Quantum algebras with straightening laws and applications.

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Introduction.

The content of this talk is essentially based on a joint work with Tom Lenagan, see [LR].

The present work is at the intersection of three main areas of algebra:
(i) classical invariant theory (as a motivation);
(ii) noncommutative algebraic geometry (as a point of view);
(iii) quantum groups (as a source of examples).

The aim is to study quantum analogues of classical rings of invariants from a geometric perspective. More precisely, we exhibit quantum algebras which are natural analogues of classical rings of invariants (and which turn out to be, in fact, rings of coinvariants) and study their regularity properties. By regularity properties, we mean homological properties such as the Cohen-Macaulay property, the Gorenstein property, *etc.* In the present talk, we will concentrate on the Cohen-Macaulay property.

The first part is a short account on the Cohen-Macaulay property in the noncommutative setting. In the second part we show how the theory of quantum groups gives rise to a natural context of invariant theory in the noncommutative setting. We then define, in the third part, the notion of *quantum graded algebras with a straightening law* and then explain why it is the convenient tool to study the regularity properties of large classes of quantum analogues of rings of invariants.

Throughout, we work over a base field $\Bbbk$.

1 The Cohen-Macaulay condition.

We first recall the Cohen-Macaulay property in the noncommutative setting, following works by Jørgensen, Van den Bergh, Yekutieli and J.J. Zhang.

The setup. We work in the graded context. Hence, $A = \oplus_{i\geq 0} A_i$ is an $\mathbb{N}$-graded, noetherian $\Bbbk$-algebra, connected in the sense that $A_0 = \Bbbk$. For $n \in \mathbb{N}$, we put $A_{\geq n} = \oplus_{i\geq n} A_i$ and $\mathfrak{m} = A_{\geq 1}$.

We let $\mathsf{GrMod}(A)$ be the category of graded left A-modules and homogeneous morphisms of degree zero. If $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a graded left A-module and if $n \in \mathbb{N}$, the gaded module $M(n)$ is the n-th shift of M. This means that $M(n) = \bigoplus_{i\in\mathbb{Z}} M(n)_i$, where $M(n)_i = M_{n+i}$ for $i \in \mathbb{Z}$. Let $\underline{\mathrm{Hom}}_A(M,N) := \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{GrMod}(A)}(M, N(m))$. The associated Ext are

$$\underline{\mathrm{Ext}}^i_A(M,N) \cong \bigoplus_{m\in\mathbb{Z}} \mathrm{Ext}^i_{\mathsf{GrMod}(A)}(M, N(m)).$$

Local cohomology functors. If M is an object of $\mathsf{GrMod}(A)$ its torsion submodule is defined as

$$\Gamma_{\mathfrak{m}}(M) := \{m \in M \mid \exists n \in \mathbb{N},\ A_{\geq n} m = 0\}.$$

This gives rise to a left exact functor from $\mathsf{GrMod}(A)$ to itself. The associated right derived functors are denoted $H^i_{\mathfrak{m}}$ and, for a graded A-module M, the groups $H^i_{\mathfrak{m}}(M)$, $i \in \mathbb{N}$, are the so-called local cohomology groups of M. It is not difficult to see that:

$$H^i_{\mathfrak{m}}(M) \cong \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, M).$$

The AS-Cohen-Macaulay condition. The following notion is inspired by works of Artin and Schelter. Hence the "AS" in its name.

Definition. The algebra A is *AS-Cohen-Macaulay* if there exits $n \in \mathbb{N}$ such that

$$\forall i \in \mathbb{N},\ i \neq n, H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{m}^\circ}(A) = 0.$$

This means that we ask for the existence of an integer n such that the i-th local cohomology group of the *left* A-module A and of the right A-module A are zero unless $i = n$. (Following standard uses, A° is the opposite ring of A and the right A-module A is seen as a left module over A°.)

