is m-accretive in* $\mathscr{H}$.

PROOF : Let us recall the definition of m-accretive :

(i) $\forall (v,w) \in D(A)$, $(A(v,w),(v,w))_{\mathscr{H}} + \lambda \|(v,w)\|^2_{\mathscr{H}} \geqslant 0$

(ii) $A + \lambda + 1$ applies $D(A)$ onto $\mathscr{H}$.

The first condition is easy to check using a Green formula. The second one is much more difficult to prove, since A is not H^1 elliptic (there is no term in $\dfrac{\partial^2 v}{\partial x_2^2}$). It is overcome by an elliptic regularization in the x_2 direction. A priori estimates independent on the regularization parameter enable to take the limit as the latter goes to zero (see [2] for details).

Using the energy estimate (4.14), we can weaken the regularity assumptions in the classical way :

THEOREM 4.3 : Assuming only that $(v_0,v_1) \in V \times H$, *the Cauchy problem* (4.9a,b) *has an unique weak solution such that*

$$v \in \mathscr{C}^1(0,T;H) \cap \mathscr{C}^0(0,T;V) \quad .$$

Moreover, if a sequence c_ε *of velocity distributions converges a.e. to* c, *the corresponding solutions* v_ε *converge weakly to the solution* v *corresponding to the velocity* c.

We now turn to the initial boundary value problem in the half-plane

$$\mathbb{R}^2_+ = \{x \ / \ x_2 > 0\}$$

(4.9a)
$$\begin{cases} v = \sigma^{\frac{1}{2}}u \\ \frac{1}{c}\ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial t \partial x_2} - \frac{1}{2}\ \frac{\partial}{\partial x_1}\left(c\ \frac{\partial v}{\partial x_1}\right) = 0 & \text{in } \mathbb{R}^2_+ \times [0,T] \end{cases}$$

with the initial data :

(4.9c)
$$\begin{cases} u(x,0) = u_0(x) & \text{in } \mathbb{R}^2_+ \\ \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in } \mathbb{R}^2_+ \end{cases}$$

and the boundary value :

(4.9d) $\quad u(x_1,0,t) = g(x_1,t) \qquad$ in $\mathbb{R} \times [0,T]$

We again denote by (v_0,v_1) the initial values of v and $\frac{\partial v}{\partial t}$. As above we define the Hilbert space V_+ as :

$$V_+ = \{v \in L^2(\mathbb{R}^2_+) \ , \ \frac{\partial v}{\partial x_1} \in L^2(\mathbb{R}^2_+)\}$$

<u>THEOREM 4.4</u> : *With the following regularity assumptions*

$$(v_0, v_1, g) \in V_+ \times L^2(\mathbb{R}^2_+) \times H^1(0,T;L^2(\mathbb{R})) \quad ,$$

the problem (4.9a,c,d) *has an unique weak solution* u *such that* :

$$t \to v(x,t) \in W^{1,\infty}(0,T;L^2(\mathbb{R}^2_+)) \cap L^\infty(0,T;L^2(\mathbb{R}^2_+))$$

$$x_2 \to v(x,t) \in L^\infty(\mathbb{R}_+, H^1(0,T;L^2(\mathbb{R})))$$

Note that the last estimate enables us to consider x_2 as an evolution direction.

4.4. PROPAGATION PROPERTIES

We first study the propagation direction of the solution t

Theorems 4.5 and 4.6 express in different ways that the s

only in the positive x_2 direction, even when the medium i.

THEOREM 4.5 : Let u *be the weak solution of* (4.9a,b). *As.*

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \mathbb{R}^2_+$$

Then for any $t > 0$, *one has*

$$\operatorname{supp} u(.,t) \subset \mathbb{R}^2_+ \quad .$$

THEOREM 4.6 : Let u^1 *(resp* u^2*) be the weak solution of ponding to the data* (u^1_0, u^1_1) *(resp* (u^2_0, u^2_1)*) and the coe (resp* (c^2, σ^2)*). Assume that*

$$\left.\begin{aligned} u^1_0 &= u^2_0 \\ u^1_1 &= u^2_1 \\ c^1 &= c^2 \\ \sigma^1 &= \sigma^2 \end{aligned}\right\} \quad \text{a.e. in } \mathbb{R}^2_-$$

Then, for any $t > 0$, *one has :*

$$u^1(.,t) = u^2(.,t) \qquad \text{in } \mathbb{R}^2_-$$

where $\mathbb{R}^2_- = \{x., x_2 < 0\}$.

