GALOIS COHOMOLOGY, QUADRATIC FORMS AND MILNOR K-THEORY.

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Abstract. These are notes for two lectures given during the summer school 'Motives and Milnor's conjecture', Jussieu, June 2011. The material presented here is classical and can be found in the literature. Some precise references are provided for the reader's convenience, but we do not intend to give a comprehensive list of references.

Contents

Introduction

The origin of Milnor's conjecture is a famous and very nice paper [Mil] published in Inventiones in 1970. In this paper, Milnor studies the relation between three graded rings, namely

- Milnor K-theory modulo 2, $k_\star^M F$,
- The graded Witt ring of quadratic forms $GW_\star F = \oplus_{n\geq 0} I^n F/I^{n+1}F$,
- Galois cohomology $H^\star(F)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$,

where the base field F has characteristic different from 2. There exist non-trivial homomorphisms, which we will describe later,

$$\begin{tikzcd} & GW_\star F \\ k_\star^M F \arrow[ur, "s_\star"] \arrow[dr, "h^\star"'] & \\ & H^\star(F) \end{tikzcd}$$

The definition of both maps is explained in Milnor's paper; the second is due to Bass-Tate. The statement that they are isomorphisms is known as Milnor's conjecture. Milnor does not formally state this as a conjecture in his paper. But he asks in question 4.3 whether the first map is an isomorphism in every degree; and he writes in the introduction that "Section 6 describes the conjecture that $k_\star^M F$ is canonically isomorphic to $H^\star(F)$". He also gives several examples of fields for which both maps are isomorphisms.

In 2003, Voevodsky [Voe] published a proof of the second statement, that is the map $h^\star : k_\star^M F \to H^\star(F)$ is an isomorphism. The result was already known in degree 1 (we will see a proof in these lectures), in degree 2 by Merkurjev [Me], and in degree 3 by Merkurjev-Suslin[MS], and independently, Rost [R]. One of the aim of the summer school is to explain Voevodsky's proof. It actually extends to a more general result, known as the Bloch-Kato conjecture, which states that K-theory modulo ℓ is isomorphic to Galois cohomology with coefficients in μ_ℓ, for every prime ℓ and every field of characteristic different from ℓ.

Using Voevodsky's result, several authors provided proofs that the other map also is an isomorphism, or equivalently that we actually have a commutative triangle of isomorphisms

$$\begin{tikzcd} & GW_\star F \arrow[dd, "\simeq"] \\ k_\star^M F \arrow[ur, "s_\star", "\simeq"'] \arrow[dr, "h^\star", "\simeq"'] & \\ & H^\star(F) \end{tikzcd}$$

(see Orlov-Vishik-Voevodsky [OVV], Morel [Mo1] and [Mo2] and Kahn-Sujatha [KS]). Note that even though they use different approaches, they all rely heavily on Voevodsky's theorem.

1. Definition of Milnor's K-theory

Throughout these notes, the base field F has characteristic different from 2. The main reference for this section is Milnor's paper [Mil]. His definition is motivated by Matsumoto's theorem, which gives a description of K_2F by generator and relations.

1.1. **The graded ring $K_\star^M F$.** In this subsection, the base field F is arbitrary. Its multiplicative group is denoted by $F^\times$. To the field F, we associate a graded ring $K_\star^M F := \oplus_{n\geq 0} K_n F$ defined as follows. First, we define $K_1 F$ to be the group $F^\times$, written additively. Its elements are denoted by $\{a\}$, for $a \in F^\times$; they satisfy $\{ab\} = \{a\} + \{b\}$. In particular, $\{1\} = 0$.

Definition 1.1. Milnor's K ring $K_\star^M F$ is the quotient of the tensor algebra

$$\oplus_{n\geq 0}(K_1 F)^{\otimes n}$$

by the ideal generated by the elements $\{a\} \otimes \{1-a\}$, for all $a \in F^\times$, $a \neq 1$.

In other words, $K_\star^M F$ is the associative ring with unit generated by the elements $\{a\}$, for all $a \in F^\times$, subject to the relations

(1) $\{ab\} = \{a\} + \{b\}$ for all $a, b \in F^\times$,
(2) $\{a\}\{1-a\} = 0$ for all $a \in F^\times$, $a \neq 1$.

The ring $K_\star^M F$ is graded. Its nth part $K_n^M F$ is generated by the tensor products $\{a_1\} \otimes \{a_2\} \otimes \ldots \otimes \{a_n\}$, for $a_1, \ldots, a_n \in F^\times$. Such an element is called an *n-symbol* and will be denoted by

$$\{a_1, \ldots, a_n\} = \{a_1\} \otimes \ldots \otimes \{a_n\}.$$

Note in particular that $\{a_1, \ldots, a_n\} = 0$ if $a_i = 1$ for some i. The defining relations for the group $K_n F$ are the following : for all $a_1, \ldots, a_n, b_i \in F^\times$,

(1) $\{a_1, \ldots, a_i b_i, \ldots a_n\} = \{a_1, \ldots, a_i, \ldots a_n\} + \{a_1, \ldots, b_i, \ldots a_n\}$,
(2) $\{a_1, \ldots, a_i, 1-a_i, \ldots, a_n\} = 0$ if $a_i \neq 1$.

1.2. **Relations on symbols.** We now describe other relations on symbols, following easily from the relations (1) and (2).

Lemma 1.2. *For all $a \in F^\times$, we have $\{a, -a\} = 0$.*

Proof. This is obvious if $a = 1$. Otherwise, we have $-a = \frac{1-a}{1-a^{-1}}$. Hence $\{-a\} = \{1-a\} - \{1-a^{-1}\}$ and multiplying on the left by $\{a\}$, we get

$$\{a, -a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = \{a^{-1}, 1-a^{-1}\} = 0.$$

□

Remark 1.3. Combining this with the relation (2), we get

$$\{a, b\} = 0 \ \text{ if } a + b = 0 \text{ or } 1.$$

Lemma 1.4. *For all $a, b \in F^\times$, we have $\{a, b\} = -\{b, a\}$.*

Proof. By bilinearity, we have

$$\{ab, -ab\} = \{a, -a\} + \{a, b\} + \{b, a\} + \{b, -b\} = \{a, b\} + \{b, a\},$$

which is trivial by the previous lemma. □

More generally, the following holds, proving that $K_\star^M F$ is graded-commutative:

Lemma 1.5. *For all $\alpha \in K_n F$ and $\beta \in K_m F$, we have*

$$\beta\alpha = (-1)^{mn}\alpha\beta.$$

Proof. Clearly, it is enough to check the relation on symbols. Since $\{a_1, \ldots, a_n\} = \{a_1\} \ldots \{a_n\}$, the result follows from the previous lemma. □

The fundamental relations $\{a, -a\} = 0$ and $\{a, 1-a\} = 0$ extend in two ways. Consider $a_1, \dots, a_n \in F^\times$.

