Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 2, 716–726
DOI: 10.1214/13-AIHP582

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Asymptotic direction of random walks in Dirichlet environment

Laurent Tournier

LAGA, CNRS, UMR 7539, Université Paris 13, Sorbonne Paris Cité, F-93430, Villetaneuse, France. E-mail: tournier@math.univ-paris13.fr

Received 28 January 2013; accepted 16 September 2013

Abstract. We prove that, on $\mathbb{Z}^d$, random walks in i.i.d. Dirichlet environment – or equivalently oriented-edge reinforced random walks – have almost surely an asymptotic direction equal to the direction of the initial drift, i.e. $\frac{X_n}{\|X_n\|}$ converges to $\frac{E_o[X_1]}{\|E_o[X_1]\|}$ as $n\to\infty$, unless this drift is zero. This is obtained by generalizing the result of directional transience from (*Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 1–8). In addition, we identify the exact value or distribution of certain probabilities, answering and generalizing a conjecture of that paper.

Résumé. On démontre que, dans $\mathbb{Z}^d$, les marches aléatoires en milieu aléatoire i.i.d. de Dirichlet – ou, de façon équivalente, les marches renforcées par arêtes orientées – ont presque sûrement une direction asymptotique égale à la direction de la dérive initiale, c'est-à-dire que $\frac{X_n}{\|X_n\|}$ converge vers $\frac{E_o[X_1]}{\|E_o[X_1]\|}$ quand $n\to\infty$, à moins que cette dérive soit nulle. Ceci est obtenu en généralisant le résultat de transience directionnelle de (*Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 1–8). De plus, on explicite la valeur ou la loi de certaines probabilités, ce qui démontre et généralise une conjecture de ce dernier article.

MSC: 60K37; 60K35

Keywords: Random walk; Random environment; Dirichlet distribution; Reinforced random walk; Asymptotic direction; Time reversal

1. Introduction

Presentation of the model. Let $d\ge 1$, and denote by $(\vec e_1,\dots,\vec e_d)$ the canonical basis of $\mathbb{R}^d$. Define the set

$$\mathcal{V}:=\{\vec e_1,-\vec e_1,\dots,\vec e_d,-\vec e_d\}\subset\mathbb{Z}^d,$$

which will be used as possible steps, and assume we are given weights $\alpha_{\vec e}>0$, for $\vec e\in\mathcal{V}$.

Consider now the directed graph $\mathbb{Z}^d$ whose oriented edges are the pairs $e=(x,y)$ such that $\vec e:=y-x$ is an element of $\mathcal{V}$, endowed with (initial) weight

$$\alpha_e:=\alpha_{\vec e}$$

and, for $x\in\mathbb{Z}^d$, define the law $P_x^{(\alpha)}$ of a random walk $(X_n)_{n\ge 0}$ on this graph in the following way: $P_x^{(\alpha)}$-a.s., $X_0=x$, and for every time $n\in\mathbb{N}$ and every edge e starting at X_n,

$$P_x^{(\alpha)}\big((X_n,X_{n+1})=e|X_0,\dots,X_n\big)=\frac{\alpha_e+N_n(e)}{\sum_{f:\underline{f}=X_n}\alpha_f+N_n(f)},\tag{1.1}$$

where for every edge e we let $e=:(\underline{e},\overline{e})$ and

$$N_n(e):=\#\big\{0\le i<n:\ (X_i,X_{i+1})=e\big\}.$$

Under $P_x^{(\alpha)}$, $(X_n)_{n\geq 0}$ is called the *oriented-edge reinforced random walk* (or more specifically the oriented-edge *linearly* reinforced random walk) *with initial weights* $(\alpha_e)_e$, *started at* x.

Due to the embedding of an independent Polya urn at each vertex and to a de Finetti property, this model admits an equivalent representation as a random walk in an i.i.d. random environment given by Dirichlet random variables. Let us give a more precise statement. An *environment* is an element $\omega = (\omega_x(\cdot))_{x\in\mathbb{Z}^d}$ of $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ where $\mathcal{P}$ is the simplex of probabilities on $\mathcal{V}$:

$$\mathcal{P} := \left\{ \big(\omega(\vec{e})\big)_{\vec{e}\in\mathcal{V}}\colon \omega(\vec{e}) \geq 0, \sum_{\vec{e}\in\mathcal{V}} \omega(\vec{e}) = 1 \right\}.$$

Given a starting point $x \in \mathbb{Z}^d$ and such an environment ω, we may view ω as a set of transition probabilities (where $\omega_z(\vec{e})$ is the transition probability from z to $z+\vec{e}$) and define P_x^ω to be the law of the Markov chain starting at x with transition probabilities given by ω: for all $n \in \mathbb{N}$ and $\vec{e} \in \mathcal{V}$,

$$P_x^\omega(X_{n+1} = X_n + \vec{e} | X_0, \dots, X_n) = \omega_{X_n}(\vec{e}).$$

Finally, recall that the Dirichlet distribution $\mathcal{D}^{(\alpha)}$ on $\mathcal{P}$ with parameters $\alpha = (\alpha_{\vec{e}})_{\vec{e}\in\mathcal{V}}$ is the continuous probability distribution on $\mathcal{P}$ given by

$$\mathcal{D}^{(\alpha)} := \frac{\Gamma(\sum_{\vec{e}\in\mathcal{V}} \alpha_{\vec{e}})}{\prod_{\vec{e}\in\mathcal{V}} \Gamma(\alpha_{\vec{e}})} \prod_{\vec{e}\in\mathcal{V}} p_{\vec{e}}^{\alpha_{\vec{e}}-1} \, \mathrm{d}\lambda(p),$$

where λ is the Lebesgue measure on the simplex $\mathcal{P}$, and denote by $\mathbb{P}^{(\alpha)} = (\mathcal{D}^{(\alpha)})^{\otimes\mathbb{Z}^d}$ the law of an environment made of i.i.d. Dirichlet marginals. Then we have the following identity (cf. [4] for instance):

$$P_x^{(\alpha)}(\cdot) = \int P_x^\omega(\cdot) \, \mathrm{d}\mathbb{P}^{(\alpha)}(\omega).$$

This representation constitutes the specificity of oriented-edge linear reinforcement and has been the starting point to prove several sharp results, in contrast to the still very partial understanding of either random walks in random environment or reinforced random walks in two and more dimensions.

