Phase transition in 3-speed ballistic annihilation

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Joint work with John HASLEGRAVE and Vladas SIDORAVICIUS

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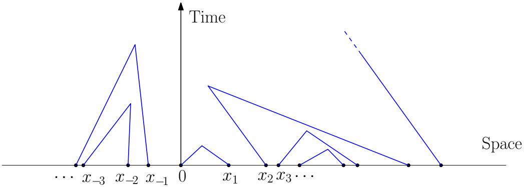
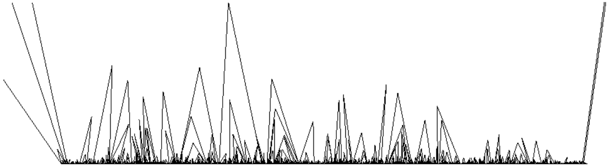
Plan of the talk – Part 1

Introduction – Ballistic annihilation

Ballistic annihilation model, from physics literature (1990's)

Let $V_n,\ n \in \mathbb{Z}$ be i.i.d. random variables, with distribution μ on $\mathbb{R}$.
From each location x_n, $n \in \mathbb{Z}$, of a point process on $\mathbb{R}$, a particle starts moving at constant speed V_n. When two particles collide, they annihilate.

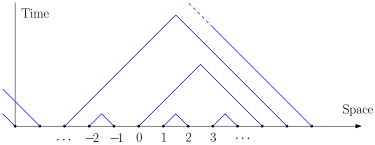




→ Speed of decay of density of particles ?
→ If μ has atoms, are there surviving particles ?

Two-speed model

From each integer, a particle is released with random speed ± 1. They annihilate upon collision.



Simple combinatorics. Density $c(t) = \mathbb{P}(\text{return time of SRW} > 2t) \sim ct^{-1/2}$
Description of "flocks of particles" : Belitzky–Ferrari '95

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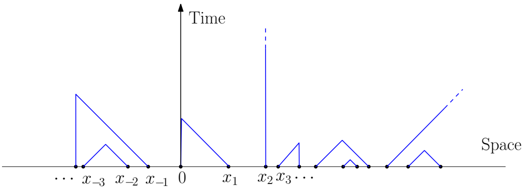
Three-speed model (Ben-Naim–Redner–Leyvraz '93, Piasecki '95)

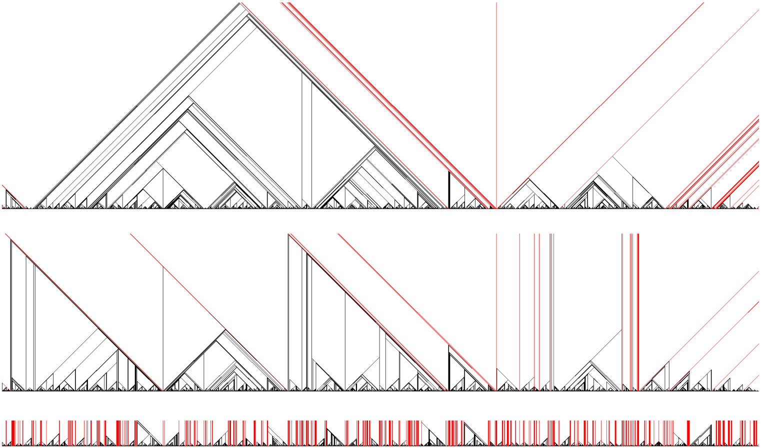
From each location of a Poisson point process, a particle starts with random speed among $-1, 0, +1$, with symmetric distribution. Annihilation upon collision.



Combinatorics become *very intricate* : no simple rule to check survival, long range dependences in both directions, dependence in interdistances, no monotonicity...

Velocities are sampled according to $\mu = \frac{1-p}{2}\delta_{-1} + p\delta_0 + \frac{1-p}{2}\delta_{+1}$.

