The Optimized Order 2 Method. Application to Convection-Diffusion Problems

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Abstract

We present an iterative, non-overlapping domain decomposition method for solving the convection-diffusion equation. A reformulation of the problem leads to an equivalent problem where the unknowns are on the boundary of the subdomains [14]. The solving of this interface problem by a Krylov type algorithm [15] is done by the solving of independant problems in each subdomain, so it permits to use efficiently parallel computation. In order to have very fast convergence, we use differential interface conditions of order 1 in the normal direction and of order 2 in the tangential direction to the interface, which are optimized approximations of Absorbing Boundary Conditions [13,8]. Numerical tests illustrate the efficiency of the method.

Key words: Domain decomposition. Optimized Order 2 method. Absorbing boundary conditions. Convection-diffusion problems.

1 The Optimized Order 2 Method

In fluid dynamics, the convection-diffusion equation models for example the concentration of a pollutant in the air. The convection-diffusion problem we consider is to find the solution $u = u(x, y)$ of

$$L(u) = cu + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} - \nu\Delta u = f \quad \text{in } \Omega \tag{1}$$

$$C(u) = g \quad \text{on } \partial\Omega \tag{2}$$

where Ω is a bounded open set of $\mathbb{R}^2$, $\mathbf{a} = (a, b)$ is the velocity field, ν is the viscosity, C is a linear operator, c is a constant which could be $c = \frac{1}{\Delta t}$ with

Δt a time step of a backward-Euler scheme for solving the time dependent convection-diffusion problem, and f, g are given functions.

The OO2 method is based on an extension of the additive Schwarz algorithm with non-overlapping subdomains :
Let $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$, with $\Omega_i \cap \Omega_j = \emptyset,\ i \neq j$. We denote by $\Gamma_{i,j}$ the common interface to Ω_i and $\Omega_j,\ i \neq j$. The outward normal from Ω_i is $\mathbf{n_i}$ and $\boldsymbol{\tau_i}$ is a tangential unit vector. Let u be the solution of problem (1)-(2), and u_i^p the approximation of u at iteration p in each subdomain $\Omega_i,\ 1 \leq i \leq N$. The additive Schwarz algorithm with non-overlapping subdomains is :

$$L(u_i^{p+1}) = f \quad \text{in } \Omega_i \tag{3}$$
$$B_i(u_i^{p+1}) = B_i(u_j^p) \quad \text{on } \Gamma_{i,j},\ i \neq j \tag{4}$$
$$C(u_i^{p+1}) = g \quad \text{on } \partial\Omega_i \cap \partial\Omega \tag{5}$$

where B_i is an interface operator. The original additive Schwarz algorithm [10], with Dirichlet interface conditions ($B_i = Id$), converges only with overlapping subdomains. In [11], the interface conditions are Robin type conditions ($B_i = \frac{\partial}{\partial \mathbf{n}_i} + c_i$, where c_i is a constant), which leads to a convergent algorithm for non-overlapping subdomains. The choice of the interface conditions is fundamental. Many methods has been proposed (see for example [4,3,2,16].

The OO2 interface conditions are based on the concept of Absorbing Boundary Conditions [5,6]. This concept enables to understand the mathematical mecanisms on the interfaces, and therefore leads to stable and efficient algorithms.
We introduce first the technique of absorbing interface conditions, then the OO2 interface conditions based on this concept, and in a third part the substructuring formulation of the method.

1.1 Absorbing interface conditions

It has been proved in [14] that the optimal interface conditions for algorithm (3)-(5) are the exact Absorbing Boundary Conditions. Unfortunately, as these conditions are not partial differential operators, they are numerically costly and difficult to use. Then, it has been proposed in [13] to use Taylor approximations of order 0 or 2, for low wave numbers, of these optimal interface conditions.

For example, the "Taylor order 0" interface operator is :

$$B_i = \frac{\partial}{\partial \mathbf{n_i}} - \frac{\mathbf{a}.\mathbf{n_i} - \sqrt{(\mathbf{a}.\mathbf{n_i})^2 + 4c\nu}}{2\nu}$$

(in [4,2], the interface conditions can be interpreted as Taylor approximations of order 0). These conditions are obtained from the artificial boundary conditions [5,6]. If $\Omega_i = \mathbb{R}^- \times \mathbb{R}$, $\Omega_j = \mathbb{R}^+ \times \mathbb{R}$, and $\Gamma_{i,j}$ is the axis $x = 0$, the artificial boundary conditions are $\partial_x - \Lambda^-$, $\partial_x - \Lambda^+$, with Λ^- the Dirichlet to Neumann operator of the right half plane defined as

$$\Lambda^- : u_0 \longrightarrow \frac{\partial w}{\partial x}(0, y) \quad \text{with } w \text{ such as} \qquad \begin{array}{ll} L(w) = 0, \ x > 0 & \\ w(0, y) = u_0(y) & \text{at } x = 0 \\ w \text{ bounded at infinity} & \end{array}$$

The Dirichlet to Neumann operator of the left half plane Λ^+ is defined in the same way. When the cœfficients of L are constants, by using Fourier transform in the y direction, we can compute the symbol λ^- of Λ^- and the symbol λ^+ of Λ^+:

$$\lambda^-(k, a, b) = \frac{a - \sqrt{a^2 + 4c\nu + 4i\nu bk + 4k^2\nu^2}}{2\nu} \tag{6}$$

$$\lambda^+(k, a, b) = \frac{a + \sqrt{a^2 + 4c\nu + 4i\nu bk + 4k^2\nu^2}}{2\nu} \tag{7}$$

where k is the Fourier variable. If we denote by Λ^+_{ap} and Λ^-_{ap} the Taylor approximations of order 0 or 2, for low wave numbers, of Λ^+ and Λ^-, they satisfy:

$$\Lambda^+_{ap} + \Lambda^-_{ap} = \Lambda^+ + \Lambda^- = \frac{a}{\nu} \tag{8}$$

Then, the interface operator for domain Ω_i is $B_i = \partial_x - \Lambda^-_{ap}$, and the one for domain Ω_j is $B_j = -(\partial_x - \Lambda^+_{ap})$. B_j can be obtained from B_i, using (8).
Numerical tests on a finite difference scheme with overlapping subdomains have shown that the Taylor order 2 interface conditions lead to very fast convergence, compared to the Taylor order 0 or Dirichlet interface conditions, except in the case of a velocity field tangential to the interface, where the convergence is very slow. So, instead of taking low wave numbers approximations, it has been proposed in [7,8] to use differential interface conditions of order 2 along the interfaces which are "good" approximations of the absorbing boundary conditions, not only for low wave numbers, but for a given range of wave numbers. This means that the interface operator is chosen in order to optimize the convergence ratio of algorithm (3)-(5). This "Optimized Order 2" interface operator is defined as follows :

1.2 OO2 interface conditions

1.2.1 Analytic analysis : 2 subdomains case and constant cœfficients

In this part, we suppose that the cœfficients of L are constants. The domain $\Omega = \mathbb{R}^2$ is decomposed in 2 subdomains $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$, and $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$, and $\Gamma_{1,2}$ is the axis $x = 0$. The interface operators are of order 2 in the tangential direction to the interface :

$$B_1 = \frac{\partial}{\partial \mathbf{n_1}} - c_1 + c_2 \frac{\partial}{\partial \boldsymbol{\tau_1}} - c_3 \frac{\partial^2}{\partial \boldsymbol{\tau_1}^2}, \quad B_2 = \frac{\partial}{\partial \mathbf{n_2}} - c_4 + c_5 \frac{\partial}{\partial \boldsymbol{\tau_2}} - c_6 \frac{\partial^2}{\partial \boldsymbol{\tau_2}^2}$$

with $c_3 \geq 0$, and are chosen as follows :