Context. Let us give a very brief account of the known results regarding transience before stating our result. We focus on the non-symmetric case, i.e. when the weights are such that the *mean drift*

$$\vec{\Delta} := E_o^{(\alpha)}[X_1] = \frac{1}{\Sigma} \sum_{\vec{e}\in\mathcal{V}} \alpha_{\vec{e}} \vec{e}$$

is non-zero, where $\Sigma = \sum_{\vec{e}\in\mathcal{V}} \alpha_{\vec{e}}$.

In any dimension, Enriquez and Sabot [5] gave the first result that was specific to Dirichlet environments, namely a sufficient ballisticity condition and bounds on the speed, later improved in [12]. On the other hand, non-ballistic cases are known to occur when weights are sufficiently small, due to the non-uniform ellipticity of the Dirichlet law (cf. [12]). Yet, under the only assumption of non-symmetry ($\vec{\Delta} \neq \vec{0}$) – and thus for both ballistic and zero-speed cases –, Sabot and the author [10] showed that the random walk is transient in the direction of a basis vector with positive probability.

In dimension at least 3, Sabot [8] proved the transience of these random walks (including in the symmetric case) and (in [9]) gave a characterization of the ballistic regime (viz., ballisticity occurs when $\vec{\Delta} \neq \vec{0}$ and the exit time from any edge is integrable, i.e. $\forall \vec{e} \in \mathcal{V}$, $2\Sigma - \alpha_{\vec{e}} - \alpha_{-\vec{e}} > 1$). Finally, Bouchet [1] recently proved that the methods of [9] extend to non-ballistic cases up to an acceleration of the walk, which implies a 0–1 law for directional transience.

1.1. *Results on directional transience and asymptotic direction*

Theorem 1. *Assume $\vec{\Delta} \neq \vec{0}$. For any $\vec{u} \in \mathbb{R}^d$ with rational slopes such that $\vec{u} \cdot \vec{\Delta} > 0$,*

$$P_o^{(\alpha)}(X_n \cdot \vec{u} \underset{n}{\longrightarrow} +\infty) > 0.$$

This theorem was proved in [10] in the case when $\vec{u} = \vec{e}_i$. The interest in the present refinement lies in the corollary below, obtained by combining the theorem with the 0–1 laws of [6] ($d = 2$) and of the recent [1] ($d \geq 3$) together with the main result of [11]. (Details follow.)

Corollary 1. *Assume $\vec{\Delta} \neq \vec{0}$. Then, $P_o^{(\alpha)}$-a.s., the walk has an asymptotic direction that is given by the direction of the mean drift*:

$$\frac{X_n}{\|X_n\|} \underset{n}{\longrightarrow} \frac{\vec{\Delta}}{\|\vec{\Delta}\|}, \quad P_o^{(\alpha)}\text{-a.s.}$$

Remarks.

- In [5], Enriquez and Sabot gave an expansion of the speed as $\gamma \to \infty$ when the parameters are $\alpha_i^{(\gamma)} := \gamma\alpha_i$, and noticed that the second order was surprisingly colinear to the first one, i.e. to $\vec{\Delta}$. This is not anymore a surprise given the above corollary; but this highlights the fact that the simplicity of the corollary comes as a surprise itself. Correlations between the transition probabilities at one site indeed affect the speed (cf. for instance [7]), and thus the speed of a random walk in random environment is typically not expected to be colinear with the mean drift, if not for symmetry reasons.
- The theorem does actually not depend on the graph structure of $\mathbb{Z}^d$ besides translation invariance, meaning that the result also holds for non-nearest neighbour models: we may enable $\mathcal{V}$ to be any finite subset of $\mathbb{Z}^d$, and the proof is written in such a way that it covers this case. The same is true for the main results of [1] and [11] with little modification, hence the corollary also generalizes in this way in dimension ≥ 3. The intersection property for planar walks used in [6] may however fail if jumps are allowed in such a way that the graph is not anymore planar. But if it is planar, then the proof carries closely. This includes in particular the case of the triangular lattice (by taking $\mathcal{V} = \{\pm\vec{e}_1, \pm\vec{e}_2, \pm(\vec{e}_1 + \vec{e}_2)\}$).
- Using the above-mentioned 0–1 laws, the probability in the theorem equals 1 and the rationality assumption is readily waived; the theorem was stated this way in order to keep its proof essentially contained in the present paper, in contrast to its corollary.
- As a complement to the theorem, note that Statement (d) of Theorem 1.8 of [3] (together with Lemma 4 of [6]) implies that in any dimension, when $\vec{u} \cdot \vec{\Delta} = 0$, $P_o^{(\alpha)}$-a.s., $\limsup_n X_n \cdot \vec{u} = +\infty$ and $\liminf_n X_n \cdot \vec{u} = -\infty$. In dimension at least 3, this is showed in [1] as well.
- Theorem 2 of [1] also implies the existence of a deterministic yet unspecified asymptotic direction in dimension at least 3. Further remarks regarding the derivation of the corollary from the theorem are deferred to the end of the proof.

1.2. *New identities*

The proof of Theorem 1 goes through proving a lower bound on the probability that the walk never leaves the half-space $\{x\colon\ x \cdot \vec{u} \geq 0\}$. In the next theorem, this lower bound is proved to be an equality.

Although the result admits a simple statement in some interesting cases (cf. (1.5) on page 720), we need to introduce further notation to deal with general directions.

Let $\vec{u} \in \mathbb{R}^d$ be a vector with rational slopes such that $\vec{u} \cdot \vec{\Delta} > 0$. Due to periodicity, the "discrete half-space" $\{x \in \mathbb{Z}^d\colon\ x \cdot \vec{u} \geq 0\}$ only has finitely many different entry points modulo translation. We shall denote by $\mathcal{H}_0$ an arbitrary set of representative entry points and by μ $(= \mu^{(\alpha,\vec{u})})$ the probability measure on $\mathcal{H}_0$ which makes $\mu(x)$ proportional to the total weight that enters vertex x from outside of the previous half-space. Let us alternatively give more formal definitions now.