PROOFS : Both results proceed from an energy identity in th

LEMMA 4.2 : Let w *be a sufficiently smooth function satis*

$$\frac{1}{c}\ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial t \partial x_2} - \frac{1}{2}\ \frac{\partial}{\partial x_1}\ (c\ \frac{\partial w}{\partial x_1}) = 0 \qquad \text{in } \mathbb{R}$$

Then the following identity is valid :

$$(4.18) \qquad \frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^2_-}\left(\left|\frac{\partial w}{\partial t}\right|^2 + \frac{c}{2}\left|\frac{\partial w}{\partial x_1}\right|^2\right)dx\right) + \int_{-\infty}^{+\infty}\left|\frac{\partial w}{\partial t}(x_1,0,t)\right|^2 dx_1 = 0$$

The identity is easily obtained after multiplying the equation by $\frac{\partial w}{\partial t}$ and integrating by parts over $\mathbb{R}^2_-$.

We apply the lemma to $w = v = \sigma^{\frac{1}{2}}u$ and get the Theorem 4.5. We then apply it to $w = \sqrt{\sigma^1}\, u^1 - \sqrt{\sigma^2}\, u^2$ and obtain the Theorem 4.6.

The last result of this section specifies an upper bound for the propagation speed.

We first introduce some notations.

. For any angle θ in $[-\pi,\pi]$, the direction $\vec{\theta}$ is defined by :

$$(4.19) \qquad \vec{\theta} = (-\sin\theta, \cos\theta)$$

. $\mathscr{E}^*(t)$ denotes the support at time t of the fundamental solution in a homogeneous medium of velocity c^* (see (2.7)) :

$$(4.20) \qquad \mathscr{E}^*(t) = \{x \in \mathbb{R}^2,\ 2x_1^2 + 4(x_2 - \frac{c^*t}{2})^2 \leqslant (c^*t)^2\}$$

<u>*THEOREM 4.7*</u> *: The solution* u *of* (4.9a,b) *propagates in any direction* $\vec{\theta}$ *with a velocity* $V(\theta)$ *bounded by*

$$(4.21) \qquad V^*(\theta) = \frac{c^*}{2}(\cos\theta + (1+\sin^2\theta)^{\frac{1}{2}})$$

If the initial values u_0 *and* u_1 *are of compact support* $\mathscr{K}$, *then at any time* t *one has :*

$$\operatorname{supp} u(.,.,t) \subset \mathscr{K} + \mathscr{E}^*(t)$$

<u>PROOF</u> : We first assume that the data are smooth, so that energy estimates hold. We also assume that $\mathscr{K}$ is the disc of center 0 and radius R.

Let Ω_θ^t be the half-plane the ingoing normal vector of which is $\vec{\theta}$ and that propagates with a given velocity $V > 0$:

$$(4.22) \qquad \Omega_\theta^t = \{ x, (x - (R + Vt)\vec{\theta}).\vec{\theta} > 0 \}$$

Γ_θ^t denotes the boundary of Ω_θ^t, $d\sigma$ is the measure on Γ_θ^t.

We shall actually prove that for any θ, the energy in Ω_θ^t is decreasing as a function of time, for all values of V such that $V \geqslant V^*(\theta)$. This will give the first part of the theorem.

The energy contained in the half-space Ω_θ^t is denoted by $E(v,\Omega_\theta^t,t)$.

$$(4.23) \qquad E(v,\Omega_\theta^t,t) = \frac{1}{2} \iint_{\Omega_\theta^t} \frac{1}{c} \left|\frac{\partial v}{\partial t}\right|^2 dx + \frac{1}{4} \iint_{\Omega_\theta^t} c \left|\frac{\partial v}{\partial x_1}\right|^2 dx$$

By a Green's formula, we can prove the following :

<u>LEMMA 4.3</u> : *For smooth data, the solution of the Cauchy problem satisfies the equality :*

$$(4.24) \qquad \frac{d}{dt} E(v,\Omega_\theta^t,t) + \frac{1}{2} \int_{\Gamma_\theta^t} \frac{\Phi}{c} \, d\sigma = 0$$

where the function Φ *is given by :*

$$(4.25) \qquad \Phi = (V - c \cos \theta) \left|\frac{\partial v}{\partial t}\right|^2 + c^2 \sin \theta \frac{\partial v}{\partial t} \frac{\partial v}{\partial x_1} + \frac{1}{2} c^2 V \left|\frac{\partial v}{\partial x_1}\right|^2$$

This function Φ can be seen as a quadratic form in the variables $\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial x_1}$. A straightforward calculation shows that it is positive if $V \geqslant V^*(\theta)$. Under that condition, the energy in Ω_θ^t is decreasing as a function of time. The second part of the theorem is easily derived by taking the intersection of all the half-spaces Ω_θ^t for $V = V^*(\theta)$.