- First we link B_1 and B_2 as in (8). This means that c_4, c_5, c_6 are obtained from c_1, c_2, c_3:
  $c_1 = c_1(\mathbf{a}\cdot\mathbf{n_1}, \mathbf{a}\cdot\boldsymbol{\tau_1})$, $c_2 = c_2(\mathbf{a}\cdot\mathbf{n_1}, \mathbf{a}\cdot\boldsymbol{\tau_1})$, $c_3 = c_3(\mathbf{a}\cdot\mathbf{n_1}, \mathbf{a}\cdot\boldsymbol{\tau_1})$, and
  $c_4 = c_1 - \frac{\mathbf{a}\cdot\mathbf{n_1}}{\nu}$, $c_5 = c_2(\mathbf{a}\cdot\mathbf{n_2}, \mathbf{a}\cdot\boldsymbol{\tau_2})$, $c_6 = c_3(\mathbf{a}\cdot\mathbf{n_2}, \mathbf{a}\cdot\boldsymbol{\tau_2})$. So we only have to determine c_1, c_2, c_3.
- Then, we choose $c_1 = \frac{\mathbf{a}\cdot\mathbf{n_1} - \sqrt{(\mathbf{a}\cdot\mathbf{n_1})^2 + 4c\nu}}{2\nu}$ so that the interface condition is exact for the lowest wave number.
- Finally, we compute c_2 and c_3 by minimizing the convergence ratio of the Schwarz algorithm in the case of 2 subdomains and constant cœfficients.

The minimization problem on c_2 and c_3 is sought in term of wave numbers k: B_1 and B_2 can be written as : $B_1 = \partial_x - \Lambda_{ap}^-$, and $B_2 = -(\partial_x - \Lambda_{ap}^+)$ where the symbol of Λ_{ap}^- is

$$\lambda_{ap}^-(k) = \lambda^-(0) - c_2 ik - c_3 k^2 \tag{9}$$

B_2 is obtained from B_1 using (8). Then the convergence ratio of the Schwarz algorithm (3)-(5) can be computed explicitly : we denote by e_i^p the error $u_i^p - u$ at the interface Γ_{12} of Ω_i at step p, $i = 1, 2$. Let $\hat{e}$ denote the partial Fourier transform in the y direction of a function e. Then, we define the convergence ratio of algorithm (3)-(5) by $\hat{e}_1^{p+2} = \rho \hat{e}_1^p, \quad p \geq 1$. The relation (8) simplify ρ in the form :

$$\rho(k, c_2, c_3) = \left(\frac{\lambda^-(k) - \lambda_{ap}^-(k)}{\lambda^+(k) - \lambda_{ap}^-(k)} \right)^2$$

Then, we minimize the maximum of the convergence ratio $k \to \rho(k, c_2, c_3)$ on the interval $|k| \leq k_{max}$ where k_{max} is a given constant, $k_{max} > 0$ (in the discrete case, $k_{max} = \frac{\pi}{h}$ where h is the mesh size in y) (see [7]). When the convection velocity is normal to the interface, we have :

Theorem 1 *Let $a \in \mathbb{R},\ a \neq 0,\ b = 0$ and $c \geq 0$ in (1). Then, for $k_{max} > 0$, there exists a unique $\bar{c_3} \geq 0$ solution of*

$$\min_{c_3 \geq 0} \max_{0 \leq k \leq k_{max}} |\rho(k, 0, c_3)| \tag{10}$$

Moreover, $\bar{c_3}$ is the unique solution in $[\frac{\lambda^-(0)-\lambda^-(k_{max})}{k_{max}^2}, \frac{\nu}{\sqrt{a^2+4c\nu}}]$ of the equation

$$\rho(k_1(c_3), 0, c_3) = \rho(k_{max}, 0, c_3)$$

where $k_1 = k_1(c_3)$ is the root of the derivative of $k \longrightarrow \rho(k, 0, c_3)$ such that $\rho(k_1, 0, c_3) \neq 0$.

Proof We consider in a first step that $c = 0$. We introduce the notations $x = (\frac{2\nu k}{a})^2$, $x_{max} = (\frac{2\nu k_{max}}{a})^2$, and $\gamma = \frac{c_3|a|}{2\nu}$. Then the convergence ratio is :

$$\rho(x, \gamma) = \left(\frac{-1 + \sqrt{1+x} - \gamma x}{1 + \sqrt{1+x} + \gamma x} \right)^2 \tag{11}$$

the minimization problem (10) is equivalent to the following one :

$$\min_{\gamma \geq 0} \max_{0 \leq x \leq x_{max}} |\rho(x, \gamma)| \tag{12}$$

The function $(x, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \rho(x, \gamma)$ is C^∞, positive, and for $\gamma \geq 0$ given, $\lim_{x \longrightarrow \infty} \rho(x, \gamma) = 1$. The research of the maximum of $x \longrightarrow \rho(x, \gamma)$ on $[0, x_{max}]$, for $\gamma \geq 0$ leads to consider the cases : $0 < \gamma < \frac{1}{2}$, $\gamma = 0$, and $\gamma \geq \frac{1}{2}$.

$\underline{0 < \gamma < \frac{1}{2}}$
Let $x_1(\gamma) = \frac{1-2\gamma}{\gamma}$ and $x_{int}(\gamma) = \frac{1-2\gamma}{\gamma^2}$. The function $x \longrightarrow \rho(x, \gamma)$ is increasing on $[0, x_1(\gamma)] \cup [x_{int}(\gamma), x_{max}]$, decreasing on $[x_1(\gamma), x_{int}(\gamma)]$. We then have three possibilities for the maximum of $x \longrightarrow \rho(x, \gamma)$ on $[0, x_{max}]$:

- if γ is such that $x_{max} \leq x_1(\gamma) \iff \gamma \leq f(x_{max})$, with $f(x) = \frac{1}{x+2}$. Then, the maximum is at x_{max},
- if γ is such that $x_1(\gamma) \leq x_{max} \leq \frac{1-2\gamma}{\gamma^2} \iff f(x_{max}) \leq \gamma \leq g(x_{max})$, with $g(x) = \frac{-1+\sqrt{1+x}}{x}$. Then, the maximum is at $x_1(\gamma)$,
- if γ is such that $x_{max} \geq \frac{1-2\gamma}{\gamma^2} \iff \gamma \geq g(x_{max})$. Then, the maximum is at $x_1(\gamma)$ or at x_{max}.

