NdT 12/12/20

Tensor product and Weights

Set up: • $\mathbb{k}$ is alg. closed of char. 0

• $(\mathfrak{g}, \mathfrak{h})$ is a pair consisting of a f.d. s.s. Lie alg. $\mathfrak{g}$ and a maximal toral subalgebra $\mathfrak{h}$

• Recall from the lecture notes the notion of weight space of a representation of $\mathfrak{g}$ (cf. V.1).

• Let V and W be representations of $\mathfrak{g}$ and let $\Pi(V)$ and $\Pi(W)$ be their resp. sets of weights. Supp. that V and W are the (direct) sum of their weight spaces:

$$V = \bigoplus_{\lambda \in \Pi(V)} V_\lambda \quad \text{and} \quad W = \bigoplus_{\mu \in \Pi(W)} W_\mu . \qquad (*)$$

1. Show that, $\forall v \in V_\lambda$, $\forall w \in W_\mu$, $\lambda, \mu \in \mathfrak{h}^*$ then $v \otimes w \in (V \otimes W)_{\lambda+\mu}$.
2. Let $\mathcal{B}$ (resp. $\mathcal{C}$) be a basis of V (resp. W) adapted to the decompositions $(*)$. Using $\mathcal{B}$ and $\mathcal{C}$, show that $V \otimes W$ is the sum of its weight spaces and describe the set of weights of $V \otimes W$ and the corresponding weight spaces.
3. Let $\lambda, \mu \in \Lambda^+$. (Here, Λ^+ is the set of dominant weights relative to the choice of a base of the root syst. attached to the pair $(\mathfrak{g}, \mathfrak{h})$.)

Put $V = V(\lambda)$ and $W = V(\mu)$.

3.1. Show that $(V \otimes W)_{\lambda+\mu}$ is a one dimensional subspace of $V \otimes W$.

3.2 Show that the subrepresentation of $V \otimes W$ generated by $(V \otimes W)_{\lambda+\mu}$ is isomorphic (as a rep.) to $V(\lambda+\mu)$!

Solution

1. By definition of the tensor product of representations (cf. Ex. I.2.6), $\forall x \in \mathfrak{g}$, $\forall v \in V$, $\forall w \in W$, we have:

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$$

Now, if λ, μ are elts of $\mathfrak{h}^*$, $v \in V_\lambda$, $w \in V_\mu$ and $x \in \mathfrak{h}$, we have:

$$\begin{aligned} x \cdot (v \otimes w) &= \lambda(x)\, v \otimes w + v \otimes \mu(x) w \\ &= (\lambda(x) + \mu(x))\, v \otimes w \\ &= (\lambda + \mu)(x)\, v \otimes w. \end{aligned}$$

Hence $v \otimes w \in (V \otimes W)_{\lambda + \mu}$.

2. Let I, J be sets and suppose $\mathcal{B} = \{v_i, i \in I\}$ and $\mathcal{C} = \{w_j, j \in J\}$ are bases of V and W resp. We know that $\{v_i \otimes w_j, (i,j) \in I \times J\}$ is a basis of $V \otimes W$. Of course, we may choose $\mathcal{B}$ adapted to the dec. $V = \bigoplus_{\lambda \in \Pi(\lambda)} V_\lambda$. This means that, $\forall i \in I$, v_i is a weight vector, whose weight we denote λ_i. (Beware that the λ_i, $i \in I$, need not be pairwise distinct.). We do the same for $\mathcal{C}$, choosing the w_j, $j \in J$, of weight μ_j. Then, the pure tensors $v_i \otimes w_j$, $(i,j) \in I \times J$ form a basis of $V \otimes W$ of weight vectors of weight $\lambda_i + \mu_j$. Hence $V \otimes W$ is the direct sum of its weight spaces.