This proves the theorem when the data are smooth and supported in $B(0,R)$. By translation, linearity and continuity, it can be extended to any support, and eventually to discontinuous data.

To finish with the propagation properties, we shall point out a result analogous to the reciprocity property for the wave equation, i.e. the solution at point B for a source located at point A is equal to the solution at point A for a source located at point B at the same time.

For the parabolic approximation, the result can be stated as follows. Let A and B be two points in $\mathbb{R}^2$ such that $B_2 > A_2$. u^A (resp u^B) is defined as the solution of an "upgoing" (resp "downgoing") equation with a source at point A (resp B). The initial values are equal to zero.

$$\begin{cases} v^A = \sigma^{\frac{1}{2}} u^A \quad ; \quad v^B = \sigma^{\frac{1}{2}} u^B \\ \frac{1}{c} \frac{\partial^2 v^A}{\partial t^2} + \frac{\partial^2 v^A}{\partial t \partial x_2} - \frac{1}{2} \frac{\partial}{\partial x_1} \left(c \frac{\partial v^A}{\partial x_1}\right) = \delta(A,t) & (4.26) \\ \frac{1}{c} \frac{\partial^2 v^B}{\partial t^2} - \frac{\partial^2 v^B}{\partial t \partial x_2} - \frac{1}{2} \frac{\partial}{\partial x_1} \left(c \frac{\partial v^B}{\partial x_1}\right) = \delta(B,t) & (4.27) \\ v^A(x,0) = \frac{\partial v^A}{\partial t}(x,0) = v^B(x,0) = \frac{\partial v^B}{\partial t}(x,0) = 0 \end{cases}$$

<u>THEOREM 4.8</u> : *At any time* $t > 0$, *one has* :

$$u^A(B,t) = u^B(A,t) \tag{4.28}$$

<u>PROOF</u> : For any $\tau > 0$, we define a new function $u^{B,\tau}$ by :

$$u^{B,\tau}(x,t) = u^B(x,\tau - t)$$

Then $v^{B,\tau} = \sigma^{\frac{1}{2}} u^{B,\tau}$ is solution of :

$$\begin{cases} \frac{1}{c} \frac{\partial^2 v^{B,\tau}}{\partial t^2} + \frac{\partial^2 v^{B,\tau}}{\partial t \partial x_2} - \frac{1}{2} \frac{\partial}{\partial x_1} \left(c \frac{\partial v^{B,\tau}}{\partial x_1}\right) = \delta(B,\tau - t) \\ v^{B,\tau}(x,\tau) = \frac{\partial v^{B,\tau}}{\partial t}(x,\tau) = 0 \end{cases} \tag{4.29}$$

We then multiply (4.26) by $v^{B,\tau}$, (4.29) by v^A, substract the latter from the former and integrate it on $[0,\tau] \times \mathbb{R}^2$.

Integrations by part easily show that the left-hand side is equal to zero.
The right-hand side is :

$$\int_0^\tau\int_{\mathbb{R}^2} \left[v^{B,\tau}\cdot\delta(A,t) - v^A\ \delta(B,t)\right]\,dx\,dt = v^B(A,\tau) - v^A(B,\tau)$$

which proves the identity (4.28).

4.5. REFLECTED AND TRANSMITTED WAVE AT A LINEAR INTERFACE

The notations are the same as in 3.2. The transmission conditions at the interface $\Gamma(\alpha)$ for equation (4.9a) are :

(4.30a) If $\alpha = 0$ $\qquad [\sigma^{\frac{1}{2}} u]_{\Gamma(0)} = 0$

(4.30b) If $\alpha \neq 0$ $\qquad \begin{cases} [\sigma^{\frac{1}{2}} u]_{\Gamma(\alpha)} = 0 \\ [c \dfrac{\partial}{\partial x_1} (\sigma^{\frac{1}{2}} u)]_{\Gamma(\alpha)} = 0 \end{cases}$

These transmission conditions are continuous with respect to α. For any incident wave u^I, there is one reflected wave if $\alpha \neq 0$, none if $\alpha = 0$. There is always one transmitted wave, which is either traveling or evanescent (see the analysis in 3.2). For simplicity we shall consider the simplest equation, that is the case where $\sigma = c^{-1}$. We introduce some notations :

. The velocity contrast q is

$$q = \frac{c^-}{c^+}$$

. The incidence angle at the interface, θ, is defined as the angle of the group velocity vector $V_G(\frac{k^I}{\omega})$ to the normal to the interface.

