Asymptotic direction of oriented-edge reinforced random walks

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Oriented-edge reinforced random walk (OERRW)

Directed graph $G = (V, E)$. Notation: $e = (\underline{e}, \overline{e}) \in E$
Let initial weights $\alpha_e > 0$ be given for $e \in E$ (oriented edges).
The *oriented-edge reinforced random walk* is defined by the following:

- probability transitions across edges are proportional to their weights;
- the weight of an (oriented) edge increases by 1 after the traversal.

In other words, for all n, if $\underline{e} = X_n$,

$$P^{(\alpha)}((X_n, X_{n+1}) = e | X_0, \ldots, X_n) = \frac{\alpha_e + N_e^{(n)}}{\sum_{\underline{f} = X_n} (\alpha_f + N_f^{(n)})},$$

where $N_f^{(n)}$ is the number of traversals of edge f before time n.

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NB: for non-oriented edges (and translation-invariant weights), positive recurrence for small weights (on $\mathbb{Z}^d$) was proved very recently by Sabot-Tarrès and Angel-Crawford-Kozma, independently.

OERRW and RWRE

Remark

For the OERRW, the successive choices of edges used to exit each vertex are given by independent Polya urns.

Two representations of a Polya urn (cf. De Finetti's theorem):

Initially:
$\alpha_1, \ldots, \alpha_r$ balls of color $1, \ldots, r$.
Then:
Reinforced urn (chosen color +1 ball).

Proportions of colors converge a.s. to

$$p = (p_1, \ldots, p_r) \sim \mathscr{D}(\alpha_1, \ldots, \alpha_r).$$

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I.i.d. draws according to p.

Dirichlet distribution:

$$\mathscr{D}\big((\alpha_i)_{i\in I}\big) = \frac{1}{Z}\prod_{i\in I} x_i^{\alpha_i - 1} d\lambda(x)$$

(λ is Lebesgue measure on a simplex of probabilities)

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The OERRW with initial weights $(\alpha_e)_{e \in E}$ has same law as a **random walk in a random environment** given by independent Dirichlet random variables $\omega_{(x,\cdot)}, x \in V$.

$$P_o^{(\alpha)}(\cdot) = \int P_{o,\omega}(\cdot)\, \mathrm{d}\mathbb{P}^{(\alpha)}(\omega)$$

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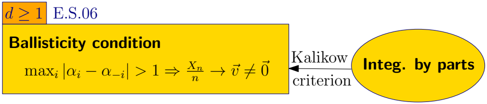
$$P_o^{(\alpha)}(\cdot)=\int P_{o,\omega}(\cdot)\,\mathrm{d}\mathbb{P}^{(\alpha)}(\omega)$$

$\Rightarrow$ on $\mathbb{Z}$, behaviour of OERRW known in details (via RWRE)

Previous results on $\mathbb{Z}^d$



E. : Enriquez
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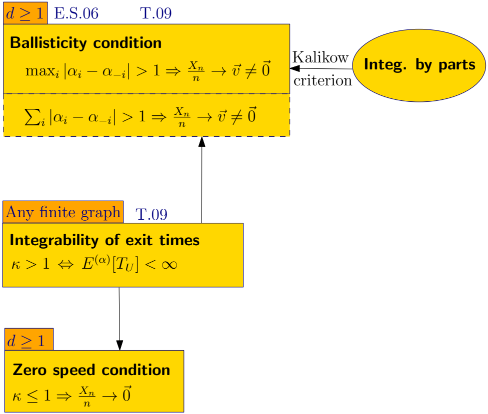

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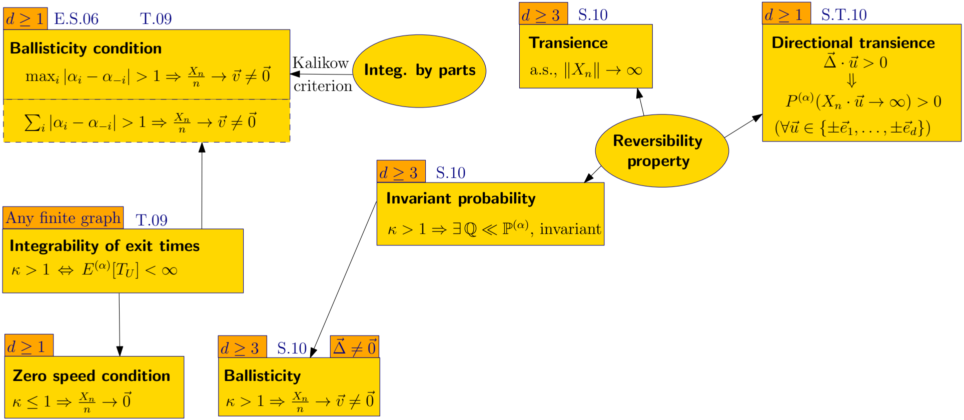
$$\kappa = \min_{\emptyset \neq B \subset E_U} \sum_{e \in \partial B} \alpha_e = 2 \sum_i (\alpha_i + \alpha_{-i}) - \max_i (\alpha_i + \alpha_{-i})$$



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$$\vec{\Delta} = E_o^{(\alpha)}[X_1] = \frac{1}{\Sigma} \sum_i (\alpha_i - \alpha_{-i}) \vec{e}_i$$