Up to multiplication by a positive number, we may assume $\vec{u} \in \mathbb{Z}^d$. We extend $\vec{u}$ into a basis $(\vec{u}, \vec{u}_2, \dots, \vec{u}_d)$ chosen in such a way that $\vec{u}_i \perp \vec{u}$ and $\vec{u}_i \in \mathbb{Z}^d$ for all i. Since the "discrete half-spaces"

$$\mathcal{A}_x := \{ y \in \mathbb{Z}^d \colon\ y \cdot \vec{u} \geq x \cdot \vec{u} \}$$

satisfy $\mathcal{A}_x = \mathcal{A}_{x \pm \vec{u}_i}$ for all $i \geq 2$, $\mathcal{A}_x$ takes only finitely many different values when x is in the "discrete hyperplane"

$$\mathcal{H} := \{ x \in \mathbb{Z}^d \colon\ \exists \vec{e} \in \mathcal{V}, (x - \vec{e}) \cdot \vec{u} < 0 \leq x \cdot \vec{u} \},$$

namely for instance each of the values obtained when x belongs to the finite set

$$\mathcal{H}_0 := \mathcal{H} \cap (\mathbb{R}_+ \vec{u} + [0, \vec{u}_2) + \dots + [0, \vec{u}_d)).$$

We then define the probability measure μ $(= \mu^{(\alpha, \vec{u})})$ on $\mathcal{H}_0$ as follows: for all $x \in \mathcal{H}_0$,

$$\mu(x) := \frac{1}{Z} \sum_{\substack{\vec{e} \in \mathcal{V}: \\ (x - \vec{e}) \cdot \vec{u} < 0}} \alpha_{\vec{e}}, \tag{1.2}$$

where Z is a normalizing constant (we have $Z > 0$ because $\vec{\Delta} \cdot \vec{u} > 0$).

Let us also define a quenched analogue to μ. We enlarge Ω by adding a component $\omega(\partial, \cdot)$ to each environment ω, where $\omega(\partial, \cdot)$ is a probability distribution on $\mathcal{H}_0$. And we extend $\mathbb{P}^{(\alpha)}$ so that $\omega(\partial, \cdot)$ is independent of $(\omega_x(\cdot))_{x \in \mathbb{Z}^d}$ and follows a Dirichlet distribution of parameters

$$\Bigg(\sum_{\substack{\vec{e} \in \mathcal{V}: \\ (x - \vec{e}) \cdot \vec{u} < 0}} \alpha_{\vec{e}} \Bigg)_{x \in \mathcal{H}_0}.$$

Note that, for $x \in \mathcal{H}_0$,

$$\mu(x) = \int \omega(\partial, x) \, \mathrm{d}\mathbb{P}^{(\alpha)}(\omega).$$

Theorem 2. *Assume that $\vec{\Delta} \neq \vec{0}$ and $\vec{u}$ is a vector of $\mathbb{R}^d$ with rational slopes such that $\vec{u} \cdot \vec{\Delta} > 0$. Then the following identity holds*:

$$P_\mu^{(\alpha)}(\forall n \geq 0, X_n \cdot \vec{u} \geq 0) = 1 - \frac{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_+]}. \tag{1.3}$$

For the walk inside the cylinder

$$C := \mathbb{Z}^d / (\mathbb{Z}\vec{u}_2 + \dots + \mathbb{Z}\vec{u}_d),$$

the previous identity also holds, *as well as the following one involving distributions*:

$$\mathcal{L}_{\mathbb{P}^{(\alpha)}}\big(P_{\omega(\partial, \cdot)}^{\omega}(\forall n \geq 0, X_n \cdot \vec{u} \geq 0)\big) = \mathrm{Beta}\Bigg(\sum_{\substack{x \in \mathcal{H}_0, \vec{e} \in \mathcal{V}: \\ (x \pm \vec{e}) \cdot \vec{u} < 0}} \mp \alpha_{\vec{e}}, \sum_{\substack{x \in \mathcal{H}_0, \vec{e} \in \mathcal{V}: \\ (x + \vec{e}) \cdot \vec{u} < 0}} \alpha_{\vec{e}} \Bigg), \tag{1.4}$$

where $\mathcal{L}_{\mathbb{P}}(X)$ denotes the law under probability $\mathbb{P}$ of a random variable X and $\mathrm{Beta}(\cdot, \cdot)$ *is the usual Beta distribution.*

Remarks.

- The distribution of X_1 under $P_o^{(\alpha)}$ is simply given by the initial weights, normalized, hence (1.3) is fully explicit. This also follows from taking the expectation of the law in (1.4) (the expectation of $\mathrm{Beta}(a,b)$ is $\frac{a}{a+b}$).
- The case $\vec{u}=\vec{e}_1$ with nearest-neighbour jumps admits a simple expression. Indeed, $\mathcal{H}_0=\{0\}$ hence the results read as follows: if $\alpha_1>\alpha_{-1}$,

$$P_o^{(\alpha)}(\forall n\ge 0, X_n\cdot\vec{e}_1\ge 0)=1-\frac{\alpha_{-1}}{\alpha_1} \tag{1.5}$$

(as conjectured in [10]), and on the cylinder $\mathbb{Z}\times\mathbb{T}$ with $\mathbb{T}=\mathbb{Z}^{d-1}/(\mathbb{Z}\vec{v}_2+\cdots+\mathbb{Z}\vec{v}_d)$ for some basis $(\vec{v}_2,\dots,\vec{v}_d)$ of $\mathbb{R}^{d-1}$ with integer coordinates,

$$\mathcal{L}_{\mathbb{P}^{(\alpha)}}\big(P^{\omega}_{\omega(\partial,\cdot)}(\forall n\ge 0, X_n\cdot\vec{e}_1\ge 0)\big)=\mathrm{Beta}\big((\alpha_1-\alpha_{-1})\#\mathbb{T},\alpha_1\#\mathbb{T}\big),$$

where, under $\mathbb{P}^{(\alpha)}$, $\omega(\partial,\cdot)$ follows a Dirichlet distribution on $\{0\}\times\mathbb{T}$ with all parameters equal to α_1.
- In dimension 1, the identities already follow from [10] in a simple way. Note that they are not trivial even in this case: the quenched identity actually dates back to [2] where it was proved in a completely different way.
- Mild variations of the proof also provide other identities, as for instance

$$E_\mu^{(\alpha)}\big[\widetilde{T}_0^{\vec{u}}\,|\,\widetilde{T}_0^{\vec{u}}<\infty\big]=E_\mu^{(\alpha)}\big[T_0^{\vec{u}}\big]+1-\frac{E_o^{(\alpha)}[(X_1\cdot\vec{u})_+]}{E_o^{(\alpha)}[(X_1\cdot\vec{u})_-]}$$

or, for all $L\in\mathbb{N}$ such that $L\|\vec{u}\|>\|\vec{e}\|$ for all $\vec{e}\in\mathcal{V}$,

$$\frac{P_\mu^{(\alpha)}(\widetilde{T}_0^{\vec{u}}<T_L^{\vec{u}})}{P_\mu^{(\alpha)}(T_0^{\vec{u}}<\widetilde{T}_{-L}^{\vec{u}})}=\frac{E_o^{(\alpha)}[(X_1\cdot\vec{u})_-]}{E_o^{(\alpha)}[(X_1\cdot\vec{u})_+]}, \tag{1.6}$$

where, for $L\in\mathbb{Z}$, we defined the projected hitting times

$$T_L^{\vec{u}}:=\inf\big\{n:\ X_n\cdot\vec{u}>L\|\vec{u}\|^2\big\}\quad\text{and}\quad\widetilde{T}_L^{\vec{u}}:=\inf\big\{n:\ X_n\cdot\vec{u}<L\|\vec{u}\|^2\big\}.$$

2. Proof of Theorem 1: Directional transience

The proof, like [10], uses a time reversal property from [8] (re-proved in a more probabilistic way in [10]). To keep the present proof more self-contained, and for the sake of introducing some notation, we recall the following very elementary (yet powerful) lemma that sums up the only aspect of this property that we will use later. This is Lemma 1 of [10].