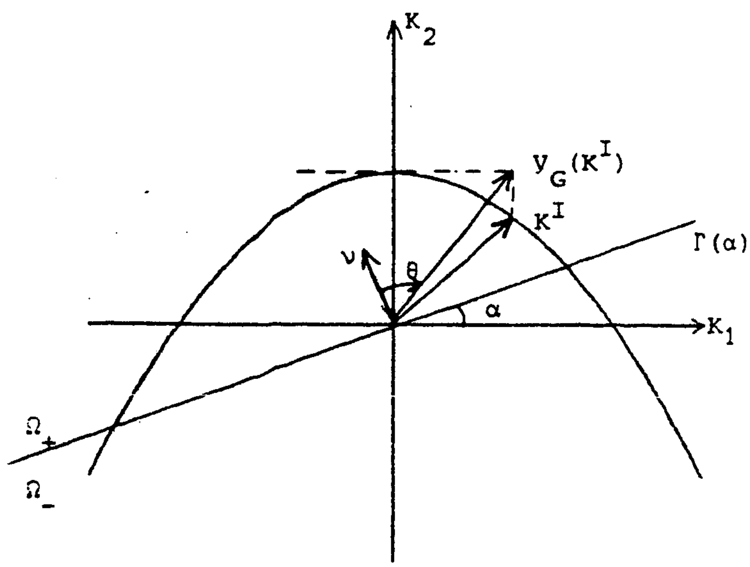


Figure 4.1 : definition of the incidence angle

The reflection and transmission coefficients depend on θ, q, and α. The following plots show them as functions of $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} - \alpha\right]$ for various values of the parameter q and α. When q is greater than 1 (i.e. when the velocity in Ω^- is lower than in Ω^+), there is a critical angle $\theta^*(q,\alpha)$: for $\theta > \theta^*$, the transmitted wave is traveling, while for $\theta < \theta^*$, it is evanescent, and we do not plot it.