So (12) reduces to

$$\min\Big(\min_{0 < \gamma \leq f(x_{max})} F(\gamma), \min_{f(x_{max}) \leq \gamma \leq g(x_{max})} G(\gamma), \min_{g(x_{max}) \leq \gamma < \frac{1}{2}} (F(\gamma), G(\gamma)) \Big)$$

with $F(\gamma) = \rho(x_{max}, \gamma)$, and $G(\gamma) = \rho(x_1(\gamma), \gamma)$, $0 < \gamma < \frac{1}{2}$.
Let $F_0(x, \gamma) = \frac{\partial \rho}{\partial \gamma}(x, \gamma)$, $x \geq 0,\ 0 < \gamma < \frac{1}{2}$. Then, we have $F'(\gamma) = F_0(x_{max}, \gamma)$,

and $G^{'}(\gamma) = F_0(x_1(\gamma), \gamma)$. As $F_0(x, \gamma) \leq 0$ if and only if $\gamma \leq g(x)$ and as $f(x_{max}) \leq g(x_{max})$, then for $0 < \gamma \leq f(x_{max})$ we have $F^{'}(\gamma) \leq 0$. So the minimum of the continuous function F on $]0, f(x_{max}]$ is at $\gamma = f(x_{max})$ (i.e. $x_{max} = x_1(\gamma)$). Moreover, $G^{'}(\gamma) \leq 0$ if and only if $\gamma \leq g(x_1(\gamma))$. Since $\gamma \leq g(x_1(\gamma)), \forall \gamma \in]0, \frac{1}{2}[$, then $G^{'}(\gamma) \leq 0,\ 0 < \gamma < \frac{1}{2}$. The minimum of the continuous function G on $[f(x_{max}), g(x_{max})]$ is then at $\gamma = g(x_{max})$. So we are lead to the problem

$$\min_{g(x_{max}) \leq \gamma < \frac{1}{2}} (\rho(x_1(\gamma), \gamma), \rho(x_{max}, \gamma))$$

Since F and G are continuous on $]0, \frac{1}{2}[$, $F^{'}(\gamma) \geq 0$ and $G^{'}(\gamma) \leq 0$ for $g(x_{max}) \leq \gamma < \frac{1}{2}$, $F(\frac{1}{2}) \leq 0$, and $G(\frac{1}{2}) = 0$, there exists a unique $\bar{\gamma}$, $g(x_{max}) \leq \bar{\gamma} < \frac{1}{2}$, such that $F(\bar{\gamma}) = G(\bar{\gamma}) = \min_{0<\gamma<\frac{1}{2}} \max_{0 \leq x \leq x_{max}} |\rho(x, \gamma)|$.

$\underline{\gamma = 0}$
The function $x \longrightarrow \rho(x, 0)$ is increasing on $[0, x_{max}]$, it's maximum is at x_{max}. A simple computation show that $\rho(x_{max}, 0) > \rho(x_{max}, \bar{\gamma})$ (using $\bar{\gamma} < \frac{1}{2}$).

$\underline{\gamma \geq \frac{1}{2}}$
The function $x \longrightarrow \rho(x, \gamma)$ is increasing on $[0, x_{max}]$, it's maximum is at x_{max}. Since $F^{'}(\gamma) \geq 0$ for $\gamma \geq \frac{1}{2}$, the minimum is at $\gamma = \frac{1}{2}$. Then, using $\bar{\gamma} < \frac{1}{2}$ we have $\rho(x_{max}, \frac{1}{2}) > \rho(x_{max}, \bar{\gamma})$.

This ends the proof in the case $c = 0$. When $c \geq 0$, we introduce $c^{'} = \frac{4\nu c}{a^2}$. The convergence ratio can be written in the form (11) with $x = (\frac{1}{1+c^{'}} \frac{2\nu k}{a})^2$, $\gamma = \frac{c_3|a|}{2\nu}\sqrt{1+c^{'}}$, The minimization problem (10) is then equivalent to the problem (12) with $x_{max} = (\frac{1}{1+c^{'}} \frac{2\nu k_{max}}{a})^2$, and the proof is the same as in the $c = 0$ case.

$\bar{c_3}$ can be obtained by minimizing ρ on wave numbers $k_{int},\ 0 < k_{int} \leq k_{max}$ such that $\rho(k_{int}, 0, c_3) = 0$:

Theorem 2 *The problem (10) is equivalent to the problem*

$$\min_{\substack{0<k_{int} \leq k_{max} \\ \rho(k_{int},0,c_3)=0}} \max_{0 \leq k \leq k_{max}} |\rho(k, 0, c_3(k_{int}))| \tag{13}$$

Then, for $k_{max} > 0$ given, there exists a unique $\bar{k_{int}} \in]0, k_{max}]$, depending on k_{max}, solution of (13). Moreover, $\bar{k_{int}}$ is the unique solution in $]0, k_{max}]$ of the equation $\rho(k_1(k_{int}), 0, c_3(k_{int})) = \rho(k_{max}, 0, c_3(k_{int}))$, where k_1 is the root in $]0, k_{int}[$ of the derivative of $k \longrightarrow \rho(k, 0, c_3(k_{int}))$.

Figure 1 illustrate the behaviour of the convergence ratio with the different interface conditions.

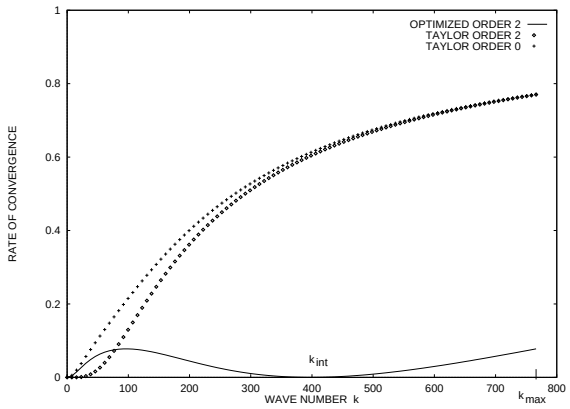

Fig. 1. Ratio of convergence versus wave numbers k
$0 \leq k \leq k_{max},\ k_{max} = \frac{\pi}{h}$
$a = 1,\ b = 1,\ \nu = 0.01,\ c = 0,\ h = \frac{1}{240}$

1.2.2 General case

In this part we give the technique to obtain the OO2 interface conditions in the case of variable cœfficients and an arbitrary decomposition.

We first find the expression of the symbol $\lambda^-(k, a, b)$ and $\lambda^+(k, a, b)$ of the Dirichlet to Neumann operator that is (6) and (7). Then the OO2 interface operator is defined by :

$$B_i = \frac{\partial}{\partial \mathbf{n_i}} - \frac{\mathbf{a}\cdot\mathbf{n_i} - \sqrt{(\mathbf{a}\cdot\mathbf{n_i})^2 + 4c\nu}}{2\nu} + c_2\frac{\partial}{\partial \boldsymbol{\tau_i}} - c_3\frac{\partial^2}{\partial \boldsymbol{\tau_i}^2}$$

where $c_2 = c_2(\mathbf{a}\cdot\mathbf{n_i}, \mathbf{a}\cdot\boldsymbol{\tau_i})$ and $c_3 = c_3(\mathbf{a}\cdot\mathbf{n_i}, \mathbf{a}\cdot\boldsymbol{\tau_i})$ are chosen in order to minimize the convergence ratio of algorithm (3)-(5). The analytic analysis in the case of 2 subdomains and constant cœfficients reduces the minimization problem to a one parameter minimization problem. This technique is extended here that is only one parameter is computed, and with this parameter we get c_2 and c_3 :

We denote by $\lambda^-(k) = \lambda^-(k, \mathbf{a}\cdot\mathbf{n_i}, \mathbf{a}\cdot\boldsymbol{\tau_i})$ and $\lambda^+(k) = \lambda^+(k, \mathbf{a}\cdot\mathbf{n_i}, \mathbf{a}\cdot\boldsymbol{\tau_i})$. We introduce for $0 \leq k \leq k_{max}$ and $0 < k_{int} \leq k_{max}$:

$$\rho(k, k_{int}) = \left(\frac{\lambda^-(k) - \lambda^-_{ap}(k, k_{int})}{\lambda^+(k) - \lambda^-_{ap}(k, k_{int})}\right)^2$$

with $\lambda^-_{ap}(k, k_{int}) = \lambda^-(0) - c_2(k_{int})ik - c_3(k_{int})k^2$, where $c_2 = c_2(k_{int})$ and $c_3 = c_3(k_{int})$ are given by $\rho(k_{int}, c_2, c_3) = 0$, that is :

$$c_2(k_{int}) = -\frac{Im(\lambda^-(k_{int}))}{k_{int}}, \quad c_3(k_{int}) = \frac{\lambda^-(0) - Re(\lambda^-(k_{int}))}{k_{int}^2}$$