Previous results on $\mathbb{Z}^d$

$d \geq 1$ E.S.06 T.09

Ballisticity condition

$\max_i |\alpha_i - \alpha_{-i}| > 1 \Rightarrow \frac{X_n}{n} \to \vec{v} \neq \vec{0}$

$\sum_i |\alpha_i - \alpha_{-i}| > 1 \Rightarrow \frac{X_n}{n} \to \vec{v} \neq \vec{0}$

Kalikow criterion

Integ. by parts

$d \geq 3$ S.10

Transience

a.s., $\|X_n\| \to \infty$

$d \geq 1$ S.T.10

Directional transience

$\vec{\Delta} \cdot \vec{u} > 0$

$\Downarrow$

$P^{(\alpha)}(X_n \cdot \vec{u} \to \infty) > 0$

$(\forall \vec{u} \in \{\pm \vec{e}_1, \ldots, \pm \vec{e}_d\})$

Reversibility property

Any finite graph T.09

Integrability of exit times

$\kappa > 1 \Leftrightarrow E^{(\alpha)}[T_U] < \infty$

$d \geq 3$ S.10 B.12

Invariant probability

$\kappa > 1 \Rightarrow \exists \mathbb{Q} \ll \mathbb{P}^{(\alpha)}$, invariant

Same result for an accelerated walk

$\kappa_\Lambda > 1 \Rightarrow \exists \mathbb{Q} \ll \mathbb{P}^{(\alpha)}$, invariant

$d \geq 1$

Zero speed condition

$\kappa \leq 1 \Rightarrow \frac{X_n}{n} \to \vec{0}$

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$\kappa \leq 1 \Rightarrow \frac{\log X_n \cdot \vec{\Delta}}{\log n} \xrightarrow{(p)} \kappa$

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Time reversal

Finite directed graph $G=(V,E)$, weights $\alpha=(\alpha_e)_{e\in E}$.
Reversed graph: $\check{G}=(V,\check{E})$ made of reversed edges $\check{e}=(\overline{e},\underline{e})$, endowed with weight $\check{\alpha}_e=\alpha_{\check{e}}$ for $e\in\check{E}$.
Reversed environment: for $e\in E$, $\quad \check{\omega}_{\check{e}}=\frac{\pi(\underline{e})}{\pi(\overline{e})}\omega_e$
(π: invariant probability for ω)
For $x\in V$, denote

$$\alpha_x=\sum_{\underline{e}=x}\alpha_e.$$

Property

Assume $\mathrm{div}(\alpha)=0$: for all $x\in V$, $\alpha_x=\check{\alpha}_x$. Then,

$$\omega\sim\mathbb{P}^{(\alpha)}\Rightarrow\check{\omega}\sim\mathbb{P}^{(\check{\alpha})}.$$

Then, for all cycle σ in G going through o,

$$P_o^{(\alpha)}((X_n)_n \text{ follows } \sigma)=P_o^{(\check{\alpha})}((X_n)_n \text{ follows } \check{\sigma}).$$

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Proof:

$$P_o^{(\alpha)}(\sigma) = \frac{\prod_{e \in E} \alpha_e(\alpha_e + 1) \cdots (\alpha_e + N_e(\sigma) - 1)}{\prod_{x \in V} \alpha_x(\alpha_x + 1) \cdots (\alpha_x + N_x(\sigma) - 1)}$$

and $\alpha_e = \check{\alpha}_{\check{e}}$, $\alpha_x = \check{\alpha}_x$, $N_e(\sigma) = N_{\check{e}}(\check{\sigma})$, $N_x(\sigma) = N_x(\check{\sigma})$.

Directional transience

Recall the mean drift

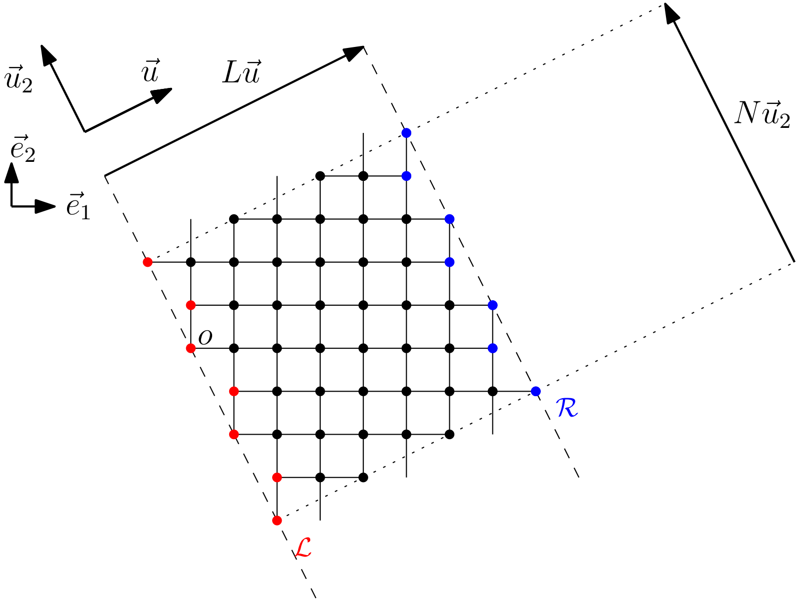
$$\vec{\Delta} = E_o^{(\alpha)}[X_1] = \frac{1}{\sum_i(\alpha_i + \alpha_{-i})} \sum_{i=1}^{d} (\alpha_i - \alpha_{-i})\vec{e}_i.$$