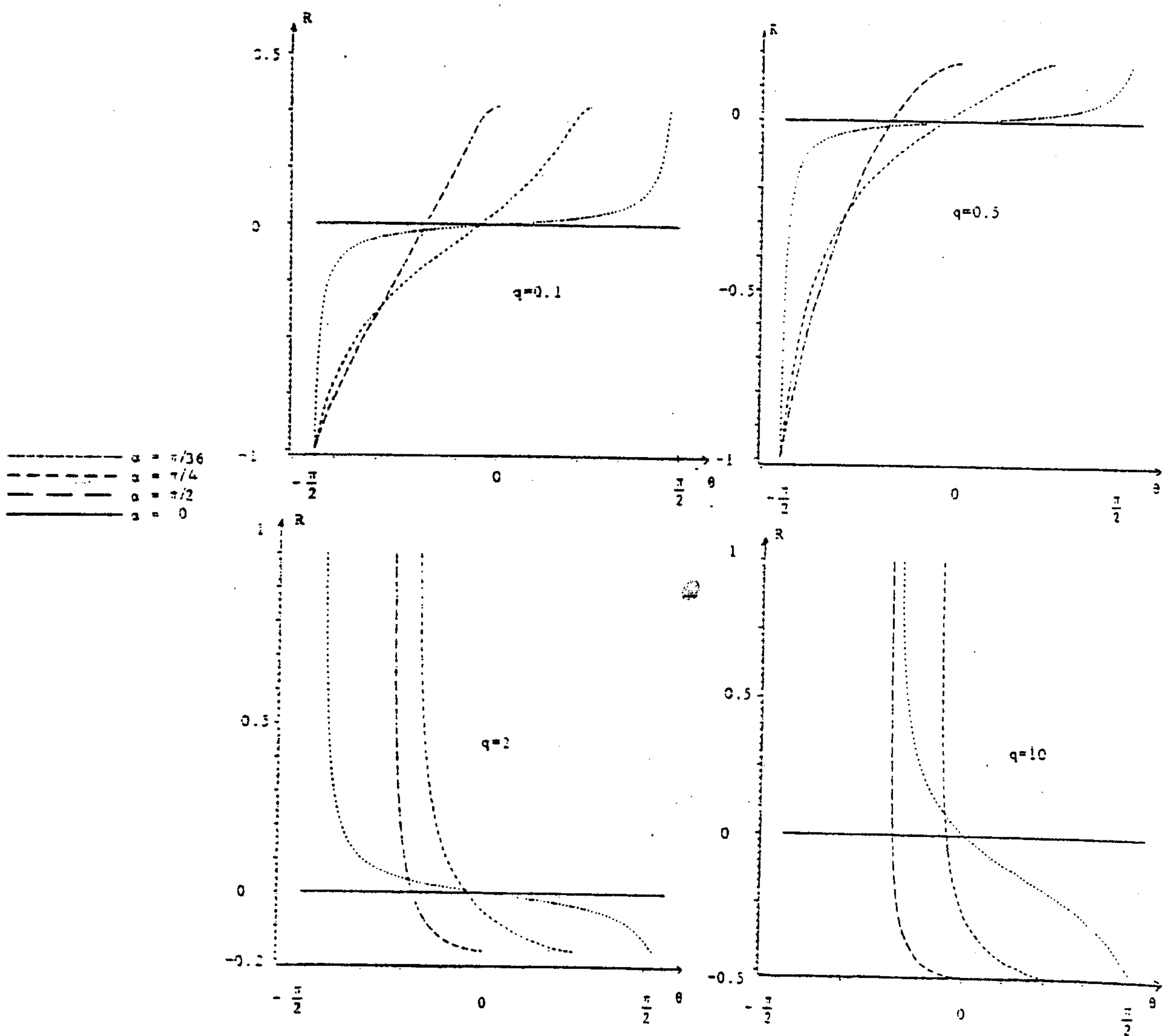
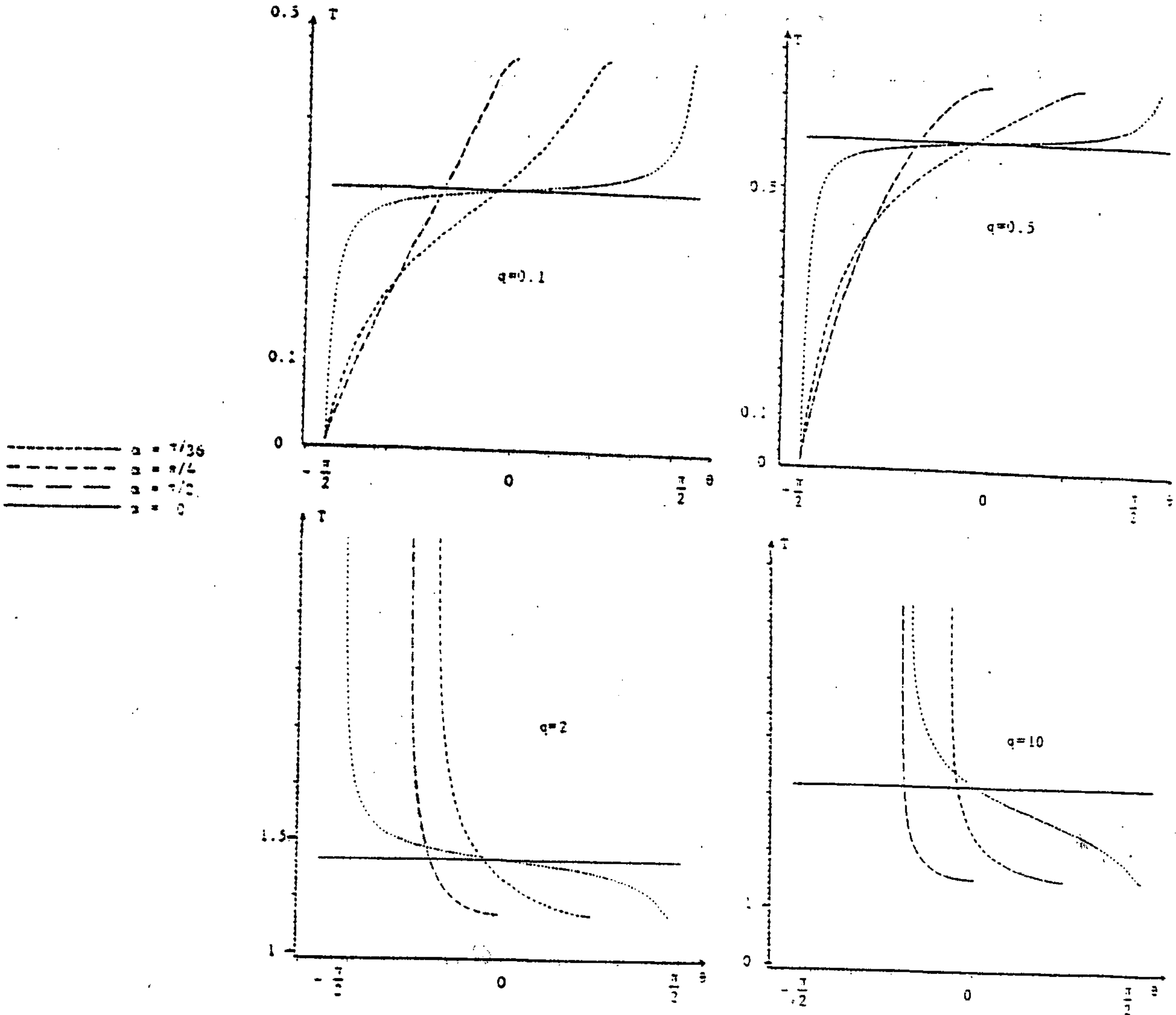