Let $k_1 = k_1(k_{int})$ be the root in $]0, k_{int}[$ of the derivative of $k \longrightarrow |\rho(k, k_{int})|$. Then, we compute the solution $\bar{k_{int}}$ in $]0, k_{max}]$ of the equation :

$$|\rho(k_1, k_{int})| = |\rho(k_{max}, k_{int})|$$

with a dichotomy algorithm. With k_{int} we can compute c_2 and c_3:

$$c_2 = c_2(\mathbf{a}\cdot\mathbf{n_i}, \mathbf{a}\cdot\boldsymbol{\tau_i}) = -\frac{Im(\lambda^-(\bar{k_{int}}))}{\bar{k_{int}}}$$

$$c_3 = c_3(\mathbf{a}\cdot\mathbf{n_i}, \mathbf{a}\cdot\boldsymbol{\tau_i}) = \frac{\lambda^-(0) - Re(\lambda^-(\bar{k_{int}}))}{(\bar{k_{int}})^2}$$

So the OO2 conditions are easy to use and not costly.

1.2.3 Convergence analysis

In the case of 2 subdomains, the convergence of algorithm (3)-(5) with the OO2 interface conditions is proved by computing explicitely the convergence ratio ρ in term of wave numbers k :

Theorem 3 *Let $\Omega = \mathbb{R}^2$ be decomposed in 2 subdomains $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$ and $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$. Then, $\forall \nu > 0,\ c > 0,\ a \in \mathbb{R},\ b \in \mathbb{R}$,*

$$\forall\ k \in \mathbb{R}, \quad |\rho(k, c_2, c_3)| < 1$$

Proof Let $\Omega = \mathbb{R}^2$ be decomposed in 2 subdomains $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$ and $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$. We use the following result from [12] :

Lemma 4 *For all $\nu > 0,\ c > 0,\ a \in \mathbb{R},\ b \in \mathbb{R},\ c_2 \in \mathbb{R},\ c_3 \geq 0$, and for all $k \in \mathbb{R}$,*

$$|\rho(k, c_2, c_3)| < 1 \iff \left(\frac{\lambda^-_{ap} - \frac{a}{2\nu}}{\lambda^- - \frac{a}{2\nu}}\right) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\} \cup \{\infty\}$$

This leads to the

Corollary 5 *For all* $\nu > 0,\ c > 0,\ a \in \mathbb{R},\ b \in \mathbb{R},\ c_2 \in \mathbb{R},\ c_3 \geq 0,$

$$\forall k \in \mathbb{R}, \quad sgn(c_2) = sgn(b) \Longrightarrow |\rho(k, c_2, c_3)| < 1$$

Proof For $z_1, z_2 \in \mathbb{C}$ we have the following relation :
$sgn(\mathrm{Re}\, z_1) = sgn(\mathrm{Re}\, z_2)$ and $sgn(\mathrm{Im}\, z_1) = sgn(\mathrm{Im}\, z_2) \Longrightarrow \mathrm{Re}(\frac{z_1}{z_2}) > 0$.
From (6), (9) and $c_3 \geq 0$, we have : $\mathrm{Re}(\lambda^- - \frac{a}{2\nu}) < 0$, $\mathrm{Re}(\bar{\lambda_{ap}^-} - \frac{a}{2\nu}) < 0$, $sgn(\mathrm{Im}(\lambda^-(k))) = -sgn(bk)$, for $k \in \mathbb{R}$ and $sgn(\mathrm{Im}(\lambda_{ap}^-(k))) = -sgn(c_2 k)$, for $k \in \mathbb{R}$. Using lemma 4 ends the proof of corollary 5.

We now prove theorem 3 : as $\bar{k_{int}} > 0$ and $sgn(Im(\lambda^-(\bar{k_{int}}))) = -sgn(b\ \bar{k_{int}})$ (from (6)), c_2 is of the same sign that b, so we can apply corallary 5.

Remark 6 *In the OO2 interface conditions, the minimization problem on* c_2 *and* c_3 *is on a set of conditions which verify (8), i.e. on a set of conditions which ensures (adding a condition on the sign of* c_2*) the convergence of the Schwarz algorithm. So an approximate minimization problem on the same set of conditions will also ensure the convergence. In the case of 2 subdomains, the convergence is proved by computing explicitely the convergence ratio. When the domain is decomposed in N subdomains (strips) the convergence ratio is estimated in function of the convergence ratio of the 2 subdomains case and the decomposition geometry. The convergence is proved in [12].*

Moreover, when the mesh size tends to 0, the condition number is asymptotically much better for OO2 than for Taylor order 0 or 2 interface conditions :

Theorem 7 *Let* $\Omega = \mathbb{R}^2$ *be decomposed in 2 subdomains* $\Omega_1 = \mathbb{R}^- \times \mathbb{R}$ *and* $\Omega_2 = \mathbb{R}^+ \times \mathbb{R}$. *Let* $a \in \mathbb{R},\ a \neq 0,\ b = 0$ *and* $c \geq 0$ *in (1)-(2). Let* h *be the mesh size, and let* $(\rho_{max})_{IC}$ *be the maximum of* ρ *on* $0 \leq k \leq \frac{\pi}{h}$ *with the interface condition* IC. *Let* $\alpha = 1 + \frac{4\nu c}{a^2}$. *Then, when* $h \to 0$ *:*

$$\begin{aligned}
(\rho_{max})_{Taylor\ order\ 0} &\approx 1 - \frac{2}{\pi}\alpha^{\frac{1}{2}}(\frac{|a|h}{\nu}) \\
(\rho_{max})_{Taylor\ order\ 2} &\approx 1 - \frac{4}{\pi}\alpha^{\frac{1}{2}}(\frac{|a|h}{\nu}) \\
(\rho_{max})_{OO2} &\approx 1 - 8\alpha^{\frac{1}{6}}(\frac{1}{4\pi}\frac{|a|h}{\nu})^{\frac{1}{3}}
\end{aligned}$$

Proof Let $a \in \mathbb{R},\ a \neq 0,\ b = 0$ and $c \geq 0$ in (1). Then the convergence ratio can be written in the form (11) with $x = (\frac{1}{1+c'}\frac{2\nu k}{a})^2$, $\gamma = \frac{c_3|a|}{2\nu}\sqrt{1+c'}$, and $c' = \frac{4\nu c}{a^2}$. For the OO2 interface conditions, we use the theorem 1. Problem (10) is equivalent to the problem (12) with $x_{max} = (\frac{1}{1+c'}\frac{2\nu k_{max}}{a})^2$ and is the

unique solution in $[\frac{-1+\sqrt{1+x_{max}}}{x_{max}}, \frac{1}{2}[$ of $\rho(x_{max},\gamma) = \rho(x_1(\gamma),\gamma)$, with $x_1(\gamma)$ the root of the derivative of $x \longrightarrow \rho(x,\gamma)$ such that $\rho(x,\gamma) \neq 0$.
The previous equation is equivalent to the following one :

$$4\gamma(1-\gamma)(1+\gamma x_{max})^2 - 1 - x_{max} = 0 \tag{14}$$

In order to prove the estimate, we first prove the two following lemmas :

Lemma 8 *The solution γ of (14) satisfies :* $\lim\limits_{x_{max}\longrightarrow+\infty}(x_{max}\gamma) = +\infty$.