Simulations for $p = 0.24$, $p = 0.25$, $p = 0.26$:



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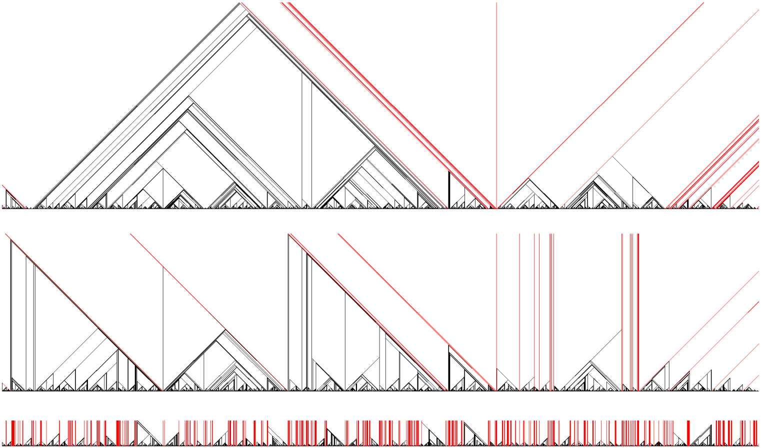


Transition at $p_c = \frac{1}{4}$ “computed” by Piasecki *et al.* ’95, and asymptotics for densities.

Plan of the talk – Part 2

Results

Consider the ballistic annihilation model, where

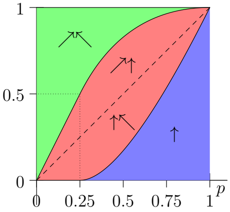
- interdistances have an atomless distribution,
- velocities are $\begin{cases} -1 & \text{with probability } (1-p)/2 \\ 0 & \text{with probability } p \\ +1 & \text{with probability } (1-p)/2. \end{cases}$

Define $\theta(p) = \mathbb{P}(\text{the particle at 0 survives indefinitely})$.

Theorem (Haslegrave-Sidoravicius-T. '18+)

The model undergoes a phase transition at $p_c = \frac{1}{4}$: $\theta(p) > 0 \Leftrightarrow p > 1/4$.
Moreover,

$$\text{for all } p > \frac{1}{4}, \qquad \theta(p) = (2\sqrt{p} - 1)^2.$$



Results – continued

Denote by $c_0(t)$ the density of stationary particles present at time t.
Denote by $c_+(t)$ the density of (+1)-particles present at time t.
Assume further that interdistances are exponentially integrable, with unit expectation.

Theorem (Haslegrave-Sidoravicius-T. '19+)

We have the following asymptotics, as $t \to \infty$: for some $c = c(p) > 0$,

$$c_0(t) = \begin{cases} \left(\frac{2p}{\pi(1-4p)} + o(1)\right) t^{-1} & \text{if } p < 1/4, \\ \left(\frac{2^{2/3}}{4\Gamma(2/3)^2} + o(1)\right) t^{-2/3} & \text{if } p = 1/4, \\ (2\sqrt{p}-1)^2 + o(e^{-ct}) & \text{if } p > 1/4, \end{cases}$$

and

$$c_+(t) = \begin{cases} \left(\frac{1}{\sqrt{\pi}}\sqrt{1-4p} + o(1)\right) t^{-1/2} & \text{if } p < 1/4, \\ \left(\frac{2^{2/3}}{8\Gamma(2/3)^2} + \frac{3}{8\Gamma(1/3)} + o(1)\right) t^{-2/3} & \text{if } p = 1/4, \\ o(e^{-ct}) & \text{if } p > 1/4. \end{cases}$$

Plan of the talk – Part 3

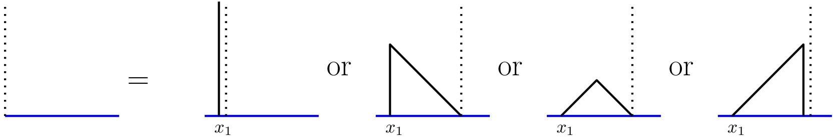
Proof – First remarks

Let us prove the first theorem : $p_c = \frac{1}{4}$ and $\theta(p) = (2\sqrt{p} - 1)_+^2$.