Figure 4.2 : Reflection coefficient as a function of the incidence angle for various values of the parameters



<u>Figure 4.3</u> : Transmission coefficient as a function of the incidence angle for various values of the parameters

5 - NUMERICAL EXPERIMENTS

In order to illustrate our theoretical results, we present here numerical experiments implemented by F. Collino at I.F.P. The time dependence is handled by Fourier transform. It leads to an evolution problem in x_2 of Schrödinger type. The corresponding equation is then semi-discretized in x_1 by P_1 finite elements. A Crank-Nicolson scheme is finally used in the x_2 direction. For further details and properties about these numerical schemes, see [11].

For each simulation, the source is quasi punctual, i.e. its support is small. Its position is indicated on the figures by the point S. Its time dependence is given by the second derivative of a gaussian function (Ricker source).

Each of the figures we present here are snapshots of the solution at a given time. This gives an image of the solution in the (x_1, x_2) plane (this representation is commonly used by geophysicists). The areas where the solution is positive are darker, the ones where it is negative are lighter.

We present on Figure 5.1 the solution in a homogeneous medium. One easily recognizes the ellipse which defines the support of the fundamental solution.

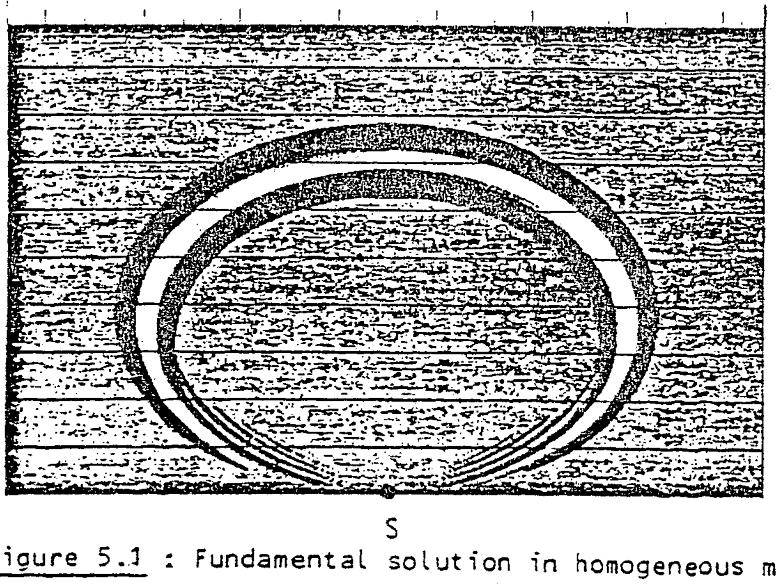


Figure 5.1 : Fundamental solution in homogeneous medium

The plots below (Figures 5.2 to 5.4) illustrate the results we gave in Sections 3 and 4 for the two-layers medium.

Figure 5.2 is a snapshot of the solution when the interface is horizontal. The ratio $\frac{c^+}{c^-}$ between the velocities in the media Ω^+ and Ω^- is equal to 2. We indicate on the figure the incident wave and the transmitted one. There is no reflected wave, and moreover there is no discontinuity between the incident front and the transmitted one.

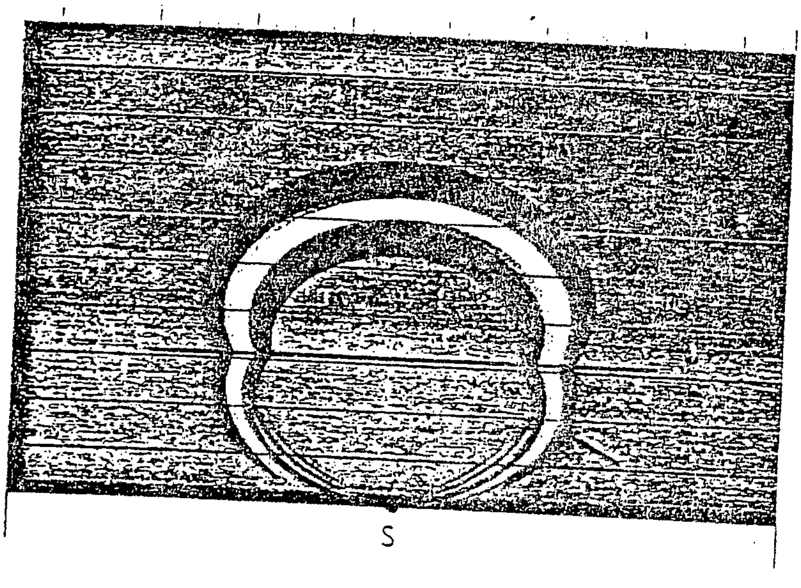


Figure 5.2 : Two-layers medium : horizontal interface

On Figure 5.3, the angle α of the interface to the x_1 direction is equal to $\frac{\pi}{4}$. The ratio $\frac{c^+}{c^-}$ is the same as in the previous example. In this case the incident wave, the transmitted wave and the reflected wave are clearly visible. Notice existence of a head wave connecting the incident wave to the transmitted one.