Proof Let $\gamma \geq 0$ be the solution of (14). We suppose that there exists a constant $M > 0$ such that $|x_{max}\gamma| \leq M,\ \forall x_{max} > 0$. Then,

$$4\gamma(1-\gamma)(1+\gamma x_{max})^2 \leq \frac{4M}{x_{max}}(1+\frac{M}{x_{max}})(1+M)^2 \tag{15}$$

As $\lim\limits_{x_{max}\longrightarrow+\infty}\frac{4M}{x_{max}}(1+\frac{M}{x_{max}})(1+M)^2 = 0$, so $\lim\limits_{x_{max}\longrightarrow+\infty} 4\gamma(1-\gamma)(1+\gamma x_{max})^2 = 0$ (from (15)), which is in contradiction with (14). So, $\lim\limits_{x_{max}\longrightarrow+\infty}(x_{max}\gamma) = +\infty$.

Lemma 9 *For $x_{max} \gg 1$ given, the solution γ_∞ of (14) satisfies :*

$$\gamma_\infty \approx \epsilon^{\frac{1}{3}} \quad \textit{with } \epsilon = \frac{1}{4x_{max}}$$

Proof Let $x_{max} \gg 1$ given, and γ_∞ the corresponding solution of (14). From lemma 8: $4\gamma_\infty(1-\gamma_\infty)(\gamma_\infty x_{max})^2 \approx x_{max}$, that is $(\gamma_\infty)^3 - (\gamma_\infty)^4 \approx \frac{1}{4x_{max}}$.
Let $\epsilon = \frac{1}{4x_{max}}$. As $0 \leq \gamma_\infty < \frac{1}{2}$, (1.2.3) leads to $\gamma_\infty \approx \epsilon^{\frac{1}{3}}$.

We can now prove theorem 7. From theorem 1, $(\rho_{max})_{OO2} = \rho(x_{max},\gamma_\infty)$ that is (using lemma 9):

$$\begin{aligned}(\rho_{max})_{OO2} &\approx \left(\frac{-1+\sqrt{1+\frac{1}{4\epsilon}} - \epsilon^{\frac{1}{3}}\frac{1}{4\epsilon}}{1+\sqrt{1+\frac{1}{4\epsilon}} + \epsilon^{\frac{1}{3}}\frac{1}{4\epsilon}}\right)^2 \\ &\approx \left(\frac{4\epsilon^{\frac{2}{3}} - 2\epsilon^{\frac{1}{6}}\sqrt{4\epsilon+1}+1}{4\epsilon^{\frac{2}{3}} + 2\epsilon^{\frac{1}{6}}\sqrt{4\epsilon+1}+1}\right)^2 \end{aligned} \tag{16}$$

Then using a Taylor expansion at $\epsilon = 0$ of (16), we get : $(\rho_{max})_{OO2} \approx 1 - 8\epsilon^{\frac{1}{6}}$.
So, when the mesh size h tends to 0 :

$$(\rho_{max})_{OO2} \approx 1 - \frac{8}{(4\pi)^{\frac{1}{3}}}(1+c')^{\frac{1}{6}}(\frac{|a|h}{\nu})^{\frac{1}{3}}$$

(we recall that $\epsilon = \frac{1}{4x_{max}}$, with $x_{max} = \frac{1}{1+c'}(\frac{2\nu k_{max}}{a})^2$, and $k_{max} = \frac{\pi}{h}$). For the Taylor order 0 or 2 interface conditions, the proof is done in a same way, and is much easier because the cœfficient c_3 is independant of x_{max}.

1.3 Substructuring formulation

In [14], the non-overlapping algorithm (3)-(5) is interpreted as a Jacobi algorithm applied to a problem where the unknowns are on the boundary of the subdomains. That is, the actual unknowns of the problem are the terms $B_i(u_i)$ on the interface $\Gamma_{i,j},\ i \neq j$. In the discrete case, this interface problem can be written in the form :

$$D\lambda = b \tag{17}$$

where λ, restricted to $\partial\Omega_i$, represents the discretization of the term $B_i(u_i)$ on the interface $\Gamma_{i,j},\ i \neq j$, and D is an interface matrix, non symmetric and full. Numerically D is never computed because it will be costly.
To accelerate convergence again, the Jacobi algorithm is replaced by a Krylov algorithm (GMRES, BICG-STAB, ...) [15]. As we use an iterative method, we need only to compute at each step the product $D\lambda$, which is the restriction to $\partial\Omega_i$ of the discretization of the jump $B_i(u_i) - B_i(u_j)$ on the interface $\Gamma_{i,j},\ i \neq j$, where the $u_i,\ 1 \leq i \leq N$, are solution of the local problem :

$$\begin{aligned} L(u_i) &= 0 \quad \text{in } \Omega_i \\ B_i(u_i) &= \lambda_{i,j} \quad \text{on } \Gamma_{i,j},\ i \neq j \\ C(u_i) &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned}$$

with $\lambda_{i,j}$ the restriction of λ on the interface $\Gamma_{i,j}$. So the solving of the interface problem (17) by a Krylov method is done by the solving of independant problems in each subdomain, that is we use efficiently parallel computation : each subdomain is assign to a processor which solve his problem independently. The interactions between subdomains are processed by the communications between processors. Once we have computed an approximate solution λ of problem (17), we get the approximate solution u_i of the solution u of problem (1)-(2) in each subdomain $\Omega_i,\ 1 \leq i \leq N$, by solving the local problem :

$$\begin{aligned} L(u_i) &= f \quad \text{in } \Omega_i \\ B_i(u_i) &= \lambda_{i,j} \quad \text{on } \Gamma_{i,j},\ i \neq j \\ C(u_i) &= g \quad \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned}$$

2 Numerical results

The convection-diffusion equation is discretized by a finite volume scheme [1] (collaboration with Matra BAe Dynamics France). The global domain Ω is decomposed in N non-overlapping subdomains. The interface problem (17) is solved by a BICG-STAB algorithm. This involves solving, at each iteration, N independant subproblems (one per subdomain) which can be performed in parallel. Each subproblem is solved by a direct method. We denote by h the mesh size. We compare the results obtained with the OO2 interface conditions and the Taylor order 0 or 2 interface conditions.

Remark : The optimized cœfficients in the OO2 method are computed in an initialisation step, that is in the computation of the local matrix. They are not computed again in the iterations of BICG-STAB. Moreover, each iteration of BICG-STAB has the same cost for all the interface conditions (Taylor order 0, Taylor order 2, OO2), because the use of order 2 conditions does not increase the bandwidth of the local matrix. So, in the BICG-STAB algorithm, the CPU time is proportional to the number of iterations.

2.1 Flow in a square

We solve the following problem :

$$\begin{aligned} L(u) &= 0, \quad 0 \le x \le 1,\ 0 \le y \le 1 \\ u(0,y) = 0,\ \frac{\partial u}{\partial x}(1,y) &= 0, \quad 0 \le y \le 1 \\ u(x,0) = 1,\ \frac{\partial u}{\partial y}(x,1) &= 0, \quad 0 \le x \le 1 \end{aligned} \tag{18}$$

We consider a cartesian mesh with constant mesh size h. The unit square is decomposed in $N_x \times N_y$ subdomains, where N_x (resp. N_y) is the number of subdomains in the x (resp. y) direction. We consider two types of convection velocity field : a shear velocity ($a = y,\ b = 0$) and a rotating velocity ($a = -\sin(\pi(y-\frac{1}{2}))\cos(\pi(x-\frac{1}{2})),\ b = \cos(\pi(y-\frac{1}{2}))\sin(\pi(x-\frac{1}{2}))$). The isovalues of the solution of problem (18) with the shear velocity are represented in figure 2, and with the rotating velocity in figure 3.