The Grothendieck vanishing theorem. The following theorem is due to Yekutieli and Zhang (see [YZ]). It is a difficult and extremely useful result which provides an analogue of the famous *Grothendieck vanishing theorem* in the noncommutative setting. It provides numerical invariants allowing to check whether or not a given graded algebra is AS-Cohen-Macaulay. One of these numerical invariants is the depth, whose definition we first recall.

Definition. *If M is a finitely generated left A-module, the depth of M is defined as*

$$\mathrm{depth}_A M = \inf\{i \in \mathbb{N} \mid \underline{\mathrm{Ext}}^i_A(\Bbbk, M) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

Theorem. [Yekutieli and Zhang, 1999]
Assume A has enough normal elements. Then, for any non-zero finitely generated graded left A-module,
(i) $\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\} \neq \emptyset$ *;*
(ii) $\inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\} = \mathrm{depth}_A M$ *;*
(iii) $\sup\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\} = \mathrm{GKdim}_A M < +\infty$.

Here, the assumption A *has enough normal elements* means that any factor of A by a homogeneous prime ideal different from $\mathfrak{m}$ contains a nonzero homogeneous normal element of positive degree. (Recall that an element x of a ring B is said to be normal if the right and left ideals of B generated by x coincide.)

Remark. If the algebra A above is commutative, then it is AS-Cohen-Macaulay if and only if it is Cohen-Macaulay in the usual sense of commutative algebra. For details on this equivalence, see [LR; Remark 2.1.10].

The Hochster-Roberts theorem. The following deep and fundamental result states that, in the commutative setting, rings of invariants are Cohen-Macaulay. For an extensive discussion of this theorem, see [BH; section 6.5].

Theorem. (Hochster and Roberts, 1974) *Let $\Bbbk$ be an algebraically closed field and G be a linearly reductive group over $\Bbbk$ acting linearly on a polynomial ring $R = \Bbbk[X_1, \dots, X_n]$. Then the ring R^G of invariants is Cohen-Macaulay.*

A systematic way to produce examples of such contexts is to start with a linearly reductive group G and a finite dimensional linear representation V of G. Then, G acts on the symmetric algebra $S(V)$ which is a polynomial ring.

Example. One typical example where the Hochster-Roberts theorem applies is the following. Let

$$\gamma \,:\, SL_m \times M_{m,n} \longrightarrow M_{m,n}$$

be the natural action of the group SL_m on the affine space of $m \times n$ matrices by left multiplication. It induces an action of SL_m on the polynomial ring $\mathcal{O}(M_{m,n})$ of regular functions on $M_{m,n}$. As is well known, the ring of invariants for this action identifies with the coordinate ring $\mathcal{O}(G_{m,n})$ on the grassmannian variety of m-dimensional subspaces of $\Bbbk^n$. Hence $\mathcal{O}(G_{m,n})$ is Cohen-Macaulay.

Other interesting (and related) Cohen-Macaulay rings are the coordinate rings of Schubert varieties in the grassmannian. All these rings have natural analogues in the quantum setting as we will see later.

2 Quantum groups and noncommutative invariant theory.

The theory of quantum groups and algebras provides noncommutative deformations of classical groups and of their natural spaces of representation. It also provides noncommutative deformations of the natural actions of the formers on the latters. To be more precise, these deformations are defined at the level of coordinate rings and, in this process, the groups and spaces are not available anymore. However, the geometric information is still present in the deformed coordinate rings. We now give a simple example of this situation.

The quantum deformation of SL_m is the quantum group $\mathcal{O}_q(SL_m)$ (it is a Hopf algebra) while the quantum deformation of $M_{m,n}$ is the quantum algebra $\mathcal{O}_q(M_{m,n})$. The action γ of SL_m on $M_{m,n}$ can also be quantized under the form of a coaction

$$\gamma_q^* \,:\, \mathcal{O}_q(M_{m,n}) \longrightarrow \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{m,n}).$$

(Here the nonzero scalar q is the deformation parameter.) Now, $\mathcal{O}_q(SL_m)$, $\mathcal{O}_q(M_{m,n})$ and γ_q^* are deformations of SL_m, $M_{m,n}$ and γ in the (naive) sense that, at $q = 1$, one recovers the classical coordinate rings $\mathcal{O}(SL_m)$ and $\mathcal{O}_q(M_{m,n})$ as well as the comorphism γ^* of the morphism γ of algebraic varieties.