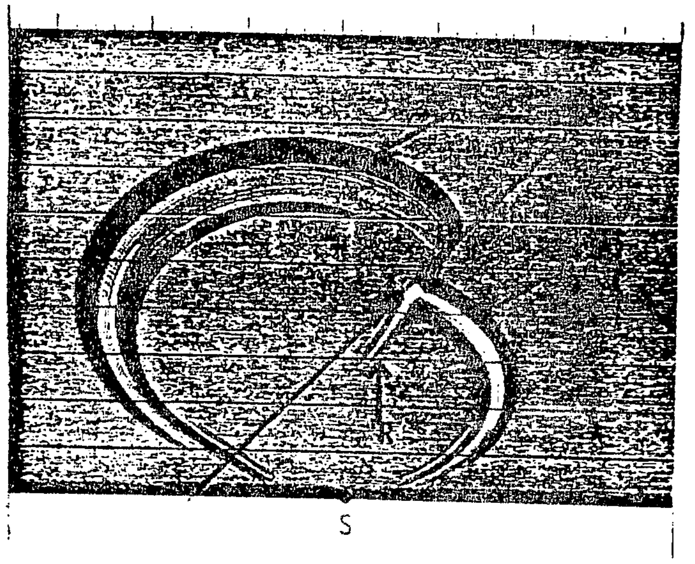


Figure 5.3 : Two-layers medium : $\alpha = \frac{\pi}{4}$

To illustrate the influence of the angle α on the behaviour of the solution, we represent on Figure 5.4 the same snapshot when α is equal to $\frac{\pi}{8}$. The reflected wave is much weaker than in the previous example, as proved in Section 4.

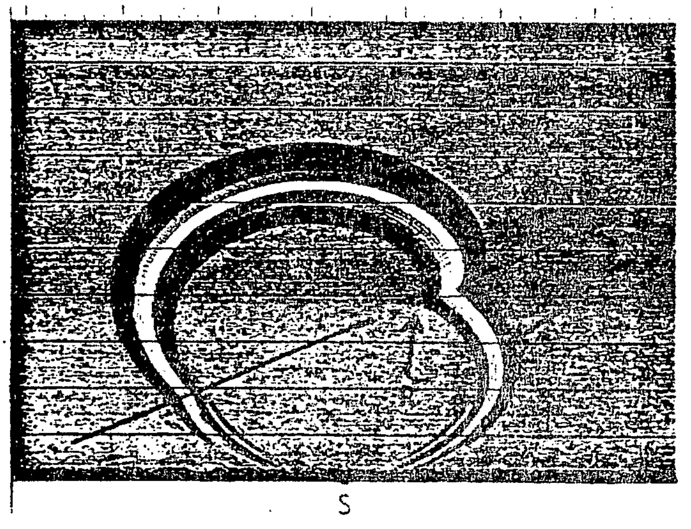


Figure 5.4 : Two-layers medium : $\alpha = \frac{\pi}{8}$

We finally present a numerical experiment concerning two media divided by a corner made of a horizontal line and a vertical one. The horizontal interface produces no reflected wave, while the vertical interface gives rise to a reflected wave.

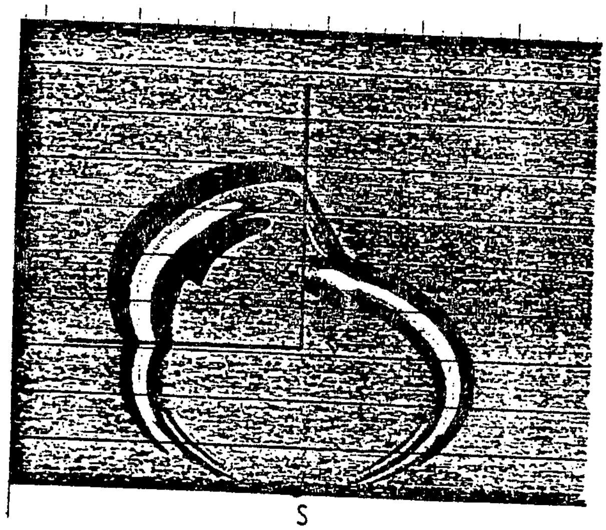


Figure 5.5 : The corner case

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REFERENCES

[1] BAMBERGER, A., ENGQUIST, B., HALPERN, L., JOLY, P., *Higher order paraxial approximations in heterogeneous media*, to appear.