Lemma 1. *Let $G=(V,E)$ be a directed graph, endowed with positive weights $(\alpha_e)_{e\in E}$. We denote by $\check{G}=(V,\check{E})$ its reversed graph, i.e. $\check{E}:=\{\check{e}:=(\overline{e},\underline{e}):\ e=(\underline{e},\overline{e})\in E\}$, endowed with the weights $\check{\alpha}_{\check{e}}:=\alpha_e$. Assume that* $\mathrm{div}(\alpha)=0$, *i.e., for every $x\in V$,*

$$\alpha_x:=\sum_{e:\underline{e}=x}\alpha_e=\sum_{e:\overline{e}=x}\alpha_e=:\check{\alpha}_x.$$

Then, for any closed path $\sigma=(x_0,x_1,\dots,x_{n-1},x_0)$ in G, letting $\check{\sigma}:=(x_0,x_{n-1},\dots,x_1,x_0)$ denote its reverse (in $\check{G}$), we have

$$P_{x_0}^{(\alpha)}\big((X_0,\dots,X_n)=\sigma\big)=P_{x_0}^{(\check{\alpha})}\big((X_0,\dots,X_n)=\check{\sigma}\big),$$

where the laws of oriented-edge reinforced random walks on G or $\check{G}$ are defined as in (1.1).

Proof. From the definition of $P_{x_0}^{(\alpha)}$ we get

$$P_{x_0}^{(\alpha)}\big((X_0,\ldots,X_n)=\sigma\big)=\frac{\prod_{e\in E}\alpha_e(\alpha_e+1)\cdots(\alpha_e+n_e(\sigma)-1)}{\prod_{x\in V}\alpha_x(\alpha_x+1)\cdots(\alpha_x+n_x(\sigma)-1)},$$

where $n_e(\sigma)$ (resp. $n_x(\sigma)$) is the number of crossings of the oriented edge e (resp. the number of visits of the vertex x) in the path σ. Cyclicity gives $n_e(\sigma)=n_{\check{e}}(\check{\sigma})$ and $n_x(\sigma)=n_x(\check{\sigma})$ for all $e\in E, x\in V$. Furthermore we have by assumption $\check{\alpha}_x=\alpha_x$ for every vertex x, and by definition $\alpha_e=\check{\alpha}_{\check{e}}$ for every edge e. This shows that the previous product matches the similar product with $\check{E}$, $\check{\alpha}$ and $\check{\sigma}$ instead of E, α and σ, hence the lemma. □

Let us turn to the proof of Theorem 1. Assume $\vec{\Delta}\neq\vec{0}$, and let $\vec{u}$ be a vector with rational slopes such that $\vec{\Delta}\cdot\vec{u}>0$.

We make use of the notations introduced before Theorem 2. As in the Introduction, up to multiplication by a constant, we may assume that $\vec{u}\in\mathbb{Z}^d$, and also that $\|\vec{u}\|\geq\|\vec{e}\|$, $\forall\vec{e}\in\mathcal{V}$ (we may have $\|\vec{e}\|>1$, cf. the second remark after the corollary). Remember that $(\vec{u},\vec{u}_2,\ldots,\vec{u}_d)$ is a basis such that $\vec{u}_i\in\mathbb{Z}^d$ and $\vec{u}_i\perp\vec{u}$ for all i.

Let us consider the event $D:=\{\forall n\geq 0, X_n\cdot\vec{u}\geq 0\}$, and define a finite graph that will enable us to bound $P_\mu^{(\alpha)}(D):=\sum_{x\in\mathcal{H}_0}\mu(x)P_x^{(\alpha)}(D)$ from below.

Let $N,L\in\mathbb{N}^*$. We first consider the cylinder

$$C_{N,L}:=\big\{x\in\mathbb{Z}^d\colon\ 0\leq x\cdot\vec{u}\leq L\|\vec{u}\|^2\big\}/(N\mathbb{Z}\vec{u}_2+\cdots+N\mathbb{Z}\vec{u}_d),$$

i.e. the slab $\{0\leq x\cdot\vec{u}\leq L\|\vec{u}\|^2\}\cap\mathbb{Z}^d$ where vertices that differ by $N\vec{u}_i$ for some $i\in\{2,\ldots,d\}$ are identified. Let $\mathcal{R}$ denote its "right" end, i.e.

$$\mathcal{R}:=\big\{x\in\mathbb{Z}^d\colon\ \exists\vec{e}\in\mathcal{V}, x\cdot\vec{u}\leq L\|\vec{u}\|^2<(x+\vec{e})\cdot\vec{u}\big\}/(N\mathbb{Z}\vec{u}_2+\cdots+N\mathbb{Z}\vec{u}_d)\subset C_{N,L}$$

(note that the inclusion holds for small L due to the constraint $\|\vec{u}\|\geq\|\vec{e}\|$) and similarly $\mathcal{L}\subset C_{N,L}$ for the "left" end. We may now define the finite graph $G_{N,L}$ (refer to Fig. 1 for an example in $\mathbb{Z}^2$). Its vertex set is

$$V_{N,L}:=C_{N,L}\cup\{R,\partial\},$$

where R and ∂ are new vertices, and the edges of $G_{N,L}$ are of the following types:

(a) edges induced by those of $\mathbb{Z}^d$ inside $C_{N,L}$;
(b) edges from (resp. to) the vertices of $\mathcal{L}$ to (resp. from) ∂, corresponding to the edges of $\mathbb{Z}^d$ exiting (resp. entering) the cylinder "through the left end";
(c) edges from (resp. to) the vertices of $\mathcal{R}$ to (resp. from) R, corresponding to the edges of $\mathbb{Z}^d$ exiting (resp. entering) the cylinder "through the right end";
(d) a new edge from R to ∂.