More precisely, if $\nu \in \mathfrak{h}^*$, then the weight space $(V \otimes W)_\nu$ is the subspace of $V \otimes W$ generated by those $v_i \otimes w_j$ such that $\lambda_i + \mu_j = \nu$. In addition: the set of weights of $V \otimes W$ is

$$\Pi(V \otimes W) = \{ \lambda + \mu,\ \lambda \in \Pi(V),\ \mu \in \Pi(W) \} \subseteq \mathfrak{h}^*.$$

3. Recall that $V(\lambda)$ and $V(\mu)$ are irreducible rep. of $\mathfrak{g}$ of highest weights λ and μ resp. Denote by v a highest weight vector of $V(\lambda)$ and w a highest weight vector of $V(\mu)$. Therefore, $V(\lambda)$ is generated (as a rep.) by v and v has weight λ and similarly for w and $V(\mu)$. In addition $\Pi(\lambda) \subseteq \lambda - \mathbb{N}\Delta$ and $\Pi(\mu) \subseteq \mu - \mathbb{N}\Delta$. 3.1) See below.

3.2) Consider the Cartan-Chevalley dec. of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Since v is a highest weight of $V(\lambda)$, by definition: $\mathfrak{n}.v = 0$. Similarly $\mathfrak{n}.w = 0$. From the def. of the tensor product of rep. it follows at once that $\mathfrak{n}.v \otimes w = 0$. So $v \otimes w$ is a highest weight vector of weight $\lambda + \mu$ of $V(\lambda) \otimes V(\mu)$. Let now $\mathcal{U}$ be the subrepresentation of $V(\lambda) \otimes V(\mu)$ generated by $v \otimes w$. By definit°, $\mathcal{U}$ is a highest weight rep. of $\mathfrak{g}$ of weight $\lambda + \mu$ and it is finite dimensional as $V(\lambda) \otimes V(\mu)$ is. By Theorem V.2.6, it must be indecomposable.

But, being indecomposable and finite dimensional, by Weyl's Theorem of complete reducibility, it must be simple. Hence $\mathcal{U}$ must be isomorphic to $V(\lambda+\mu)$.

3.1) By question 1, $v \otimes w$ is a weight vector of weight $\lambda+\mu$. (Notice that $v \otimes w \neq 0$ since we are tensoring <u>vector spaces</u>.). We can consider bases $\mathcal{B}$ and $\mathcal{E}$, as in question 2, with v in $\mathcal{B}$ and $w \in \mathcal{E}$. Let v' be a vector in $\mathcal{B}$ with $v' \neq v$. By Theorem V.2.6, v' has weight $\lambda' = \lambda - \sum_{\alpha \in \Delta} n_\alpha \alpha$, where n_α, $\alpha \in \Delta$, are elts of $\mathbb{N}$, not all equal to zero. Likewise, any $w' \neq w$ in $\mathcal{E}$ has weight $\mu' = \mu - \sum_{\alpha \in \Delta} m_\alpha \alpha$. Therefore $v' \otimes w'$ has weight $\lambda + \mu - \sum_{\alpha \in \Delta} (n_\alpha + m_\alpha)\alpha \neq \lambda + \mu$ (because Δ is linearly independent). And the same holds for $v \otimes w'$ and $v' \otimes w$. All in all, all the elts of the basis of $V(\lambda) \otimes V(\mu)$ built out of $\mathcal{B}$ and $\mathcal{E}$ have weights distinct from $\lambda+\mu$. This shows that $(V(\lambda) \otimes V(\mu))_{\lambda+\mu}$ is a one dimensional vector space generated by $v \otimes w$.

Complement:

a) Questions 1 and 2 still hold in a larger context: no hypoth. on $\Bbbk$ is needed and it is enough to have a weight decomp. of the rep. of $\mathfrak{g}$ w.r.t. some adequate Lie subalg. $\mathfrak{h}$.

b) Question 1 and 2 extends verbatim with more than two representations ...

c) Let $\varpi_1, \dots, \varpi_\ell$ be the fundamental weights of the pair $(\mathfrak{g}, \mathfrak{h})$ associated to Δ. Suppose we are given the corresponding irred. rep. $V(\varpi_1), \dots, V(\varpi_\ell)$.

Let $\lambda \in \Lambda^+$ be a dominant weight. By Lemma III.9.10, $\lambda = \sum_{i=1}^{\ell} n_i \varpi_i$, where $n_1, \dots, n_\ell \in \mathbb{N}$. Taking b) above into consideration, this gives a way to realise $V(\lambda)$ as a subrepresentation of the tensor product $V(\varpi_1)^{\otimes n_1} \otimes \dots \otimes V(\varpi_\ell)^{\otimes n_\ell}$. (This is Exercise 8 p 117 of [Humphreys].)