Quantum groups and algebras (of type A). Let us now give more precise definitions of $\mathcal{O}_q(SL_m)$, $\mathcal{O}_q(M_{m,n})$, *etc.*

The algebra of $m \times n$ quantum matrices, denoted $\mathcal{O}_q(M_{m,n})$, is defined by generators and relations as follows.

Generators | Relations.

$$\mathbf{X} = \begin{pmatrix} & \vdots & & \vdots & \\ \dots & X_{ij} & \dots & X_{il} & \dots \\ & \vdots & & \vdots & \\ \dots & X_{kj} & \dots & X_{kl} & \dots \\ & \vdots & & \vdots & \end{pmatrix} \qquad \begin{array}{c} X_{ij}X_{il} = qX_{il}X_{ij}, \\ X_{ij}X_{kj} = qX_{kj}X_{ij}, \\ X_{ij}X_{kl} = X_{kl}X_{ij}, \\ X_{ij}X_{kl} - X_{kl}X_{ij} = (q-q^{-1})X_{il}X_{kj}. \end{array}$$

For $1 \leq t \leq \min\{m,n\}$, $I = \{i_1 < \ldots < i_t\} \subseteq \{1,\ldots,m\}$ and $J = \{j_1 < \ldots < j_t\} \subseteq \{1,\ldots,n\}$, one defines the $t \times t$ *quantum minor*:

$$[I|J] = \sum_{\sigma \in \mathfrak{S}_t} (-q)^{\ell(\sigma)} X_{i_{\sigma(1)}j_1} \ldots X_{i_{\sigma(t)}j_t}.$$

In the case where $m = n$, $\mathcal{O}_q(M_n) := \mathcal{O}_q(M_{m,n})$ is a bialgebra (quantum monoid) whose unique $n \times n$ quantum minor, $\Delta_q := [1,\ldots,n|1,\ldots,n]$ is central.

Hence, we can define the linear quantum groups

$$\mathcal{O}_q(GL_n) := \mathcal{O}_q(M_n)[\Delta_q^{-1}] \quad \text{and} \quad \mathcal{O}_q(SL_n) := \mathcal{O}_q(M_n)/\langle \Delta_q - 1\rangle.$$

These are Hopf algebras.

The quantization of the standard action of SL_m on $M_{m,n}$ is then given by the algebra morphism

$$\begin{array}{ccccc} \mathcal{O}_q(M_{m,n}) & \stackrel{\lambda_q^*}{\to} & \mathcal{O}_q(GL_m) \otimes \mathcal{O}_q(M_{m,n}) & \stackrel{\pi\otimes\mathrm{id}}{\to} & \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{m,n}) \\ X_{ij} & \mapsto & \sum_{k=1}^m X_{ik} \otimes X_{kj} & & \end{array}$$

which endows $\mathcal{O}_q(M_{m,n})$ with the structure of a left $\mathcal{O}_q(SL_m)$-module algebra. (Here, π is the canonical projection $\pi : \mathcal{O}_q(GL_m) \longrightarrow \mathcal{O}_q(SL_m)$.)

Coinvariants. In the context of Hopf algebras coactions on rings, there is a notion that generalizes the notion of invariants. It is defined as follows. Let H be a Hopf algebra and A an algebra on which H coacts by a morphism of algebras

$$\gamma^* : A \longrightarrow H \otimes A.$$

The subring of coinvariants of A is defined by

$$A^{\mathrm{co}-H} := \{a \in A \,|\, \gamma^*(a) = 1 \otimes a\}.$$

Example. One can then identify the subring of $\mathcal{O}_q(M_{m,n})$ of coinvariants for the coaction γ_q^* of $\mathcal{O}_q(SL_m)$ mentioned above. It is the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by the $m \times m$ quantum minors; it is denoted $\mathcal{O}_q(G_{m,n})$. This result is far from being obvious, however we won't make any further comment here (for details see [GLR] and [KLR]). For this reason (and for others), the ring $\mathcal{O}_q(G_{m,n})$ is called the quantum grassmannian (of m-dimensional spaces in $\mathbb{k}^n$).