[2] BAMBERGER, A., ENGQUIST, B., HALPERN, L., JOLY, P., *Construction et analyse d'approximations paraxiales en milieu hétérogène. I Approximation parabolique*, Rapport interne CMAP Ecole Polytechnique n° 114 (octobre 1984).

[3] BAMBERGER, A., ENGQUIST, B., HALPERN, L., JOLY, P., *Construction et analyse d'approximations paraxiales en milieu hétérogène. II Approximation d'ordre supérieur*, Rapport interne CMAP Ecole Polytechnique n° 128 (mars 1985).

[4] BREZIS, H., *Analyse fonctionnelle, théorie et applications*, Masson (1983).

[5] BROCK, H.K., BUCHAL, R.N., and SPOFFORD, C.W., *Modifying the soundspeed profile to improve the accuracy of the parabolic equation technique*, J. Acoust. Soc. Am., Vol. 62, pp. 543-552 (1977).

[6] CLAERBOUT, J.F., *Coarse grid calculation of waves in inhomogeneous media with application to delineation of complicated seismic structure*, Geophysics, Vol. 35, pp. 407-418 (1970).

[7] CLAERBOUT, J.F., *Numerical holography*, in Acoustical Holography, Vol. 3, ed. Metherell, A.F., Plenum Press, New York, pp. 273-283 (1971).

[8] CLAERBOUT, J.F., *Fundamentals of geophysical data processing*, Mc Graw-Hill, New York (1976).

[9] CLAYTON, R., and ENGQUIST, B., *Absorbing boundary conditions for acoustic and elastic wave equations*, Bull. Seism. Soc. Am., Vol. 67, pp. 1529-1540 (1977).

[10] COLE, J.D., *Modern development in transonic flow*, SIAM J. Appl. Math., Vol. 29, pp. 763-787 (1975).

[11] COLLINO, F., Thèse de 3e cycle, Paris (1986).

[12] DE SANTO, J.A., *Relation between the solutions of the Helmholtz and parabolic equations for sound propagation*, J. Acoust. Soc. Am., Vol. 62, pp. 295-297 (1977).

[13] ENGQUIST, B., and MAJDA, A., *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp., Vol. 31, pp. 629-651 (1977).

[14] HALPERN, L., *The paraxial approximation for the wave equation : some new results*, Proceedings of the Fifth IMACS International Symposium on Computer Methods for PDE, Lehigh University, PA, USA (June 1984).

[15] HASEGAWA, A., and TAPPERT, F., *Transmission of stationary non linear optical pulses in dispersive dielectric fibers. I Anomalous disper*. Appl. Phys. Lett., Vol. 23, p. 142 (1973).

[16] HASEGAWA, A., and TAPPERT, F., *Transmission of stationary non linear optical pulses in dispersive dielectric fibers. II Normal dispersion*, Appl. Phys. Lett., Vol. 23, p. 171 (1973).

[17] HUDSON, J.A., *A parabolic approximation for elastic waves*, Wave Motion, Vol. 2, p. 207 (1980).

[18] JOLY, P., *Etude mathématique de l'approximation parabolique de l'équation des ondes en milieu stratifié*, Rapport INRIA n° 299 (mai 1984).

[19] LANDERS, T., and CLAERBOUT, J.F., *Numerical calculations of elastic waves in laterally inhomogeneous media*, J. Geophys. Res., Vol. 77, pp. 1476-1482 (1972).

[20] LEONTOVICH, M., and FOCK, V., *Solution of the problem of electromagnetic waves along the earths surface by the method of parabolic equation*, J. Phys., Vol. 10, pp. 13-24 (1946).

[21] LIONS, J.L., and MAGENES, E., *Problèmes aux limites non homogènes et applications*, Dunod (1968).

[22] Mc COY, J.J., *A parabolic theory of stress wave propagation through inhomogeneous linearly elastic solids*, J. Appl. Mech., Vol. 44, pp. 462-468 (1977).

[23] Mc DANIEL, S.T., *Propagation of normal modes in the parabolic approximation*, J. Acoust. Soc. Am., Vol. 57, pp. 307-311 (1975).

[24] Mc DANIEL, S.T., *Parabolic approximations for underwater sound propagation*, J. Acoust. Soc. Am., Vol. 58, pp. 1178-1185 (1975).

[25] NIRENBERG, L., *Lectures on linear partial differential equations*, Conference Board of the mathematical sciences, n° 17.

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