Lemma 1.6. *If $a_i + a_j$ is 0 or 1 for two distinct indices $1 \leq i, j \leq n$, then $\{a_1, \dots, a_n\} = 0$.*

Proof. Rearranging the factors using lemma 1.4, we get

$$\{a_1, \dots, a_n\} = \pm\{a_i\}\{a_j\}\{a_1\} \dots \{a_n\} = 0.$$

□

Lemma 1.7. *If the sum $a_1 + \dots + a_n$ is 0 or 1, then $\{a_1, \dots, a_n\} = 0$.*

Proof. We argue by induction on the number n of factors. The result is known if $n = 2$. For $n \geq 3$, consider $\{a_1, \dots, a_n\}$ with $a_1 + \dots + a_n = 0$ or $a_1 + \dots + a_n = 1$. If $a_1 + a_2 = 0$, then $\{a_1, a_2\} = 0$, and we are done. Otherwise, we have $\frac{a_1}{a_1+a_2} + \frac{a_2}{a_1+a_2} = 1$, so

$$\big(\{a_1\} - \{a_1 + a_2\}\big)\big(\{a_2\} - \{a_1 + a_2\}\big) = 0,$$

that is

$$\{a_1, a_2\} - \{a_1, a_1 + a_2\} + \{a_2, a_1 + a_2\} + \{a_1 + a_2, a_1 + a_2\} = 0.$$

Multiplying on the right by $\{a_3, \dots, a_n\}$, we get a sum of four terms. The last three terms are multiples of $\{a_1 + a_2, a_3, \dots, a_n\}$, which is trivial by the induction hypothesis. So the first term $\{a_1, \dots, a_n\}$ also is trivial. □

We finish with a result that proves useful in computations:

Lemma 1.8. *For all $a \in F^\times$, we have $\{a\}^2 = \{a, -1\} = \{-1, a\}$.*

Proof. Again, this follows from lemma 1.2. Indeed, we have

$$\{a, a\} = \{a, (-1)(-a)\} = \{a, -1\}.$$

The second equality can be proven similarly. It does not contradict lemma 1.4 since $\{-1, a\}$ has order 2,

$$\{-1, a\} + \{-1, a\} = \{(-1)^2, a\} = 0.$$

□

1.3. **Milnor's K-theory modulo 2.** In these lectures, we are mostly interested in K-theory modulo 2.

Definition 1.9. The K-theory modulo 2 of F is the quotient

$$k_\star^M F = K_\star^M F / 2K_\star^M F.$$

In particular, we have $k_0 F = \mathbb{Z}/2\mathbb{Z}$, $k_1 F \simeq F^\times / F^{\times 2}$, and more generally, $k_n F = K_n F / 2K_n F$. Note that by Lemma 1.5, the product is commutative in $k_\star^M F$. The structure of graded ring of $k_\star^M F$ induces a structure of graded $\mathbb{Z}/2\mathbb{Z}$-algebra on $k_\star^M F$.

We use the same notation $\{a_1, \dots, a_n\}$ for a symbol in $K_n^M F$ and its class modulo $2K_n^M F$. The presentation of $K_\star^M F$ by generators and relations extends to Milnor's K-theory modulo 2 as follows. The classes $\{a\}$, with $a \in F^\times$, generate the graded ring $k_\star^M F$; they are subject to the following relations:

(1) $\{ab\} = \{a\} + \{b\}$;
(2) $\{a, 1-a\} = 0$ if $a \neq 1$;

(3) $2\{a\} = 0$.

In particular, we have

$$\{a_1, \dots, a_i b^2, \dots a_n\} = \{a_1, \dots, a_n\} + 2\{a_1, \dots, b, \dots, a_n\} = \{a_1, \dots, a_n\} \in k_n^M F,$$

so that symbols $\{a_1, \dots, a_n\}$ with $a_i \in F^\times / F^{\times 2}$ are well defined in Milnor's K-theory modulo 2. Moreover, for all a, b, c with $b^2 - ac^2 \neq 0$, the symbol $\{a, b^2 - ac^2\}$ vanishes in $k_2^M F$. This is clear if $b = 0$; if $b \neq 0$, it follows from the following:

$$\{a, b^2 - ac^2\} = \{a, 1 - a\big(c/b\big)^2\} = \{a\big(c/b\big)^2, 1 - a\big(c/b\big)^2\} = 0 \in k_2^M F.$$

In particular, if $a \notin F^{\times 2}$, we have

$$\{a, b\} = 0 \in k_2^M F \text{ if } \ b \in N_{F(\sqrt{a})/F}\big(F(\sqrt{a})^\times\big).$$

2. Connections with quadratic forms

2.1. **The Witt ring $W(F)$.** In this subsection, we recall the definition of the Witt ring, introduced by E. Witt in 1937 [W]. He endows the set of isometry classes of anisotropic quadratic forms over a given field F with a ring structure, using direct orthogonal sum and tensor product. We do not include proofs, and refer the reader to [Kahn], [Lam] or [Sch] for a detailed exposition.

All the quadratic forms considered here are assumed to be non-degenerate. For every such form $\varphi : V \to F$, we let $\langle a_1, \dots, a_n\rangle$ be a diagonalization. The vector $v \in V$ is *isotropic* if $v \neq 0$ and $\varphi(v) = 0$. The quadratic form φ is said to be isotropic if it admits an isotropic vector and anisotropic otherwise. Recall the following classical and elementary result on quadratic forms:

Proposition 2.1. *Let $\varphi : V \to F$ be a non degenerate quadratic form over F.*

(1) *There exists $v \in V$ such that $\varphi(v) = a \in F^\times$ if and only if φ admits a diagonalization $\varphi \simeq \langle a, a_2, \dots, a_n\rangle$. When these two assertions hold, we say that φ represents the value a.*
(2) *The form φ is isotropic if and only if it admits a diagonalization $\varphi \simeq \langle 1, -1, a_3, \dots, a_n\rangle$.*

The 2-dimensional subspace $\mathbb{H} = \langle 1, -1\rangle$ is called a hyperbolic plane. A quadratic form is said to be hyperbolic if it is an orthogonal sum of hyperbolic planes. Using this proposition, one may easily split a quadratic form φ as a hyperbolic part and an anisotropic part. It is a remarkable fact that this decomposition is unique, that is:

Theorem and Definition 2.2. Let φ be a non-degenerate quadratic form over F. There exist a unique integer i and an anisotropic quadratic form φ_{an} over F, uniquely defined up to isomorphism, such that φ is the direct orthogonal sum $\varphi \simeq i\mathbb{H} + \varphi_{an}$. The integer i is the *Witt index* of φ and the form φ_{an} is its *anisotropic part.*

The Witt-index can be computed as follows: it is the maximal dimension of totally isotropic subspaces, that is subspaces over which φ is identically zero. The uniqueness part of the statement is a straightforward consequence of the following result, known as Witt-cancellation theorem:

Theorem 2.3. *Let φ_1, φ_2 and φ_3 be three quadratic forms over F. If the orthogonal sums $\varphi_1 + \varphi_3$ and $\varphi_2 + \varphi_3$ are isometric, then so are φ_1 and φ_2.*

With this in hand, we may now consider the following equivalence relation on quadratic forms:

Definition 2.4. The quadratic forms φ and ψ are *Witt equivalent* if their anisotropic parts φ_{an} and ψ_{an} are isomorphic.

If the forms φ and ψ are Witt-equivalent, their dimensions differ by some even integer, say $\dim(\varphi) = \dim(\psi) + 2r$. By Witt cancellation theorem, we then have $\varphi \simeq \psi + r\mathbb{H}$. Using this, one may easily check that direct sum and tensor product are compatible with the Witt-equivalence relation. Note in particular that the tensor product of a hyperbolic plane with any quadratic form is hyperbolic. Hence we get

Theorem and Definition 2.5. The direct orthogonal sum and the tensor product induce well-defined operations on the set $W(F)$ of equivalence classes of quadratic forms for Witt-equivalence. With these operations, $W(F)$ is a ring called the *Witt ring* of the field F.

Remark 2.6. Alternately, one may define the Witt group as the quotient of the Grothendieck group of the category of quadratic forms, with direct orthogonal sum, by the classes of hyperbolic forms.

Remark 2.7. Since quadratic forms are diagonalizable, the group $W(F)$ is generated by the one-dimensional forms $\langle a\rangle$. Moreover, they satisfy the following relations[1]

(1) $\langle ab^2\rangle = \langle a\rangle$,
(2) $\langle a\rangle + \langle -a\rangle = 0$.

In particular, we have the following equality in $W(F)$:

$$\langle a, b\rangle = \langle 1, a\rangle - \langle 1, -b\rangle. \tag{1}$$

2.2. **Augmentation ideal and the graded Witt ring $GW_\star F$.** The dimension modulo 2 is an augmentation map for $W(F)$:

$$\overline{\dim} : W(F) \to \mathbb{Z}/2\mathbb{Z}.$$

We let $IF = \ker(\overline{\dim})$ be the augmentation ideal, that is the ideal of Witt-classes of even dimensional quadratic forms. It is called the *fundamental* ideal. The Witt ring $W(F)$ is filtered by powers of IF,

$$W(F) = I^0F \supset IF \supset I^2F \cdots \supset I^nF \supset \dots.$$

Moreover, by the celebrated Arason-Pfister Hauptsatz, which asserts that an anisotropic quadratic form with Witt class in I^nF has dimension at least 2^n, we have

$$\bigcap_{n=1}^{\infty} I^nF = \{0\}.$$

Definition 2.8. The graded Witt-ring $GW_\star F$ is the graded ring associated to this filtration,

$$GW_\star F = \oplus_{n\geq 0} I^nF/I^{n+1}F.$$

In particular, $GW_0F = W(F)/IF \simeq \mathbb{Z}/2\mathbb{Z}$. In general, the fundamental ideal and its powers have the following set of generators:

[1] This is not a presentation of $W(F)$; see [Kahn, 1.3.5] for a description of all required relations.

Lemma 2.9. (1) *The fundamental ideal IF is generated by the binary forms representing* 1,
$$\langle\langle a\rangle\rangle = \langle 1, -a\rangle, \quad a \in F^\times;$$
(2) *The nth power I^nF is generated by the so-called n-fold Pfister forms*
$$\langle\langle a_1, \dots, a_n\rangle\rangle = \otimes_{1\leq i\leq n}\langle\langle a_i\rangle\rangle = \otimes_{1\leq i\leq n}\langle 1, -a_i\rangle, \quad a_1, \dots, a_n \in F^\times.$$

Proof. Clearly, it suffices to prove the first assertion, which follows easily from equation (1). □

Among quadratic forms, Pfister forms have exceptional properties, as we shall now recall.

2.3. **Properties of Pfister forms.** Let $\pi = \langle\langle a_1, \dots, a_n\rangle\rangle$ be an n-fold Pfister form. Clearly, it decomposes as $\pi = \langle 1\rangle \oplus \pi'$ for some quadratic form π', uniquely defined up to isomorphism, and called the pure subform of π. Since the form π' diagonalizes as
$$\pi' = \langle -a_1, -a_2, \dots, -a_n, a_1a_2, \dots, (-1)^n a_1 \dots a_n\rangle,$$
it represents the value $-a_1$. By induction, one may prove that, conversely, the following holds (see for instance [Kahn, 2.1.7]):

Proposition 2.10. *Let π be an n-fold Pfister form, and consider $a \in F^\times$ such that $-a$ is represented by the pure subform π' of π. Then, there exists $a_2, \dots, a_n \in F^\times$ such that $\pi = \langle\langle a, a_2, \dots, a_n\rangle\rangle$.*

The main properties of Pfister forms follow quite easily from this proposition. Indeed, we have:

Theorem 2.11. *An isotropic Pfister form is hyperbolic.*

Proof. Let $\pi = \langle 1\rangle \oplus \pi'$ be an n-fold Pfister form, with pure subform π'. Assume the form π is isotropic; we claim its pure subform π' represents -1. Indeed, there exists a non trivial pair $(x, v) \in F \times F^{2^n-1}$ such that $x^2 + \pi'(v) = 0$. If x is non trivial, we get $\pi'(x^{-1}v) = -1$. Otherwise, π' is isotropic, so it clearly represents -1. Applying Prop 2.10, we now get
$$\pi = \langle\langle 1, a_2, \dots a_n\rangle\rangle = \langle 1, -1\rangle \otimes \langle\langle a_2, \dots, a_n\rangle\rangle$$
is hyperbolic. □