Notice in addition that there exists a family of quotients of $\mathcal{O}_q(G_{m,n})$ which are quantum analogues of coordinate rings of Schubert varieties in the grassmannian. These rings will be defined later.

3 Quantum graded algebras with a straightening law.

The above developments lead to a very natural question: are these quantum deformations of rings of invariants (quantum grassmannians and their associated quantum Schubert varieties) AS-Cohen-Macaulay? In fact the same question can be asked for many other classes of rings that we won't discuss here.

One can even ask a more general question: is there any quantum analogue of the Hochster-Roberts theorem?

As to our knowledge, no such analogue is known at the moment. In fact, even a reasonable conjecture seems difficult to formulate due to the lack of a good notion of symmetric algebra in the quantum setting.

This is why we are lead to use a different approach in order to prove the AS-Cohen-Macaulay property for rings of coinvariants arising as deformations of classical rings of invariants. This approach is based on the notion of *quantum graded algebra with a straightening law*; it is inspired by works of De Concini, Eisenbud and Procesi in the commutative setting. The aim of this section is to describe this approach.

The definition. Quantum graded algebras with a straightening law form a large class of algebras endowed with nice combinatorial properties and which includes all the rings of coinvariants we are interested in. Here is the definition.

Definition. *Let A be an $\mathbb{N}$-graded $\mathbb{k}$-algebra. Let Π be an ordered finite subset of homogeneous elements of A of positive degree, which generates A as a $\mathbb{k}$-algebra. We say that A is a quantum graded algebra with a straightening law on Π if the following conditions are satisfied.*
(1) The set of standard monomials on Π is linearly independent (a standard monomial on Π is an ordered product of elements of Π).
(2) For uncomparable elements $\alpha, \beta \in \Pi$, $\alpha\beta$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\mathrm{st}} \mu$ and $\lambda <_{\mathrm{st}} \alpha, \beta$.
(3) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha\beta} \in \mathbb{k}^*$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\mathrm{st}} \mu$ and $\lambda <_{\mathrm{st}} \alpha, \beta$.

The relations required by condition (2) above are called *straightening relations* while relations required by condition (3) will be called *commutation relations*.

The notion of Π-ideal will be very useful later. What we call a Π-ideal is a subset of Π with the property that, if it contains an element α, then it must contain any element of Π smaller than α.

Example: quantum deformations of grassmannians and their Schubert varieties. Recall that the quantum grassmannian is the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by the $m \times m$ quantum minors. Such a quantum minor is labelled by an m-elements subset I of $\{1, \dots, n\}$ (the set of indices of the involved columns) and is then denoted $[I]$. Then $m \times m$ quantum minors are ordered as follows: if $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_m\}$, then

$$[I] \leq [J] \qquad \text{iff} \qquad i_\ell \leq j_\ell, \quad \forall \ell \in \{1, \dots, m\}.$$

We can then define quantum Schubert varieties as follows. To any element $\gamma \in \Pi$, we associate the Π-ideal $\Omega_\gamma = \{\pi \in \Pi \,|\, \pi \not\geq \gamma\}$ and we put

$$\mathcal{O}_q(G_{m,n})_\gamma := \mathcal{O}_q(G_{m,n})/\langle \Omega_\gamma \rangle.$$

We define quantum Schubert varieties as the quotient rings $\mathcal{O}_q(G_{m,n})_\gamma$, $\gamma \in \Pi$.

The following theorem is proved in [LR]. Its proof is rather long and difficult. Among other arguments, it uses a big amount of knowledge from the representation theory of the quantum enveloping algebra $U_q(\mathfrak{sl}_n)$.