Proposition

For any direction $\vec{u} \in \mathbb{Z}^d$ such that $\vec{u} \cdot \vec{\Delta} > 0$,

$$P^{(\alpha)}(X_n \cdot \vec{u} \to +\infty) > 0.$$

NB: Result actually holds for non nearest-neighbour walks with finite range.
(Hence for the triangular lattice for instance)



$\vec{u}_2$
$\vec{u}$
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$\mathcal{R}$
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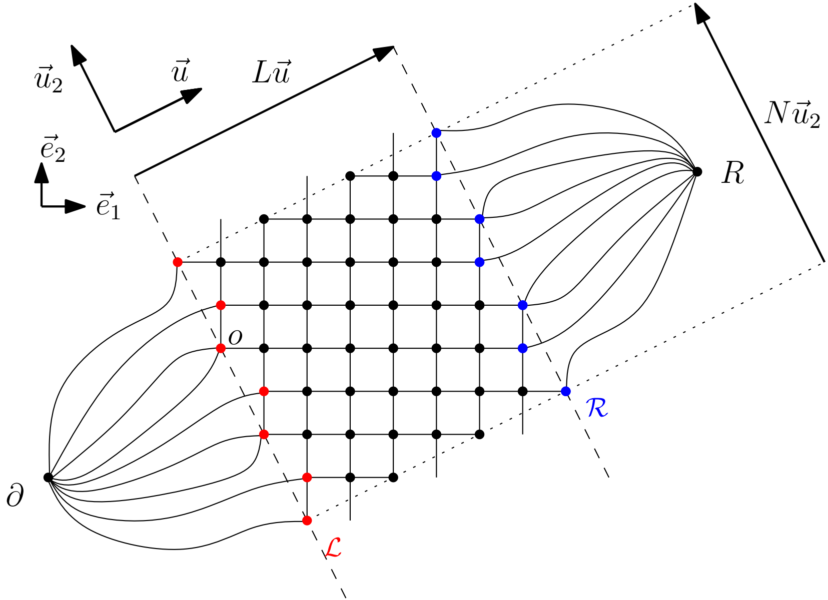

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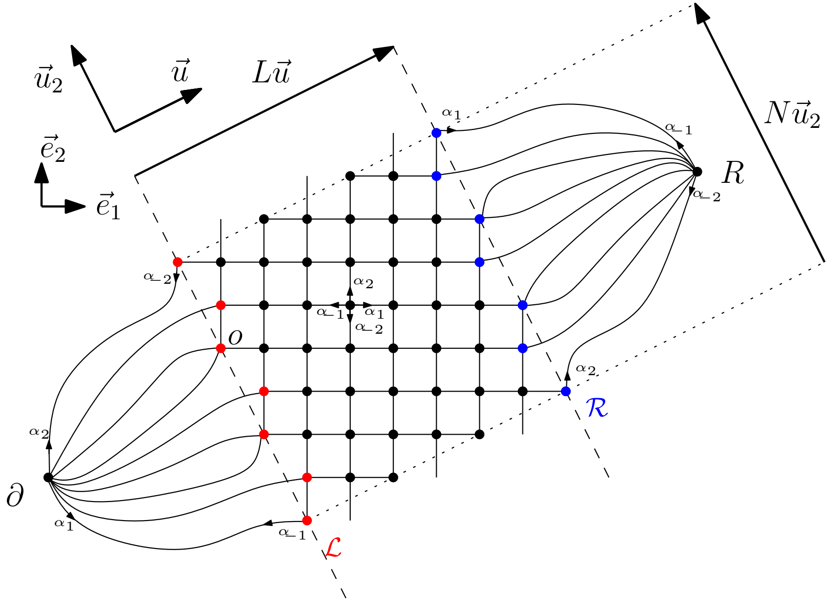
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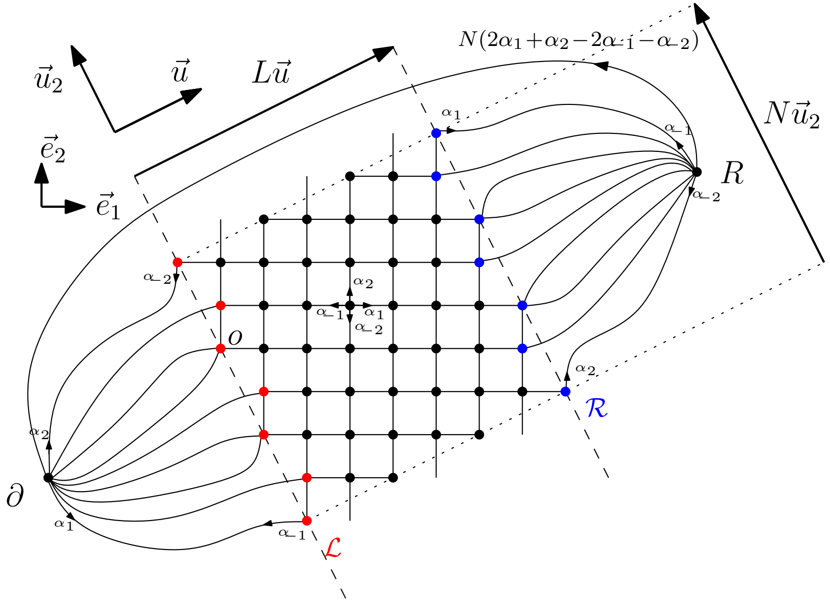
α_{-2}