The second result concerns similarity factors of Pfister forms, that is scalars $\lambda \in F^\times$ such that $\pi \simeq \langle\lambda\rangle \otimes \pi$. A quadratic form representing the value 1 also represents its similarity factors. For Pfister forms, the converse also holds:

Theorem 2.12. *Let π be a Pfister form, and let $\alpha \in F^\times$. The following are equivalent:*
(1) *λ is represented by π;*
(2) *λ is a similarity factor for π, that is $\pi \simeq \langle\lambda\rangle \otimes \pi$.*

Proof. As already mentioned, one direction is clear since the form π represents 1. To prove the converse, assume λ is represented by π, that is $\lambda = x^2 - a$ for some elements $x \in F^\times$ and $-a$ represented by the pure subform π' of π. If a is zero, the result is obvious. Otherwise, we can write $\pi = \langle\langle a, a_2, \dots, a_n\rangle\rangle = \langle\langle a\rangle\rangle\langle\langle a_2, \dots, a_n\rangle\rangle$ for some $a_2, \dots, a_n \in F^\times$. Note that the form $\pi_1 = \langle\langle a\rangle\rangle = \langle 1, -a\rangle$ is the norm form of the quadratic étale extension $F[x]/(x^2-a)$. Hence it is multiplicative, that is it satisfies $\pi_1(vw)=\pi_1(v)\pi_1(w)$ for all vectors $v,w\in F[x]/(x^2-a)$. So, in particular, for every vector v with $\pi_1(v)\neq 0$, we have $\pi_1\simeq\langle\pi_1(v)\rangle\otimes\pi_1$, that is every non trivial element represented by π_1 is a similarity factor for π_1. This applies in particular to $\lambda=x^2-a$, so that

$$\langle\lambda\rangle\otimes\pi=\langle\lambda\rangle\otimes\pi_1\otimes\langle\langle a_2,\dots,a_n\rangle\rangle=\pi_1\otimes\langle\langle a_2,\dots,a_n\rangle\rangle=\pi.$$

□

Corollary 2.13. *The set of non trivial values represented by a Pfister form is a group.*

Proof. It coincides with the set of similarity factors of π, which is a subgroup of $F^\times$. □

2.4. **K-theory modulo 2 and quadratic forms.** We are now in position to prove

Theorem 2.14 ([Mil])**.** *There is a homomorphism*

$$s_\star:\, k_\star F\to GW_\star F,$$

mapping the symbol $\{a\}$ to the 1-fold Pfister form $\langle\langle a\rangle\rangle$. Moreover, the map $s_\star$ is surjective.

Proof. If such a homomorphism exists, it maps the n-symbol $\{a_1,\dots,a_n\}$ to the n-fold Pfister form $\langle\langle a_1,\dots,a_n\rangle\rangle$. Since I^nF is generated by n-fold Pfister forms, surjectivity is clear and we only have to prove that the map $s_\star$ is well defined. In view of the presentation of $k_\star^MF$ by generators and relations, it is enough to check that the relations (1), (2) and (3) of § 1.3 also hold in the graded Witt ring. Consider $a,b\in F^\times$. If $a\neq 1$, the form

$$\langle\langle a,1-a\rangle\rangle=\langle 1,-a,-(1-a),a(1-a)\rangle$$

is isotropic, hence hyperbolic by Theorem 2.11. So we have $\langle\langle a,1-a\rangle\rangle=0$ in the Witt ring $W(F)$. The other relations do not hold in the Witt ring, but they are valid in $GW_\star F$. Indeed, one may easily check that

$$\langle\langle a\rangle\rangle+\langle\langle b\rangle\rangle=\langle\langle ab\rangle\rangle+\langle\langle a,b\rangle\rangle\in W(F).$$

Since $\langle\langle a,b\rangle\rangle\in I^2F$, we get $\langle\langle a\rangle\rangle+\langle\langle b\rangle\rangle=\langle\langle ab\rangle\rangle\in IF/I^2F$ as required. Similarly, one has $2\langle\langle a\rangle\rangle=\langle\langle a,-1\rangle\rangle\in I^2F$, so that $2\langle\langle a\rangle\rangle=0\in IF/I^2F$. □

In his paper, Milnor proves that s_1 and s_2 are bijective. In fact, using an ad-hoc version of Stiefel-Whitney invariant, he constructs a 'stable' section for s_n, which is a section if $n=1,2$. Using properties of Pfister forms, one can prove the following partial injectivity result, which says that s_n induces a bijection between n-symbols modulo 2 and n-fold Pfister forms modulo $I^{n+1}F$ (see [Kahn, §9.4]):

Proposition 2.15. *Restricted to symbols, the map s_n is injective.*

This result is of course much weaker than the injectivity of s_n. In these notes, we prove an even weaker result, which is enough for the norm quadric proposition below:

Lemma 2.16. *If the Pfister form $\langle\langle a_1,\dots,a_n\rangle\rangle$ belongs to $I^{n+1}F$, then the corresponding symbol $\{a_1,\dots,a_n\}=0\in k_n^MF$.*

Proof. By the Arason-Pfister Haupsatz mentioned above, every anisotropic form with Witt class in $I^{n+1}F$ has dimension at least 2^{n+1}. Hence, if the Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$ belongs to $I^{n+1}F$, then it is hyperbolic. So we want to prove that $\{a_1, \dots, a_n\} = 0 \in k_n^M F$ if the form $\langle\langle a_1, \dots, a_n \rangle\rangle$ is hyperbolic. We proceed by induction on n, and we let $\pi_n = \langle\langle a_1, \dots, a_n \rangle\rangle$. If $n = 1$, and if the Pfister form $\pi_1 = \langle\langle a_1 \rangle\rangle$ is hyperbolic then $a_1 \in F^{\times 2}$, so we get $\{a_1\} = 0 \in k_1^M F$. In general, let us assume that π_n is hyperbolic. Since $\pi_n = \pi_{n-1} - \langle a_n \rangle \otimes \pi_{n-1}$, the hypothesis says that a_n is a similarity factor for π_{n-1}. So by theorem 2.12, a_n is represented by π_{n-1}; hence it decomposes as $a_n = x - a_{n-1}y$ for some x and y represented by π_{n-2}.