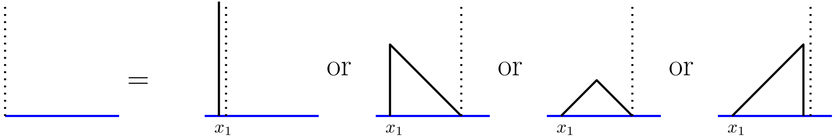
A few remarks :

- by symmetry and independence, it suffices to consider the system on $(0, +\infty)$ and to evaluate $q = \mathbb{P}_{(0,\infty)}(0 \leftarrow \overleftarrow{\bullet})$; then we have $\theta(p) = p(1-q)^2$.
- a (-1)-particle is never caught by a particle on its right. Therefore, for all $k \in \mathbb{N}$, the event $\mathbb{P}_{(0,\infty)}(0 \leftarrow \overleftarrow{\bullet}_k)$ only depends on the finite system of the first k particles.
- the distribution of the system is invariant under mirroring the piece of configuration between particles k and l (for any $k < l$)
- a $(+1)$-particle almost surely collides with another particle : if not, then (by ergodicity) almost surely infinitely many would survive forever in the process on $\mathbb{R}$; but by symmetry the same holds for (-1)-particles...

Proof – Identities in pictures



$$1 - q \quad = \quad p(1-q) \;+\; pq(1-q) \;+\; \alpha(1-q) \;+$$



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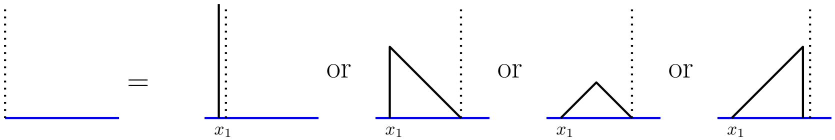

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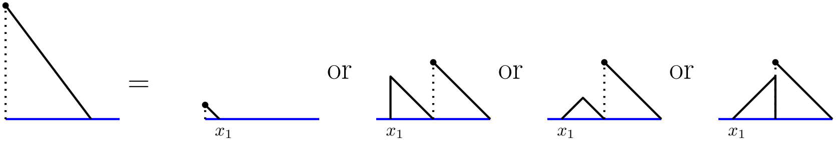

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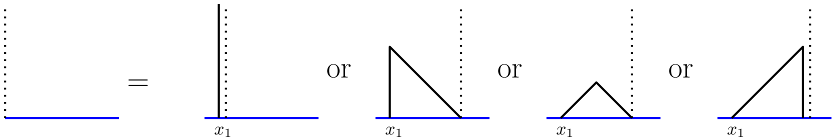

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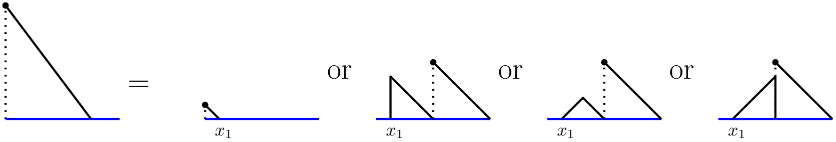

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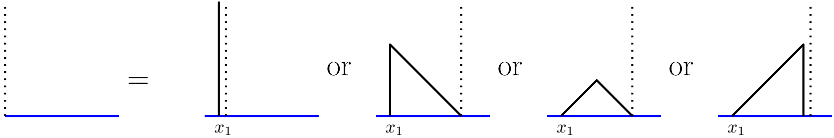

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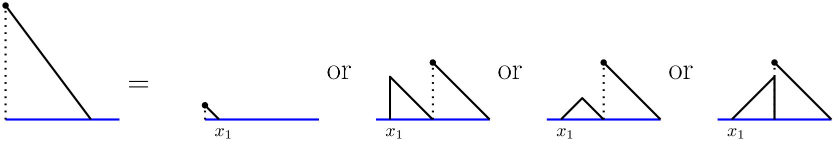


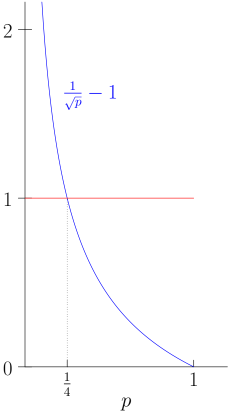
$$q \;=\; \frac{1-p}{2} + pq^2 + \alpha q + \frac{1}{2}pq^2$$

$\boxed{\text{If } q \neq 1}$, 1st equation gives α. Inject into 2nd, use $q \neq 1$. Get $\boxed{q = \frac{1}{\sqrt{p}} - 1}$.