Note that in (b) and (c) several edges may connect two vertices, and that in (d) no edge goes from ∂ to R. We also introduce weights $\alpha_e^{N,L}$ on the edges of $G_{N,L}$ as follows (invoking the translation invariance of the weights in $\mathbb{Z}^d$):

– edges defined in (a), (b) and (c) have the weight of the corresponding edge in $\mathbb{Z}^d$;
– the edge from R to ∂ has weight

$$\alpha_{(R,\partial)}^{N,L}:=\bigg(\sum_{\substack{x\in\mathcal{R},\vec{e}\in\mathcal{V}:\\ x+\vec{e}\notin C_{N,L}}}\alpha_{\vec{e}}\bigg)-\bigg(\sum_{\substack{x\in\mathcal{L},\vec{e}\in\mathcal{V}:\\ x+\vec{e}\notin C_{N,L}}}\alpha_{\vec{e}}\bigg).$$

By construction, we have $\operatorname{div}\alpha^{N,L}=0$. The main point to check however is that $\alpha_{(R,\partial)}^{N,L}$ is positive.

Due to periodicity, both $\mathcal{L}$ and $\mathcal{R}$ decompose into N^{d-1} subsets which are mapped to $\mathcal{H}_0$ by translations and we have, by gathering terms by value of the index $\vec{e}$,

$$\alpha_{(R,\partial)}^{N,L}=N^{d-1}\operatorname{Area}(\vec{u}_2,\ldots,\vec{u}_d)\sum_{\vec{e}\in\mathcal{V}}\big(\Phi_{\vec{u}}(\vec{e})-\Phi_{-\vec{u}}(\vec{e})\big)\alpha_{\vec{e}},$$

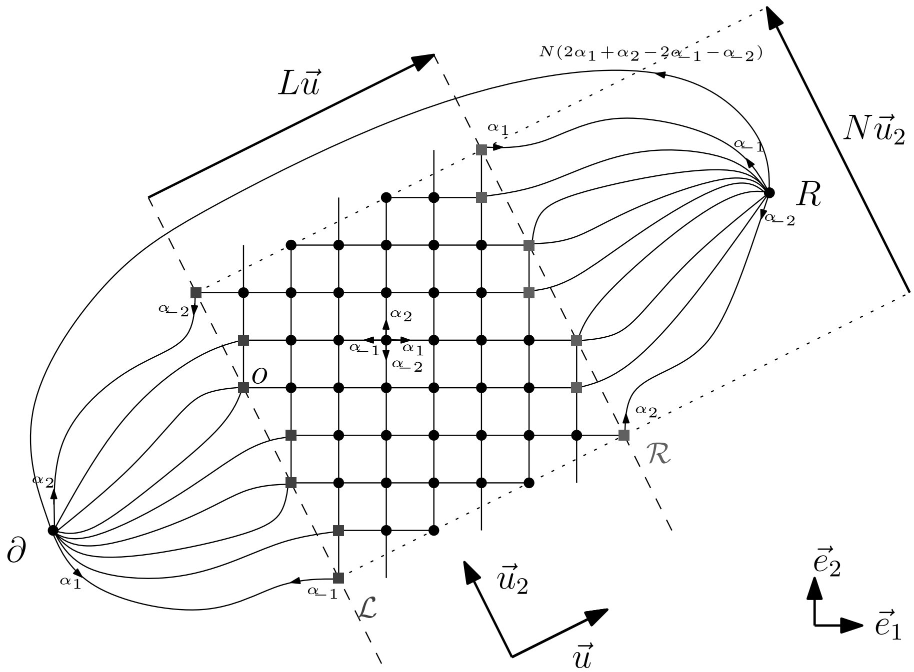

Fig. 1. Graph $G_{N,L}$ for $\vec{u} = 2\vec{e}_1 + \vec{e}_2$ (boundary conditions in direction $\vec{u}_2$ are periodic).

where $\text{Area}(\vec{u}_2, \dots, \vec{u}_d) = \frac{|\det(\vec{u}, \vec{u}_2, \dots, \vec{u}_d)|}{\|\vec{u}\|}$ is the $(d-1)$-volume of the hypersurface $[0, \vec{u}_2] + \cdots + [0, \vec{u}_d]$ and $\Phi_{\vec{u}}(\vec{e})$ is the flux of $\vec{e}$ through the oriented hyperplane $\vec{u}^{\perp}$:

$$\begin{aligned}\Phi_{\vec{u}}(\vec{e}) &:= \frac{1}{N^{d-1}\,\text{Area}(\vec{u}_2, \dots, \vec{u}_d)} \#\{x \in \mathcal{R}\colon\ x + \vec{e} \notin C_{N,L}\} \\ &= \frac{1}{\text{Area}(\vec{u}_2, \dots, \vec{u}_d)} \#\big(\big\{x \in \mathbb{Z}^d\colon\ x \cdot \vec{u} \le 0 < (x + \vec{e}) \cdot \vec{u}\big\} / (\mathbb{Z}\vec{u}_2 + \cdots + \mathbb{Z}\vec{u}_d)\big).\end{aligned}$$

Clearly $\Phi_{\vec{u}}(\vec{e})$ is zero if $\vec{u} \cdot \vec{e} \le 0$ and otherwise it is a simple geometric fact that the last cardinality above equals the volume of the parallelotope on the vectors $\vec{e}, \vec{u}_2, \dots, \vec{u}_d$. Indeed, this cardinality is also the number of lattice points in the torus $\mathbb{R}^d / (\mathbb{Z}\vec{e} + \mathbb{Z}\vec{u}_2 + \cdots + \mathbb{Z}\vec{u}_d)$, and this torus can be partitioned into the unit cubes $x + [0, 1)^d$ indexed by the lattice points x in it. Hence in any case

$$\Phi_{\vec{u}}(\vec{e}) = \frac{\text{Vol}(\vec{e}, \vec{u}_2, \dots, \vec{u}_d)}{\text{Area}(\vec{u}_2, \dots, \vec{u}_d)} \mathbf{1}_{(\vec{u}\cdot\vec{e}>0)} = \left(\frac{\vec{u}}{\|\vec{u}\|} \cdot \vec{e}\right)_+.$$

This gives

$$\begin{aligned}\alpha^{N,L}_{(R,\partial)} &= N^{d-1}\,\text{Area}(\vec{u}_2, \dots, \vec{u}_d) \sum_{\vec{e} \in \mathcal{V}} \left(\left(\frac{\vec{u}}{\|\vec{u}\|} \cdot \vec{e}\right)_+ - \left(-\frac{\vec{u}}{\|\vec{u}\|} \cdot \vec{e}\right)_+\right) \alpha_{\vec{e}} \\ &= N^{d-1}\,\text{Area}(\vec{u}_2, \dots, \vec{u}_d) \sum_{\vec{e} \in \mathcal{V}} \left(\frac{\vec{u}}{\|\vec{u}\|} \cdot \vec{e}\right) \alpha_{\vec{e}} \\ &= N^{d-1}\,\text{Area}(\vec{u}_2, \dots, \vec{u}_d) \frac{\vec{u}}{\|\vec{u}\|} \cdot \Sigma\vec{\Delta}\end{aligned}$$

therefore finally $\alpha^{N,L}_{(R,\partial)} > 0$ since $\vec{u} \cdot \vec{\Delta} > 0$, as expected.