If $y = 0$, then $a_n = x$ is represented by π_{n-2}, so the Pfister form

$$\pi_{n-2} \otimes \langle\langle a_n \rangle\rangle = \pi_{n-2} \oplus \langle -a_n \rangle \pi_{n-2}$$

is isotropic, hence hyperbolic by Theorem 2.11. By the induction hypothesis we get $\{a_1, \dots, a_{n-2}, a_n\} = 0 \in k_{n-1}^M F$ which implies the conclusion. If $x = 0$, the same argument shows that $\{a_1, \dots, a_{n-2}, -a_{n-1}a_n\} = 0 \in k_{n-1}^M F$. Multiplying on the right by $\{a_n\}$, and using the fact that $\{-a_n, a_n\} = 0 \in k_2^M F$, we get

$$\{a_1, \dots, a_{n-2}, -a_{n-1}a_n, a_n\} = \{a_1, \dots, a_n\} = 0 \in k_n^M F.$$

In general, one may write $a_n = x - a_{n-1}y = x(1 - a_{n-1}z)$, where $z = y/x$ also is represented by π_{n-2} by corollary 2.13. So we have

$$\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-1}, x\} + \{a_1, \dots, a_{n-1}, 1 - a_{n-1}z\}.$$

The same argument as before shows that $\{a_1, \dots, a_{n-2}, x\}$ and $\{a_1, \dots, a_{n-2}, z\}$ are both trivial in $k_n^M F$. In particular, we have

$$\{a_1, \dots, a_{n-1}\} = \{a_1, \dots, a_{n-1}z\},$$

so that $\{a_1, \dots, a_{n-1}, 1 - a_{n-1}z\} = \{a_1, \dots, a_{n-1}z, 1 - a_{n-1}z\} = 0 \in k_n^M F$. □

Finally, we get the following:

Proposition 2.17 (Norm quadric)**.** *Let $a_1, \dots, a_n$ be non trivial elements of F. The following assertions are equivalent:*

(1) *The quadratic form $\varphi = \langle\langle a_1, \dots, a_{n-1} \rangle\rangle + \langle -a_n \rangle$ is isotropic.*
(2) *The Pfister form $\pi = \langle\langle a_1, \dots, a_n \rangle\rangle$ is isotropic.*
(3) *The Pfister form $\pi = \langle\langle a_1, \dots, a_n \rangle\rangle$ is hyperbolic.*
(4) *The symbol $\{a_1, \dots, a_n\} = 0 \in k_n^M F$.*

Proof. Clearly, (1) implies (2) and (2) implies (3) by 2.11. Assume now that (3) holds. Then, the form φ, which is a subform of π of dimension strictly larger than half the dimension of π has to be isotropic. The equivalence between (3) and (4) is given by lemma 2.16. □

Remark 2.18. Let X_φ be the projective quadric associated to the quadratic form φ. Rephrasing the previous proposition, we get that the symbol $\{a_1, \dots, a_n\}$ vanishes in Milnor's K-theory modulo 2 if and only if X_φ has a rational point. We say that X_φ is a splitting variety for this symbol.

3. Functorial properties of K-theory

3.1. **Restriction map.** It is clear from the definition that K-theory is functorial. That is, for every field extension L/F, the inclusion $F \to L$ induces morphisms $\mathrm{res}_{L/F} : K^M_\star F \to K^M_\star L$, and $k^M_\star F \to k^M_\star L$, mapping a symbol $\{a_1, \dots, a_n\}$ to itself, viewed as an element of $K^M_\star L$ or $k^M_\star L$. We will use the notation $\alpha_L = \mathrm{res}_{L/F}(\alpha)$, for every $\alpha \in K^M_\star F$ or $k^M_\star F$. Via this restriction map, one may endow $K^M_\star L$ with a structure of $K^M_\star F$-module.

3.2. **Residue map.** Assume F has a discrete valuation $v : F^\times \to \mathbb{Z}$. Since v is a group homomorphism, it can be thought of as a morphism

$$K^M_1 F \mapsto K^M_0 \bar{F},$$

where $\bar{F}$ is the residue field. This map extends to a residue homomorphism

$$\partial_v : K_n F \mapsto K^M_{n-1} \bar{F}$$

as we now proceed to show. Let $\mathcal{O}$ be the valuation ring, $\mathcal{O}^\times$ the set of units, and $\pi \in \mathcal{O}$ a prime element, that is $v(\pi) = 1$. For every unit $u \in \mathcal{O}$, we let $\bar{u}$ be its image in the residue field $\bar{F}$.

Proposition 3.1. *For all $n \geq 1$, there is a unique morphism, called the residue map,*

$$\partial_v : K^M_n F \mapsto K^M_{n-1} \bar{F},$$

satisfying

$$\partial_v(\{a, u_2, \dots, u_n\}) = v(a)\{\bar{u}_2, \dots, \bar{u}_n\},$$

for all $a \in F^\times$ and units $u_2, \dots, u_n \in \mathcal{O}^\times$.

Proof. If such a map exists, it satisfies in particular

$$\begin{aligned} &\partial_v(\{\pi, u_2, \dots, u_n\}) = \{\bar{u}_2, \dots, \bar{u}_n\}, \text{ and} \\ &\partial_v\big(\{u_1, \dots, u_n\}\big) = 0, \end{aligned}$$

for all prime elements π and units $u_1, \dots, u_n$. Moreover, since every $a \in F^\times$ can be written uniquely $a = \pi^i u$, where $i = v(a) \in \mathbb{Z}$, for some unit $u \in \mathcal{O}^\times$, one may check using linearity and the identity $\{\pi\}\{\pi\} = \{\pi\}\{-1\}$ that $K^M_n F$ is generated by symbols $\{\pi, u_2, \dots, u_n\}$, and $\{u_1, \dots, u_n\}$ for units $u_1, \dots, u_n \in \mathcal{O}^\times$. Hence, if such a morphism exists, the condition guarantees its uniqueness.

The residue map is constructed in Milnor's paper. The argument presented here is due to Serre. Consider the ring $L(\bar{F}) = K^M_\star \bar{F}[\xi]$, generated by $K^M_\star \bar{F}$ and by an additional element ξ satisfying:

$$\xi^2 = \{-1\}\xi \text{ and } \xi\alpha = -\alpha\xi \text{ for all } \alpha \in K_1 \bar{F}.$$

We let ξ be of degree 1, so that L is graded and satisfies

$$L_n \bar{F} = K_n \bar{F} \oplus \xi K_{n-1} \bar{F}.$$

Let us fix a prime element π, and consider the map

$$d_\pi : K^M_1 F \mapsto L_1(\bar{F}), \quad \{\pi^i u\} \mapsto \{\bar{u}\} + \xi i$$

It clearly induces a morphism $K^M_1 F^{\otimes n} \to L_n(\bar{F})$. Moreover, we have:

Lemma 3.2. *For all $a \in F^\times$, $a \neq 1$, the map $d_\pi^{\otimes 2}$ maps $\{a\} \otimes \{1-a\}$ to zero.*