Consequences

$\rightsquigarrow$ Either $q = 1$ or $q = \frac{1}{\sqrt{p}} - 1$.

If $p \leq \frac{1}{4}$, then necessarily $q = 1$. Also, clearly $q = 0$ at $p = 1$.



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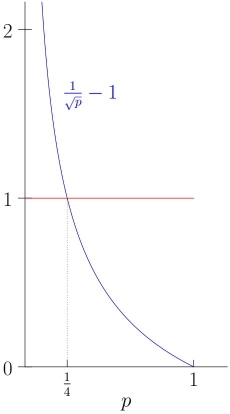

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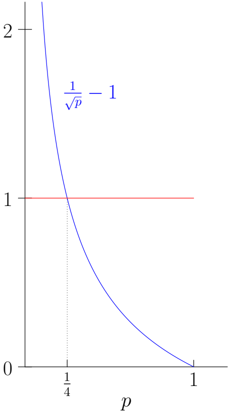

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$$\{p > \frac{1}{4} : q = 1\} = \{p > \frac{1}{4} : q > \frac{1}{\sqrt{p}} - 1\} \text{ is open}$$

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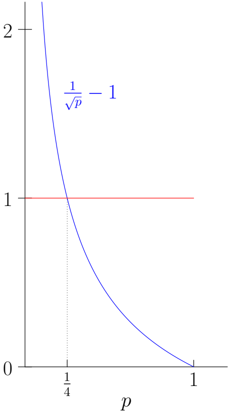

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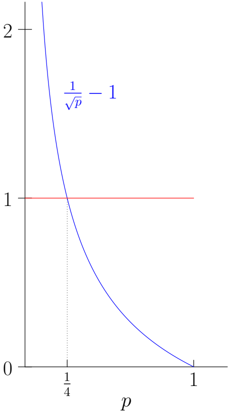

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- These together imply by connectedness that the supercritical phase covers the whole interval $(\frac{1}{4}, 1]$. QED.

Effective characterization of the survival phase

Explore the configuration from left to right, one particle at a time.

Count $+1$ (resp. -1) for each “fresh” $\uparrow$ (resp. $\nwarrow$)

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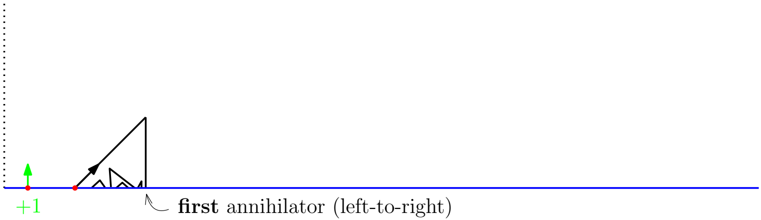

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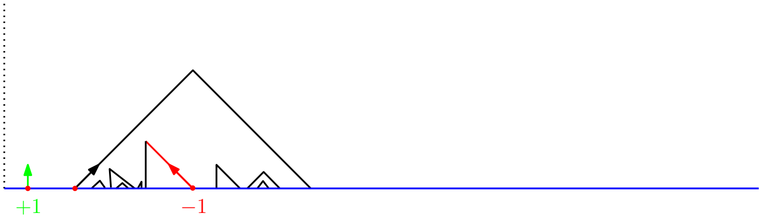

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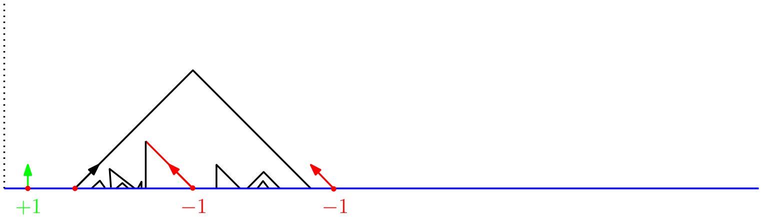