$\vec{u}_2$
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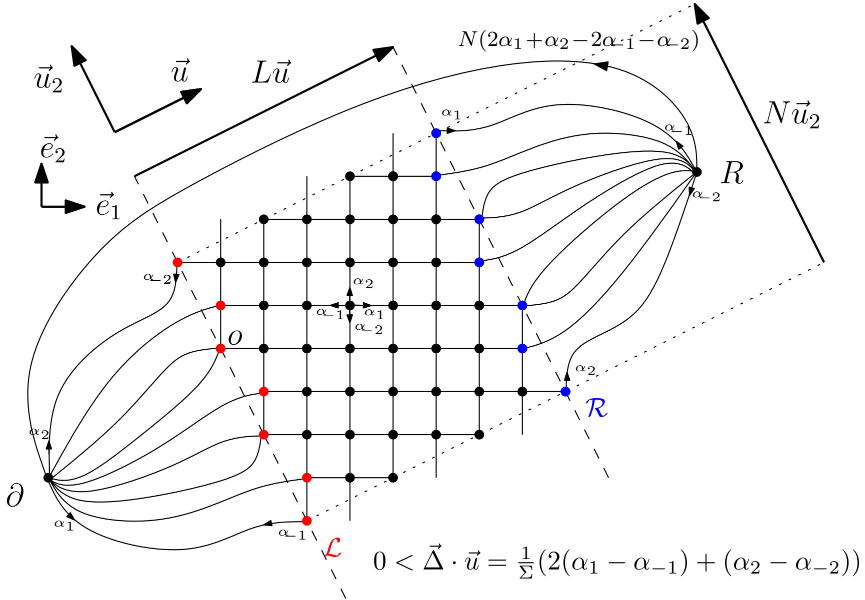

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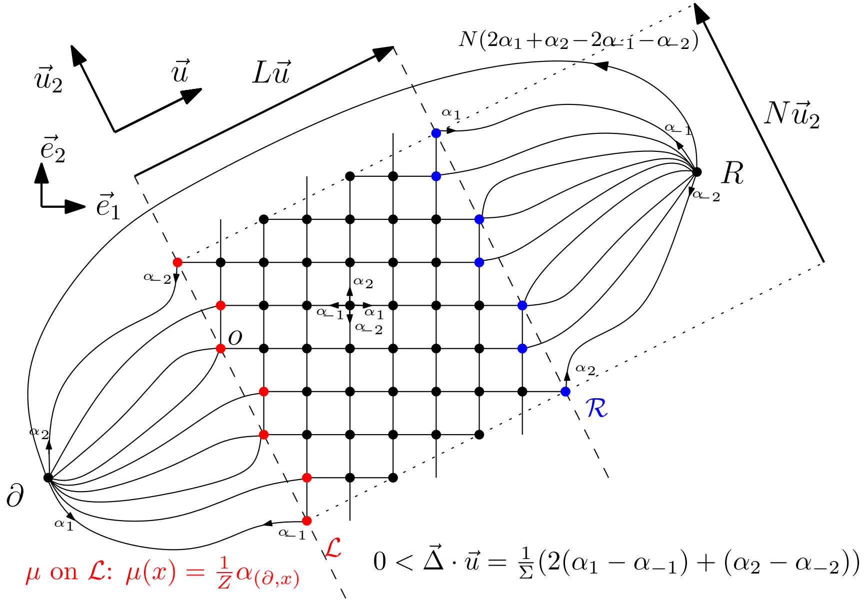

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Corollary

We got: $\vec{u} \in \mathbb{Q}^d, \vec{u} \cdot \vec{\Delta} > 0 \Rightarrow P_o^{(\alpha)}(X_n \cdot \vec{u} \to +\infty) \geq P_\mu^{(\alpha)}(\forall \text{n}, X_n \cdot \vec{u} \geq 0) > c$

Corollary

Assume $\vec{\Delta} \neq \vec{0}$.

- $P^{(\alpha)}$-a.s., $\vec{u} \cdot \vec{\Delta} > 0 \Rightarrow X_n \cdot \vec{u} \to +\infty$.

- ($d = 2$) 0-1 law of Zerner-Merkl(01): for elliptic RWRE, $P(A_\ell) \in \{0, 1\}$ where
$$A_\ell = \{X_n \cdot \ell \xrightarrow[n]{} +\infty\}$$
- ($d \geq 3$) 0-1 law of Bouchet (12): for Dirichlet RWRE, $P(A_\ell) \in \{0, 1\}$.