Proof. The element $a \in F^\times$ can be written $a = \pi^i u$ where $i = v(a) \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$. If $i > 0$, then $1 - a = 1 - \pi^i u$ is a unit with residue $\overline{1-a} = \bar{1}$. Hence $d_\pi(1-a) = \{\bar{1}\} = 0 \in K_1(\overline{F})$, and the result follows in this case. Assume now that $i < 0$. We have $1 - a = 1 - \pi^i u = \pi^i(\pi^{-i} - u)$, where the second factor is a unit, with residue $\overline{\pi^{-i} - u} = -\bar{u}$. Hence we have

$$d_\pi^{\otimes 2}(\{a\} \otimes \{1-a\}) = (\{\bar{u}\} + \xi i)(\{-\bar{u}\} + \xi i).$$

Since $\{\bar{u}, -\bar{u}\} = 0$ and $\xi\{\bar{u}\} + \{\bar{u}\}\xi = 0$, we get

$$d_\pi^{\otimes 2}(\{a\} \otimes \{1-a\}) = (i - i^2)\xi\{-1\} = \xi\{(-1)^{i-i^2}\} = 0 \in L_2(\bar{F}).$$

The remaining case is $i = 0$, that is $a = u$ is a unit. If so, $1 - a$ has a non negative valuation. If $v(1-a) > 0$, the result follows from the first case, exchanging the roles of a and $1 - a$. Finally, if both a and $1 - a$ are units, we have

$$d_\pi^{\otimes 2}(\{a\} \otimes \{1-a\}) = \{\bar{u}\}\{\overline{1-u}\} = \{\bar{u}, 1 - \bar{u}\} = 0 \in K_2^M \bar{F}.$$

□

Therefore, the morphism $d_\pi^{\otimes n}$ factors through $K_n^M F$. Composing with the projection

$$L_n(\bar{F}) = K_n^M \bar{F} \oplus \xi K_{n-1}^M \bar{F} \mapsto K_{n-1}^M \bar{F},$$

we get a well-defined morphism

$$\partial_\pi : K_n^M F \mapsto K_{n-1}^M \bar{F}.$$

Moreover, since $d_\pi(\{\pi^i u_1\}) = \{\bar{u}_1\} + \xi i$, and $d_\pi(\{u_i\}) = \{\bar{u}_i\}$ for all $u_1, \ldots, u_n \in \mathcal{O}^\times$, an easy computation shows $\partial_\pi(\{a, u_2, \ldots, u_n\}) = v(a)\{\bar{u}_2, \ldots, \bar{u}_n\}$, for all $a \in F^\times$. In particular, it follows from this formula that ∂_π does not depend on the choice of π, and can be denoted by ∂_v. □

3.3. **Norm map.** Let L/F be a field extension, of finite degree $[L : F]$. The norm homomorphism $L^\times \to F^\times$ can be viewed as a morphism of additive groups

$$N_{L/F} : K_1^M L \to K_1^M F.$$

Using the residue map that we just defined, and a theorem of Milnor computing Milnor's K-theory of a function field in one variable, one may extend this map to a norm homomorphism defined in any degree

$$N_{L/F} : K_\star^M L \to K_\star^M F.$$

The description of this map, at least for simple field extensions, is rather elementary; as opposed to this, the fact the norm map is canonically defined, that is does not depend on any choice is an important and rather technical result due to Kato. In these notes, we only mention two properties of the norm map and refer the reader to [FV] or [GS, Chap 7] for a detailed exposition.

Theorem 3.3. *For every finite field extension L/F, there exists a norm map*

$$N_{L/F} : K_\star^M L \to K_\star^M F,$$

which is multiplication by $[L : F]$ in degree 0 and the usual norm in degree 1. Moreover, it satisfies the following properties:

(1) *If E/L and L/F are finite field extensions, then we have*

$$N_{E/F} = N_{L/F} \circ N_{E/L}.$$

(2) *Projection formula: for all* $\alpha \in K^M_\star F$ *and* $\beta \in K^M_\star L$*, we have*
$$N_{L/F}(\alpha_L \cdot \beta) = \alpha \cdot N_{L/F}(\beta) \in K^M_\star F.$$

4. Galois cohomology

This section, included for the reader's convenience, is purely expository.

4.1. **Definition.** Consider the absolute Galois group $\Gamma_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ of the field F. A *Galois module over* F is a discrete Γ_F-module, that is an abelian group M, endowed with a continuous action of the profinite group Γ_F.

Example 4.1. (1) Every abelian group, endowed with the trivial Γ_F action, is a Galois-module. For instance, $\mathbb{Z}/2\mathbb{Z}$ is a Galois-module.
(2) Endowed with its natural Γ_F-action, the multiplicative group $F_{\mathrm{sep}}^\times$ is a Galois-module.

We use the following notation for the cohomology groups of Γ_F with coefficients in M:
$$H^n(F, M) = H^n(\Gamma_F, M).$$
In degree 0, it consists of invariant elements $H^0(F, M) = M^{\Gamma_F}$. In particular, we have $H^0(F, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H^0(F, F_{\mathrm{sep}}^\times) = F^\times$.

In degree 1, the group $H^1(F, M)$ can be explicitly described as follows. A 1-cocycle is a continuous map $\Gamma_F \to M$, $\gamma \mapsto a_\gamma$ satisfying $a_{\gamma\tau} = a_\gamma + \gamma(a_\tau)$. Two cocycles a and b are cohomologous if they differ by a coboundary, that is $a_\gamma = b_\gamma + \gamma(m) - m$ for some $m \in M$. The group $H^1(F, M)$ is the group of equivalence classes of 1-cocycles for this relation. In particular, if Γ_F acts trivially on M, then $H^1(F, M)$ is the group of continuous homomorphisms $\Gamma_F \to M$.

4.2. **Long exact sequence.** Let M, M' and M'' be Galois modules over F. We assume that the exact sequence
$$0 \to M' \to M \to M'' \to 0$$
is an exact sequence of Galois-modules, that is the maps are morphisms of Γ_F-groups. If so, this sequence induces an infinite long exact sequence of cohomology groups
$$0 \to H^0(F, M') \to H^0(F, M) \to H^0(F, M'') \to H^1(F, M') \to H^1(F, M) \to \dots$$

4.3. **Restriction and corestriction.** Let M be a Galois module over F. For every field extension L/F, we can choose separable closures so that $F_{\mathrm{sep}} \subset L_{\mathrm{sep}}$. Hence, restricting automorphisms, we get a continuous morphism $\Gamma_L \to \Gamma_F$, and M also is a Galois module over L. Moreover, we have restriction morphisms
$$\mathrm{res}_{L/F} : H^n(F, M) \to H^n(L, M).$$
In degree 0, it coincides with the natural map $M^{\Gamma_F} \to M^{\Gamma_L}$. We will frequently use the notation $\alpha_L = \mathrm{res}_{L/F}(\alpha)$, for $\alpha \in H^n(F, M)$.