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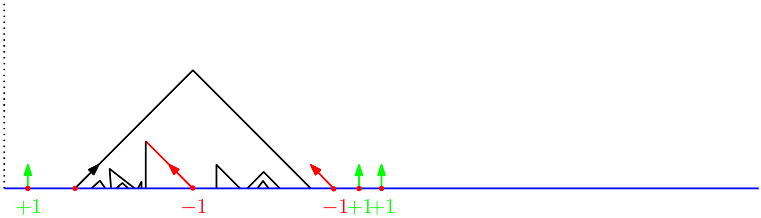

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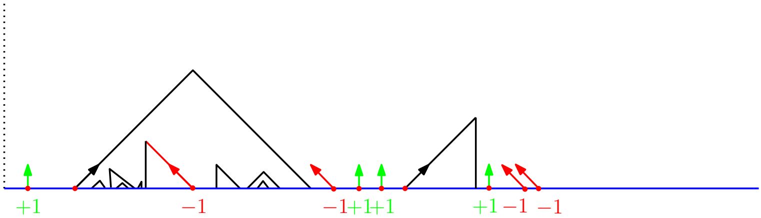

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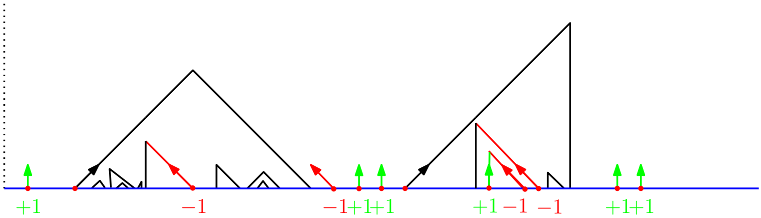
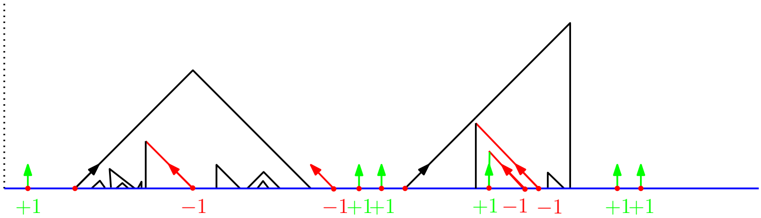


Explore the configuration from left to right, one particle at a time.

Count $+1$ (resp. -1) for each "fresh" ↑ (resp. ↖)



As long as ↑ outnumber ↖, 0 is not hit. Thus, $p > \frac{1}{3}$ implies $q < 1$.

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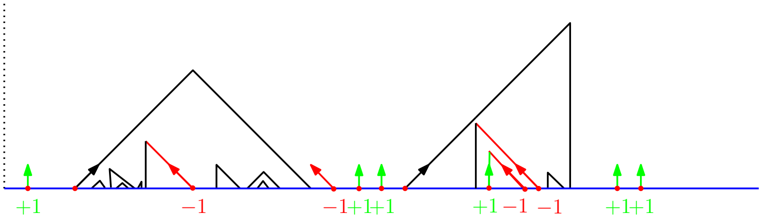

As long as $\uparrow$ outnumber $\nwarrow$, 0 is not hit. Thus, $p > \frac{1}{3}$ implies $q < 1$.
More generally,
Explore the configuration from left to right, k particles at a time.
First “resolve” the inner interactions of these k particles, then explore until the first “fresh” site in an analogous sense, and repeat.
As long as

$$\varphi_k = \mathbb{E}[\#(\text{surviving} \uparrow \text{in } k \text{ particles}) - \#(\text{surviving} \nwarrow \text{in } k \text{ particles})] > 0,$$

0 has positive chance not to be hit : $q < 1$.

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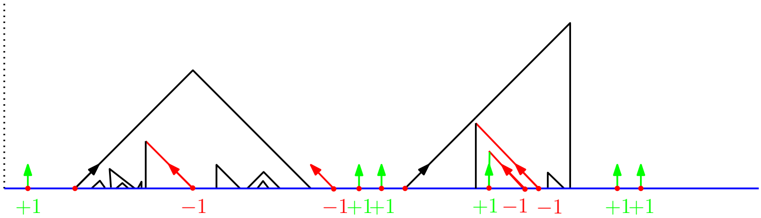

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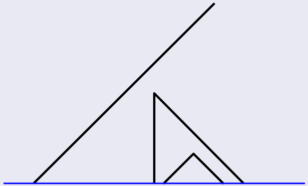
$\rightarrow$ In fact, this is equivalent : $q < 1 \Leftrightarrow \exists k,\ \varphi_k > 0$.