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- (any d) Result of Simenhaus (06): for elliptic RWRE,
$$P(A_\ell) = 1 \text{ for every } \ell \text{ in an open set} \Rightarrow \frac{X_n}{\|X_n\|} \xrightarrow[n]{} \vec{v}$$
where $\vec{v}$ is deterministic.
- (any d) Result of Drewitz-Ramírez (09): for elliptic RWRE,
$$P(A_\ell) > 0 \text{ for every } \ell \text{ in an open set} \Rightarrow \frac{X_n}{\|X_n\|} \xrightarrow[n]{} (\mathbf{1}_A - \mathbf{1}_{A^c})\vec{v}.$$

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$$P(A_\ell) > 0 \text{ for every } \ell \text{ in an open set} \Rightarrow \frac{X_n}{\|X_n\|} \xrightarrow[n]{} (\mathbf{1}_A - \mathbf{1}_{A^c})\vec{v}.$$

Asymptotic direction (renewal time)

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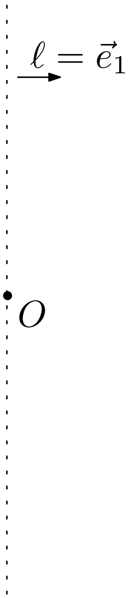

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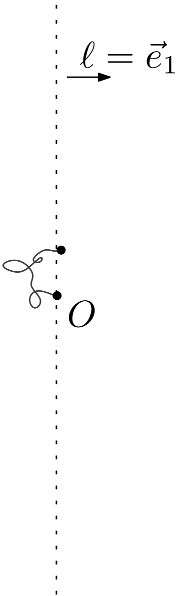

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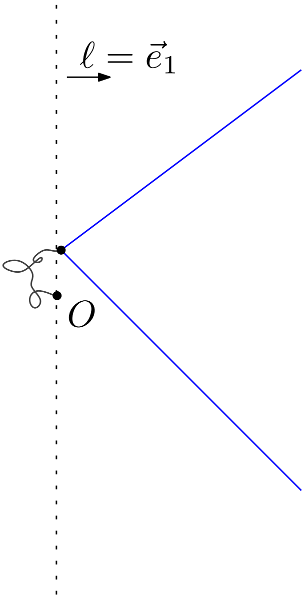

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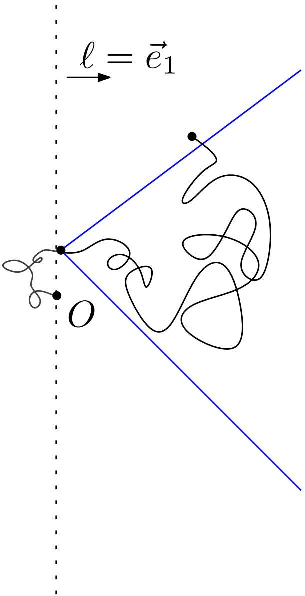

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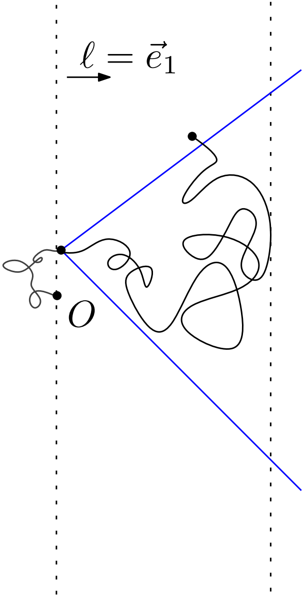

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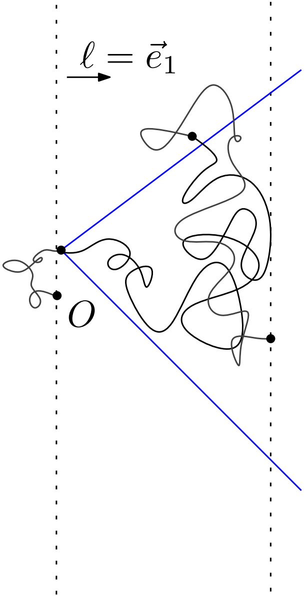

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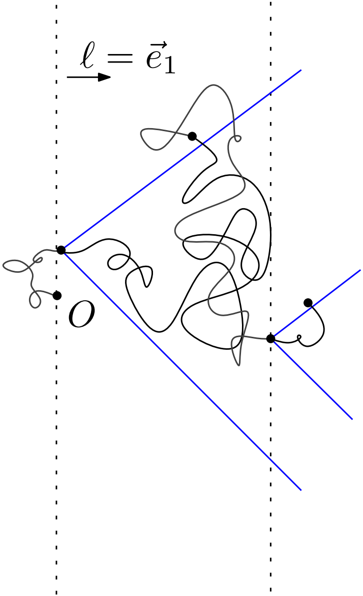

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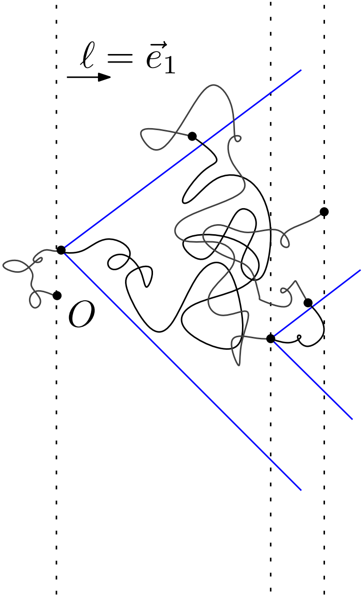