NB. The above computation also shows that, introducing a new notation,

$$\alpha^{N,L}_{(\mathcal{L},\partial)} := \sum_{x\in\mathcal{L}} \alpha^{N,L}_{(x,\partial)} = \sum_{\substack{x\in\mathcal{L},\vec{e}\in\mathcal{V}:\\ x-\vec{e}\notin C_{N,L}}} \alpha_{\vec{e}} = N^{d-1}\operatorname{Area}(\vec{u}_2,\ldots,\vec{u}_d)\sum_{\vec{e}\in\mathcal{V}}\left(-\frac{\vec{u}}{\|\vec{u}\|}\cdot\vec{e}\right)_+ \alpha_{\vec{e}},$$

hence in particular

$$\frac{\alpha^{N,L}_{(\mathcal{L},\partial)}}{\alpha^{N,L}_{(R,\partial)}} = \frac{\sum_{\vec{e}\in\mathcal{V}}(-\vec{u}\cdot\vec{e})_+\alpha_{\vec{e}}}{\sum_{\vec{e}\in\mathcal{V}}(\vec{u}\cdot\vec{e})\alpha_{\vec{e}}} = \frac{E_o^{(\alpha)}[(X_1\cdot\vec{u})_-]}{E_o^{(\alpha)}[X_1\cdot\vec{u}]}. \tag{2.1}$$

Since the weighted graph $G_{N,L}$ is now well defined, let us resume the proof. If the walk starts from $X_0=\partial$, then we have $X_1 = Z(\operatorname{mod}\vec{u}_2,\ldots,\vec{u}_d)$ where Z has law μ (defined in (1.2)). Using translation invariance with respect to vectors $\vec{u}_2,\ldots,\vec{u}_d$, and the fact that, starting at ∂, the event $\{H_R < H_\partial^+\}$ (where H stands for hitting time and H^+ for positive hitting time) only depends on the walk before its first return in ∂ – and thus not on the reinforcement of the very first edge – we deduce, considering μ as a law on (a subset of) $\mathcal{L}$,

$$P_\mu^{(\alpha^{N,L})}(H_R < H_\partial) = P_\partial^{(\alpha^{N,L})}(H_R\circ\theta_1 < H_\partial\circ\theta_1) = P_\partial^{(\alpha^{N,L})}\big(H_R < H_\partial^+\big) \tag{2.2}$$

(using θ to denote time shift) and thus, by simple inclusion of events,

$$P_\mu^{(\alpha^{N,L})}(H_R < H_\partial) \ge P_\partial^{(\alpha^{N,L})}(X_{H_\partial-1} = R).$$

The last event is the probability that the walk follows a cycle in a given family (namely cycles going through ∂ only once and containing the edge (R,∂)). Applying Lemma 1 to every such cycle and summing up, we get (using (2.1) for the last equality)

$$\begin{aligned} P_\partial^{(\alpha^{N,L})}(X_{H_\partial-1}=R) &= P_\partial^{(\check{\alpha}^{N,L})}(X_1=R)\\ &= \frac{\alpha^{N,L}_{(R,\partial)}}{\alpha^{N,L}_{(R,\partial)}+\alpha^{N,L}_{(\mathcal{L},\partial)}}\\ &= \frac{E_o^{(\alpha)}[X_1\cdot\vec{u}]}{E_o^{(\alpha)}[(X_1\cdot\vec{u})_+]}. \end{aligned}$$

This lower bound is positive and uniform with respect to L and N. We may rewrite the result as

$$P_\mu^{(\alpha^{N,L})}(H_R < H_\partial) \ge 1 - \frac{E_o^{(\alpha)}[(X_1\cdot\vec{u})_-]}{E_o^{(\alpha)}[(X_1\cdot\vec{u})_+]}.$$

Letting N and then L go to infinity as in [10] (applied to each of the finitely many possible values of X_0 in $\mathcal{H}_0$) we get, in $\mathbb{Z}^d$,

$$P_\mu^{(\alpha)}(\forall n\ge 0, X_n\cdot\vec{u}\ge 0) \ge 1 - \frac{E_o^{(\alpha)}[(X_1\cdot\vec{u})_-]}{E_o^{(\alpha)}[(X_1\cdot\vec{u})_+]}$$

hence, by translation invariance of $P_o^{(\alpha)}$ and then Kalikow's 0–1 law together with Lemma 4 of [6] (which shows that the walk cannot stay in a slab),

$$P_o^{(\alpha)}(X_n\cdot\vec{u}\underset{n}{\longrightarrow}+\infty) = P_\mu^{(\alpha)}(X_n\cdot\vec{u}\underset{n}{\longrightarrow}+\infty) \ge P_\mu^{(\alpha)}(\forall n\ge 0, X_n\cdot\vec{u}\ge 0) \ge 1 - \frac{E_o^{(\alpha)}[(X_1\cdot\vec{u})_-]}{E_o^{(\alpha)}[(X_1\cdot\vec{u})_+]} > 0.$$

This is the content of Theorem 1.

3. Proof of the Corollary 1: Asymptotic direction

Recall from the introduction that oriented-edge reinforced random walks are also random walks in Dirichlet environment. Due to the 0–1 law of Zerner and Merkl [6] (cf. also [13]) in dimension 2 (for random walks in elliptic random environment), and of Bouchet [1] in dimension at least 3 (for random walks in Dirichlet environment), the result of Theorem 1 turns into: for any $\vec{u} \in \mathbb{R}^d$ with rational slopes and such that $\vec{u} \cdot \vec{\Delta} > 0$,

$$X_n \cdot \vec{u} \underset{n}{\longrightarrow} +\infty, \quad P_o^{(\alpha)}\text{-a.s.} \tag{3.1}$$

Note that the set of directions $\vec{u} \in \mathbb{R}^d$ such that (3.1) holds also has to be convex, therefore it contains the whole half-space $\{\vec{u} \in \mathbb{R}^d \colon \vec{u} \cdot \vec{\Delta} > 0\}$.