Plan of the talk – Part 4

Let us denote by m the distribution of interdistances in the initial configuration.

- The proof, hence the result, doesn't depend on m, besides being atomless
- Yet,

The model genuinely depends on m : not only probabilities of configurations can vary, but even some configurations are possible or not, depending on m.



(consider exponential distribution, vs. uniform distribution on $[1,2]$)
$\rightsquigarrow$ no possible coupling between models for different choices of m

$\rightarrow$ Even though the law of the pairing **does** depend on m, sub/supercriticality doesn't !

A universal distribution (given p)

A stronger universality property holds true.
Denote by A the **index** (in $\mathbb{N}$) of the first particle hitting 0 on $(0,\infty)$, if there is any, and let $A = \infty$ otherwise.

Theorem (Haslegrave-Sidoravicius-T. '19+)

The distribution of A does not depend on m (provided m is atomless).

It can even be "computed" : for $0 \le p \le 1$*, for* $x \in [-1,1]$*, the generating series*

$$f_p(x) = \mathbb{E}[x^A \mathbf{1}_{\{A<\infty\}}] = \sum_{n=1}^{\infty} \mathbb{P}(A=n)x^n$$

satisfies

$$pxf_p(x)^4 - (1+2p)xf_p(x)^2 + 2f_p(x) - (1-p)x = 0. \tag{1}$$

NB. Since $q = \mathbb{P}_{(0,\infty)}(A < \infty) = f_p(1)$ we recover a polynomial equation for q.

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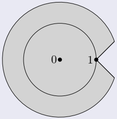
NB. Since $q = \mathbb{P}_{(0,\infty)}(A<\infty) = f_p(1)$ we recover a polynomial equation for q.

In particular, we can extend f analytically to the whole plane $\mathbb{C}$ except for slits ; and we can then find asymptotics for $\mathbb{P}(A=n)$ as $n\to\infty$ by **singularity analysis**.

Main theorem (from Flajolet & Sedgewick's *Analytic Combinatorics*)

Let f be a holomorphic function on the unit disk. Assume that 1 is the unique singularity of f on the unit circle. Assume furthermore that f can be extended analytically to a Δ-domain :



For $\alpha \in \mathbb{R} \setminus \{0,1,2,3,\ldots\}$, $C \in \mathbb{C}^*$,

$$f(z) - f(1) \underset{\substack{z\to 1\\ z\in\Delta}}{\sim} C(1-z)^{\alpha} \qquad \Rightarrow \qquad [z^n]f(z) \underset{n\to\infty}{\sim} \frac{C}{\Gamma(-\alpha)} n^{-(\alpha+1)}$$

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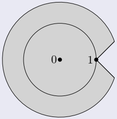

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$$f(z) - f(1) \underset{\substack{z \to 1 \\ z \in \Delta}}{\sim} C(1-z)^{\alpha} \qquad \Rightarrow \qquad [z^n]f(z) \underset{n \to \infty}{\sim} \frac{C}{\Gamma(-\alpha)} n^{-(\alpha+1)}$$

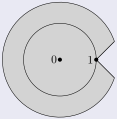
In our case,

- We need to find the singularities of f_p ;
- Existence of analytic extension follows standardly (monodromy theorem).

Singularity analysis

Main theorem (from Flajolet & Sedgewick's *Analytic Combinatorics*)

Let f be a holomorphic function on the unit disk. Assume that 1 is the unique singularity of f on the unit circle. Assume furthermore that f can be extended analytically to a Δ-domain :



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In our case,

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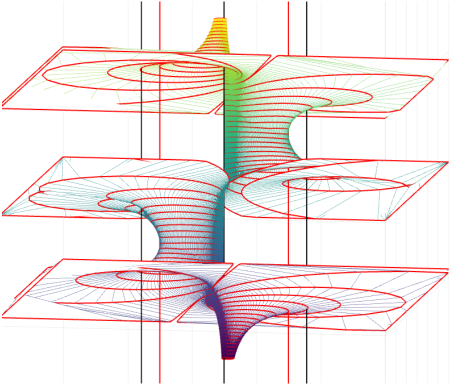
The implicit equation $F(z,f(z)) = 0$ actually defines a multivalued analytic function $\mathfrak{f}(z)$, or an analytic function on the algebraic Riemann surface $\{F(\cdot,\cdot) = 0\}$.
One easily finds singularities of $\mathfrak{f}$; but are they singularities of f ?