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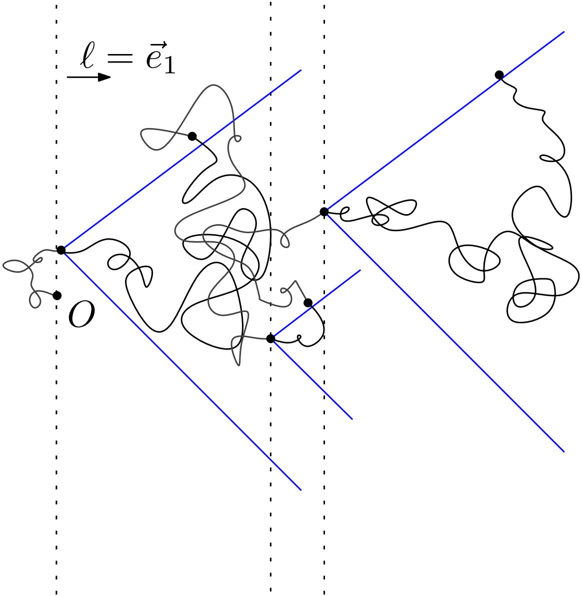

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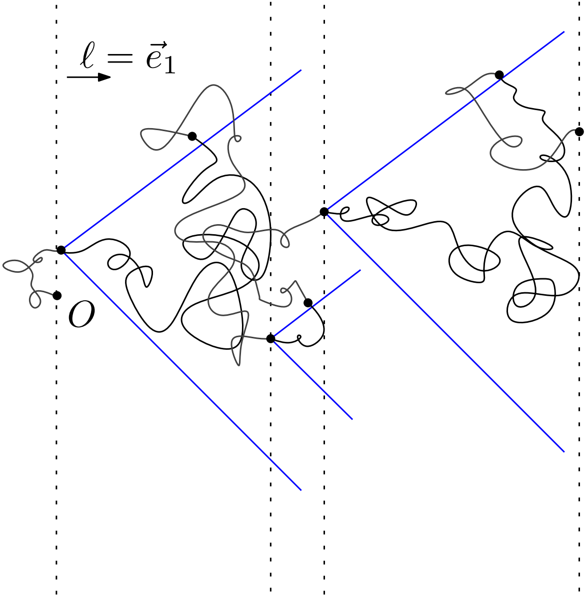

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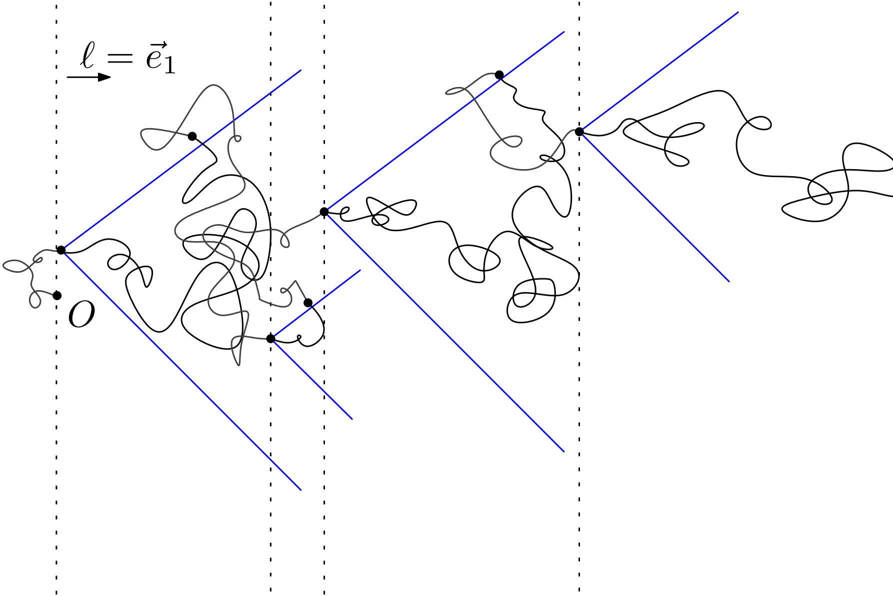

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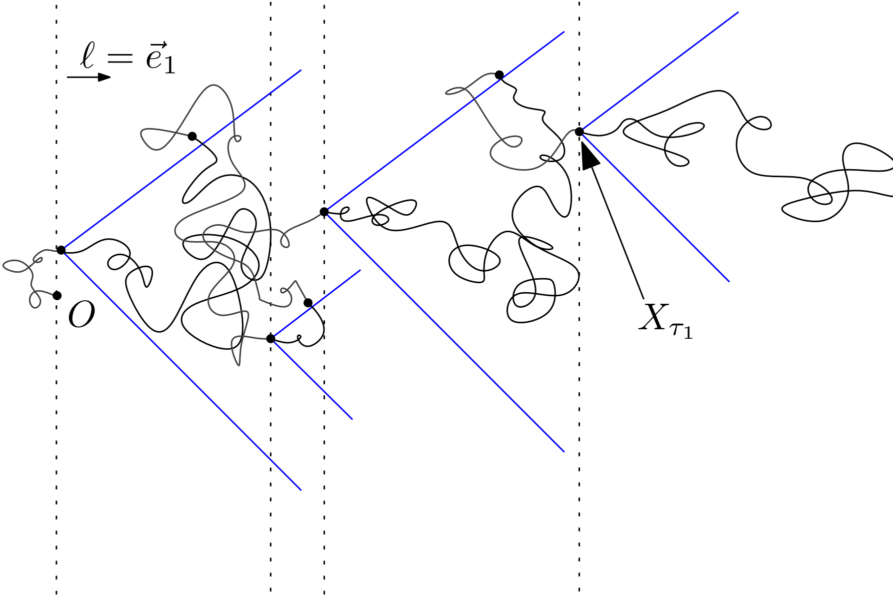
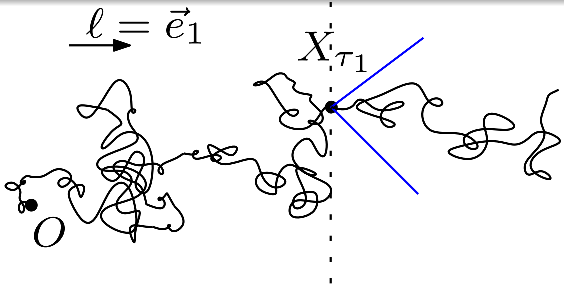