Theorem.
(i) The quantum grassmannian $\mathcal{O}_q(G_{m,n})$ is a quantum graded A.S.L. on the set of $m \times m$ quantum minors ordered as above.
(ii) Quantum Schubert varieties are quantum graded A.S.L.

Notice that (ii) follows from (i) by general properties of quantum graded A.S.L. (see property 5 below).

Let us look more closely at an example in low dimension: $\mathcal{O}_q(G_{3,6})$. The generic matrix of $\mathcal{O}_q(M_{3,6})$ is

$$\mathbf{X} = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \end{pmatrix}.$$

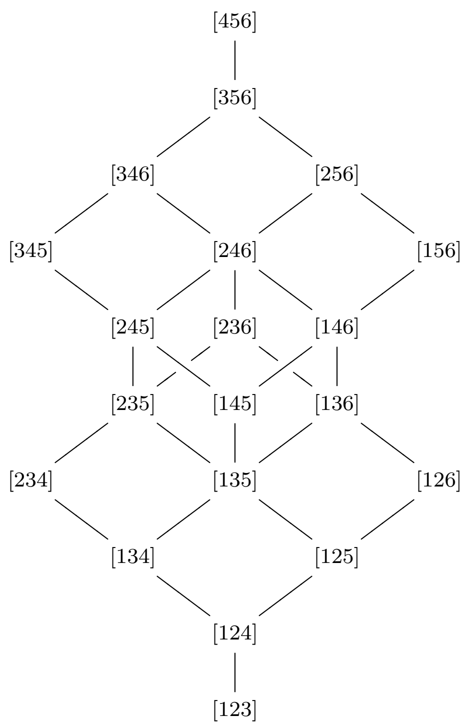
The canonical generators of $\mathcal{O}_q(G_{3,6})$ are the twenty 3×3 quantum minors of $\mathbf{X}$. We order them as follows:
Here is now an example of straightening relation:

$$[156][345] = q^{-1}[145][356] - q^{-2}[135][456];$$

and an example of commutation relation:

$$[125][346] - q^3[346][125] = q^{-1}(q - q^{-1})[123][456] - (q - q^{-1})[124][356].$$

Main properties of quantum graded A.S.L. If A is a quantum graded A.S.L. on the poset Π, the following properties hold.
1. The set of standard monomials on Π is a basis of A.
2. The Gelfand-Kirillov dimension of A is the maximum cardinality of a totally ordered subset of Π.



3. Any element $\alpha \in \Pi$ is normal modulo the ideal generated by the elements of Π smaller than α.
4. A is noetherian.
5. For any Π-ideal Ω, the quotient ring $\Pi/\langle\Omega\rangle$ is a quantum graded algebra on the set $\Pi \setminus \Omega$.

Quantum graded A.S.L. and the AS-Cohen-Macaulay property. The relevance of the class of quantum graded A.S.L. is shown by the following theorem.

Theorem. *If A is a quantum graded A.S.L. on Π and if Π is a distributive lattice, then A is AS-Cohen-Macaulay.*

Roughly speaking, the proof of this theorem is based on the following lemma, by means of an induction on the cardinality of the underlying poset.

Lemma. *Let A be a connected $\mathbb{N}$-graded noetherian $\Bbbk$-algebra with enough normal elements.*
(1) Suppose A has polynomial growth. Let $x \in A$, normal, homogeneous of positive degree and regular. Then A is AS-Cohen-Macaulay if and only if $A/\langle x\rangle$ is.
(2) Let I and J be homogeneous ideals of A such that
(i) $\mathrm{GKdim}_A A/I = \mathrm{GKdim}_A A/J = \mathrm{GKdim}_A A$,
(ii) $\mathrm{GKdim}_A(A/I+J) = \mathrm{GKdim}_A A - 1$,
(iii) $I \cap J = (0)$,
(iv) A/I and A/J are AS-Cohen-Macaulay.
Then, A is AS-Cohen-Macaulay if and only if $A/(I+J)$ is.

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