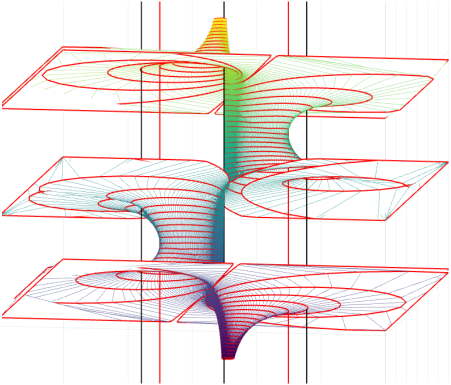
Looking for the singularities of f



$\mathfrak{f}$ has singularities at $(z,w) = (0,\infty)$, $(\pm 1, \pm 1)$ and $(\pm R, \pm W)$ where $R = \sqrt{\frac{3p}{1-p}}$.

- if $p < 1/4$, then $R < 1$: f is smooth at R. Thus ± 1 are singularities of f ;
- if $p > 1/4$, then $R > 1$, and $f(1) = q < 1$ (first theorem) so 1 is not a singularity of f. Thus $\pm R$ are singularities of f.

$\rightsquigarrow$ Standard computations then give the asymptotics for $\mathbb{P}(A = n)$.



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NB. Asymptotics for densities need an extra approximation (*indices* $\rightarrow$ *distances*).

Concluding remarks

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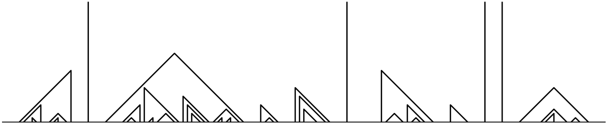

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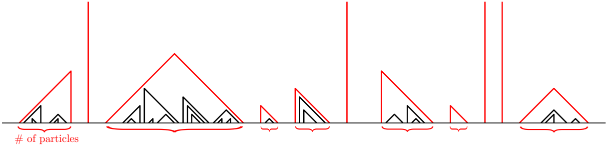

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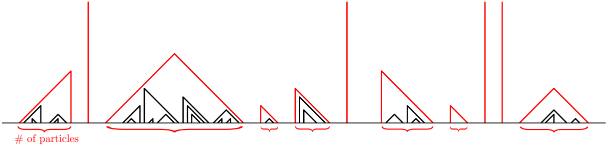

- For distributions with atoms, **triple collisions** may happen.
  - Assume they resolve by **total annihilation**. The arguments still go through, but identities involve $\sigma(p) = \mathbb{P}(\text{triple collision at } 0)$, apparently not explicit. If $m = \delta_1$, extinction holds for $p < 0.2347$ and survival for $p > 0.2405$.

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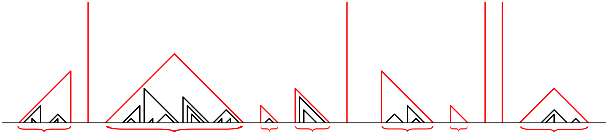

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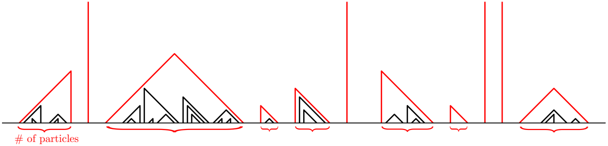

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  - Assume triple collisions resolve **uniformly at random among** ± 1. Then the model still observes universality, and in particular changes phase at $1/4$.
- If distribution of speed is not symmetric (but still takes 3 values), then some of the analysis carries over (with adaptations) but involves too many unknowns to get uniqueness of phase transition ; still gives extinction below $1/4$. Results due to Junge–Lyu '18.