Asymptotic direction (renewal structure)



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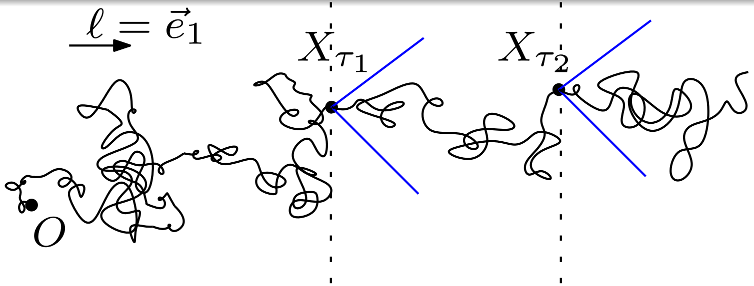

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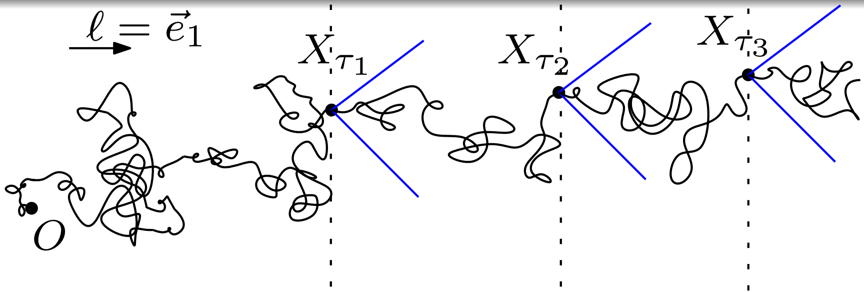

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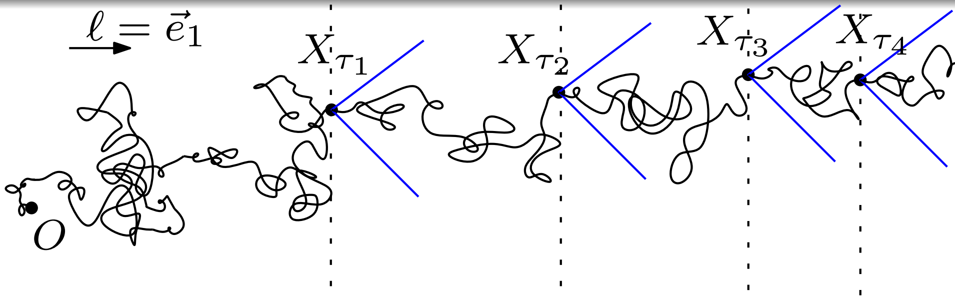

Expected density of points in $\{X_{\tau_k} \cdot \vec{e}_1 : k \geq 1\} \subset \mathbb{N}$:

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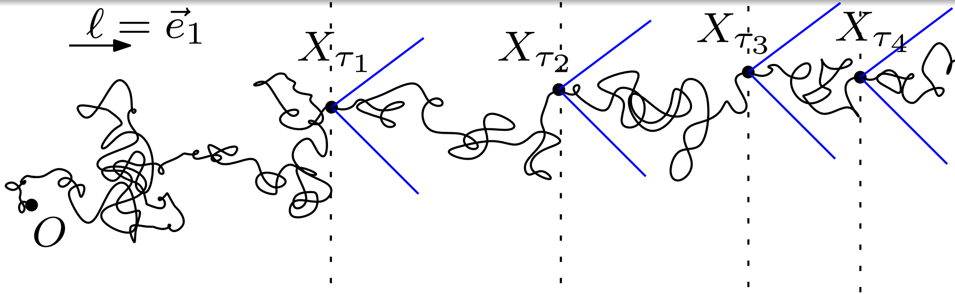

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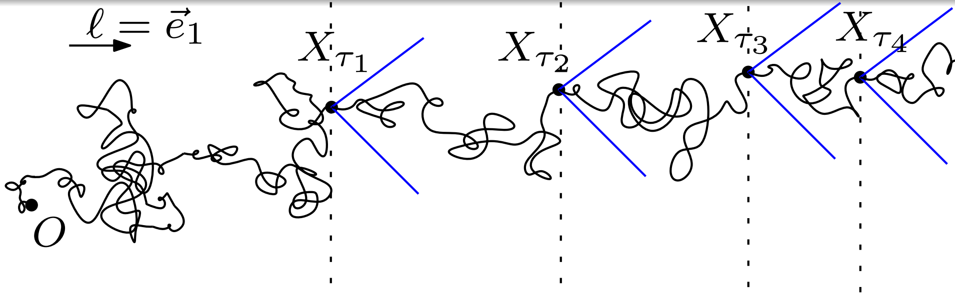