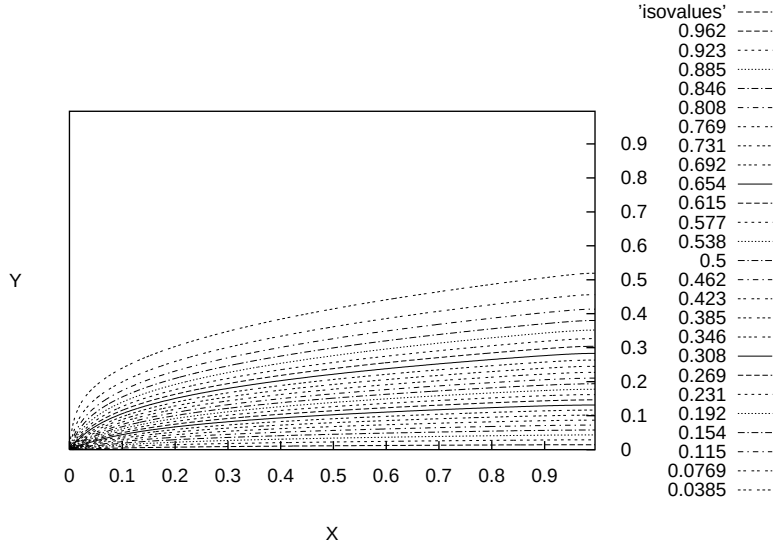

Fig. 2. Isovalues of the solution u, shear velocity
$a = y,\ b = 0,\ \nu = 1.d-2,\ CFL = 1.d9,\ h = \frac{1}{241}$

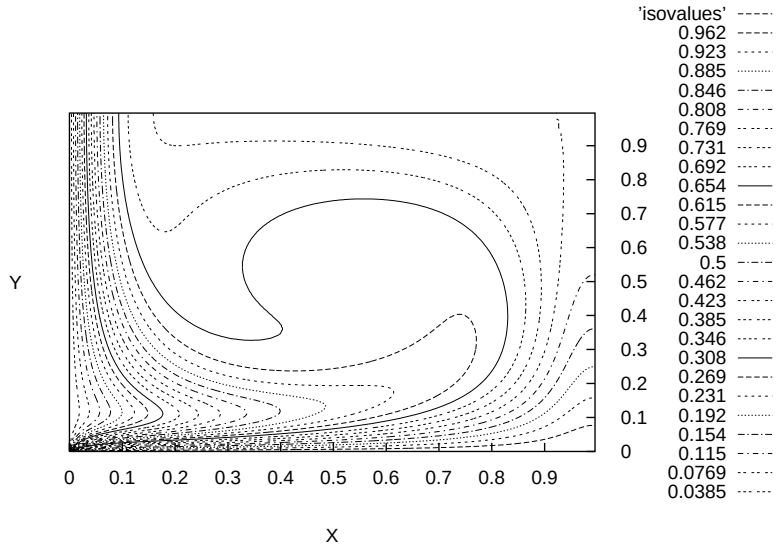

Fig. 3. Isovalues of the solution u, rotating velocity
$a = -\sin(\pi(y-\frac{1}{2}))\cos(\pi(x-\frac{1}{2})),\ b = \cos(\pi(y-\frac{1}{2}))\sin(\pi(x-\frac{1}{2}))$
$\nu = 1.d-2,\ CFL = 1.d9,\ h = \frac{1}{241}$

In table 1, we take a decomposition in strips in order to observe the influence on the convergence of the convection velocity angle to the interfaces. We observe that the OO2 interface conditions give a significantly better convergence which is independant of the convection velocity angle to the interfaces. One of the advantages is that for a given number of subdomains, the decomposition of the domain doesn't affect the convergence. Particularly here, for 16 subdomains, the decomposition in strips (table 1) or in squares (figure 4) doesn't affect the convergence.

Decomposition of the domain	OO2	Taylor order 2	Taylor order 0
normal velocity to the interface $\mathbf{16 \times 1}$ subdomains	15	123	141
tangential velocity to the interface $\mathbf{1 \times 16}$ subdomains	21	not convergent	86

Table 1: Number of iterations versus the convection velocity's angle
$a = y,\ b = 0,\ \nu = 1.d-2,\ CFL = 1.d9,\ h = \frac{1}{241},\ \log_{10}(Error) < 1.d-6$

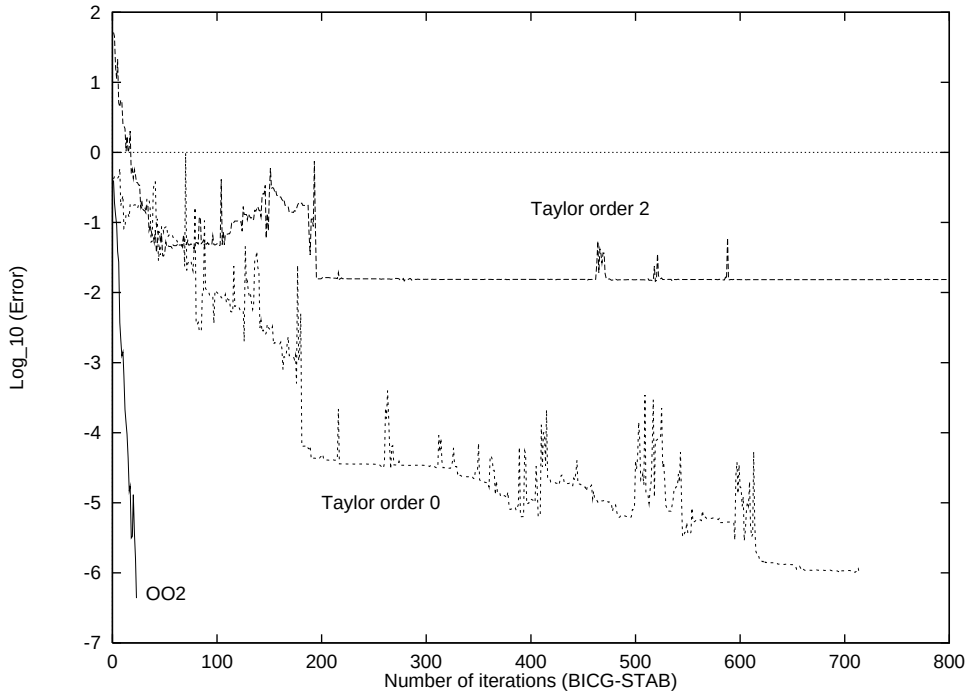

Fig. 4. Error versus the number of iterations
$\mathbf{4 \times 4}$ subdomains, shear velocity
$a = y,\ b = 0,\ \nu = 1.d-2,\ CFL = 1.d9,\ h = \frac{1}{241}$

Figure 5 shows that the convergence with the OO2 interface conditions is significantly better for a more general convection velocity (the rotating velocity) and decomposition (in 4×8 subdomains).