Assume now that L/F is finite separable, so that Γ_L is an open subgroup of finite index in Γ_F. Every Galois module M over F also is a Galois module over L, and there are natural corestriction morphisms
$$N_{L/F} : H^n(L, M) \to H^n(F, M).$$
In degree 0, it is given by $m \in M^{\Gamma_L} \mapsto \Sigma\gamma(m)$, where γ runs over a complete set of representatives of the quotient-set Γ_F/Γ_L.

It is clear in degree 0, and it is true in general that the composition $N_{L/F} \circ \mathrm{res}_{L/F}$ is multiplication by the degree $[L:F]$ of the extension L/F.

4.4. **Cup-products and projection formula.** Let M and N be two Galois-modules over F. The tensor product $M \otimes_{\mathbb{Z}} N$, endowed with the diagonal action of Γ_F, also is a Galois module. Moreover, there is a bilinear map, called the cup-product:

$$H^p(F, M) \times H^q(F, N) \to H^{p+q}(M \otimes_{\mathbb{Z}} N),\ (\alpha, \beta) \mapsto \alpha \cdot \beta$$

If $p = q = 0$, it coincides with the natural map

$$M^{\Gamma_F} \times N^{\Gamma_F} \to (M \otimes N)^{\Gamma_F}.$$

The cup product is associative, and graded commutative. Moreover, the following projection formula holds:

$$N_{L/F}(\alpha_L \cdot \beta) = \alpha \cdot N_{L/F}(\beta) \text{ for all } \alpha \in H^p(F, M) \text{ and } \beta \in H^q(L, N).$$

5. K-theory and Galois cohomology

5.1. **Galois cohomology and Hilbert theorem 90.** Originally, Hilbert 90 is the following statement:

Theorem 5.1 (Hilbert)**.** *Let L/F be a finite cyclic Galois extension, and let θ be a generator of its Galois group. For every element $a \in L^\times$ such that $N_{L/F}(a) = 1$, there exists $b \in L^\times$ such that $a = \frac{b}{\theta(b)}$.*

Proof. Let n be the degree of L/F, that is the order of θ. The norm map is given by

$$N_{L/F}(a) = \prod_{i=0}^{n-1} \theta^i(a).$$

Therefore, the hypothesis on a gives $a\,\theta(a) \dots \theta^{n-1}(a) = 1$, and it follows that for every $\lambda \in F$, the element

$$b = \lambda + a\theta(\lambda) + a\theta(a)\theta^2(\lambda) + \cdots + a\theta(a) \dots \theta^{n-2}(a)\theta^{n-1}(\lambda)$$

satisfies $a\theta(b) = b$. The theorem now follows from Artin's theorem on characters, which guarantees that $b \neq 0$ for a suitable choice of λ. □

This classical result has a very nice interpretation in Galois cohomology, also known as Hilbert 90, as we now proceed to show:

Theorem 5.2. *The Galois cohomology group $H^1(F, F_{\mathrm{sep}}^\times)$ is trivial.*

Remark 5.3. More generally, one may prove that $H^1(F, \mathrm{GL}_1(A)) = 0$ for every separable and associative F-algebra A (see for instance [KMRT]).

Proof. Since $H^1(F, F_{\mathrm{sep}}^\times)$ is the direct limit of $H^1(\mathrm{Gal}(L/F), L^\times)$ for all finite Galois extensions L/F, it is enough to prove that $H^1(\mathrm{Gal}(L/F), L^\times) = 0$ (see [Se, Chap. X, §3]). Consider a 1-cocycle $a : \mathrm{Gal}(L/F) \to L^\times$. It satisfies $a_{\gamma\sigma} = a_\gamma\, \gamma(a_\sigma)$ for all $\gamma, \sigma \in \mathrm{Gal}(L/F)$. Again by Artin's theorem, there exist $\lambda \in L^\star$ such that

$$b = \sum_{\sigma \in G} a_\sigma \sigma(\lambda) \neq 0.$$

For every $\gamma \in \mathrm{Gal}(L/F)$, we have

$$\gamma(b) = \sum_{\sigma\in G} \gamma(a_\sigma)\,(\gamma\sigma)(\lambda) = \sum_{\sigma\in G} a_\gamma^{-1} a_{\gamma\sigma}\,(\gamma\sigma)(\lambda) = a_\gamma^{-1} b.$$

Hence the cocycle a is a coboundary $a_\gamma = \frac{b}{\gamma(b)}$, and its cohomology class is trivial. □

Remark 5.4. The link between the two Hilbert 90 theorems mentioned here is pretty obvious from the proof. It can be made even more explicit as follows. Assume the extension L/F is cyclic of order n, and pick a generator θ of $\mathrm{Gal}(L/F)$. A 1-cocycle $a : \mathrm{Gal}(L/F) \mapsto L^\times$ is uniquely determined by a_θ, which is a norm 1 element. Indeed, the cocycle relation shows that $a_{\theta^i} = a_\theta \theta(a_\theta)\dots\theta^{i-1}(a_\theta)$. In particular $a_1 = a_{\theta^n} = N_{L/F}(a_\theta) = 1$. One may then easily check that if $a_\theta = \frac{b}{\theta(b)}$, then the cocycle a is a coboundary.

5.2. **The norm residue homomorphism.** Recall that the base field F has characteristic different from 2. We are now interested in the Galois cohomology groups of F with coefficients in the Galois module $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$, with trivial action of Γ_F, for which we use the following notation:

$$H^n(F) = H^n(F, \mathbb{Z}/2\mathbb{Z}).$$

In particular, $H^0(F) = \mathbb{Z}/2\mathbb{Z}$. The map $F_{\mathrm{sep}}^\times \to F_{\mathrm{sep}}^\times$, $x \mapsto x^2$ has kernel $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$. So it induces an exact sequence of Galois modules

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow F_{\mathrm{sep}}^\times \xrightarrow{2} F_{\mathrm{sep}}^\times \longrightarrow 1$$

The long exact sequence associated in Galois cohomology starts as follows

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow F^\times \xrightarrow{2} F^\times \longrightarrow H^1(F) \longrightarrow H^1(F, F_{\mathrm{sep}}^\times).$$

By Hilbert 90, $H^1(F, F_{\mathrm{sep}}^\times) = 0$, so that the coboundary map $F^\times \to H^1(F)$ is surjective. Therefore, it induces an isomorphism

$$F^\times/F^{\times 2} \to H^1(F).$$

Identifying $F^\times/F^{\times 2}$ with $k_1^M F$, we get an isomorphism

$$h^1 : k_1^M F = F^\times/F^{\times 2} \to H^1(F).$$

For all $a \in F^\times$, we denote by

$$(a) = h^1(\{a\}) \in H^1(F)$$

the corresponding element.