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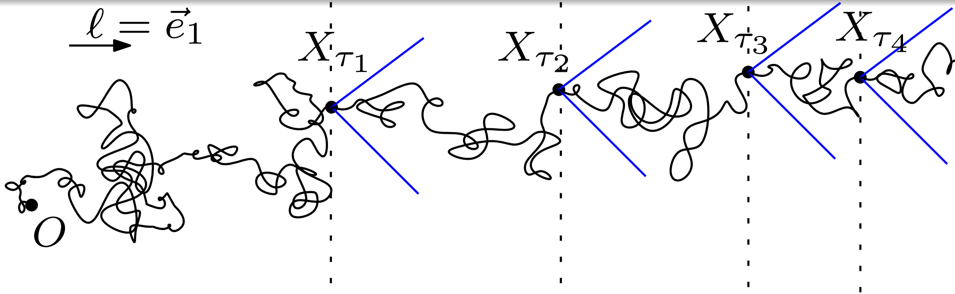

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Law of large numbers: $\frac{X_{\tau_k}}{k} \to v \neq 0$, hence $\frac{X_{\tau_k}}{\|X_{\tau_k}\|} \to \frac{v}{\|v\|}$. And $\frac{X_n}{\|X_n\|} \to \frac{v}{\|v\|}$.

Conclusion – Open questions

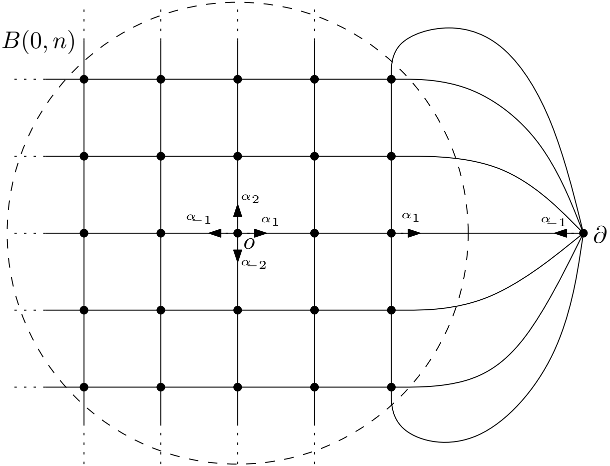
- ($d \geq 2$, $\kappa \leq 1$) Limit law? ($\|X_n\| \simeq n^\kappa$)
- ($d \geq 2$, $\kappa > 1$) Central limit theorem?
- ($d = 2$, $\vec{\Delta} \neq 0$) Optimal criterion of ballisticity? ($\kappa > 1$?)
- ($d = 2$, $\vec{\Delta} = 0$) Recurrence?

Transience if $d \geq 3$

Theorem (Sabot)

If $d \geq 3$,

$$|X_n| \to +\infty \quad P^{(\alpha)}\text{-a.s.}$$



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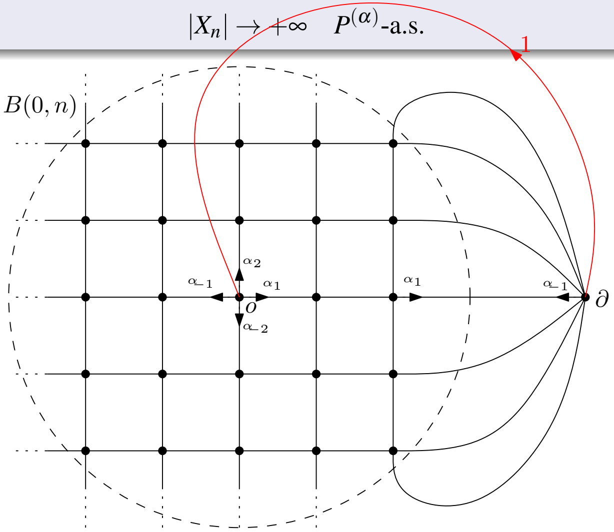

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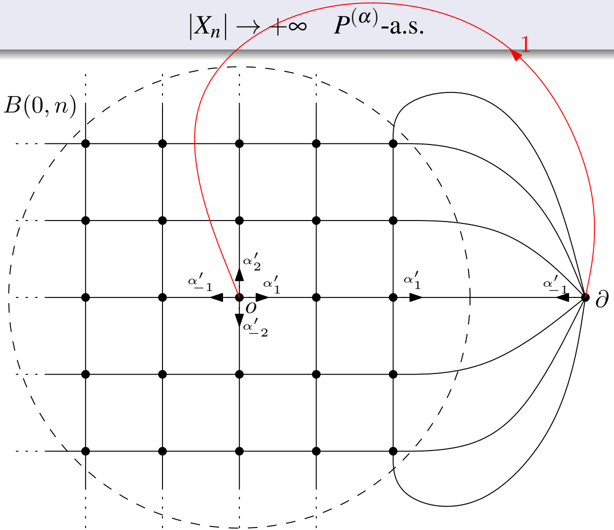

Transience if $d \geq 3$

0-1 law for transience

For any random walk in elliptic random environment on $\mathbb{Z}^d$,

$$P(|X_n| \to \infty) \in \{0, 1\}.$$

Proof: ergodicity.

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