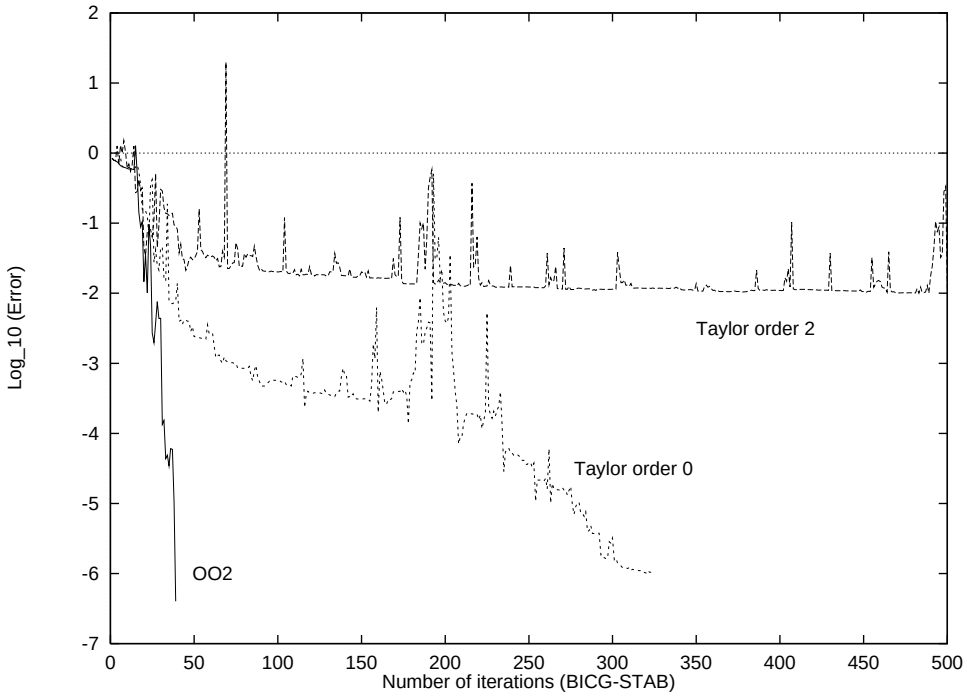

Fig. 5. Error versus the number of iterations
$\mathbf{4 \times 8}$ subdomains, rotating velocity
$a = -\sin(\pi(y-\frac{1}{2}))\cos(\pi(x-\frac{1}{2}))$, $b = \cos(\pi(y-\frac{1}{2}))\sin(\pi(x-\frac{1}{2}))$
$\nu = 1.d-2$, $CFL = 1.d9$, $h = \frac{1}{241}$

The convergence with the OO2 interface conditions, for the studied numerical cases, is also nearly independant of the mesh size (see table 2). We practically fit to the theoretical estimates of theorem 7.

grid	65×65	129×129	241×241
OO2	25	26	30
Taylor order 0	76	130	224

Table 2: Number of iterations versus the mesh size
$\mathbf{4 \times 4}$ subdomains, rotating velocity
$a = -\sin(\pi(y-\frac{1}{2}))\cos(\pi(x-\frac{1}{2}))$, $b = \cos(\pi(y-\frac{1}{2}))\sin(\pi(x-\frac{1}{2}))$
$\nu = 1.d-2$, $CFL = 1.d9$, $\log_{10}(Error) < 1.d-6$

The convergence with the OO2 interface conditions is also very little sensible to the variations of the CFL, as it shown on table 3.

	$CFL = 1.d0$	$CFL = 1.d3$	$CFL = 1.d5$	$CFL = 1.d9$
OO2	3	12	15	15
Taylor ordre 2	2	21	58	123
Taylor ordre 0	3	18	48	141

Table 3: Number of iterations versus the CFL, $\mathbf{16 \times 1}$ subdomains
$a = y,\ b = 0,\ \nu = 1.d-2,\ CFL = 1.d9,\ h = \frac{1}{241},\ \log_{10}(Error) < 1.d-6$

Figure 6 shows the *speed-up* $= \frac{CPU\ time\ (1\ domain)}{CPU\ time\ (N\ subdomains)}$ of the method. Let i_{max} (resp. j_{max}) be the number of grid points in the x (resp. y) direction, for the global domain. We note N_{it} the number of BICG-STAB iterations. For a decomposition of the domain in $N_x \times N_y$ subdomains, the total cost can be estimated by : $\alpha_1(\frac{i_{max}}{N_x})^3 \frac{j_{max}}{N_y} + \alpha_2 N_{it}(\frac{i_{max}}{N_x})^2 \frac{j_{max}}{N_y}$, where α_1 and α_2 are constants. Figure 6 shows that for a small number of subdomains, the first term (arising from the LU factorization of the local matrix) is predominant. Then, the second term (arising from the BICG-STAB algorithm) become predominant. After 32 subdomains, the estimate is no more valid, because of the communication costs which can not be neglected.

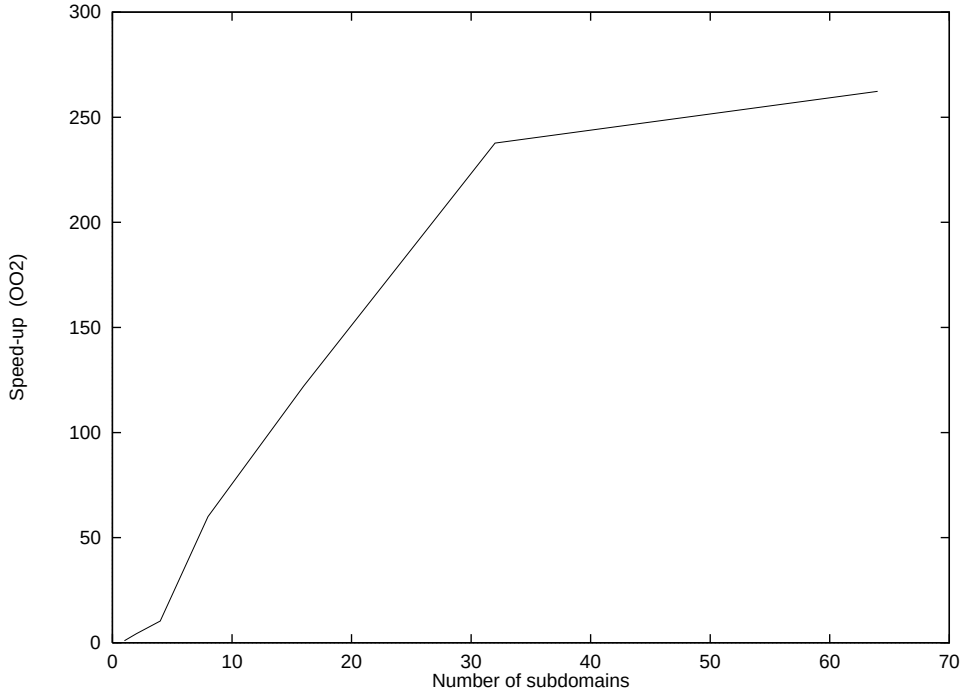

Fig. 6. Speed-up versus the number of subdomains, for decompositions in 2×1, 2×2, 4×2, 4×4, 8×4 and 8×8 subdomains
shear velocity
$a = y,\ b = 0,\ \nu = 1.d-2,\ CFL = 1.d9,\ h = \frac{1}{241},\ Residual < 1.d-9$

2.2 Flow around a cylinder, issued from a Navier-Stokes computation

The convection velocity field is a Navier-Stokes incompressible flow, with Reynolds number $Re = 10000$, around a cylinder. This velocity field is from a computation performed with the AEROLOG software of the aerodynamic department at Matra BAe Dynamics France. The domain is defined by $\Omega = \{(x,y) = (r\cos(\theta), r\sin(\theta)),\ 1 \leq r \leq R,\ 0 \leq \theta \leq 2\pi\}$ with $R > 1$ given. We solve the following problem :

$$\begin{aligned} L(u) &= 0 \quad \text{in } \Omega \\ u &= 1 \quad \text{on } \{(x,y) = (\cos(\theta), \sin(\theta)),\ 0 \leq \theta \leq 2\pi\} \\ u &= 0 \quad \text{on } \{(x,y) = (R\cos(\theta), R\sin(\theta)),\ 0 \leq \theta \leq 2\pi\} \end{aligned} \tag{19}$$

The grid is $\{(x_i, y_j) = (r_i\cos(\theta_j), r_i\sin(\theta_j)),\ 1 \leq\ i,j\ \leq\ 65\}$, and is refined around the cylinder and in the direction of the flow (see figure 9). The isovalues of the solution of problem (19) are represented in figure 8 (without the grid) and in figure 9 (with the grid). We note N_{max} the number of points on the boundary of a subdomain multiply by the number of subdomains.