By Theorem 1 of [11], there exists a direction $\vec{\nu} \in \mathbb{S}^{d-1}$ such that

$$\frac{X_n}{\|X_n\|} \underset{n}{\longrightarrow} \vec{\nu}, \quad P_o^{(\alpha)}\text{-a.s.}$$

On the other hand, this direction must satisfy $\vec{\nu} \cdot \vec{u} \geq 0$ for every $\vec{u}$ that satisfies (3.1), hence in particular for every $\vec{u}$ such that $\vec{\Delta} \cdot \vec{u} > 0$. This fully characterizes $\vec{\nu}$, which therefore has to be

$$\vec{\nu} = \frac{\vec{\Delta}}{\|\vec{\Delta}\|}.$$

Remarks.

– We may derive a much weaker result in dimension at least 3 without resorting to the Dirichlet 0–1 law from [1], namely that an asymptotic direction exists, although it remains unidentified, and possibly random (two-valued). Indeed, by the 0–1 law of Kalikow (in its elliptic version proved in [6]) and Theorem 1.8 of [3], there exists $\vec{\nu} \in \mathbb{S}^{d-1}$ and an event A such that, almost-surely,

$$\frac{X_n}{\|X_n\|} \underset{n}{\longrightarrow} (\mathbf{1}_A - \mathbf{1}_{A^c})\vec{\nu}$$

but identifying $\vec{\nu}$ from Theorem 1 is hindered by the restriction to rational slopes due to the possible non-convexity of the set of directions $\vec{u}$ of transience (i.e. satisfying the theorem).

– In dimension at least 3, since [1] already proves the existence of an asymptotic direction, an alternative derivation of the corollary without [11] would consist in using Theorem 1 in the proof of Theorem 2 of [1] instead of referring to [10].

4. Proof of Theorem 2: Identities

4.1. *Annealed identity*

Let $N, L \in \mathbb{N}^*$. Let us make use of the definitions involved in the proof of Theorem 1, in particular the graph $G_{N,L}$, and apply Lemma 1 to a different family of cycles.

As in (2.2), we have

$$P_\mu^{(\alpha^{N,L})}(H_R < H_\partial) = P_\partial^{(\alpha^{N,L})}\big(H_R < H_\partial^+\big) = 1 - P_\partial^{(\alpha^{N,L})}\big(H_\partial^+ < H_R\big).$$

The last event is the probability that the walk follows a cycle in a given family (namely cycles that pass through ∂ exactly once and don't visit R). Note that this set of cycles is globally invariant by change of orientation. Thus, applying Lemma 1 to every such cycle and summing up, we get

$$P_\partial^{(\alpha^{N,L})}\big(H_\partial^+ < H_R\big) = P_\partial^{(\check{\alpha}^{N,L})}\big(H_\partial^+ < H_R\big).$$

The edge (∂, R) is in $\check{G}$, hence we may decompose the event on the right as follows: the first step is different from R, and then the walk comes back to ∂ before reaching R. Since the edge (∂, X_1) is not involved in the second part, these events are independent and we have

$$P_\partial^{(\check{\alpha}^{N,L})}\big(H_\partial^+ < H_R\big) = P_\partial^{(\check{\alpha}^{N,L})}(X_1 \neq R) P_{\check{\mu}}^{(\check{\alpha}^{N,L})}(H_\partial < H_R),$$

where $\check{\mu}$ is defined like μ with respect to $\check{\alpha}$ instead of α. First, using (2.1) for the last equality,

$$P_\partial^{(\check{\alpha}^{N,L})}(X_1 \neq R) = 1 - \frac{\alpha_{(R,\partial)}}{\alpha_{(R,\partial)} + \alpha_{(\mathcal{L},\partial)}} = \frac{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_+]}.$$

Gathering everything, we obtain

$$P_\mu^{(\alpha^{N,L})}(H_R < H_\partial) = 1 - \frac{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_+]}\big(1 - P_{\check{\mu}}^{(\check{\alpha}^{N,L})}(H_R < H_\partial)\big).$$

Letting N, then L go to infinity as in [10], we get

$$P_\mu^{(\alpha)}(\forall n \geq 0, X_n \cdot \vec{u} \geq 0) = 1 - \frac{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_+]}\big(1 - P_{\check{\mu}}^{(\check{\alpha})}(\forall n \geq 0, X_n \cdot \vec{u} \geq 0)\big).$$

However, by central symmetry,

$$P_{\check{\mu}}^{(\check{\alpha})}(\forall n \geq 0, X_n \cdot \vec{u} \geq 0) = P_{\check{\mu}}^{(\alpha)}(\forall n \geq 0, X_n \cdot \vec{u} \leq 0)$$

and the latter event has probability 0 because of Theorem 1 combined to a 0–1 law (namely that of [6] in dimension 2, and of [1] in dimension ≥ 3). This concludes.

4.2. *Quenched identity on a cylinder*

Let us first recall the quenched version of Lemma 1, for which we refer to [8] or [10].

Lemma 2. *Let $G = (V, E)$ be a finite directed graph, endowed with positive weights $(\alpha_e)_{e \in E}$. Recall notations from Lemma 1. To any environment ω on G, we also associate its reverse $\check{\omega}$ defined by $\check{\omega}_{\check{e}} = \frac{\pi(\underline{e})}{\pi(\overline{e})}\omega_e$ for all $e \in E$, where π is the invariant measure for ω. Assume that* $\operatorname{div}(\alpha) = 0$. *Then $\mathcal{L}_{\mathbb{P}^{(\alpha)}}(\check{\omega}) = \mathbb{P}^{(\check{\alpha})}$.*

Let us follow the same steps as for the annealed identity, now with fixed $N = 1$, which we omit from indices.

Let $L \in \mathbb{N}^*$. We have, for any environment ω on G_L,

$$P_{\omega(\partial,\cdot)}^\omega(H_R < H_\partial) = P_\partial^\omega(H_R < H_\partial) = 1 - P_\partial^\omega\big(H_\partial^+ < H_R\big).$$

Furthermore, the latter event involves cycles and, considering the reversed cycles, we have

$$P_\partial^\omega\big(H_\partial^+ < H_R\big) = P_\partial^{\check{\omega}}\big(H_\partial^+ < H_R\big),$$

as a consequence of two facts: first, the set of cycles in the left event is globally unchanged after reversal, and second the probability of a cycle in ω is equal to the probability of its reverse in $\check{\omega}$, as a consequence of the definition of $\check{\omega}$.