The element (a) can be explicitly described as follows: let $\delta \in F_{\mathrm{sep}}^\times$ be one of the two square roots of a. The map $\Gamma_F \to \mathbb{Z}/2\mathbb{Z}$, $\gamma \mapsto \varepsilon_\gamma$ defined by $(-1)^{\varepsilon_\gamma} = \frac{\gamma(\delta)}{\delta}$ is a 1-cocycle. Its cohomology class precisely is (a). In particular, this cocycle is trivial if $a \in F^{\times 2}$, since we can choose $\delta \in F$, so that $\gamma(\delta) = \delta$ for all $\gamma \in \Gamma_F$.

Since $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ is canonically isomorphic to $\mathbb{Z}/2\mathbb{Z}$, the cup product induces a map $H^i(F) \times H^j(F) \to H^{i+j}(F)$. This map defines a product on $H^\star(F) = \oplus_{n\geq 0} H^n(F)$, which now is a graded ring, and even a graded $\mathbb{Z}/2\mathbb{Z}$-algebra. We use the notation

$$(a_1, \dots, a_n) = (a_1)\cdot(a_2)\cdot\dots\cdot(a_n) \in H^n(F);$$

such an element is called a symbol. With this in hand, we can define the norm residue map (also called Galois symbol) as follows:

Theorem 5.5 (Bass-Tate)**.** *The isomorphism* $h^1 : k_1^M F \to H^1(F)$ *extends uniquely to a morphism of graded rings*

$$h : k_\star^M F \to H^\star(F),$$

called the norm residue homomorphism.

Proof. Clearly, the isomorphism h^1 induces a map $k_1^M F^{\otimes n} \to H^n(F)$, satisfying $\{a_1, \ldots, a_n\} \mapsto (a_1, \ldots, a_n)$. As before, we only have to check that the relations (1), (2) and (3) of section 1.3 hold in Galois cohomology. Since h_1 is an isomorphism, we clearly have $(ab) = (a) + (b)$ and $2(a) = (a^2) = 0$ in $H^1(F)$, so that relations (1) and (3) hold. It only remains to prove that $(a) \cdot (1 - a) = 0$ for all $a \in F^\times$. If $a \in F^{\times 2}$, this is clear. Otherwise, let $\delta \in F_{\text{sep}}^\times$ be a square root of a and $L = F(\delta)$. We have $\delta^2 = a$, and $N_{L/F}(1 - \delta) = (1 - \delta)(1 + \delta) = 1 - a$. Therefore, by the projection formula,

$$(a) \cdot (1 - a) = (a) \cdot N_{L/F}(1 - \delta) = N_{L/F}((a)_L \cdot (1 - \delta)) = N_{L/F}((\delta^2) \cdot (1 - \delta)) = 0.$$

□

So we have constructed the norm residue map $k_\star^M F \to H^\star(F)$, and also proved, using Hilbert 90 theorem, that it is an isomorphism in degree 1. This is the starting point of Voevodsky's proof, which is by induction on n.

References

[FV] Fesenko, I. B. and Vostokov, S. V. Local fields and their extensions. Second edition. Translations of Mathematical Monographs, 121. American Mathematical Society, Providence, RI, 2002.

[GS] Gille, P. and Szamuely, T. Central simple algebras and Galois cohomology. Cambridge Studies in Advanced Mathematics, 101. Cambridge University Press, Cambridge, 2006.

[Kahn] Kahn, B . Formes quadratiques sur un corps. (French) [Quadratic forms over fields] Cours Spécialisés [Specialized Courses], 15. Société Mathématique de France, Paris, 2008.

[KS] Kahn, Bruno; Sujatha, R. Motivic cohomology and unramified cohomology of quadrics. J. Eur. Math. Soc. (JEMS) 2 (2000), no. 2, 145–177.

[KMRT] Knus, M.-A., Merkurjev A., Rost, M. and Tignol, J.-P. The book of involutions. American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI, 1998.

[Lam] Lam, T. Y. Introduction to quadratic forms over fields. Graduate Studies in Mathematics, 67. American Mathematical Society, Providence, RI, 2005.

[Me] Merkurjev, A. S. On the norm residue symbol of degree 2. (Russian) Dokl. Akad. Nauk SSSR 261 (1981), no. 3, 542–547; translation in Soviet Math. Dokl. 24 (1981),no. 3, 546?551.

[MS] Merkurjev, A. S. and Suslin, A. A. The norm residue homomorphism of degree three. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 2, 339–356; translation in Math. USSR-Izv. 36 (1991), no. 2, 349–367.

[Mil] Milnor, John. Algebraic K-theory and quadratic forms. Invent. Math. 9 1969/1970 318–344.

[Mo1] Morel, Fabien. Milnor's conjecture on quadratic forms and mod 2 motivic complexes. Rend. Sem. Mat. Univ. Padova 114 (2005), 63–101 (2006).

[Mo2] Morel, Fabien. Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques. (French) [Adams spectral sequence and cohomological invariants of quadratic forms] C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), no. 11, 963–968.

[OVV] Orlov, D.; Vishik, A.; Voevodsky, V. An exact sequence for $K_\star^M/2$ with applications to quadratic forms. Ann. of Math. (2) 165 (2007), no. 1, 1–13.

[R] Rost M. Hilbert's theorem 90 for K_3^M for degree-two extensions, prépublication,Univ. Regensburg, 1986.

[Sch] Scharlau, W. Quadratic and Hermitian forms. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 270. Springer-Verlag, Berlin, 1985.

[Se] Serre, J.-P. Corps locaux. Publications de l'Université de Nancago, No. VIII. Hermann, Paris, 1968.

[Voe] Voevodsky, Vladimir. Motivic cohomology with Z/2-coefficients. Publ. Math. Inst. Hautes Études Sci. No. 98 (2003), 59–104.

[W] Witt E. Theorie des quadratischen Formen in beliebigen Korpern. J. Reine Angew. Math. 176 (1937), p. 31-44.

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