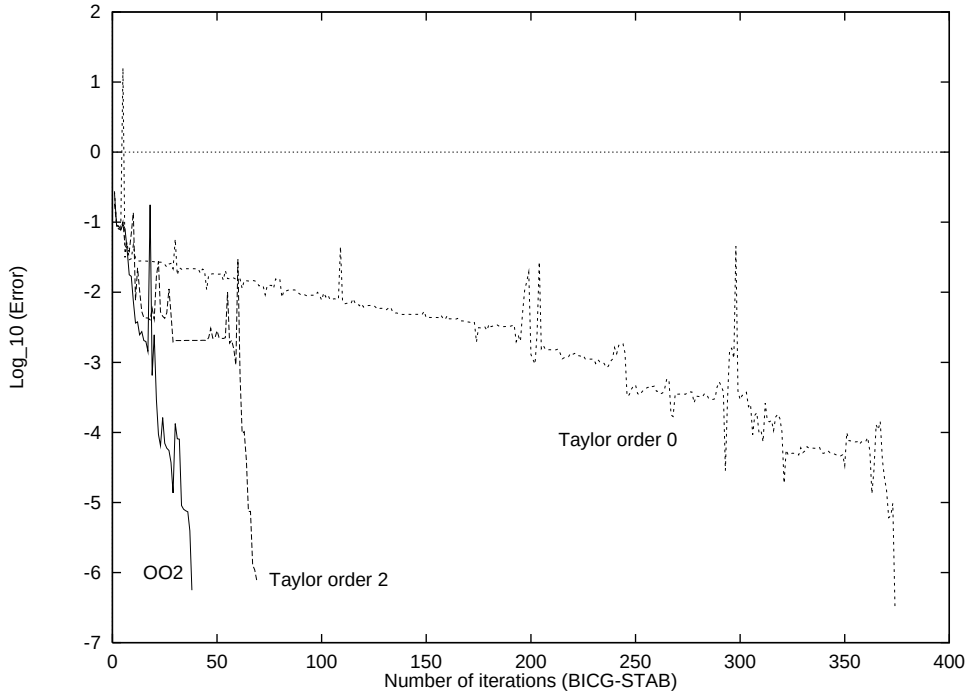

Fig. 7. Error versus the number of iterations
$\mathbf{4 \times 2}$ subdomains, Navier-Stokes flow velocity, $\nu = 1.d-4,\ CFL = 1.d9$

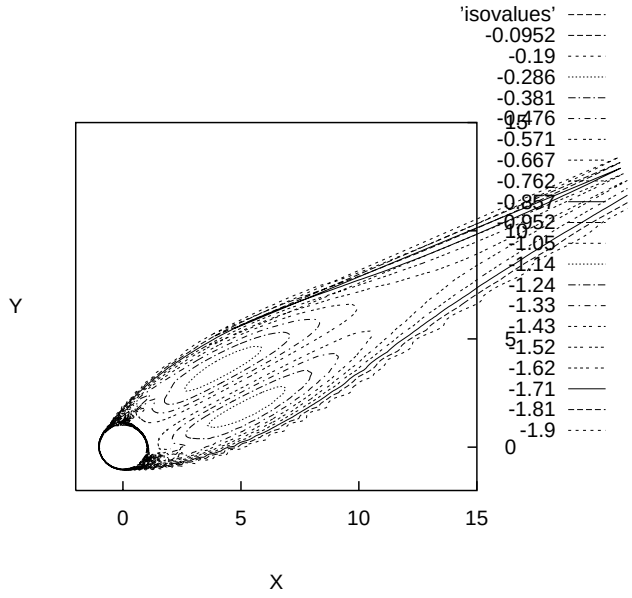

Fig. 8. Isovalues of the solution u
Navier-Stokes flow velocity, $\nu = 1.d-4,\ CFL = 1.d9$

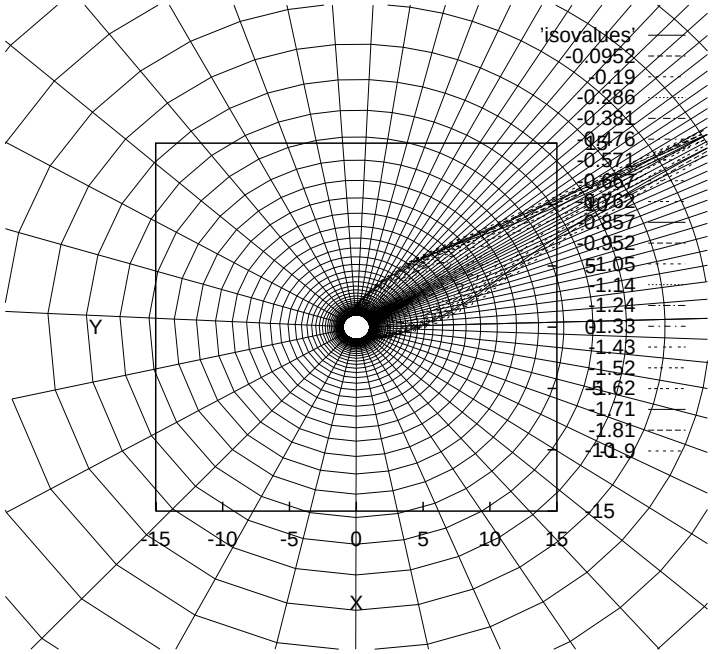

Fig. 9. Grid and isovalues of the solution u
Navier-Stokes flow velocity, $\nu = 1.d-4,\ CFL = 1.d9$

The OO2 interface conditions give also significantly better convergence in that case (figure 7). We observe in table 4 that the convergence is practically independant of the viscosity ν.

	OO2	Taylor order 2	Taylor order 0
$\nu = 1.d-5$	56	41	119
$\nu = 1.d-4$	43	121	374
$\nu = 1.d-3$	32	$N_{max} = 768$ $log_{10}(Error) = -5.52$	$N_{max} = 768$ $log_{10}(Error) = -2.44$

Table 4: Number of iterations versus the viscosity
$\mathbf{4 \times 2}$ subdomains, Navier-Stokes flow velocity
$\nu = 1.d-4,\ CFL = 1.d9,\ \log_{10}(Error) < 1.d-6$

Remark 10 *Numerically, the convergence ratio of the method is nearly linear upon the number of subdomains. So it is necessarly to send global information between subdomains, in order to have a convergence ratio independant of the number of subdomains. To tackle this problem, in [9], a "low wave number" preconditioner is applied to the OO2 method.*

3 Conclusion

The OO2 method applied to convection-diffusion problems appears to be a very efficient method. Its main advantage is that it is a general domain decomposition technique, with no a priori knowledge of the boundary layers or the recirculation zones location. The convergence ratio is numerically nearly independant both of the physical parameters and the discretization parameters.

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