For any environment ω on G_L, $\check{\omega}$ is an environment on $\check{G}_L$ hence we may decompose as before, applying Markov property at time 1,

$$P_\partial^{\check{\omega}}\big(H_\partial^+ < H_R\big) = (1 - \check{\omega}_{(\partial,R)}) P_{\check{\omega}(\partial,\cdot)_{|\mathcal{L}}}^{\check{\omega}}(H_\partial < H_R),$$

where $\check{\omega}(\partial,\cdot)_{|\mathcal{L}}$ is the law of X_1 under $P_\partial^{\check{\omega}}$ conditioned on $\{X_1 \neq R\}$. By Lemma 2, under $\mathbb{P}^{(\alpha^L)}$, $\check{\omega} \sim \mathbb{P}^{(\check{\alpha}^L)}$. As a consequence, and because of the "restriction property" of Dirichlet distribution (cf. for instance [12]), under $\mathbb{P}^{(\alpha^L)}$, $1-\check{\omega}_{(\partial,R)}$ is independent of $\check{\omega}(\partial,\cdot)_{|\mathcal{L}}$ and the latter follows a Dirichlet distribution with parameters $\check{\alpha}_{(\partial,\cdot)}$. On the other hand, under $\mathbb{P}^{(\alpha^L)}$,

$$1-\check{\omega}_{(\partial,R)} \sim \mathrm{Beta}(\check{\alpha}_{(\partial,\mathcal{L})},\check{\alpha}_{(\partial,R)}) = \mathrm{Beta}(\alpha_{(\mathcal{L},\partial)},\alpha_{(R,\partial)}),$$

which is the distribution in (1.4).

Gathering everything, we obtain that the law under $\mathbb{P}^{(\alpha^L)}$ of $P^\omega_{\omega(\partial,\cdot)}(H_R < H_\partial)$ is the same as the law of

$$1-(1-\omega_{(R,\partial)})\bigl(1-P^\omega_{\omega(\partial,\cdot)}(H_R<H_\partial)\bigr) \tag{4.1}$$

under $\mathbb{P}^{(\check{\alpha}^L)}$ (note that here ω is an environment on $\check{G}$). Although this is not necessary to conclude, we may note that the two factors are independent, because the paths involved in the last event don't go out of vertex R.

As was noticed in the annealed proof, when L goes to infinity, the expectation under $\mathbb{P}^{(\check{\alpha}^L)}$ of the last probability in (4.1) goes to $P^{(\alpha)}_{\check{\mu}}(\forall n, X_n\cdot\vec{u}\le 0)=0$, hence the last probability under $\mathbb{P}^{(\check{\alpha}^L)}$ goes to 0 in L^1 and thus in law. On the other hand, the law of $\omega_{(R,\partial)}$ under $\mathbb{P}^{(\check{\alpha}^L)}$ was shown above to be the Beta distribution from (1.4), and thus does not depend on L.

We conclude that the law under $\mathbb{P}^{(\alpha^L)}$ of $P^\omega_{\omega(\partial,\cdot)}(H_R<H_\partial)$ converges to the Beta distribution given in (1.4). This is the expected conclusion since, on the other hand, these quenched probabilities for growing L can be expressed on the same cylinder C and thus seen to converge as $L\to\infty$:

$$P^\omega_{\omega(\partial,\cdot)}(H_R<H_\partial) = P^\omega_{\omega(\partial,\cdot)}\bigl(T_L^{\vec{u}}<\widetilde{T}_0^{\vec{u}}\bigr) \underset{L\to\infty}{\longrightarrow} P^\omega_{\omega(\partial,\cdot)}\Bigl(\forall n, X_n\cdot\vec{u}\ge 0, \text{and } \limsup_n X_n\cdot\vec{u}=+\infty\Bigr).$$

As before, the event $\{\limsup_n X_n\cdot\vec{u}=+\infty\}$ is $\mathbb{P}^{(\alpha)}$-a.s. included in $\{\forall n, X_n\cdot\vec{u}\ge 0\}$ because of Lemma 4 of [6].

Acknowledgement

This work was partly supported by the french ANR project MEMEMO2 ANR-10-BLAN-0125.

References

[1] E. Bouchet. Sub-ballistic random walk in Dirichlet environment. *Electron. J. Probab.* **18** (58) (2013) 1–25 (electronic). MR3068389
[2] J.-F. Chamayou and G. Letac. Explicit stationary distributions for compositions of random functions and products of random matrices. *J. Theoret. Probab.* **4** (1991) 3–36. MR1088391
[3] A. Drewitz and A. Ramírez. Asymptotic direction in random walks in random environment revisited. *Braz. J. Probab. Stat.* **24** (2) (2010) 212–225. MR2643564
[4] N. Enriquez and C. Sabot. Edge oriented reinforced random walks and RWRE. *C. R. Math. Acad. Sci. Paris* **335** (11) (2002) 941–946. MR1952554
[5] N. Enriquez and C. Sabot. Random walks in a Dirichlet environment. *Electron. J. Probab.* **11** (31) (2006) 802–817 (electronic). MR2242664
[6] M. Zerner and F. Merkl. A zero–one law for planar random walks in random environment. *Ann. Probab.* **29** (4) (2001) 1716–1732. MR1880239
[7] C. Sabot. Ballistic random walks in random environments at low disorder. *Ann. Probab.* **32** (4) (2004) 2996–3023. MR2094437
[8] C. Sabot. Random walks in random Dirichlet environment are transient in dimension $d\ge 3$. *Probab. Theory Related Fields* **151** (1–2) (2009) 297–317. MR2834720
[9] C. Sabot. Random Dirichlet environment viewed from the particle in dimension $d\ge 3$. *Ann. Probab.* **41** (2) (2013) 722–743. MR3077524
[10] C. Sabot and L. Tournier. Reversed Dirichlet environment and directional transience of random walks in Dirichlet environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (1) (2011) 1–8. MR2779393
[11] F. Simenhaus. Asymptotic direction for random walks in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **43** (6) (2007) 751–761. MR3252429
[12] L. Tournier. Integrability of exit times and ballisticity for random walks in Dirichlet environment. *Electron. J. Probab.* **14** (16) (2009) 431–451 (electronic). MR2480548
[13] M. Zerner. The zero–one law for planar random walks in i.i.d. random environments revisited. *Electron. Commun. Probab.* **12** (2007) 326–335 (electronic). MR2342711